# The large $c$ limit of $\det(\mathbf{A} + c\mathbf{B})$, where $\mathbf{B}$ is singular

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### Abstract

In the Miller Group meeting on 5 June 2008, Prof. Miller provided us with the following mathematical challenge:

What is the value of $\det(\mathbf{A} + c\mathbf{B})$ as $c \to \infty$, when $\mathbf{B}$ is a matrix with one zero eigenvalue?

The following is the solution I developed that afternoon.

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First, let's rewrite the determinant in an obvious way. For an $F$-dimensional system, we have:

$$\det\left(\mathbf{A} + c\mathbf{B}\right) = c^F \det\left(\frac{1}{c}\mathbf{A} + \mathbf{B}\right) \tag{1}$$

Now let's transform this into the basis in which the matrix $\mathbf{B}$ is diagonal. We will write the representations of $\mathbf{A}$ and $\mathbf{B}$ in this basis as $\mathbf{A}^{(B)}$ and $\mathbf{B}^{(B)}$, respectively. From the proposition, $\mathbf{B}^{(B)}$ is diagonal with one diagonal element with the value zero. We order the indices such that $\mathbf{B}_{11}^{(B)} = 0$.

Since the change of basis is a unitary transformation, the determinant is not changed by the transformation. So

$$\det\left(\mathbf{A} + c\mathbf{B}\right) = c^F \det\left(\frac{1}{c}\mathbf{A}^{(B)} + \mathbf{B}^{(B)}\right) \tag{2}$$

Let's define $\tilde{\mathbf{B}} = \frac{1}{c}\mathbf{A}^{(B)} + \mathbf{B}^{(B)}$. While $\mathbf{B}^{(B)}$ is diagonal, $\mathbf{A}^{(B)}$ is, in general, not diagonal. In the limit that $c \to \infty$, we have

$$\tilde{\mathbf{B}}_{ij} \to \begin{cases} \mathbf{B}_{ij}^{(B)} & \text{if } \mathbf{B}_{ij}^{(B)} \neq 0 \text{ (namely, for the nonzero diagonal elements of } \mathbf{B}^{(B)}), \\ \frac{1}{c}\mathbf{A}_{ij}^{(B)} & \text{otherwise} \end{cases} \tag{3}$$

Essentially, this means that the matrix $\tilde{\mathbf{B}}$ is the same as the matrix $\frac{1}{c}\mathbf{A}^{(B)}$, but replacing the diagonal elements (except 1,1) with the eigenvalues of $\mathbf{B}$.

Next, we recall the definition of the determinant (for an $F$-dimensional matrix):

$$\det\left(\mathbf{M}\right) = \sum_{\sigma \in S_F} s(\sigma) \prod_{k=1}^{F} \mathbf{M}_{\sigma(k)k} \tag{4}$$

where $S_F$ is the set of permutations, and $s(\sigma)$ is the signature of the permutation $\sigma$.

We'll define $\sigma_1(k)$ as the identity permutation: that is, $\sigma_1(k) = k$. We'll separate this permutation from the sum over the permutations in the definition of the determinant.

$$\det\left(\tilde{\mathbf{B}}\right) = \prod_{k=1}^{F} \tilde{\mathbf{B}}_{kk} + \sum_{\sigma \in \{S_F \setminus \sigma_1\}} s(\sigma) \prod_{k=1}^{F} \tilde{\mathbf{B}}_{\sigma(k)k} \tag{5}$$

$$= \frac{1}{c} \mathbf{A}_{11}^{(B)} \prod_{k=2}^{F} \mathbf{B}_{kk}^{(B)} + \mathcal{O}\left(\left(\frac{1}{c}\right)^2\right) \tag{6}$$

The trick is to recognize that all permutations except the identity permutation will contribute more than one factor of $\frac{1}{c}$, and therefore they will make vanishingly small contributions to the determinant. Putting all of this together, we have

$$\det\left(\mathbf{A} + c\mathbf{B}\right) \to c^{F-1} \mathbf{A}_{11}^{(B)} \prod_{k=2}^{F} \mathbf{B}_{kk}^{(B)} \tag{7}$$

Finally, let's compare this to our intuition. We expect to have a factor of $c^{F-1}$ since we expect a factor of $c$ to show up from each degree of freedom except the one where $\mathbf{B}$ has a zero eigenvalue. Also, this result immediately simplifies down to the expected answer if we look only at the subspace in which $\mathbf{B}$ has non-zero eigenvalues.

The idea developed here can be trivially generalized to any singular matrix $\mathbf{B}$ (except the zero matrix, where terms contributed by any permutation would have the same number of factors of $\frac{1}{c}$). It does require the additional condition that $\mathbf{A}_{11}^{(B)}$ be nonzero. If this is not the case, then the determinant will be of order $c^{F-2}$. In that situation, a similar analysis to the one outlined above should be possible, but it would require calculating terms from the perturbations whose products contribute two factors of $\frac{1}{c}$.