Probability and Discrete Probability Distributions

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Review of Probability Concepts

- Classical (theoretical) approach:

$$\frac{\text{No. Ways Event } A \text{ Can Occur}}{\text{Total Number of Events}}$$

process has to be known!

- Empirical approach (relative frequency):

$$\frac{\text{No. Times Result } A \text{ Occurred in the Experiment}}{\text{Total Number of Observations}}$$

- The relative frequency converges to the probability for a large number of experiments.

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Review of Probability Rules

1. A probability is a number between 0 and 1 assigned to an event that is the outcome of an experiment:
$$P[A] \in [0,1]$$
2. Complement of event A.
$$P[A] = 1 - P[\overline{A}]$$
3. If events A and B are mutually exclusive then $$P[A \text{ or } B] = P[A] + P[B]$$
$$P[A \text{ and } B] = 0$$

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Review of Probability Rules (cont'd)

4. If events $A_1, \ldots, A_N$ are mutually exclusive and collectively exhaustive then:
$$\sum_{i=1}^{N} P[A_i] = 1$$
5. If events A and B are not mutually exclusive then: $P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$
6. Conditional Probability:
$$P[A \mid B] = \frac{P[A \text{ and } B]}{P[B]} = \frac{P[B \mid A]P[A]}{P[B]}$$

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Review of Probability Rules (cont'd)

7. If events A and B are independent (i.e., P[A] = P[A|B] and P[B]=P[B|A]) then:

$$P[A \text{ and } B] = P[A] \times P[B]$$

8. If events A and B are not independent then

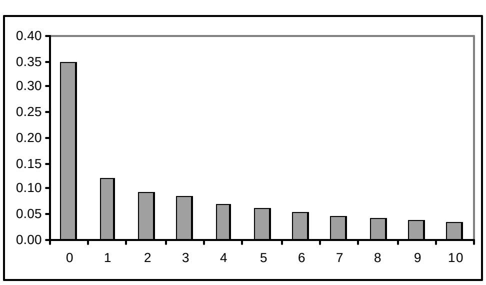
$$P[A \text{ and } B] = P[A \mid B]P[B] = P[B \mid A]P[A]$$

9. Theorem of Total Probability: if events A_1, …, A_N are mutually exclusive and collectively exhaustive then

$$P[B] = \sum_{i=1}^{N} P[B \mid A_i]P[A_i]$$

Discrete Probability Distribution

- Distribution: set of all possible values and their probabilities.



Number of I/Os per Transaction	Probability
0	0.350
1	0.120
2	0.095
3	0.085
4	0.070
5	0.060
6	0.054
7	0.048
8	0.043
9	0.040
10	0.035
	1.000

Moments of a Discrete Random Variable

- Expected Value:

$$m = E[X] = \sum_{\forall i} X_i \times P[X_i]$$

- k-th moment:

$$m = E[X^k] = \sum_{\forall i} X_i^k \times P[X_i]$$

Number of I/Os per Transaction	Probability	For First Moment (average)	For Second Moment
0	0.350	0.000	0.000
1	0.120	0.120	0.120
2	0.095	0.190	0.380
3	0.085	0.255	0.765
4	0.070	0.280	1.120
5	0.060	0.300	1.500
6	0.054	0.324	1.944
7	0.048	0.336	2.352
8	0.043	0.344	2.752
9	0.040	0.360	3.240
10	0.035	0.350	3.500
	1.000	**2.859**	**17.673**

mean

second moment

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Central Moments of a Discrete Random Variable

- k-th central moment:

$$E[(X - \overline{X})^k] = \sum_{\forall i} (X_i - \overline{X})^k \times P[X_i]$$

- The variance is the second central moment:

$$s^2 = E[(X - \overline{X})^2] = E[X^2 + (\overline{X})^2 - 2X\overline{X}]$$
$$= E[X^2] + (\overline{X})^2 - 2(\overline{X})^2 =$$
$$= E[X^2] - (\overline{X})^2$$

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Central Moments of a Discrete Random Variable

Number of I/Os per Transaction	Probability	For First Moment (average)	For Second Moment	For Second Central Moment
0	0.350	0.000	0.000	2.8609
1	0.120	0.120	0.120	0.4147
2	0.095	0.190	0.380	0.0701
3	0.085	0.255	0.765	0.0017
4	0.070	0.280	1.120	0.0911
5	0.060	0.300	1.500	0.2750
6	0.054	0.324	1.944	0.5328
7	0.048	0.336	2.352	0.8231
8	0.043	0.344	2.752	1.1365
9	0.040	0.360	3.240	1.5085
10	0.035	0.350	3.500	1.7848
	1.000	**2.859**	**17.673**	**9.4991**

average *variance*

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Properties of the Mean

- The mean of the sum is the sum of the means.

$$E[X+Y]=E[X]+E[Y]$$

- If X and Y are independent random variables, then the mean of the product is the product of the means.

$$E[XY]=E[X]E[Y]$$

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Discrete Random Variables

- Binomial
- Hypergeometric
- Negative Binomial
- Geometric
- Poisson

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The Binomial Distribution

- Distribution: based on carrying out independent experiments with two possible outcomes:
  - Success with probability p and
  - Failure with probability $(1\text{-}p)$.
- A binomial r.v. counts the number of successes in n trials.

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

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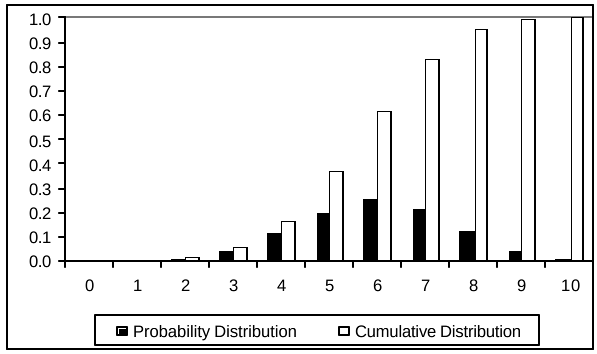
The Binomial Distribution

Success Probability **0.6 (p)**
Number of Attempts **10 (n)**

Number of Attempts (k)	Probability k successful attempts in n	Cumulative
0	0.000105	0.000105
1	0.001573	0.001678
2	0.010617	0.012295
3	0.042467	0.054762
4	0.111477	0.166239
5	0.200658	0.366897
6	0.250823	0.617719
7	0.214991	0.832710
8	0.120932	0.953643
9	0.040311	0.993953
10	0.006047	1.000000



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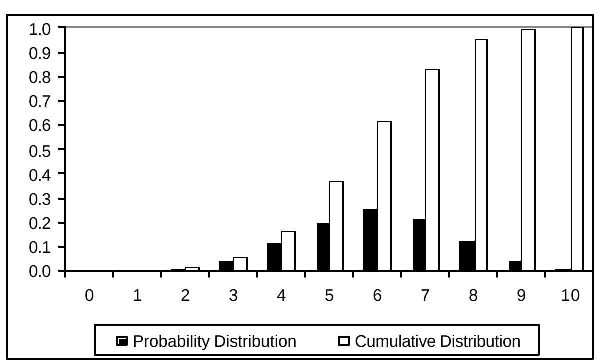
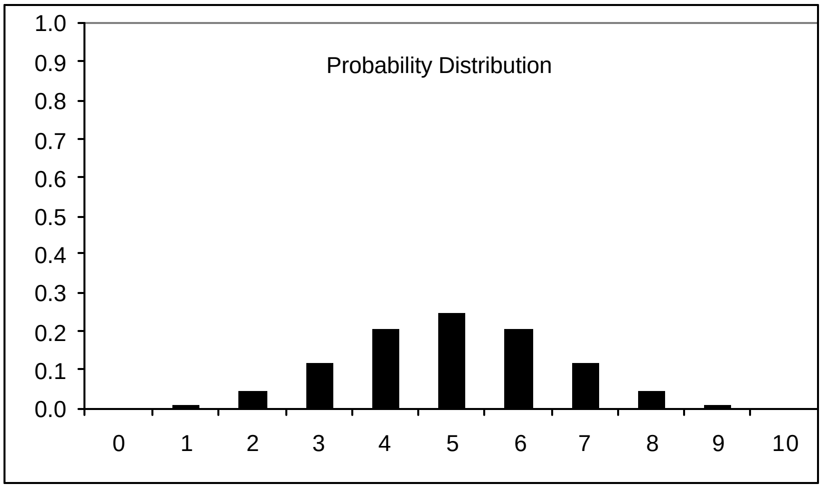
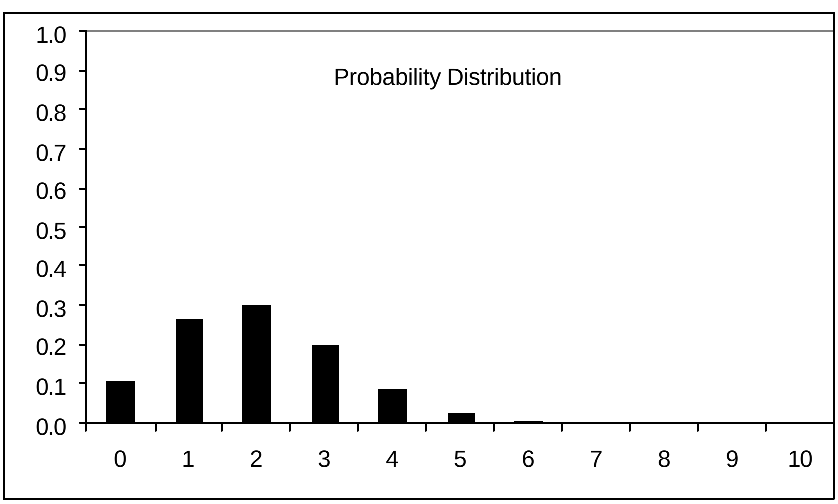


Shape of the Binomial Distribution



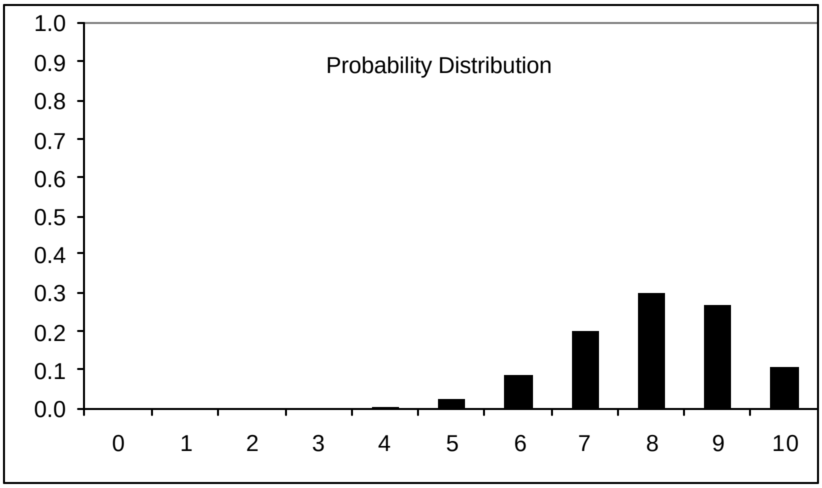
$p = 0.5$ symmetric for any n.

Shape of the Binomial Distribution



$p = 0.2$ right skewed

Shape of the Binomial Distribution



p = 0.8 left skewed

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Moments of the Binomial Distribution

- Average: $n\,p$
- Variance: $np(1-p)$
- Standard Deviation: $\sqrt{np(1-p)}$
- Coefficient of Variation:

$$\frac{\sqrt{np(1-p)}}{np} = \sqrt{\frac{1-p}{np}}$$

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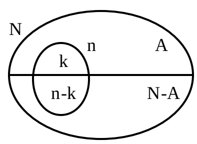
Hypergeometric Distribution

- Binomial was based on experiments with equal success probability.
- Hypergeometric: not all experiments have the same success probability.
- Given a sample size of *n* out of a population of size *N* with *A* known successes in the population, the probability of k successes is

choose k successes out of A successes in the population

choose (n-k) failures from N-A failures in the population



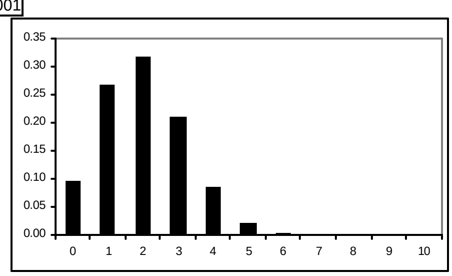
$$P[X=k]=\frac{\binom{A}{k}\binom{N-A}{n-k}}{\binom{N}{n}}$$

total # of possible samples

Hypergeometric Distribution

No. successes in sample k	sample size n	no. successes in population A	population size N	
0	20	10	100	0.09511627
1	20	10	100	0.26793316
2	20	10	100	0.31817063
3	20	10	100	0.20920809
4	20	10	100	0.08410730
5	20	10	100	0.02153147
6	20	10	100	0.00354136
7	20	10	100	0.00036793
8	20	10	100	0.00002300
9	20	10	100	0.00000078
10	20	10	100	0.00000001

In Excel:
Pr[X=k]=HYPGEOMDIST (k,n,A,N)



Moments of the Hypergeometric

- Average: $\frac{nA}{N}$
- Standard Deviation: $\sqrt{\frac{nA(N-A)}{N^2}}\sqrt{\frac{N-n}{N-1}}$

- If the sample size is less than 5% of the population, the binomial is a good approximation for the hypergeometric.

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Negative Binomial Distribution

- Probability of success is equal to *p* and is the same on all trials.
- Random variable X counts the number of trials until the *k*-th success is observed.

$$P[X = n] = \binom{n-1}{k-1}(1-p)^{n-k}\, p^k$$

S	F	F	S	. . .	F	S
1	2	3	4		n-1	n

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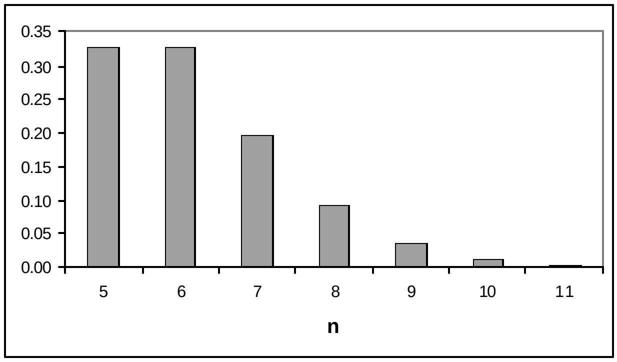
Negative Binomial Distribution

Success probability	0.8

k	n	Prob[X=n]
1	1	0.800000
1	2	0.160000
1	3	0.032000
1	4	0.006400
5	5	0.327680
5	6	0.327680
5	7	0.196608
5	8	0.091750
5	9	0.036700
5	10	0.013212
5	11	0.004404



In Excel:
Pr [X=n] = NEGBINOMDIST (n-k,k,p)

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Moments of the Negative Binomial Distribution

- Average: $\dfrac{k}{p}$
- Standard Deviation: $\sqrt{\dfrac{k(1-p)}{p^2}}$
- Coefficient of Variation: $\sqrt{\dfrac{1-p}{k}}$

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Geometric Distribution

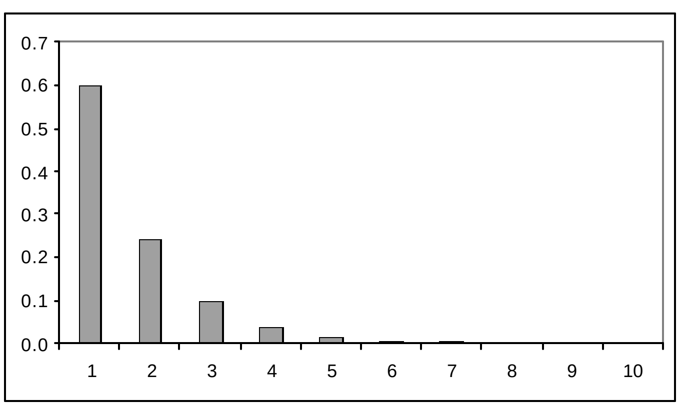
- Special case of the negative binomial with *k*=1.
- Probability that the first success occurs after *n* trials is

$$p[X = n] = p(1 - p)^{n-1} \qquad n = 1,2,\ldots$$

Geometric Distribution

Success probability	0.6

n	P[X=n]
1	0.6000
2	0.2400
3	0.0960
4	0.0384
5	0.0154
6	0.0061
7	0.0025
8	0.0010
9	0.0004
10	0.0002



Moments of the Geometric Distribution

- Average: $\dfrac{1}{p}$
- Standard Deviation: $\sqrt{\dfrac{1-p}{p^2}}$
- Coefficient of Variation: $\sqrt{1-p} \leq 1$

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Poisson Distribution

- Used to model the number of arrivals over a given interval, e.g.,
  - Number of requests to a server
  - Number of failures of a component
  - Number of queries to the database.
- A Poisson distribution usually arises when arrivals come from a large number of independent sources.

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Poisson Distribution

- Distribution: $P[X = k] = \frac{l^k e^{-l}}{k!} \quad k = 0,1,\ldots,\infty$

- Counting arrivals in an interval of duration *t*:

$$P[k \text{ arrivals in } [0, \text{t})] = \frac{(lt)^k e^{-lt}}{k!} \quad k = 0,1,\ldots,\infty$$

- Average=Variance=λ

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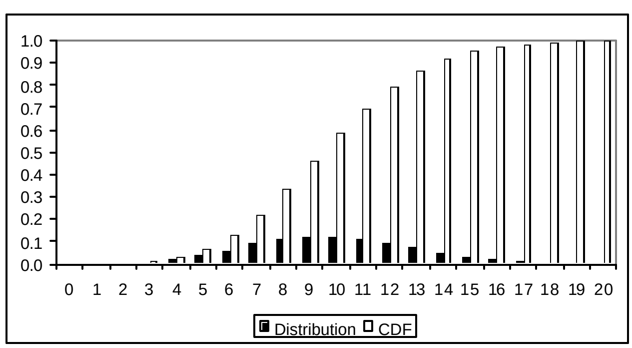
Poisson Distribution

Lambda	10	
K	Poisson Distribution	CDF
0	0.00005	0.0000
1	0.00045	0.0005
2	0.00227	0.0028
3	0.00757	0.0103
4	0.01892	0.0293
5	0.03783	0.0671
6	0.06306	0.1301
7	0.09008	0.2202
8	0.11260	0.3328
9	0.12511	0.4579
10	0.12511	0.5830
11	0.11374	0.6968
12	0.09478	0.7916
13	0.07291	0.8645
14	0.05208	0.9165
15	0.03472	0.9513
16	0.02170	0.9730
17	0.01276	0.9857
18	0.00709	0.9928
19	0.00373	0.9965
20	0.00187	0.9984



In Excel:
P[X=k] = POISSON (k,λ,FALSE)
P[X=k] = POISSON (k,λ,TRUE)

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