Design of Experiments: Factorial Designs

Prof. Daniel A. Menasce
Dept. of Computer Science
George Mason University

1

Basic Concepts

- Factorial design: more than one factor is studied simultaneously.

$$2^{k}$$

number of factors

number of levels of each factor

2^3 design: three factors, each with two levels. Total of 8 (2^3) combinations

2

Two-factor Design with Equal Number of Replicates (n')

		Factor B			
		1	2	...	c
Factor A	1	X111 X112 ... X11n'	X121 X122 ... X12n'	... ...	X1c1 X1c2 ... X1cn'
	2	X211 X212 ... X21n'	X221 X222 ... X22n'	... ...	X2c1 X2c2 ... X2cn'
	...	...	...	...	...
	r	Xr11 Xr12 ... Xr1n'	Xr21 Xr22 ... Xr2n'	... ...	Xrc1 Xrc2 ... Xrcn'

Notation

r : number of levels of factor A

c : number of levels of factor B

n' : number of replications for each cell

n : total number of observations (n = rcn')

X_{ijk} : k-th observation for level i of factor A
and level j of factor B

Means

$$\bar{\bar{X}} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n'} X_{ijk}}{rcn'}$$ (overall or grand mean)

$$\bar{X}_{i..} = \frac{\sum_{j=1}^{c}\sum_{k=1}^{n'} X_{ijk}}{cn'}$$ (mean of i-th level of factor A)

$$\bar{X}_{.j.} = \frac{\sum_{i=1}^{r}\sum_{k=1}^{n'} X_{ijk}}{rn'}$$ (mean of j-th level of factor B)

$$\bar{X}_{ij.} = \sum_{k=1}^{n'} \frac{X_{ijk}}{n'}$$ (mean of cell i,j)

5

Area for Grand Mean

$$\bar{\bar{X}} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n'} X_{ijk}}{rcn'}$$

		Factor B			
		1	**2**	**…**	**c**
Factor A	**1**	X111 X112 … X11n'	X121 X122 … X12n'	… … … …	X1c1 X1c2 … X1cn'
	2	X211 X212 … X21n'	X221 X222 … X22n'	… … … …	X2c1 X2c2 … X2cn'
	…	…	…	…	…
	r	Xr11 Xr12 … Xr1n'	Xr21 Xr22 … Xr2n'	… … … …	Xrc1 Xrc2 … Xrcn'

6

Area for Mean of a Level of Factor A

$$\bar{X}_{i..} = \frac{\sum_{j=1}^{c}\sum_{k=1}^{n'} X_{ijk}}{cn'}$$

i

		Factor B			
		1	**2**	**…**	**c**
Factor A	**1**	X111 X112 … X11n'	X121 X122 … X12n'	… … … …	X1c1 X1c2 … X1cn'
	2	X211 X212 … X21n'	X221 X222 … X22n'	… … … …	X2c1 X2c2 … X2cn'
	…	…	…	…	…
	r	Xr11 Xr12 … Xr1n'	Xr21 Xr22 … Xr2n'	… … … …	Xrc1 Xrc2 … Xrcn'

Area for Mean of a Level of Factor B

j

$$\bar{X}_{.j.} = \frac{\sum_{i=1}^{r}\sum_{k=1}^{n'} X_{ijk}}{rn'}$$

		Factor B			
		1	**2**	**…**	**c**
Factor A	**1**	X111 X112 … X11n'	X121 X122 … X12n'	… … … …	X1c1 X1c2 … X1cn'
	2	X211 X212 … X21n'	X221 X222 … X22n'	… … … …	X2c1 X2c2 … X2cn'
	…	…	…	…	…
	r	Xr11 Xr12 … Xr1n'	Xr21 Xr22 … Xr2n'	… … … …	Xrc1 Xrc2 … Xrcn'

Area for Mean of a Cell

$$\overline{X}_{ij.} = \sum_{k=1}^{n'} \frac{X_{ijk}}{n'}$$

		Factor B			
		1	**2**	**…**	**c**
Factor A	**1**	X111 X112 … X11n'	X121 X122 … X12n'	… … … …	X1c1 X1c2 … X1cn'
	2	X211 X212 … X21n'	X221 X222 … X22n'	… … … …	X2c1 X2c2 … X2cn'
	…	…	…	…	…
	r	Xr11 Xr12 … Xr1n'	Xr21 Xr22 … Xr2n'	… … … …	Xrc1 Xrc2 … Xrcn'

9

Partitioning the Variation

Total Variation (SST)
df = n - 1

- Factor A Variation (SSA)
  df = r - 1
- Factor B Variation (SSB)
  df = c - 1
- Interaction (SSAB)
  df = (r – 1)(c-1)
- Random Variation (SSE)
  df = rc(n’ – 1)

10

Partitioning the Variation

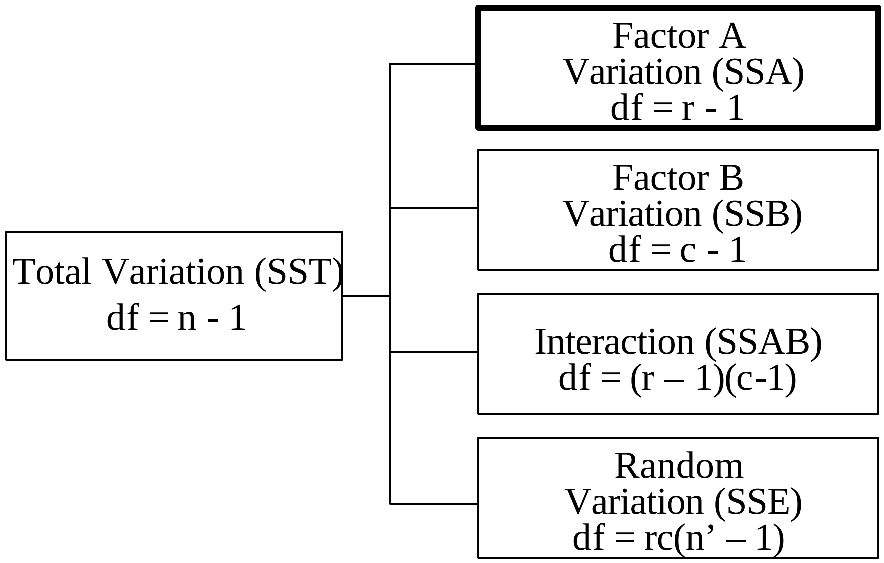
Total Variation (SST)
df = n - 1

- Factor A Variation (SSA)
  df = r - 1
- Factor B Variation (SSB)
  df = c - 1
- Interaction (SSAB)
  df = (r – 1)(c-1)
- Random Variation (SSE)
  df = rc(n’ – 1)

Total Variation (SST)

$$SST = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n'} \left(X_{ijk} - \overline{\overline{X}} \right)^2$$

Partitioning the Variation



13

Factor A Variation (SSA)

$$SSA = cn' \sum_{i=1}^{r} \left(\overline{X}_{i..} - \overline{\overline{X}} \right)^2$$

14

Partitioning the Variation

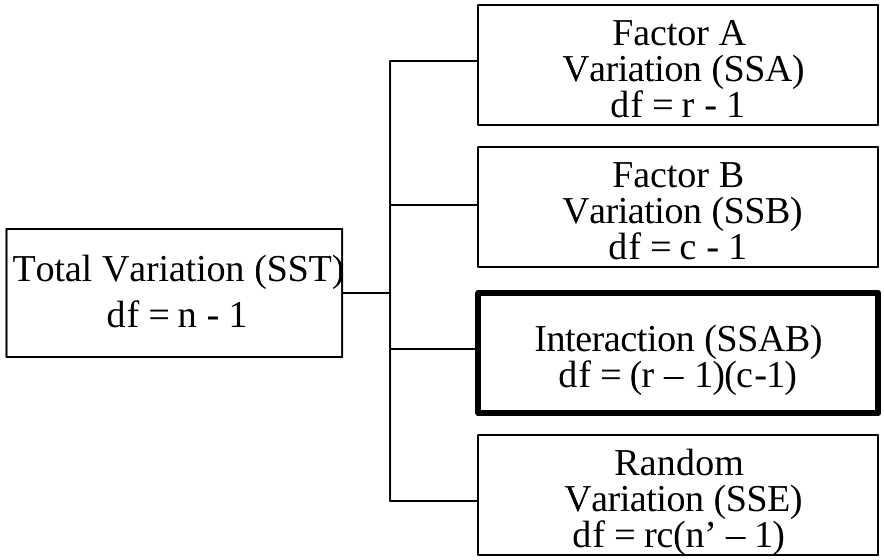
Total Variation (SST)
df = n - 1

- Factor A Variation (SSA)
  df = r - 1
- **Factor B Variation (SSB)**
  df = c - 1
- Interaction (SSAB)
  df = (r – 1)(c-1)
- Random Variation (SSE)
  df = rc(n’ – 1)

Factor B Variation (SSB)

$$SSB = rn' \sum_{j=1}^{c} \left(\bar{X}_{.j.} - \bar{\bar{X}} \right)^2$$

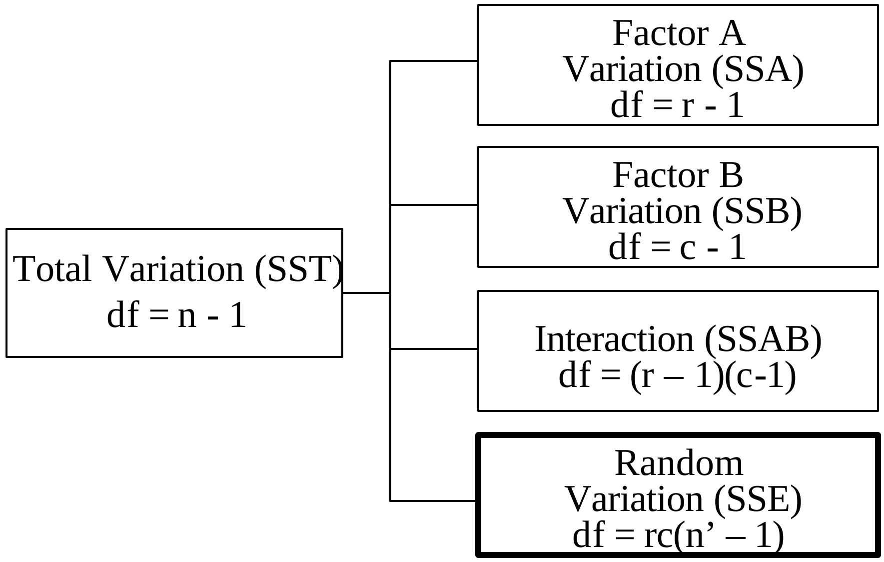
Partitioning the Variation



Variation due to Interaction (SSAB)

$$SSAB = n' \sum_{i=1}^{r} \sum_{j=1}^{c} \left(\overline{X}_{ij.} - \overline{X}_{i..} - \overline{X}_{.j.} + \overline{\overline{X}} \right)^2$$

Partitioning the Variation



Random Error (SSE)

$$SSE = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n'} \left(X_{ijk} - \overline{X}_{ij.} \right)^2$$

Mean Squares

$$MSA = \frac{SSA}{r-1}$$

$$MSB = \frac{SSB}{c-1}$$

$$MSAB = \frac{SSAB}{(r-1)(c-1)}$$

$$MSE = \frac{SSE}{rc(n^{'}-1)}$$

21

Two-Factor ANOVA Model No Difference Due to Factor A

$$H_0 : m_{1..} = m_{2..} = \ldots = m_{r..}$$

$$H_1 : \text{Not all } m_{i..} \ (i = 1, \ldots, r) \text{ are equal.}$$

F-Test statistic for Factor A: $F = \frac{MSA}{MSE}$

The F-test statistic follows an F distribution with (r-1) degrees of freedom in the numerator and rc(n’-1) in the denominator.

Reject H_0 if $F > Fu$

22

Two-Factor ANOVA Model
No Difference Due to Factor B

$$H_0 : m_{.1.} m_{.2.} = \ldots = m_{.c.}$$

$$H_1 : \text{Not all} \, m_{.j.} \; (j = 1, \ldots, c) \text{ are equal.}$$

F-Test statistic for Factor B: $F = \dfrac{MSB}{MSE}$

The F-test statistic follows an F distribution with (c-1) degrees of freedom in the numerator and rc(n’-1) in the denominator.

Reject H_0 if $F > Fu$

Two-Factor ANOVA Model
No Interaction of Factors A and B

$$H_0 : \text{the interaction of A and B is 0.}$$

$$H_1 : \text{the interaction of A and B} \neq 0.$$

F-Test statistic for the interaction: $F = \dfrac{MSAB}{MSE}$

The F-test statistic follows an F distribution with (r-1)(c-1) degrees of freedom in the numerator and rc(n’-1) in the denominator.

Reject H_0 if $F > Fu$

Example of Two Factor Design Analysis

Response time (in msec) of a Web Site.

	1 CPU	2 CPUs
1 Server	101.0	98.0
1 Server	103.0	97.5
1 Server	102.4	99.3
1 Server	104.0	100.0
2 Servers	43.0	41.0
2 Servers	46.0	44.0
2 Servers	45.0	42.0
2 Servers	49.0	46.0

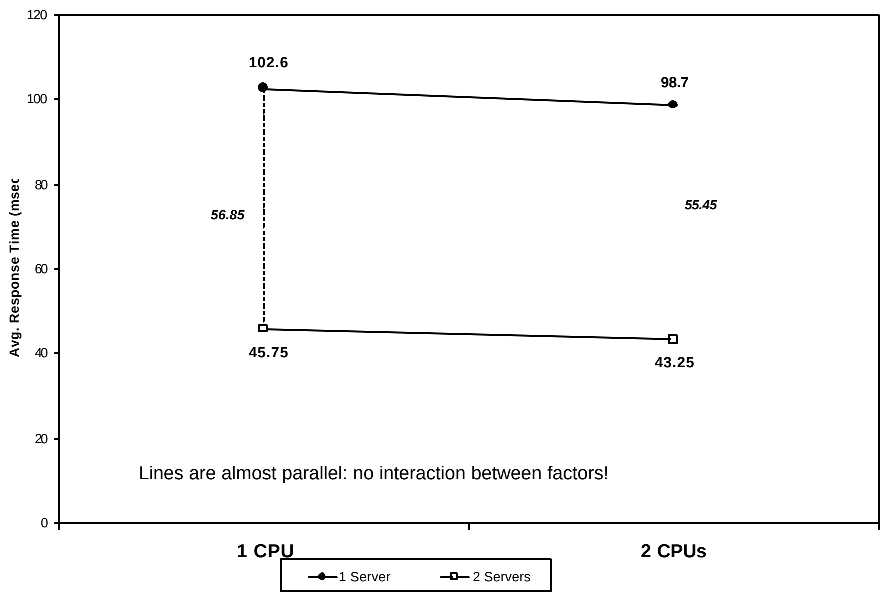
Anova: Two-Factor With Replication

SUMMARY	1 CPU	2 CPUs	Total
1 Server			
Count	4	4	8
Sum	410.4	394.8	805.2
Average	102.6	98.7	100.65
Variance	1.573333	1.326667	5.588571
2 Servers			
Count	4	4	8
Sum	183	173	356
Average	45.75	43.25	44.5
Variance	6.25	4.916667	6.571429
Total			
Count	8	8	
Sum	593.4	567.8	
Average	74.175	70.975	
Variance	926.7593	881.1621	

ANOVA

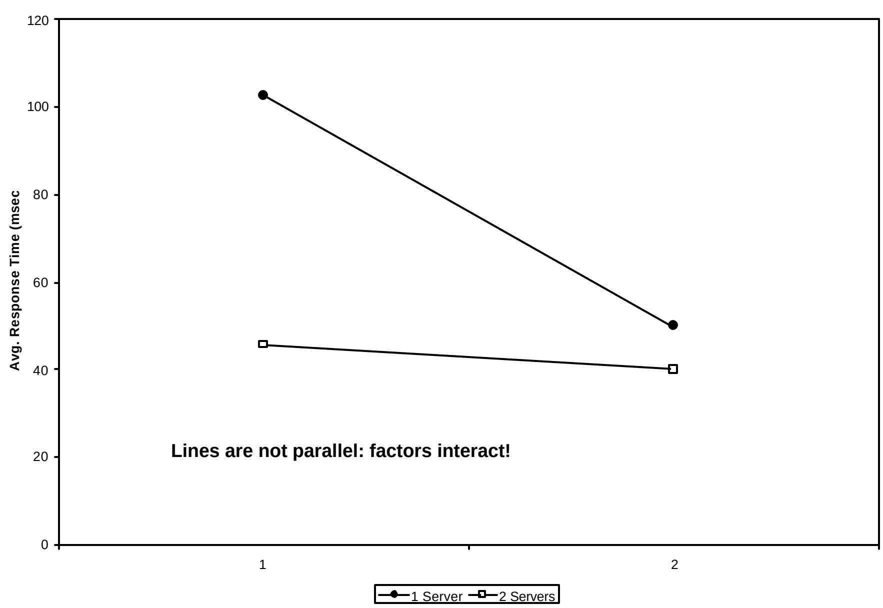
Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Number of Servers	12611.29	1	12611.29	3586.149	3.11E-16	4.747221
Number of CPUs	40.96	1	40.96	11.64739	0.005146	4.747221
Interaction	1.96	1	1.96	0.557346	0.469706	4.747221
Within	42.2	12	3.516667			
Total	12696.41	15				

Reject Hypothesis that there is no difference due to number of servers.
Reject Hypothesis that there is no difference due to number of CPUs.
Accept hypothesis that there is no interaction between number of servers and number of CPUs.



120
100
80
60
40
20
0
Avg. Response Time (msec
102.6
98.7
56.85
55.45
45.75
43.25
Lines are almost parallel: no interaction between factors!
1 CPU
2 CPUs
1 Server
2 Servers

27



120
100
80
60
40
20
0
Avg. Response Time (msec
Lines are not parallel: factors interact!
1
2
1 Server
2 Servers

28