# 1. Constitutive models of materials

1.1. **Starin measures.** Assume cell wall is a smooth surface $S \subset \mathbf{R}^3$, $\boldsymbol{n}$ is the exterior normal vector to $S$. Define the deformation is $\boldsymbol{u} \in \mathbf{H}^1(S)$, its gradient is denoted by $\mathbb{F} := \nabla \boldsymbol{u}$. **Green-Lagrangian strain tensor** is defined by

$$\mathbb{E} = \frac{1}{2}\left(\mathbb{F}^\top \mathbb{F} - \mathbb{I}\right).$$

The Green strain succeeds in discarding the rotational degrees of freedom, which have no bearing on the serverity of deformation, and retains the stretch/shear information in the 6-DOF symmetric factor. We can construct a **linear** approximation by forming a Taylor expansion around the undeformed configuration $\mathbb{F} = \mathbb{I}$.

$$\mathbb{E}(\mathbb{F}) \approx \mathbb{E}(\mathbb{I}) + \frac{\partial \mathbb{E}}{\partial \mathbb{F}}\Big|_{\mathbb{F}=\mathbb{I}} : (\mathbb{F} - \mathbb{I}) = \frac{\partial \mathbb{E}}{\partial \mathbb{F}}\Big|_{\mathbb{F}=\mathbb{I}} : (\mathbb{F} - \mathbb{I}),$$

where $:$ is the double dot product defined by $\mathbb{A} : \mathbb{B} = \mathrm{tr}(\mathbb{A}^\top \mathbb{B})$. The derivative $\partial \mathbb{E}/\partial \mathbb{F}$ is most conveniently defined via the differetial $\delta \mathbb{E}$ :

$$\frac{\partial \mathbb{E}}{\partial \mathbb{F}} : \delta \mathbb{F} = \delta \mathbb{E} = \frac{1}{2}\left(\delta \mathbb{F}^\top \mathbb{F} + \mathbb{F}^\top \delta \mathbb{F}\right).$$

Thus

$$\frac{\partial \mathbb{E}}{\partial \mathbb{F}}\Big|_{\mathbb{F}=\mathbb{I}} : (\mathbb{F} - \mathbb{I}) = \frac{1}{2}\left[(\mathbb{F} - \mathbb{I})^\top \mathbb{I} + \mathbb{I}^\top (\mathbb{F} - \mathbb{I})\right] = \frac{1}{2}\left(\mathbb{F}^\top + \mathbb{F}\right) - \mathbb{I}.$$

The matrix resulting from this linear approximation of $\mathbb{E}(\mathbb{F})$ is denoted by $\boldsymbol{\epsilon}$, where:

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbb{F}^\top + \mathbb{F}\right) - \mathbb{I}$$

and called the **small strain tensor**, or the **infinitesimal strain tensor**.

1.2. **linear elasticity.** The simplest practical constitutive model is **linear elasticity**, defined in terms of the strain energy density as:

$$\Psi(\mathbb{F}) = \mu \boldsymbol{\epsilon} : \boldsymbol{\epsilon} + \frac{\lambda}{2}\mathrm{tr}^2(\boldsymbol{\epsilon}),$$

where $\boldsymbol{\epsilon}$ is the small strain tensor, and $\mu, \lambda$ are the **Lamé coefficients**, which are related to the material properties of **Young's modulus** $k$ (a measure of stretch resistance) and **Poisson's ratio** $\nu$ (a measure of incompressibility) as:

$$\mu = \frac{k}{2(1+\nu)} \qquad \lambda = \frac{k\nu}{(1+\nu)(1-2\nu)}.$$

The relation between the **first Piola stress** $\mathbb{P}$ and $\mathbb{F}$ can be derivaed as follows:

$$\mathbb{P} = \frac{\delta \Psi}{\delta \mathbb{F}} = 2\mu \boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathbb{I},$$

or, after one final substitution for $\boldsymbol{\epsilon}$ (and a few algebraic reductions):

$$\mathbb{P}(\mathbb{F}) = \mu\left(\mathbb{F} + \mathbb{F}^\top - 2\mathbb{I}\right) + \lambda \mathrm{tr}(\mathbb{F} - \mathbb{I})\mathbb{I}.$$

In the Voigt matrix form, the above relation can be written as

$$\begin{bmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_3 \\ \mathbb{P}_4 \\ \mathbb{P}_5 \\ \mathbb{P}_6 \end{bmatrix}$$

The motified St. Venant-Kirchhoff model is given by

$$\mathcal{E} = \int_S \mathbb{E} : (\mathbb{C})\, \mathbb{E} + \int_S ((\nabla \boldsymbol{u} - P\mathbb{I})\boldsymbol{n})^2 \,, \tag{1.1}$$

where $\mathbb{C} = \mathbb{C}_g + \mathbb{C}_f$ $\mathbb{C}_g, \mathbb{C}_f$ respectively are the stiffness matrixes for the gel and fiber,

$$\mathbb{C}_g = Y_g \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}, \quad \mathbb{C}_f = \frac{\pi Y_f \rho_0}{16} \begin{pmatrix} 3 + \frac{\rho_2 + 4\rho_1}{\rho_0} & 1 - \frac{\rho_2}{\rho_0} & \frac{2\widetilde{\rho_1} + \widetilde{\rho_2}}{\rho_0} \\ 1 - \frac{\rho_2}{\rho_0} & 3 + \frac{\rho_2 - 4\rho_1}{\rho_0} & \frac{2\widetilde{\rho_1} - \widetilde{\rho_2}}{\rho_0} \\ \frac{2\widetilde{\rho_1} + \widetilde{\rho_2}}{\rho_0} & \frac{2\widetilde{\rho_1} - \widetilde{\rho_2}}{\rho_0} & 1 - \frac{\rho_2}{\rho_0} \end{pmatrix},$$

$P$ is the pressure per unit volume applied to the surface, $\nu, Y_g, Y_f$ are constants, $\rho(\theta)$ is $\pi$-periodic angular microfibril distribution function, $\{\widehat{\rho_n}(t)\}_{n=0}^2$ are the Fourier transform coefficients given by

$$\widehat{\rho_n} = \frac{1}{\pi} \int_0^\pi \rho(t,\theta) e^{-2in\theta} d\theta, \quad (\rho_n, \widetilde{\rho_n}) = 2\left(\mathrm{Re}\left(\widehat{\rho_h}\right), -\mathrm{Im}\left(\widehat{\rho_n}\right)\right).$$

$\phi(\theta)$ is $\pi$-periodic angular microtubule distribution function. We have the following evolution and equilibrium equation for angular microfibril and microtubul distribution.

$$\frac{d\rho(\theta)}{dt} = k_\rho \frac{\phi(\theta)}{\int_0^\pi \phi(\theta')d\theta'} - k'_\rho \rho(t,\theta), \tag{1.2}$$

$$\phi(\theta) = \frac{c_0 k_\phi e^{\gamma f(\mathbb{CE},\theta)}}{1 + k_\phi \int_0^\pi e^{\gamma f(\mathbb{CE},\theta')} d\theta'}, \quad f(\mathbb{CE},\theta) = \boldsymbol{e}_\theta^\top (\mathbb{CE})\, \boldsymbol{e}_\theta. \tag{1.3}$$

where $\mathbb{C} = \mathbb{C}_g + \mathbb{C}_f$, $c_0, \gamma, k_\phi, k_\rho, k'_\rho$ are suitable positive constants. By Fourier transform, we have

$$\frac{d\widehat{\rho_n}}{dt} = \frac{k_\rho}{\pi} \frac{\widehat{\phi_n}}{\phi_0} - k'_\rho \widehat{\rho_n}, \quad \widehat{\phi_n} = \frac{1}{\pi} \int_0^\pi \phi(\theta) e^{-2in\theta} d\theta, \tag{1.4}$$

$$\widehat{\phi_n} = \frac{c_0}{\pi} \frac{I_n\left(2\gamma \left|\widehat{f_1}\right|\right)}{e^{-\gamma f_0}/(\pi k_\phi) + I_0\left(2\gamma\left|\widehat{f_1}\right|\right)} e^{-2in\theta^*}, \quad \widehat{f_1} = \frac{1}{\pi}\int_0^\pi f(\mathbb{CE},\theta) e^{-2i\theta} d\theta, \tag{1.5}$$

where $\theta^*$ is the direction of the main stress, $I_n$ is the following modified Bessel function of the first kind.

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x\cos(\theta)} \cos(n\theta) d\theta \quad \forall x \in \mathbf{R},$$

## 2. Weak formulation

For a bounded surface domain $S \subset \mathbf{R}^3$, let the inner product on $L^2(S)$ be denoted by

$$(u, v)_S = \int_S uv, \quad \forall u, v \in L^2(S).$$

For $\Gamma \subset \partial S$, the sub-space of $H^1(S)$ with homogeneous boundary conditions on $\Gamma$ is denoted by

$$H^1_\Gamma(S) = \left\{ v \in H^1(S) : v = 0 \quad \text{on} \quad \Gamma \right\}.$$

Moreover, vector-valued and matrix-valed quantities will be denoted by boldface and hollow notations, respectively, such as $\boldsymbol{L}^2(S) = \left(L^2(S)\right)^3, \mathbb{L}^2(S) = \left(L^2(S)\right)^{3\times 3}$.

The weak formulation of problem