1. Constitutive models of materials

1.1. **Starin measures.** Assume cell wall is a smooth surface $S \subset \mathbf{R}^3$, $\boldsymbol{n}$ is the exterior normal vector to S. Define the deformation is $\boldsymbol{u} \in \mathbf{H}^1(S)$, its gradient is denoted by $\mathbb{F} := \nabla \boldsymbol{u}$. **Green-Lagrangian strain tensor** is defined by

$$\mathbb{E} = \frac{1}{2}\left(\mathbb{F}^\top \mathbb{F} - \mathbb{I}\right).$$

The Green strain succeeds in discarding the rotational degrees of freedom, which have no bearing on the serverity of deformation, and retains the stretch/shear information in the 6-DOF symmetric factor. We can construct a **linear** approximation by forming a Taylor expansion around the undeformed configuration $\mathbb{F} = \mathbb{I}$.

$$\mathbb{E}\left(\mathbb{F}\right) \approx \mathbb{E}\left(\mathbb{I}\right) + \frac{\partial \mathbb{E}}{\partial \mathbb{F}}\Big|_{\mathbb{F}=\mathbb{I}} : \left(\mathbb{F} - \mathbb{I}\right) = \frac{\partial \mathbb{E}}{\partial \mathbb{F}}\Big|_{\mathbb{F}=\mathbb{I}} : \left(\mathbb{F} - \mathbb{I}\right),$$

where $:$ is the double dot product defined by $\mathbb{A} : \mathbb{B} = \mathrm{tr}(\mathbb{A}^\top \mathbb{B})$. The derivative $\partial \mathbb{E}/\partial \mathbb{F}$ is most conveniently defined via the differetial $\delta \mathbb{E}$:

$$\frac{\partial \mathbb{E}}{\partial \mathbb{F}} : \delta \mathbb{F} = \delta \mathbb{E} = \frac{1}{2}\left(\delta \mathbb{F}^\top \mathbb{F} + \mathbb{F}^\top \delta \mathbb{F}\right).$$

Thus

$$\frac{\partial \mathbb{E}}{\partial \mathbb{F}}\Big|_{\mathbb{F}=\mathbb{I}} : \left(\mathbb{F} - \mathbb{I}\right) = \frac{1}{2}\left[\left(\mathbb{F} - \mathbb{I}\right)^\top \mathbb{I} + \mathbb{I}^\top \left(\mathbb{F} - \mathbb{I}\right)\right] = \frac{1}{2}\left(\mathbb{F}^\top + \mathbb{F}\right) - \mathbb{I}.$$

The matrix resulting from this linear approximation of $\mathbb{E}(\mathbb{F})$ is denoted by $\boldsymbol{\epsilon}$, where:

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbb{F}^\top + \mathbb{F}\right) - \mathbb{I}$$

and called the **small strain tensor**, or the **infinitesimal strain tensor**.

1.2. **linear elasticity.** The simplest practical constitutive model is **linear elasticity**, defined in terms of the strain energy density as:

$$\Psi\left(\mathbb{F}\right) = \mu \boldsymbol{\epsilon} : \boldsymbol{\epsilon} + \frac{\lambda}{2}\mathrm{tr}^2\left(\boldsymbol{\epsilon}\right),$$

where $\boldsymbol{\epsilon}$ is the small strain tensor, and μ, λ are the **Lamé coefficients**, which are related to the material properties of **Young's modulus** k (a measure of stretch resistance) and **Poisson's ratio** ν (a measure of incompressibility) as:

$$\mu = \frac{k}{2\left(1+\nu\right)} \quad \lambda = \frac{k\nu}{\left(1+\nu\right)\left(1-2\nu\right)}.$$

The relation between the **first Piola stress** $\mathbb{P}$ and $\mathbb{F}$ can be derivaed as follows:

$$\mathbb{P} = \frac{\delta \Psi}{\delta \mathbb{F}} = 2\mu \boldsymbol{\epsilon} + \lambda \mathrm{tr}\left(\boldsymbol{\epsilon}\right)\mathbb{I},$$

or, after one final substitution for $\boldsymbol{\epsilon}$ (and a few algebraic reductions):

$$\mathbb{P}\left(\mathbb{F}\right) = \mu\left(\mathbb{F} + \mathbb{F}^\top - 2\mathbb{I}\right) + \lambda \mathrm{tr}\left(\mathbb{F} - \mathbb{I}\right)\mathbb{I}.$$

In the Voigt matrix form, the above relation can be written as

$$\begin{bmatrix} \mathbb{P}_{11} \\ \mathbb{P}_{22} \\ \mathbb{P}_{33} \\ \mathbb{P}_{23} \\ \mathbb{P}_{13} \\ \mathbb{P}_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{11} \\ \boldsymbol{\epsilon}_{22} \\ \boldsymbol{\epsilon}_{33} \\ 2\boldsymbol{\epsilon}_{23} \\ 2\boldsymbol{\epsilon}_{13} \\ 2\boldsymbol{\epsilon}_{12} \end{bmatrix}$$

where the transition matrix in the above equality can written as:

$$\frac{k}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

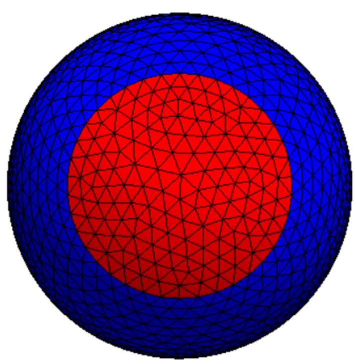
1.3. **The PDE form of linear elasticity.** Assume an externally applied force distribution $\boldsymbol{f}_{\text{ext}}$. When the object has settled to an equilibrium (rest) configuration, the deformation function will satisfy:

$$\text{div}\mathbb{P} = \boldsymbol{f}_{\text{ext}}, \tag{1.1}$$

which can lead to the following formulation:

$$-\mu\Delta\boldsymbol{u} - (\mu+\lambda)\nabla(\text{div}\boldsymbol{u}) = \boldsymbol{f}_{\text{ext}}. \tag{1.2}$$

Numerical experiments:



$$\nu = 0.5, k = \begin{cases} 1 & \text{in} \quad \text{CZ}, \\ 0.25 & \text{in} \quad \text{PZ}, \end{cases} \qquad \boldsymbol{f} = 0.2\boldsymbol{n}.$$