CS 251 Homework 1

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September 24, 2023

Problem 1.

Resources used: None

Collaborators: None

We need to prove that $log(n!) = \Theta(nlog(n))$

This can be done by showing that the upper bound and lower bound are both $\Theta(nlog(n))$

We can refactor the original expression to look like so:

$log(n!) = \Theta(log(n^n))$

because

$$\begin{aligned} log(n!) &= log(n(n-1)(n-2)...(1)) \\ &= log(n) + log(n-1) + log(n-2) + ... + log(1) \end{aligned}$$

We can see that the above expression is the same as the summation of log(n) from 1 to n.

Similarly,

Problem 2.

Resources used: None

Collaborators: None

a)

Base case: $F_13 = 233$, $F_14 = 377$

Inductive hypothesis: For all $n \geq 13$, $F_n \geq 2^{0.6n}$ and $F_{n-1} \geq 2^{0.6(n-1)}$

Inductive step:

$$\begin{aligned} F_{n+1} &= 2^{0.6(n+1)} \\ F_n + F_{n-1} &\geq F_{n+1} \end{aligned}$$

We can infer that the minimum value of FN and F n-1 is greater than or equal to the value of F n+1.

$$\begin{aligned} 2^{0.6n} + 2^{0.6(n-1)} &\geq 2^{0.6(n+1)} \\ 2^{0.6} + 1 &\geq 2^{0.6 \cdot 2} \\ 2.52... &\geq 2.3... \end{aligned}$$

We have proven the hypothesis with strong induction.

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b)

Base case: $F_0 = 0$, $F_1 = 1$

Inductive hypothesis: For all $n \geq 0$, $F_n \leq 2^{0.7n}$ and $F_{n-1} \leq 2^{0.7(n-1)}$

Inductive step:

$$\begin{aligned} F_{n+1} &= 2^{0.7(n+1)} \\ F_n + F_{n-1} &\leq F_{n+1} \end{aligned}$$

We can infer that the maximum value of FN and F n-1 is less than or equal to the value of F n+1.

$$\begin{aligned} 2^{0.7n} + 2^{0.7(n-1)} &\leq 2^{0.7(n+1)} \\ 2^{0.7} + 1 &\leq 2^{0.7\cdot 2} \\ 2.62... &\leq 2.63... \end{aligned}$$

We have proven the hypothesis with strong induction.

Problem 3.

Resources used: None

Collaborators: None

Base Case: 2, $T(2) = 2log_2(2) = 2$

Inductive hypothesis: For all $k > 0$, $T(n) = nlog(n)$ if $n = 2^k$

Inductive step:

$$\begin{aligned}
T(2^{k+1}) &= 2T(2^{k+1}/2) + 2^{k+1} \\
&= 2T(2^{k+1}/2^1) + 2^{k+1} \\
&= 2T(2^k) + 2^{k+1} \\
\text{Replace } T(2^k) \text{ with } 2^k log(2^k) & \\
&= 2(2^k log(2^k)) + 2^{k+1} \\
&= 2^{k+1} log(2^k) + 2^{k+1} \\
&= 2^{k+1}(log(2^k) + 1) \\
&= 2^{k+1} log(2^{k+1})
\end{aligned}$$

We have proven the hypothesis with induction.

Problem 4.

Resources used: None —- Collaborators: None

(a) 21 compares, 21 swaps.

Problem 4 Insertion sort.

Compare

Swap

7	6	5	4	3	2	1
6	7	5	4	3	2	1
6	5	7	4	3	2	1
5	6	7	4	3	2	1
5	6	4	7	3	2	1
5	4	6	7	3	2	1
4	5	6	7	3	2	1
4	5	6	3	7	2	1
4	5	3	6	7	2	1
4	3	5	6	7	2	1
3	4	5	6	7	2	1
3	4	5	6	2	7	1
3	4	5	2	6	7	1
3	4	2	5	6	7	1
3	2	4	5	6	7	1
2	3	4	5	6	7	1
2	3	4	5	6	1	7
2	3	4	5	1	6	7
2	3	4	1	5	6	7
2	3	1	4	5	6	7
2	1	3	4	5	6	7
1	2	3	4	5	6	7

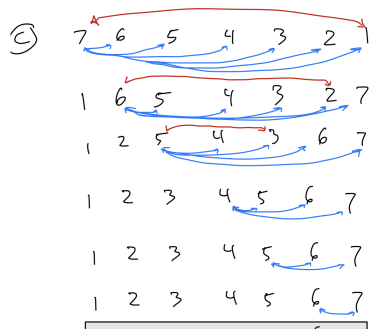
(b) The number of swaps and compars for Insertion sort on an array [n, n - 1, ..., 2, 1] would be as follows:

For first element, its compared to the previous, which results in 1 compare and one swap. The next one is compared to the previous two, which results in 2 compares and 2 swaps. This can be written with a summation.

$$\begin{aligned}\sum_{i=1}^{n-1} i &= \frac{n(n-1)}{2} \\ &= \frac{n^2-n}{2} \\ &= \frac{n^2}{2} - \frac{n}{2}\end{aligned}$$

For swaps it is the exact same thing, since every compare results in a swap.

(c) 21 compares, 3 swaps.



(d) Selection sort on an array [n, n - 1, ..., 2, 1] would have compares and swaps like this:

First run: Compares all elements from n to 1 to find the smallest one - n compares. Swaps the smallest element with the first element - 1 swap.

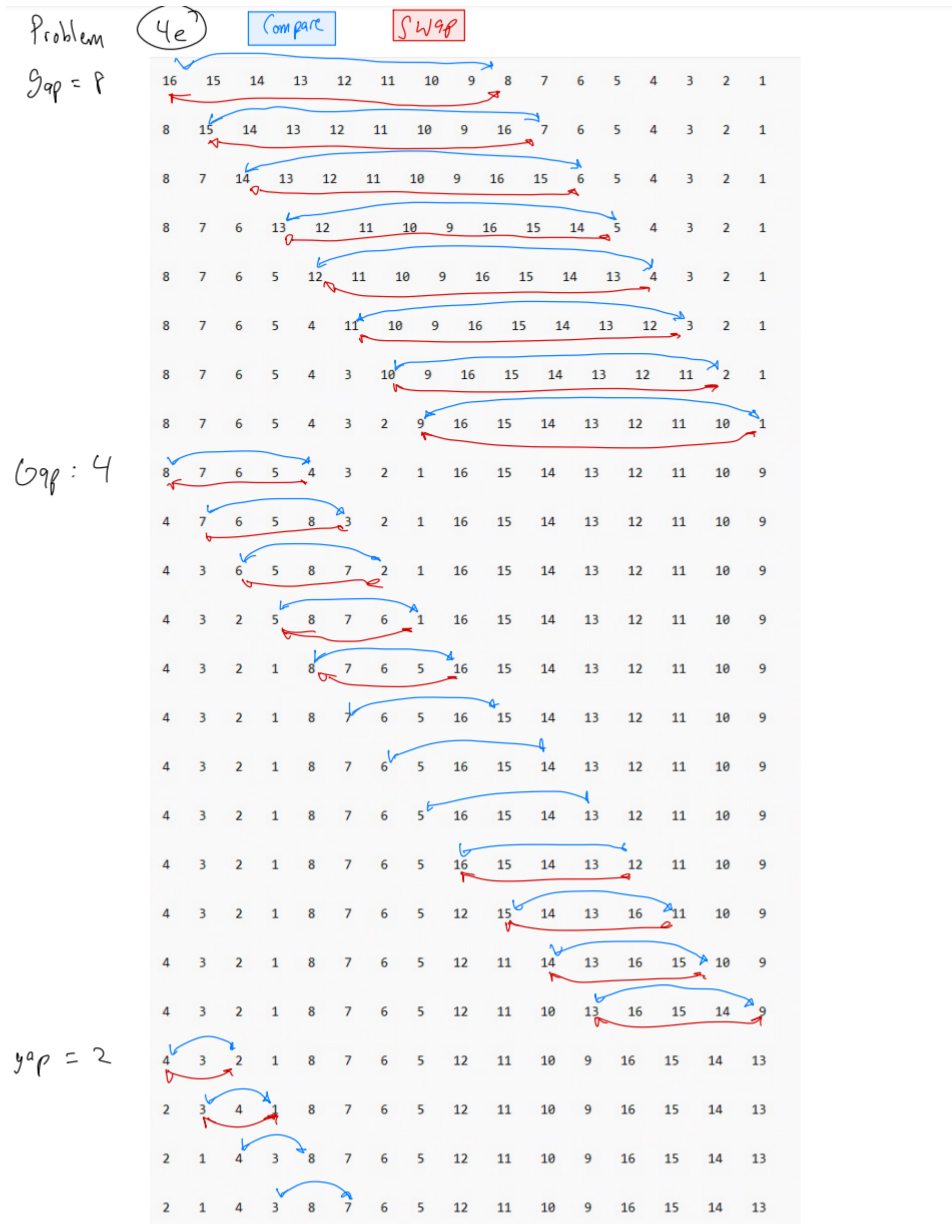
Second run - n - 1 compares, 1 swap.

The summation for compares:

$$\begin{aligned}\sum_{i=1}^{n-1} i &= \frac{n(n-1)}{2} \\ &= \frac{n^2 - n}{2} \\ &= \frac{n^2}{2} - \frac{n}{2}\end{aligned}$$

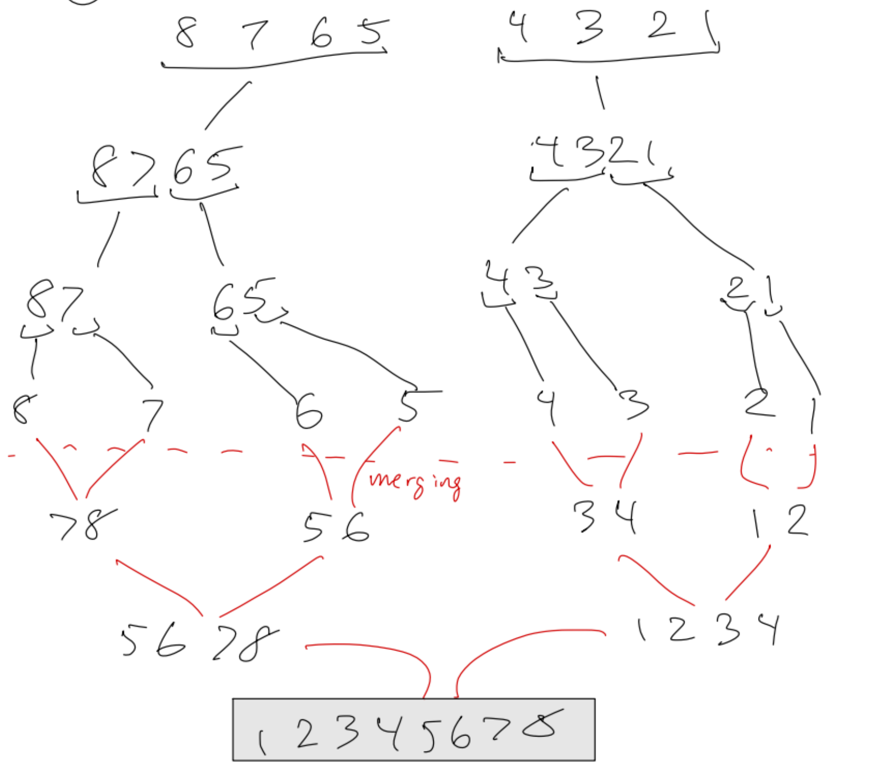
The summation for swaps is just n, because every run only has one swap, the smallest element to it's correct position.

(e) 49 compares, 33 swaps



	2	1	4	3	8	7	6	5	12	11	10	9	16	15	14	13
	2	1	4	3	6	7	8	5	12	11	10	9	16	15	14	13
	2	1	4	3	6	5	8	7	12	11	10	9	16	15	14	13
	2	1	4	3	6	5	8	7	12	11	10	9	16	15	14	13
	2	1	4	3	6	5	8	7	12	11	10	9	16	15	14	13
	2	1	4	3	6	5	8	7	10	11	12	9	16	15	14	13
	2	1	4	3	6	5	8	7	10	9	12	11	16	15	14	13
	2	1	4	3	6	5	8	7	10	9	12	11	16	15	14	13
	2	1	4	3	6	5	8	7	10	9	12	11	16	15	14	13
	2	1	4	3	6	5	8	7	10	9	12	11	14	15	16	13
gap = 1	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
	1	2	4	3	6	5	8	7	10	9	12	11	14	13	16	15
	1	2	4	3	6	5	8	7	10	9	12	11	14	13	16	15
	1	2	3	4	6	5	8	7	10	9	12	11	14	13	16	15
	1	2	3	4	6	5	8	7	10	9	12	11	14	13	16	15
	1	2	3	4	5	6	8	7	10	9	12	11	14	13	16	15
	1	2	3	4	5	6	8	7	10	9	12	11	14	13	16	15
	1	2	3	4	5	6	7	8	10	9	12	11	14	13	16	15
	1	2	3	4	5	6	7	8	10	9	12	11	14	13	16	15
	1	2	3	4	5	6	7	8	9	10	12	11	14	13	16	15
	1	2	3	4	5	6	7	8	9	10	12	11	14	13	16	15
	1	2	3	4	5	6	7	8	9	10	11	12	14	13	16	15
	1	2	3	4	5	6	7	8	9	10	11	12	14	13	16	15
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	15
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	15
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

4 (f)


8 7 6 5
4 3 2 1
8765
4321
87
65
43
21
8
7
6
5
4
3
2
1
merging
78
56
34
12
5678
1234
12345678

(f)

Problem 5.

Resources used: None

Collaborators: None

(a) Inversion pairs:

(0, 4), (1, 4), (2, 3), (2, 4), (3, 4) - 5 pairs

(b) A reversed array has the most inversions. It has n(n - 1)/2 inversions.

(c) As inversion count increases, the runtime for insertion sort increases, Insertuion sort is based on swapping a previous larger item with a smaller item that is after. The more inversions, the more swaps.

(d) Runtime analysis: It is the exact same as mergesort, because the only overhead that we are adding is memory related. The

Using the hint in the bottom of the question, I used the algorithm written in the slides and modified it for this purpose.

```
algorithm MergeSort(A, l, r)
  inversions = 0                       // this is the inversion counter
                                       // that will be incremented.
  if (l < r)
    m = (l + r) / 2
    inversions += MergeSort(A, l, m)
    inversions += MergeSort(A, m + 1, r)
    inversions += merge(A, l, m, r)
  end if

  return A, inversions
end algorithm


algorithm merge(A, l, m, r)
  n1 = m - l + 1
  n2 = r - m

  inv = 0 // counter for inversions
```

```
    let L be an array of size n1 + 1
    let R be an array of size n2 + 1

      for (i = 0 to n1 - 1)
        L[i] = A[l + i]
      end for

      for (j = 0 to n2 - 1)
        R[j] = A[m + j + 1]
      end for

      L[n1] = ∞, R[n2] = ∞
      i = 0, j = 0

      for (k = l to r - 1)
        if (L[i] <= R[j])
          A[k] = L[i]
          i = i + 1
        else                  // this is the inversion
          A[k] = R[j]         // case, when right > left
          j = j + 1
          inv += 1
        end if
      end for

    return A, inv
  end algorithm
```