Numerical Solutions of the 3-Body Problem

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1 Introduction

The motions of free bodies in space can be modelled using Verlet integration.

Two bodies in free space will accelerate towards each other as a result of the gravitational attraction between them, given by

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mathbf{r}}{|r|^3}. \tag{1}$$

This equation can then be split into a pair of coupled 1^{st} order ordinary differential equations,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \text{and} \quad \frac{d\mathbf{v}}{dr} = -\frac{GM\mathbf{r}}{|r|^3}, \tag{2}$$

to be solved by Verlet integration.

2 Verlet Integration

In molecular dynamics the most commonly used time integration algorithm is the Verlet Algorithm [3]. One can write two third order Taylor expansions for the positions $\mathbf{r}(t)$, one forward in time and one backwards in time. Summing these two quantities gives an expression for the position of a particle at time $t + \Delta t$, given it's positions at time t and $t - \Delta t$.

$$\mathbf{r}(t + \Delta t) = 2\mathbf{r}(t) - \mathbf{r}(t - \Delta t) + \mathbf{a}(t)(\Delta t)^2 \tag{3}$$

The system can then be adjusted to explicitly incorporate velocity to produce the Velocity Verlet Algorithm [1, 2, 3]:

$$\mathbf{v}_i^{new} = \mathbf{v}_i^{old} + \mathbf{a}_i \frac{\Delta t}{2} \tag{4}$$

$$\mathbf{r}_i^{new} = \mathbf{r}_i^{old} + \mathbf{v}_i^{new} \Delta t \tag{5}$$

$$\mathbf{a}_i = -\sum_{j=1;j\neq i}^{N} \frac{Gm_j}{(r_{ij}^2 + \epsilon^2)} \mathbf{r}_{ij} \tag{6}$$

$$\mathbf{v}_i^{new} = \mathbf{v}_i^{new} + \mathbf{a}_i \frac{\Delta t}{2} \tag{7}$$

These equations must then be used for each dimension and each particle, giving $N \times D$ positions, velocities and accelerations, where N is the number of particles and D is the number of dimensions.

2.1 Stable Orbits

There are a few know stable orbits for three bodies. The Sun, Earth, Moon system is one of them, and produces Figure 1. The z-axis for this plot is significantly more zoomed in than the x and y axes, and accentuates the motion of the moon in the z-direction, when in reality the moon only orbits at $\sim 5°$ elevation to the Earth's plane of orbit. The initial conditions for this system were found on NASA's horizons tool [4]. The moon does not return to its initial position because it's orbit ($\sim$ 28days) is not divisible by 1 year.

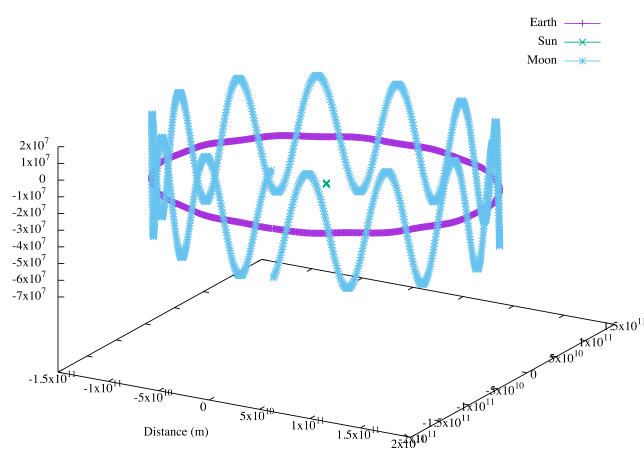

Figure 1: Graph showing the orbit of the Sun, Moon and Earth using the Verlet Algorithm. Note the z-axis is significantly more zoomed in than x or y.

The figure of eight is also a stable orbit [5], shown in Figure 2. Here the three bodies are of equal mass of 0.9×10^{30}kg, and only orbit in 2-Dimensions. There is no z-component required to keep stability. Other orbit patterns also exist.

Chaotic orbits are orbits that are unpredicatble, and therefore the position cannot be perfectly determined in the future. An example of these mechanisms, is the close approach of a planet. When two asteroids pass near a planet, even for very similar initial conditions, the orbits can diverge drastically [6].

2.2 Errors

The gravitational potential energy of orbiting objects is given by [7]

$$E_T = -\frac{GMm}{2R} \tag{8}$$

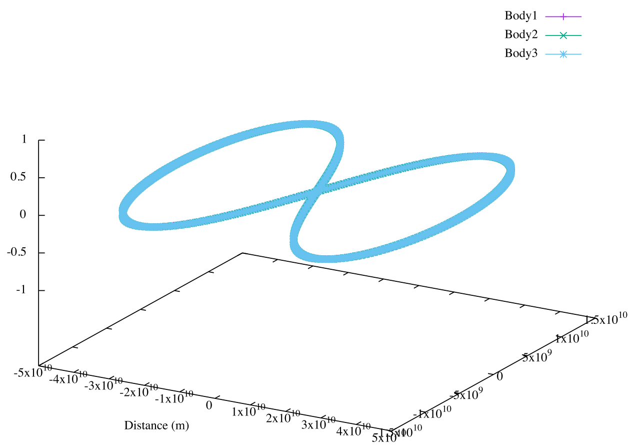

Figure 2: Graph showing the stable orbit of three bodies of equal mass of 0.9×10^{30}kg in a figure of eight shape.

Body	Absolute Energy Loss (%)
Sun	7.54503e-07
Earth	4.36966e-05
Moon	3.678343-03

Table 1: Output of Nbody.c, showing the absolute energy loss over the course of the simulation.

Evaluating the total energy of each body at the beginning of the simulation and at the end gives a discrepancy shown in Table 1. The reasons for this inaccuracy is due to the nature of the Verlet algorithm. Equation (3) is produced from a sum of two Taylor expansions. Higher order terms have been truncated for this algorithm, and thus provide the inaccuracies for the method. Table 1 shows that the absolute error for heavier bodies is less than that for lighter bodies. This is as expected, since the error in calculation is likely to be similar for each body, but will account for a larger proportion of the lighter body's energy.

3 Conclusions

For this system of 1^{st} order ODEs the Verlet method appears to be very effective. The relative energy losses seen in Table 1 are all less than 0.005%, with larger relative energy losses for smaller bodies, as expected. Unfortunetly this cannot be compared to the RKF method, but given that it employs an error on the order of $O(\Delta t^4)$ [9], which is the same as the truncated term in the Taylor expansion for the Verlet algorithm, we would expect the two models to be relatively similar in errors.

References

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[9] https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods