

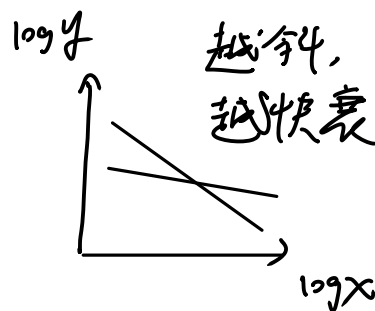
1. Power law  $f(x) = a x^{-b}$

$$\begin{aligned}\textcircled{1} \quad f(cx) &= a (cx)^{-b} \\ &= c^{-b} a x^{-b} \\ &= c^{-b} f(x)\end{aligned}$$

$$\frac{f(x)}{f(cx)} = c^b$$

$$\textcircled{2} \quad \log f(x) = \log a - b \log x$$

$$\log f(x) \propto \log x$$



$\textcircled{3}$   $b$  determines how fast to decay,  
how thin the tail is

E.g.  $b=2$

$$\frac{f(x)}{f(2x)} = 2^2 = 4$$

$$\frac{f(x)}{f(3x)} = 3^2 = 9$$

$b=3$

$$2^3 = 8$$

$$3^3 = 27$$

$b=4$

$$2^4 = 16$$

$$3^4 = 81$$

2. decay : exponential decay  $a e^{-bx}$   
 power decay  $a x^{-b}$   
 exponential squared decay  $a e^{-bx^2}$



13.]  $e^{-x}$   $e^{-x^2}$   $x^{-2}$   $x^{-3}$   $x^{-4}$

$\frac{f(x)}{f(2x)}$

$e^x$   $e^{3x^2}$  4 8 16  
 when  $x=1$ ,  
 $e = 2.7$   $e^3 = 20$   
 when  $x=2$

$\frac{f(x)}{f(3x)}$

$e^{2x}$   $e^{8x^2}$  9 27 81  
 $e^2 = 7.38$   $e^8 = 2980$

$e^4 = 54.6$   $e^{32} = 7896296018$

3. Where to observe power law?

① upper tail of lognormal

详见 5.

② Student - t

$$f(x) = \text{const} \cdot \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

$$\log f(x) = \log(\text{const}) - \frac{\nu+1}{2} \log\left(1 + \frac{x^2}{\nu}\right)$$

当  $x$  足够大, "1" 可以忽略.

$$\log\left(1 + \frac{x^2}{\nu}\right) \approx \log\left(\frac{x^2}{\nu}\right)$$

$$= 2 \log x - \text{const}$$

$$\boxed{\log f(x) \propto \log x}$$

log-log plot

4. exponential squared decay  $a e^{-bx^2}$

例: 正态分布

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\log f(x) = \text{const} - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\log f(x) \propto x^2$$

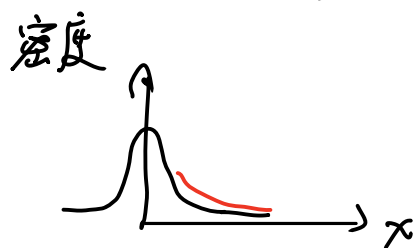
衰减得太快

导致出现极端事件概率太小

不符合现实

增加极端事件发生的概率

↳ 肥尾



$$* \quad E[e^{tx}]$$

$$= \int e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int e^{-\frac{(x-\mu)^2}{2\sigma^2} + tx} dx$$

$$I = -\frac{(x-\mu)^2}{2\sigma^2} + tx$$

$$= -\frac{(x-\mu)^2 - 2t\sigma^2 x}{2\sigma^2}$$

$$= -\frac{x^2 + \mu^2 - 2\mu x - 2t\sigma^2 x}{2\sigma^2}$$

$$x^2 - 2(\mu + t\sigma^2)x + (\mu + t\sigma^2)^2 - (\mu + t\sigma^2)^2 + \mu^2$$

$$= -\frac{x^2 - 2(\mu + t\sigma^2)x + (\mu + t\sigma^2)^2 - (\mu + t\sigma^2)^2 + \mu^2}{2\sigma^2}$$

$$= - \frac{(x - \mu - t\sigma^2)^2 - t^2\sigma^4 - 2\mu t\sigma^2}{2\sigma^2}$$

$$= - \frac{(x - \mu + t\sigma^2)^2}{2\sigma^2} + \frac{t^2\sigma^2}{2} + \mu t$$

$$E[e^{tx}]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int e^I dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi}\sigma} \int e^{-\frac{(x - \mu^*)^2}{2\sigma^2}} dx}_{\text{所有概率之和为1}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

5. log normal distribution

$$Y = e^X, \quad X \sim \mathcal{N}(\mu, \sigma^2)$$

$$\ln Y = X \sim \mathcal{N}(\mu, \sigma^2)$$

$X$  Normal  
 $Y$  log normal.

$$\begin{aligned} E[Y] &= E[e^X] \\ &= e^{\mu + \frac{\sigma^2}{2}}, \quad t=1 \\ &= e^{\mu + \frac{\sigma^2}{2}} \end{aligned}$$

$$\begin{aligned} E[Y^2] &= E[e^{2X}] \\ &= e^{2\mu + 2\sigma^2} \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= E[Y^2] - (E[Y])^2 \\ &= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \\ &= e^{2\mu + \sigma^2} (e^{\sigma^2} - 1) \end{aligned}$$



$$\text{pdf ?} \quad \text{pdf} = \frac{d}{dx} \text{cdf}$$

$$f(x) = \frac{d}{dx} P(Y \leq x)$$

$$= \frac{d}{dx} P(\ln Y \leq \ln x)$$

$$= \frac{d}{dx} P(X \leq \ln x), \quad X \sim N(\mu, \sigma^2)$$

$$= \frac{d}{dx} N(\ln x)$$

$$= N'(\ln x) \frac{d}{dx} (\ln x)$$

$$= \underbrace{N'(\ln x)}_{\text{Normal part}} \cdot \frac{1}{x}$$

$$= \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \cdot \frac{1}{x}$$

$$= \frac{1}{x \sigma \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$



$x$  足够大时 power decay

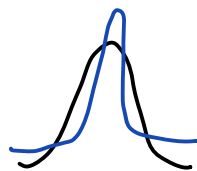
$$\rightarrow \log - \log$$

$$f(x) = \frac{1}{x \sigma \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

$$\log f(x) = -\log x - \log \sigma \sqrt{2\pi} - \frac{(\ln x - \mu)^2}{2\sigma^2}$$

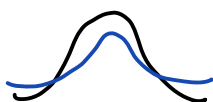
(?)

6. 目标: 收益率 不满足正态分布



kurtosis  $> 3$ .  
(四阶矩)

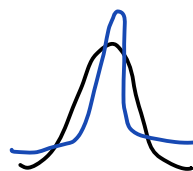
Student-t 不太行 只有肥尾



作为进阶, 考虑 power law 相关

另外一个思路:

$$50\% \cdot \underbrace{N(0, 1.5)}_{\substack{\uparrow \\ \text{peak}}} + 50\% \cdot \underbrace{N(0, 5)}_{\substack{\uparrow \\ \text{fat tail}}}$$



Expectation  
Maximization



Mixture Model  $\rightarrow$  Gaussian  
Mixture  
Model

$$P(A) P(X_1 | A) \\ + P(B) P(X_2 | B)$$

作为比较,  
有一个完全不同的概念

multivariate normal

$$Y = X_1 + X_2$$

$$X_1 \sim N(1, 2^2)$$

$$X_2 \sim N(2, 3^2)$$

Y 是否也符合正态分布?

如果 Y 是正态分布.

$$E[Y] = E[X_1] + E[X_2] = 3$$

$$\text{Var}(Y) = \text{Cov}(Y, Y)$$

$$= \text{Cov}(X_1 + X_2, X_1 + X_2)$$

$$= \text{Var}(X_1) + \text{Var}(X_2)$$

$$+ 2\text{Cov}(X_1, X_2)$$

$$= 4 + 9 + 2 \cdot 2 \cdot 3 \cdot \rho$$