Testing Rational Hypotheses in Signaling Games*

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December 20, 2021

Abstract

In this paper, we introduce a solution concept for signaling games, called Rational Hypothesis Testing Equilibrium (HTE). This equilibrium notion incorporates the Hypothesis Testing model of Ortoleva (2012) that allows for belief updating on information sets with a zero probability. Hypotheses are beliefs about rational strategies of the sender. For each message, the receiver selects a hypothesis in the maximum-likelihood fashion, and updates it by Bayes' rule. Each Rational HTE is a Perfect Bayesian Equilibrium, but not vice versa. In Rational HTE, beliefs feature the following properties: First, beliefs are structurally consistent in the spirit of Kreps and Wilson (1982). Second, beliefs are derived from rational strategies and thus, they are consistent with mutual knowledge of rationality. We prove existence of a Rational HTE for the class of monotone signaling games. We relate Rational HTE with the well-known Intuitive Criterion, and apply it to solve a finite version of the educational game in Spence (1973).

Keywords: Signaling games, Perfect Bayesian Equilibrium, Hypothesis Testing Equilibrium, Bayesian updating, Maximum-likelihood updating, Off-path beliefs, Intuitive Criterion.
JEL Classification: C72, D81, D83

*The authors are very grateful to Hans Haller, George Mailath, Gerelt Tserenjigmid, Matthew Kovach, Kevin He, and the audiences of the 28th International Conference on Game Theory and the 88th Southern Economic Association (SEA) Meetings, and seminar participants at the Australian National University for their valuable comments and fruitful discussions. This paper is based on Dongwoo Lee's Doctoral Dissertation (Chapter 3) written at the Department of Economics at Virginia Tech.

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1 Introduction

Signaling games are an important class of dynamic games with incomplete information. Signaling refers to interactive situations in which one party uses observable actions of an informed opponent to make inferences about hidden information. Signaling games have been applied to explain a variety of economic phenomena, including job search (Spence, 1973), advertising (Nelson, 1974; Milgrom and Roberts, 1986), dividends (Bhattacharya, 1979; John and Williams, 1985), product quality (Miller and Plott, 1985), warranties (Gal-Or, 1989), limit pricing (Milgrom and Roberts, 1982), elections (Banks, 1990), social norms (Bernheim, 1994), or lobbying (Lohmann, 1995).[1]

A Perfect Bayesian Equilibrium (PBE) is a standard solution concept for signaling games. A *Sender* observes his type, and chooses an optimal message. A *Receiver* observes the message but not the type, forms her posterior about the Sender's types, and best responds with an action. As the name indicates, posteriors are derived via Bayes' rule, whenever possible. However, this standard updating procedure has a serious limitation. Since Bayes' rule does not specify how beliefs are derived at information sets with zero probability, PBE admits arbitrary out-of-equilibrium beliefs.

To justify off-path beliefs, Kreps and Wilson (1982) argued that equilibrium beliefs should be *structurally consistent*. That is, at each information set, each belief should be derived from a single strategy that governed the previous moves via Bayesian updating. In another work, Kreps and Ramey (1987, p.1332) provided the following interpretation of structural consistency:

> "*[...] the player who is moving should posit some single strategy combination which, in his view, has determined moves prior to his information set, and that his beliefs should be Bayes-consistent with this hypothesis. If the information set is reached with positive probability in equilibrium, then beliefs are formed using the equilibrium strategy. If, however, the information set lies off the equilibrium path, then the player must form some single "alternative hypothesis" as to the strategy governing prior play, such that under the hypothesis the information set is reached with positive probability.*"

In this paper, we introduce a solution concept, called *Focused Hypothesis Testing Equilibrium* (HTE), that formally incorporates structural consistency into signaling games via the Hypothesis Testing model axiomatized by Ortoleva (2012). This model allows for belief updating for *all* messages, including out-of-equilibrium messages. Beliefs are derived by selecting and updating hypotheses about strategic behavior of the Sender. More precisely, a hypothesis is defined as a belief over the Sender's strategies, combined with the prior information about types. For each message, the Receiver selects a hypothesis that is consistent with the message, and updates it via Bayes' rule.

[1] Riley (2001) provides a comprehensive survey of economic applications of signaling games.

In Focused HTE, hypotheses are selected in the maximum-likelihood fashion. That is, the Receiver holds a prior over a set of hypotheses, called *second-order prior*. On the equilibrium path, the Receiver chooses the most likely (initial) hypothesis with respect to her second-order prior. The initial hypothesis reflects the Sender's equilibrium behavior, and is updated by Bayes' rule. However, if an out-of-equilibrium message is observed, the initial hypothesis is rejected. The Receiver updates her second-order prior via Bayes' rule, selects a new hypothesis (i.e., the most likely hypothesis according to the updated second-order prior), and updates it by Bayes' rule. The new hypothesis is about a strategy for the Sender that generates the observed message. This updating procedure provides a system of posteriors that are well-defined for all messages.[2]

By definition, Focused HTE is a PBE with beliefs that are structurally consistent.[3] We show that the converse is also true. For each belief of a PBE, there exists a hypothesis that induces the belief. Thus, Focused HTE is equivalent to PBE. This result indicates that structural consistency, invoked as an argument to justify off-path beliefs, is very weak in the context of signaling games.

For this reason, we strengthen structural consistency by requiring beliefs to be derived from rational strategies. To this end, we suggest a stronger solution concept, called *Rational Hypothesis Testing Equilibrium*. A rational hypothesis is a belief about (second-order) rational strategies for the Sender (i.e., strategies that best respond to rational strategies for the Receiver), combined with the prior information about types. Each Rational HTE is a PBE, but not vice versa.

In Rational HTE, posteriors are consistent with mutual knowledge of the players' rationality. For this reason, we use the Rational HTE supporting a given PBE as an argument in favor of the equilibrium. To illustrate the scope of such refinement, we show that Rational HTE substantially reduces the number of PBEs in the educational signaling game of Spence (1973). The only Rational HTE is a pooling PBE in which the rational employer offers the highest wage for each effort level off the path that only the high-productivity (rational) worker has an incentive to choose.

We compare Rational HTE with the Intuitive Criterion introduced by Cho and Kreps (1987). Roughly speaking, the Intuitive Criterion eliminates beliefs that assign (strictly) positive probabilities to types that cannot benefit from sending an off-path message. In general, Intuitive PBE and Rational HTE are not nested.[4] However, if, for each off-path message of an Intuitive PBE, there is a non-empty set of types that can benefit from sending the message and the PBE belief is the Bayesian update of the prior conditional on this set, then the Intuitive PBE is a Rational HTE.

Last but not least, we show that Rational HTE can account for the empirical findings in Brandts and Holt (1992). In a series of experiments, Brandts and Holt tested the predictions by the Intuitive

[2] In Ortoleva's (2012) model, an agent rejects the initial hypothesis and selects a new one if, according to this hypothesis, the observed event has probability above $\epsilon \geq 0$. In our equilibrium concept, the threshold is zero.

[3] Note that our notion is stronger as we require beliefs to be consistent with the prior information about types.

[4] We use the term *Intuitive PBE* for a PBE that passes the Intuitive Criterion.

Criterion.[5] In one of their treatments, a significant majority of subjects played consistently with a PBE that fails the Intuitive Criterion. The reported behavior can be explained by a Rational HTE.

This paper is organized as follows. In Section 2, we recapitulate the Hypothesis Testing model, and introduce Focused HTE. In Section 3, we introduce Rational HTE, a refinement of Focused HTE. In Section 4, we compare Rational HTE with the Intuitive Criterion. In Section 5, we solve the Spence game. In Section 6, we explain the experimental findings in Brandts and Holt (1992). In Section 7, we provide final remarks. Appendix A collects all proofs.

2 Hypothesis Testing Equilibrium

2.1 Signaling Games

A signaling game consists of two players, called the *Sender* (he) and the *Receiver* (she). Nature draws a type for the Sender from a finite set of types Θ according to a prior probability distribution p on Θ. We assume that p has full support (i.e., $supp(p) = \Theta$), and is known by the players. The Sender observes his type, and chooses a message m from a finite set $\mathcal{M}$. The Receiver observes the message, but not the type, chooses an action a from a finite set $\mathcal{A}$, and the game ends. Payoffs are given by $u_S, u_R : \Theta \times \mathcal{M} \times \mathcal{A} \to \mathbb{R}$. The class of finite signaling games is denoted by $\mathcal{G}$.

A behavioral strategy for the Sender is a collection of type-contingent mixtures over messages, denoted by $b_S := (b_S(\cdot|\theta))_{\theta\in\Theta}$ (i.e., $\sum_{m\in\mathcal{M}} b_S(m|\theta) = 1$ for each $\theta \in \Theta$, where $b_S(m|\theta)$ denotes the probability that θ sends m), and $\mathcal{B}_S = [\Delta(\mathcal{M})]^\Theta$ is the set of all strategies for the Sender. If b_S is degenerate (i.e., for each $\theta \in \Theta$, $b_S(m|\theta) = 1$ for some $m \in \mathcal{M}$), he follows a pure strategy.

A behavioral strategy for the Receiver is a collection of message-contingent mixtures over actions, denoted by $b_R := (b_R(\cdot|m))_{m\in\mathcal{M}}$ (i.e., $\sum_{a\in\mathcal{A}} b_R(a|m) = 1$ for each $m \in \mathcal{M}$, where $b_R(a|m)$ is the probability that a is played in response to m). $\mathcal{B}_R = [\Delta(\mathcal{A})]^{\mathcal{M}}$ is the set of all strategies for the Receiver. If b_R is degenerate (i.e., for each $m \in \mathcal{M}$, $b_R(a|m) = 1$ for some $a \in \mathcal{A}$), she plays a pure strategy.

For each message $m \in \mathcal{M}$, $\mu(\cdot|m)$ denotes a conditional belief (posterior) over types, given m (i.e., a probability distribution over Θ). A family of posteriors is denoted by $\mu := \{\mu(\cdot|m)\}_{m\in\mathcal{M}}$.

2.2 Hypothesis Testing Model

Let us recapitulate Ortoleva's (2012) Hypothesis Testing model in the setup of signaling games.

The main component of Ortoleva's model is a set of hypotheses. In our setup, a hypothesis is a belief about strategic behavior of the Sender. We denote by $\beta_R := (\beta_R(\cdot|\theta))_{\theta\in\Theta}$ a system of

[5] In this paper, the games implemented by Brandts and Holt (1992) are presented in Figure 1 and Figure 4.

type-contingent probability distributions on $\mathcal{M}$. For each $\theta \in \Theta$, $\beta_R(\cdot|\theta)$ represents the Receiver's belief about messages chosen by type θ. Note that β_R is an element of $\mathcal{B}_S$. A system of beliefs (in short, *belief*) β_R combined with the prior information about types, p, defines a *hypothesis*.

Definition 1 (Hypothesis) *A* hypothesis π *is the probability distribution on* $\mathcal{M} \times \Theta$ *induced by a belief* $\beta_R \in \mathcal{B}_S$ *and the prior probability distribution* $p \in \Delta(\Theta)$*; i.e., for each* $(m, \theta) \in \mathcal{M} \times \Theta$*:*

$$\pi(m, \theta) = \beta_R(m|\theta)p(\theta). \tag{1}$$

A hypothesis π ascribes probability $\pi(m, \theta)$ to the state: "type θ signals m." A hypothesis π is consistent with m if $\pi(m, \theta) = \beta_R(m|\theta)p(\theta) > 0$ for some θ; i.e., the Receiver believes that her opponent plays a strategy according to which type θ signals m with a strictly positive probability.[6] Note that, by construction, each hypothesis is consistent with the prior information p. That is,

$$\pi(\mathcal{M}, \theta) = \sum_{m \in \mathcal{M}} \pi(m, \theta) = \sum_{m \in \mathcal{M}} \beta_R(m|\theta)p(\theta) = p(\theta). \tag{2}$$

A hypothesis π is called *simple* if β_R is a system of degenerate beliefs (i.e., for each $\theta \in \Theta$, $\beta_R(m|\theta) = 1$ for $m \in \mathcal{M}$). In this case, the Receiver believes that the Sender plays a pure strategy.

We assume that the Receiver selects and updates hypotheses in the following way. For a signaling game in $\mathcal{G}$, we denote by $\Delta(\mathcal{M} \times \Theta)$ the set of all probability measures on $\mathcal{M} \times \Theta$. Let $\Pi \subset \Delta(\mathcal{M} \times \Theta)$ be the set of all hypotheses associated with the game. The Receiver holds a *second-order prior* over Π, denoted by ρ. The support of ρ is finite (i.e., $|supp(\rho)| \in \mathbb{N}$). We assume that ρ induces a strict partial order over $supp(\rho)$. Before any information is revealed, the Receiver selects an initial hypothesis π^*, which is the most likely hypothesis with respect to ρ, i.e.,

$$\{\pi^*\} := \underset{\pi \in supp(\rho)}{\arg\max}\, \rho(\pi). \tag{3}$$

Upon arrival of a message m, the Receiver conducts a test. If π^* is consistent with m, she accepts π^*, and updates it via Bayes' rule. However, if π^* is inconsistent with m (i.e., $\pi^*(m, \Theta) = 0$), the Receiver rejects π^*. Then, she updates her second-order prior ρ via Bayes' rule. We assume that ρ_m, the Bayesian update of ρ given m is a strict partial order over $supp(\rho_m)$ for each $m \in \mathcal{M}$. The Receiver selects a new hypothesis π^{**}_m which is the most likely hypothesis according to ρ_m; i.e.

$$\{\pi^{**}_m\} := \underset{\pi \in supp(\rho)}{\arg\max}\, \rho_m(\pi) \quad \text{where} \quad \rho_m(\pi) = \frac{\pi(m, \Theta)\rho(\pi)}{\displaystyle\sum_{\pi' \in supp(\rho)} \pi'(m, \Theta)\rho(\pi')}, \tag{4}$$

[6]Recall that we assume $p(\theta) > 0$ for each $\theta \in \Theta$.

and updates it via Bayes' rule to determine her posterior over Θ. Posteriors are well-defined if for each $m \in \mathcal{M}$, there exists a hypothesis $\pi \in supp(\rho)$ that is consistent with m (i.e., $\pi(m, \Theta) > 0$).

A second-order prior ρ is called *focused* if its support contains only hypotheses that are used. That is,

$$supp(\rho) := \{\pi^*\} \cup \bigcup_{\substack{m \in \mathcal{M} \ s.t. \\ \pi^*(m,\Theta)=0}} \{\pi^{**}_m\}, \tag{5}$$

where π^* is the most likely hypothesis with respect to ρ and π^{**}_m is the most likely hypothesis with respect to ρ_m for a zero-probability message m according to π^*. This is the essence of Ortoleva's Hypothesis Testing model,[7] which we incorporate into a solution concept in the next subsection.[8]

2.3 Focused Equilibrium

A Focused HTE consists of a strategy profile (b^*_S, b^*_R), a (focused) second-order prior ρ, and a family of posteriors $\mu^*_\rho = \{\mu^*_\rho(\cdot|m)\}_{m \in \mathcal{M}}$ derived via the Hypothesis Testing model.

Definition 2 (Focused HTE) $(b^*_S, b^*_R, \rho, \mu^*_\rho)$ *is a Focused Hypothesis Testing Equilibrium if:*

$$(i) \;\; b^*_S(m|\theta) > 0 \textit{ implies } m \in \arg\max_{m' \in \mathcal{M}} \sum_{a \in \mathcal{A}} u_S(\theta, m', a) b^*_R(a|m') \textit{ for each } \theta \in \Theta,$$

$$(ii) \;\; b^*_R(a|m) > 0 \textit{ implies } a \in \arg\max_{a' \in \mathcal{A}} \sum_{\theta \in \Theta} u_R(\theta, m, a') \mu^*_\rho(\theta|m) \textit{ for each } m \in \mathcal{M},$$

$$(iii) \;\; \mu^*_\rho(\theta|m) = \frac{\pi^*(m,\theta)}{\pi^*(m,\Theta)} \textit{ if } \pi^*(m,\Theta) > 0, \textit{ where } \{\pi^*\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi), \textit{ and}$$

$$\pi^*(m,\theta) = \begin{cases} \beta^*_R(m|\theta)p(\theta), & \textit{if } \beta^*_R(m|\theta) > 0 \textit{ where } \beta^*_R = b^*_S, \\ 0, & \textit{otherwise}, \end{cases}$$

$$(iv) \;\; \mu^*_\rho(\theta|m) = \frac{\pi^{**}_m(m,\theta)}{\pi^{**}_m(m,\Theta)} \textit{ if } \pi^*(m,\Theta) = 0, \textit{ where } \{\pi^{**}_m\} := \arg\max_{\pi \in supp(\rho)} \rho_m(\pi),$$

$$\pi^{**}(m,\theta) = \begin{cases} \beta_R(m|\theta)p(\theta), & \textit{if } \beta_R(m|\theta) > 0 \textit{ where } \beta_R \in \mathcal{B}_S, \\ 0, & \textit{otherwise}. \end{cases}$$

[7] Strictly speaking, we consider a special case of his model. In the general version, the initial hypothesis is rejected if the conditioning event has a probability above $\epsilon \geq 0$. Such a model is called *minimal* if any $\epsilon' < \epsilon$ leads to different decisions than under ϵ (Ortoleva, 2012, Definition 3). In our setup, the initial hypothesis is rejected at information sets with zero probability ($\epsilon = 0$). Thus, we assume the Minimal Focused Hypothesis Testing model (Ortoleva, 2012, Definition 4). The advantage of this model specification is its unique representation (Ortoleva, 2012, Proposition 2).

[8] Galperti (2019) applies the Hypothesis Testing model to study optimal persuasion under strategic information design. The Sender can confirm or disconfirm the Receiver's understanding of a prior. The author explores different ways how the Receiver forms a new prior in light of unexpected signals of the Sender.

Conditions (i) and (ii) ensure sequential rationality, while (iii) and (iv) ensure that posteriors are well-defined. For each message, the Receiver best replies with respect to the posterior derived from a hypothesis about the opponent's strategies that in her view generate the message. Note that conditions $(i), (ii)$, and (iii) define a Perfect Bayesian Equilibrium (PBE), expect that off-path beliefs are determined arbitrarily (i.e., (iv) is weaker in PBE). Thus, each Focused HTE is a PBE.

In Focused HTE, beliefs are structurally consistent, as advocated by Kreps and Wilson (1982). For each information set, a belief is *structurally consistent* if there exists a single behavioral strategy for the opponent under which the information set can be reached with a positive probability. In fact, our notion of structural consistency is slightly stronger since we additionally require that posteriors have to be consistent with the prior information about types (Definition 1).

Remark 1. We assumed focused second-order priors for two reasons. First, a focused ρ is what matters for behavior. An analyst can only observe the Receiver's beliefs after different messages (on-path and off-path). However, the analyst cannot empirically test whether the Receiver has a focused or unfocused second-order prior. Thus, if we start with an equilibrium with an unfocused second-order prior, one could construct another equilibrium with the same strategy profile and the same belief for each message, such that the new equilibrium has a focused second-order prior. Thus, it is reasonable to assume that the Receiver behaves *as if* she had a focused ρ.

Second, a focused ρ, which is a strict partial order, guarantees that beliefs associated with an equilibrium $(b^*_S, b^*_R, \rho, \mu^*_\rho)$ are unique in the following sense. There exists a strict partial order $\triangleright$ on Π, such that for any other (focused) second-order prior ρ' on Π with $supp(\rho) = supp(\rho')$, $\pi \triangleright \pi'$ implies $\rho(\pi) < \rho(\pi')$ and $\rho'(\pi) < \rho'(\pi')$. Therefore, the system of posteriors associated with ρ' is the same as the system of posteriors associated with ρ; that is, $\{\mu_\rho(\cdot|m)\}_{m\in\mathcal{M}} = \{\mu_{\rho'}(\cdot|m)\}_{m\in\mathcal{M}}$. In this way, we avoid multiplicity of equilibria due to multiplicity of strict partial orders.

Example 1. Consider the game depicted in Figure 1. A worker has either type θ_L (low skill) or type θ_H (high skill). Knowing his type, the worker decides whether to invest in education (E) or no education (N). Given the signal, an employer either assigns the worker to an executive job (e) or to a manual job (m). The prior probabilities over types are given by $p(\theta_L) = 1/3$ and $p(\theta_H) = 2/3$. Note that each worker type prefers the executive job regardless of his education status. Moreover, education is more costly for the low-skilled worker. For the employer, education is not productive since her payoff is unaffected by the signal. Thus, the employer prefers to match type θ_H with the executive job and type θ_L with the manual job. There are two pooling Focused HTEs.

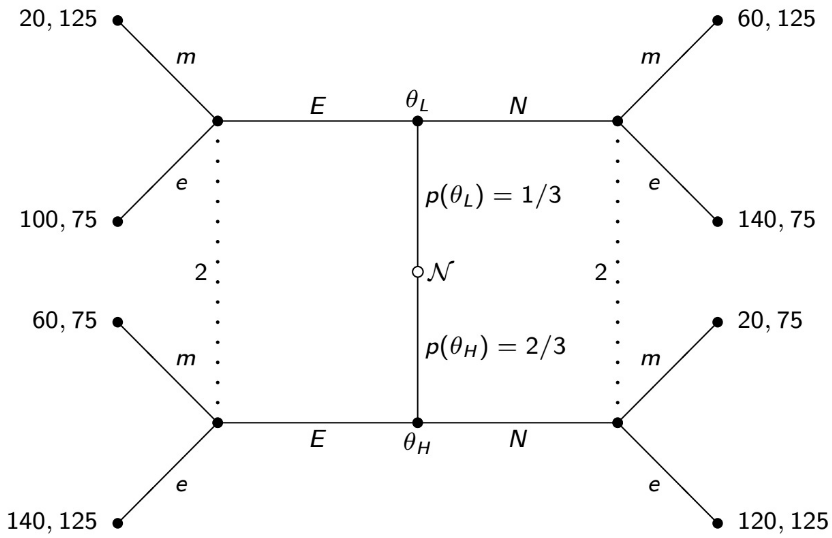

Figure 1: Labor-Market Game 1 from Brandts and Holt (1992).

HTE-1: In the first Focused HTE, each type signals education, i.e.,

$$b_S^*(E|\theta_L) = b_S^*(E|\theta_H) = 1,\ b_R^*(e|E) = b_R^*(m|N) = 1,$$
$$supp(\rho) = \{\pi_1, \pi_1'\} \text{ such that } \rho(\pi_1') < \rho(\pi_1),\ \ \mu_\rho^*(\theta_L|E) = 1/3 \ \text{ and } \ \mu_\rho^*(\theta_L|N) = 1, \text{ where}$$
$$\pi_1 := \{\pi_1(E, \theta_L) = 1/3, \pi_1(E, \theta_H) = 2/3\} \text{ where } \beta_R = (\beta_R(E|\theta_L) = 1,\ \beta_R(E|\theta_H) = 1)\,.$$
$$\pi_1' := \{\pi_1'(N, \theta_L) = 1/3, \pi_1'(E, \theta_H) = 2/3\} \text{ where } \beta_R' = (\beta_R'(N|\theta_L) = 1,\ \beta_R'(E|\theta_H) = 1)\,.$$

HTE-2: In the second Focused HTE, each type signals no education, i.e.,

$$b_S^*(N|\theta_L) = b_S^*(N|\theta_H) = 1,\ b_R^*(m|E) = b_R^*(e|N) = 1,$$
$$supp(\rho) = \{\pi_2, \pi_2'\} \text{ such that } \rho(\pi_2') < \rho(\pi_2),\ \ \mu_\rho^*(\theta_L|E) = 1 \ \text{ and } \ \mu_\rho^*(\theta_L|N) = 1/3, \text{ where}$$
$$\pi_2 := \{\pi_2(N, \theta_L) = 1/3, \pi_2(N, \theta_H) = 2/3\} \text{ where } \beta_R = (\beta_R(N|\theta_L) = 1,\ \beta_R(N|\theta_H) = 1)\,,$$
$$\pi_2' := \{\pi_2'(E, \theta_L) = 1/3, \pi_2'(N, \theta_H) = 2/3\} \ \text{ where } \beta_R' = (\beta_R'(E|\theta_L) = 1,\ \beta_R'(N|\theta_H) = 1)\,.$$

As a matter of fact, each PBE is a Focused HTE. That is, each PBE belief can be derived from a hypothesis about strategic behavior of the Sender, leading us to the following equivalence result.

Theorem 1 *Focused HTE and PBE are equivalent solution concepts.*

This result confirms that PBE beliefs are structurally consistent in the spirit of Kreps and Wilson (1982).[9] There is one important remark with respect to how structural consistency is incorporated.

[9]For more general extensive-form games, Kreps and Ramey (1987) showed that sequential-equilibrium beliefs do not need to be structurally consistent.

Remark 2. To model structural consistency, Kreps and Wilson (1982) informally suggested to use a sequence of hypotheses with a lexicographic order *à la* Blume, Brandenburger, and Dekel (1991). It is worth noting, however, that the lexicographic probability system (LPS) is very restrictive when it comes to modeling strategic behavior in signaling games. The reason is the following: In general, LPS lacks a well-defined procedure how to choose among the higher-order probabilities (hypotheses), unless all probability distributions have disjoint supports. In this special case, referred to as the *lexicographic conditional probability system*, one can think of the following procedure: For each off-path message, the Receiver rejects the first-order (initial) hypothesis, and selects a higher-order hypothesis whose support entails the message. However, hypotheses with non-overlapping supports are restrictive. For instance, in the signaling game of Figure 1, if both types pool on a message in equilibrium, the only possible explanation for the off-path message is pooling behavior. Thus, LPS eliminates the two pooling PBEs in this game. To encompass a broader behavior, the supports of hypotheses need to overlap. However, in this case, it is not clear how to select among, possibly many, higher-order hypotheses that are consistent with the same off-path message. In Focused HTE, new hypotheses are uniquely selected in the maximum-likelihood fashion, allowing for hypotheses with overlapping supports. For this reason, we conclude that Ortoleva's Hypothesis Testing model provides a convenient approach to modeling structural consistency in games.

In the next section, we strengthen structural consistency by imposing an additional restriction on hypotheses, Our goal is to eliminate off-path beliefs that are "irrational."

3 Rational Hypothesis Testing Equilibrium

In this section, we introduce a stronger solution concept, called *Rational Hypothesis Testing Equilibrium*. The idea is to derive off-path beliefs from hypotheses that reflect rational behavior.

In Focused HTE, the initial hypothesis reflects (second-order) rational behavior. According to this hypothesis, the Sender plays the (equilibrium) strategy that best responds to the Receiver's (equilibrium) strategy. However, alternative hypotheses do not need to reflect best-responding behavior. Since strategies that generate off-path behavior must differ from the Sender's equilibrium strategy, some of them may be irrational. However, if players are rational and believe that their opponents are rational, they will best respond with respect to beliefs that assign strictly positive probabilities to rational strategies of the opponents, thus justifying "rational" beliefs. To model "rational" structural consistency, we require that hypotheses have to be about rational strategies.

A behavioral strategy $b_R \in \mathcal{B}_R$ for the Receiver is (first-order) rational if for each $m \in \mathcal{M}$, any

$a \in \mathcal{A}$ with $b_R(a|m) > 0$ is a best response with respect to some belief $\mu(\cdot|m)$ over Θ; that is,

$$b_R(a|m) > 0 \text{ implies } a \in BR(\Theta, m) := \bigcup_{\mu(\cdot|m)\in\Delta(\Theta)} BR(\mu, m).^{10} \tag{6}$$

Denote by $\mathcal{B}_R^\bullet$ the set of (first-order) rational strategies for the Receiver.

A behavioral strategy $b_S \in \mathcal{B}_S$ for the Sender is (second-order) rational if it is a best response to some strategy $b_R \in \mathcal{B}_R^\bullet$; that is, for each $\theta \in \Theta$ and $m \in \mathcal{M}$,

$$b_S(m|\theta) > 0 \text{ implies } m \in \arg\max_{m' \in \mathcal{M}} \sum_{a \in \mathcal{A}} u_S\left(\theta, m', a\right) b_R(a|m'). \tag{7}$$

Denote by $\mathcal{B}_S^\bullet$ the set of (second-order) rational strategies for the Sender.

A message m is rational if there is a strategy $b_S \in \mathcal{B}_S^\bullet$ such that $b_S(m|\theta) > 0$ for some $\theta \in \Theta$. A message $m^d \in \mathcal{M}^d$ is (strictly) dominated if for each $\theta \in \Theta$, choosing m^d is a never-best response.

A *rational hypothesis* is a belief about strategies for the Sender that best respond to some rational strategies of the Receiver. Denote by $\bar{\beta}_R := (\bar{\beta}_R(\cdot|\theta))_{\theta\in\Theta} \in \mathcal{B}_S^\bullet$ a system of type-contingent beliefs. Each belief $\bar{\beta}_R \in \mathcal{B}_S^\bullet$ combined with the prior p defines a rational hypothesis.

Definition 3 (Rational Hypothesis) *A rational hypothesis is the probability distribution π on $\mathcal{M} \times \Theta$ induced by a belief $\beta_R \in \mathcal{B}_S^\bullet$ and the prior $p \in \Delta(\Theta)$; i.e., for every $(m, \theta) \in \mathcal{M} \times \Theta$,*

$$\pi(m, \theta) = \bar{\beta}_R(m|\theta)p(\theta). \tag{8}$$

A *Rational HTE* is a Focused HTE with $supp(\rho)$ that contains only rational hypotheses.

[10] $BR(\mu, m) := \arg\max_{a\in\mathcal{A}} \sum_{\theta\in\Theta} \mu(\theta|m)u_R(\theta, m, a)$.

Definition 4 (Rational HTE) $(b_S^*, b_R^*, \rho, \mu_\rho^*)$ *is a Rational Hypothesis Testing Equilibrium if:*

$$(i)\;\; b_S^*(m|\theta) > 0 \textit{ implies } m \in \arg\max_{m' \in \mathcal{M}} \sum_{a \in \mathcal{A}} u_S(\theta, m', a) b_R^*(a|m') \textit{ for each } \theta \in \Theta,$$

$$(ii)\;\; b_R^*(a|m) > 0 \textit{ implies } a \in \arg\max_{a' \in \mathcal{A}} \sum_{\theta \in \Theta} u_R(\theta, m, a') \mu_\rho^*(\theta|m) \textit{ for each } m \in \mathcal{M},$$

$$(iii)\;\; \mu_\rho^*(\theta|m) = \frac{\pi^*(m,\theta)}{\pi^*(m,\Theta)} \textit{ if } \pi^*(m,\Theta) > 0, \textit{ where } \{\pi^*\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi), \textit{ and}$$

$$\pi^*(m,\theta) = \begin{cases} \bar\beta_R^*(m|\theta)p(\theta), & \textit{if } \bar\beta_R^*(m|\theta) > 0 \textit{ where } \bar\beta_R^* = b_S^*, \\ 0, & \textit{otherwise}, \end{cases}$$

$$(iv)\;\; \mu_\rho^*(\theta|m) = \frac{\pi_m^{**}(m,\theta)}{\pi_m^{**}(m,\Theta)} \textit{ if } \pi^*(m,\Theta) = 0, \textit{ where } \{\pi_m^{**}\} := \arg\max_{\pi \in supp(\rho)} \rho_m(\pi),$$

$$\pi^{**}(m,\theta) = \begin{cases} \bar\beta_R(m|\theta)p(\theta), & \textit{if } \bar\beta_R(m|\theta) > 0 \textit{ where } \bar\beta_R \in \mathcal{B}_S^\bullet, \\ 0, & \textit{otherwise}, \end{cases}$$

$$(v)\;\; \mu_\rho^*(\cdot|m) \textit{ is an arbitrary probability distribution on } \Theta \textit{ if } m \textit{ is strictly dominated}.$$

Conditions (i) to (iv) are the same as in Definition 2, except that the Receiver focuses on rational hypotheses. Note that there might be a very costly message that none of the types will ever play. In this case, there is no rational strategy that generates the message. Since (strictly dominated) messages do not affect equilibrium behavior, condition (v) admits arbitrary beliefs for any message $m^d \in \mathcal{M}^d$, allowing us to apply Rational HTE to a broader family of signaling games. We denote by $\mathcal{M}^\circ$ the set of all out-of-equilibrium messages that are not strictly dominated.

Rational HTE is consistent with mutual knowledge of rationality, provided that $\mathcal{M}^d = \emptyset$. That is, each player believes that the opponent follows strategies that best reply to rational strategies. Thus, the Receiver believes that each message is generated by her opponent's rational strategy.[11]

Clearly, Rational HTE refines PBE, provided that it exists. For signaling games for which $\mathcal{B}_R^\bullet = \mathcal{B}_R$, a Rational HTE always exists, and is equivalent to Focused HTE (by Theorem 1). However, if $\mathcal{B}_R^\bullet \subset \mathcal{B}_R$, a Rational HTE may not exist. By invoking standard conditions, we can show that a Rational HTE exists for each (finite) monotone signaling game (see Appendix B).

[11]One could require higher-order rationality to construct hypotheses. The resulting equilibrium would provide a stronger refinement criterion. However, many refinement concepts in the economic literature such as the Intuitive Criterion of Cho and Kreps (1987) and the novel Rationality Compatible Equilibrium of Fudenberg and He (2020) implicitly assume (second-order) rationality. Our goal is to compare Rational HTE with the Intuitive Criterion (see Section 4). Moreover, we are unaware of any interesting examples where the additional level of rationality would play a role. Therefore, we consider second-order rationality. Readers interested in a solution concept for signaling games that rely on rationalizability are referred to Sobel, Stole, and Zapater (1990) and Battigalli (2006).

Note that a PBE might fail to be a Rational HTE for two reasons. Let us denote by Π_{m° the set of rational hypotheses consistent with m° (i.e. $\pi(m^\circ, \Theta) > 0$). First, there is an out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$, which is not strictly dominated and yet, $\Pi_{m^\circ} = \emptyset$. Second, $\Pi_{m^\circ} \neq \emptyset$, but none of the rational hypotheses induces the PBE belief (i.e., for all $\pi \in \Pi_{m^\circ}$, $\mu_\rho(\cdot|m^\circ) \neq \mu^*(\cdot|m^\circ)$). For instance, the example below shows that HTE-2 fails to be a Rational HTE for the second reason.

To illustrate the solution concept, we again consider the equilibria presented in Example 1.

HTE-1: The equilibrium with pooling on E is a Rational HTE. The initial hypothesis π_1 is rational since it describes the job-applicant equilibrium behavior. Consider the alternative hypothesis, π_1'. A strategy where type θ_L signals N, while type θ_H signals E, best responds to the employer's strategy that offers the executive job for each signal. Hence, π_1' is rational too.

HTE-2: Pooling on N fails to be a Rational HTE. Although the initial hypothesis, π_2, is rational, there is no rational strategy, including π_2', that could justify the PBE belief conditional on E. Consider a strategy $b_S(\gamma)$ for which type θ_L signals N with a strictly positive probability, i.e.,

$$b_S(E|\theta_L) = 1 - \gamma,\ b_S(N|\theta_L) = \gamma \in (0, 1],\ \text{and}\ b_S(E|\theta_H) = 1.$$

It is rational since it best responds to $b_R = (b_R(e|E) = 1,\ b_R(m|N) = 1/2,\ b_R(e|N) = 1/2)$.

The employer's belief $\bar{\beta}_R = b_S(\gamma)$, together with p, induces the following rational hypothesis:[12]

$$\pi_\gamma := \big\{\pi(N, \theta_L) = \gamma 1/3,\ \pi(E, \theta_L) = (1 - \gamma)1/3,\ \pi(N, \theta_H) = 0,\ \pi(E, \theta_H) = 2/3\big\}.$$

By updating π_γ given E, we obtain

$$\mu_\rho(\theta_L|E) = \frac{\pi_\gamma(E, \theta_L)}{\pi_\gamma(E, \Theta)} = \frac{(1-\gamma)\frac{1}{3}}{1 - \gamma\frac{1}{3}} < \frac{1}{3}\ \text{ for each }\ \gamma \in (0, 1].$$

The employer infers that E is more likely to be chosen by the high-skilled worker, and assigns this worker to the executive job e instead of m. Thus, pooling on N cannot be a Rational HTE.

Finally, let us remark on the conceptual difference between Rational HTE and Ortoleva's HTE.

Remark 3. Ortoleva (2012) suggested an equilibrium called *Hypothesis Testing Equilibrium*. As in our equilibrium, Ortoleva assumed that the initial hypothesis is rejected when a zero-probability message arrives (i.e., $\epsilon = 0$). Besides this, there are two substantial differences to our model. First, he considered equilibria in pure strategies, and assumed simple hypotheses. In contrast, we allow for mixed equilibrium behavior and non-simple hypotheses. Second, and more importantly, in Ortoleva (2012), hypotheses reflect first-order rational behavior, and thus are weaker than ours. His hypothesis notion is about a (single) pure strategy for the Sender that best responds to *some*

[12]Note that π_2' is not a rational hypothesis.

(not necessarily rational) pure strategy of the Receiver (Ortoleva, 2012, Section IV). Thus, off-path beliefs are admitted that are derived from strategies that only best respond to never-best responses. We eliminate such beliefs by imposing second-order rationality.[13] For this reason, all formal results of the following sections apply to Ortoleva's HTE. Recently, Sun (2019) extended Ortoleva's solution concept by allowing for $\epsilon \geq 0$. Sun showed that a strictly positive threshold may lead to behaviors that are inconsistent with the sequential equilibrium of Kreps and Wilson (1982).

4 Rational HTE versus Intuitive PBE

In this section, we compare Rational HTE with PBE that passes the well-known Intuitive Criterion introduced by Cho and Kreps (1987). Contrary to our approach, the Intuitive Criterion is a refinement procedure that does not build on a theory of belief updating. Instead, the refinement is payoff-based. That is, the idea is to eliminate off-path beliefs that assign a (strictly) positive probability to types that do not have any incentive to deviate from their equilibrium strategy.

Let us briefly recall the Intuitive Criterion. For a PBE, (b_S^*, b_S^*, μ^*), denote by $u_S^*(\theta)$ the expected equilibrium payoff for type θ. For an out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$, let

$$BR(\Theta, m^\circ) := \bigcup_{\mu(\cdot|m^\circ)\in\Delta(\Theta)} BR(\mu, m^\circ) \tag{9}$$

be the set of best responses for the Receiver to m° with respect to beliefs over Θ.[14] Let $T(m^\circ) \subseteq \Theta$ be the set of types that cannot improve upon their equilibrium payoff by playing m°, no matter how the Receiver responds. That is, for each type $\theta \in T(m^\circ)$ and all best responses $a \in BR\,(\Theta, m^\circ)$:

$$u_S^*(\theta) > u_S(\theta, m^\circ, a). \tag{10}$$

Each type in $I(m^\circ) := \Theta \setminus T(m^\circ)$ could be better off than his equilibrium payoff by playing m°.

A PBE fails the Intuitive Criterion if there exists a type that would be better off by choosing m°, provided that the Receiver best responds to m° with respect to beliefs that assign a zero probability to each type that has no incentive to deviate; otherwise the PBE survives the Intuitive Criterion.

Definition 5 (Intuitive Criterion) *A PBE, (b_S^*, b_R^*, μ^*), fails the Intuitive Criterion if for some out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$, there is a type $\theta \in I(m^\circ)$ such that for all $a \in BR\,(I(m^\circ), m^\circ)$,*

$$u_S^*(\theta) < u_S(\theta, m^\circ, a), \tag{11}$$

[13] In Section 5, we show that Rational HTE and Ortoleva's HTE yield different predictions for the Spence game.

[14] $BR(\mu, m^\circ) := \arg\max_{a\in\mathcal{A}} \sum_{\theta\in\Theta} u_R(\theta, m^\circ, a)\mu(\theta|m^\circ)$.

where $BR(I(m^\circ), m^\circ)$ is the set of best responses induced by the beliefs concentrated on $I(m^\circ)$.

If a PBE passes the Intuitive Criterion, for each type, there is a best reply for the Receiver with respect to a posterior over $I(m^\circ)$, which makes all types better off by playing their equilibrium strategies (i.e., there is $a \in BR\left(I(m^\circ), m^\circ\right)$ such that $u_S^*(\theta) \geq u_S(\theta, m^\circ, a)$ for each θ). For a singleton set $I(m^\circ) = \{\theta_{m^\circ}\}$, the Receiver learns the single type that could benefit by playing m°. In this case, the Intuitive Criterion outcome is the PBE with $\mu(\theta_{m^\circ}|m^\circ) = 1$ for each $m^\circ \in \mathcal{M}^\circ$. We call *Intuitive* (resp., *Unintuitive*) PBE a PBE that passes (resp., fails) the Intuitive Criterion.

The example below illustrates how the Intuitive Criterion operates in the game of Figure 1.

Example 2 Consider the Rational HTE-1 (Example 1). Given the equilibrium payoff, type θ_L could be better off by playing N if he expects to obtain the executive job. That is, $I(N) = \{\theta_L\}$ and $T(N) = \{\theta_H\}$. As long as the employer believes that N could be played only by type θ_L, there is no type that has an incentive to play N. Thus, this equilibrium passes the Intuitive Criterion. However, HTE-2 fails the Intuitive Criterion. Only the high-skilled type could benefit by playing the out-of-equilibrium message E. The employer learns that E is chosen by θ_H (i.e., $\mu^*(\theta_H|E) = 1$), and best responds with e. Therefore, the high-skilled worker will signal E instead of N.

In general, Rational HTE and Intuitive PBE are not nested. There is a PBE that is Rational HTE but it fails the Intuitive Criterion, and there is an Intuitive PBE that is not a Rational HTE.

Proposition 1 *Rational HTE and Intuitive PBE are not nested.*

To relate Rational HTE and Intuitive PBE, we focus on games in which there exists at least one rational hypothesis for each message. As we show below, this existence condition is guaranteed by the non-emptiness of $I(m^\circ)$. That is, for each off-path message $m^\circ \in \mathcal{M}^\circ$ of a PBE, there exists a rational hypothesis consistent with m° as long as there is at least one type that has an incentive to signal m° (i.e., $I(m^\circ) \neq \emptyset$), no matter whether the PBE passes the Intuitive Criterion or not.

Lemma 1 *If (b_S^*, b_R^*, μ^*) is a PBE such that $I(m^\circ) \neq \emptyset$ for each out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$, then there exists a rational hypothesis that is consistent with m° (i.e., $\Pi_{m^\circ} \neq \emptyset$).*

For a non-singleton set $I(m^\circ)$, the Intuitive Criterion admits arbitrary beliefs over $I(m^\circ)$. We show that some beliefs in $\Delta(I(m^\circ))$ can be justified by rational hypotheses. To this end, we need an additional notation. For an off-path message $m^\circ \in \mathcal{M}^\circ$ and $a \in BR(\Theta, m^\circ)$, we denote by

$$I(m^\circ; a) \equiv \{\theta \mid u_S^*(\theta) \leq u_S(\theta, m^\circ, a)\} \tag{12}$$

the set of types that have an incentive to deviate to m° from the equilibrium message if the Receiver plays a. Note that for each PBE that passes the Intuitive Criterion, there exists at least one a for which $I(m^\circ; a) \neq \emptyset$.[15] The Receiver can reason that the types in $I(m^\circ; a)$ pool on m° in response to a, and consequently, she updates the prior p conditional on $I(m^\circ; a)$. As the next result shows, if the PBE belief is such a Bayesian update, then the Intuitive PBE is in fact a Rational HTE.

Theorem 2 *Let (b_S^*, b_R^*, μ^*) be an Intuitive PBE. Suppose that for each out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$, there is a best response $a \in BR(\Theta, m^\circ)$ for which $I(m^\circ; a) \neq \emptyset$ and*

$$\mu^*(\theta|m^\circ) = \begin{cases} \frac{p(\theta)}{\sum\limits_{\theta' \in I(m^\circ;a)} p(\theta')} & \text{if } \theta \in I(m^\circ; a), \\ 0 & \text{if } \theta \notin I(m^\circ; a). \end{cases} \tag{13}$$

Then, there exists a Rational HTE that explains the PBE.

In other words, an Intuitive PBE fails to be a Rational HTE only if its off-path beliefs are inconsistent with the Bayesian update of p conditional on the set of potentially deviating types, $I(m^\circ; a)$.

Theorem 2 contains a special case that deserves a remark. If for each $m^\circ \in \mathcal{M}^\circ$, there is a single type that could benefit from choosing m° (i.e., $I(m^\circ) = \{\theta_{m^\circ}\}$ for some $\theta_{m^\circ} \in \Theta$), then there exists a Rational HTE supporting the Intuitive PBE in which the Receiver learns type θ_{m°.

Corollary 1 *If (b_S^*, b_R^*, μ^*) is an Intuitive PBE with a singleton $I(m^\circ)$ for each $m^\circ \in \mathcal{M}^\circ$, then there exists a Rational HTE that supports the PBE with $\mu^*(\theta_{m^\circ}|m^\circ) = 1$ where $\{\theta_{m^\circ}\} = I(m^\circ)$.*

5 Educational Signaling Game

In this section, we solve the education signaling game of Spence (1973), which is known as having a continuum of PBEs. We show that Rational HTE can substantially reduce the number of PBEs.

We consider a finite version of the Spence model. As in previous games, there is a worker (he) and an employer (she). The worker has either low (L) or high (H) productivity (i.e., $\Theta = \{\theta_L, \theta_H\}$, where $\theta_L < \theta_H$). The prior probability distribution p on Θ is $p(\theta_L) = 1-\alpha$ and $p(\theta_H) = \alpha \in (0, 1)$.

The worker knows his type θ, and chooses an education level e from $\mathcal{M} = \{e_0, e_1, \ldots, e_N\}$, where $e_0 := 0 < e_1 < \ldots < e_N$.[16] The worker's payoff is given by

$$u_S(\theta, e, w) = w - \frac{e}{\theta} \quad \text{for } \theta \in \Theta, \tag{14}$$

[15] Note that $I(m^\circ; a) \subseteq I(m^\circ)$.

[16] To simplify our analysis, we assume that $\mathcal{M}$ is sufficiently large, yet finite. In particular, we assume that $\mathcal{M}$ contains education levels $e_n := \theta_L(\theta_H - \theta_L)$ and $e_N := \theta_H(\theta_H - \theta_L)$. For an analysis of the Spence model with $\mathcal{M} = \mathbb{R}_+$, the reader is referred to Fudenberg and Tirole (1991, Chapter 8).

where w denotes the wage and $\frac{e}{\theta}$ is the cost of choosing e by type θ. Education is more costly to the low-productivity type. It is assumed that the worker can always find a job at wage $w = \theta_L$. Figure 2 depicts type-dependent indifference curves (the red one for θ_L and the blue one for θ_H).

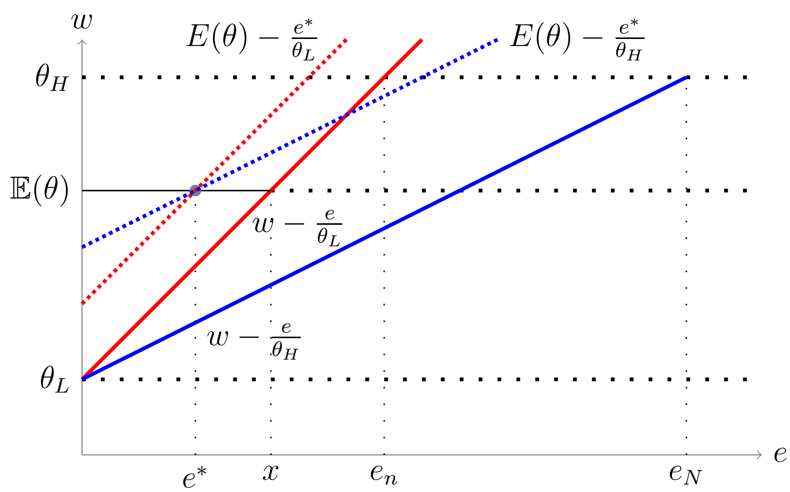

Figure 2: Pooling PBE with education level e^* and wage $w^* = \mathbb{E}(\theta)$.

The employer observes the education level e but not the worker's productivity, and offers a wage w. Her payoff is given by

$$u_R(\theta, e, w) = -(\theta - w)^2 \quad \text{for } \theta \in \Theta. \tag{15}$$

The rational employer offers a wage that is equal to expected productivity. That is, the best response for each e is given by $w(e) := \mathbb{E}(\theta|e) = \mu(\theta_H|e)\theta_H + \big(1 - \mu(\theta_H|e)\big)\theta_L$, where $\mu(\cdot|e)$ denotes her posterior belief over θ.[17] Note that $\mathcal{A} = \mathbb{R}_+$. However, $w(e) \in [\theta_L, \theta_H]$. We denote by $\mathbb{E}(\theta) := \alpha\theta_H + (1-\alpha)\theta_L$ the average productivity when $\mu(\cdot|e)$ coincides with the prior p. To simplify notation, we write $w(e)$ as a short form denoting the pure strategy $b_R(w(e)|e) = 1$.

5.1 Pooling PBE

In a Pooling PBE, both worker types choose the same education level e^*. Note that the payoff of the low-productivity type from signaling e^* at the average-productivity $w^* = \mathbb{E}(\theta)$ must be greater

[17]One can justify $w(e) = \mathbb{E}(\theta|e)$ by considering a perfect competition in a market with many rational employers. Jeong (2019) studies imperfect competition among employers in the context of job market signaling. In particular, he investigates how wage offers change, depending on the degree of competition.

than his payoff from signaling no education at the minimum wage $w = \theta_L$. Thus, e^* must satisfy

$$\mathbb{E}(\theta) - \frac{e^*}{\theta_L} \geq \theta_L, \quad \text{or equivalently,} \quad e^* \leq \underbrace{\alpha(\theta_H - \theta_L)\theta_L}_{:=x}, \tag{16}$$

specifying an upper bound x for education levels that can be explained by a Pooling PBE.

For $e^* \leq x$, neither type has an incentive to deviate from e^* at $w^* = \mathbb{E}(\theta)$ as long as the wages paid off the equilibrium paths satisfy the following conditions: For each $e < e^*$, $w(e)$ satisfies

$$\mathbb{E}(\theta) - \frac{e^*}{\theta_L} \geq w(e) - \frac{e}{\theta_L}, \tag{17}$$

whereas for each $e' > e^*$, $w(e')$ satisfies

$$\mathbb{E}(\theta) - \frac{e^*}{\theta_H} \geq w(e') - \frac{e'}{\theta_H}. \tag{18}$$

Hence, there is a continuum of equilibria. First, there are multiple levels of education that can be supported by a pooling equilibrium. Second, each Pooling PBE admits various wage schemes due to arbitrariness of the off-path beliefs. We will focus on the following family of Pooling PBEs:

Observation 1 *For each education level e such that $0 \leq e^* \leq x := \alpha(\theta_H - \theta_L)\theta_L$, the strategy profile (b^*_S, w^*_R) and beliefs $\mu^* := (\mu^*(\cdot|e)_{e \in \mathcal{M}})$ such that*

(i) $b^*_S(e^*|\theta_L) = b^*_S(e^*|\theta_H) = 1$,

(ii) $w^*(e) = \begin{cases} \theta_L & \text{if } 0 \leq e < e^*, \\ \mathbb{E}(\theta) & \text{if } e^* \leq e < e_n, \\ \mathbb{E}(\theta|e) & \text{if } e_n \leq e \leq e_N, \end{cases}$ *and* $\mu^*(\theta_H|e) = \begin{cases} 0 & \text{if } 0 \leq e < e^*, \\ \alpha & \text{if } e^* \leq e < e_n, \\ [0,1] & \text{if } e_n \leq e \leq e_N, \end{cases}$

where $\mathbb{E}(\theta|e) = \mu^*(\theta_H|e)\theta_H + (1 - \mu^*(\theta_H|e))\theta_L$, constitute a Pooling PBE in the Spence model.

In each PBE, the employer believes that any education below the equilibrium level e^* is chosen by the low-productivity type, and the employer offers the minimum wage. Any education equal to, or larger than e^*, but below e_n, does not convey any information about types. Thus, the employer pays the average productivity. The employer believes that any education above e_n is chosen by the high-productivity type with a probability ranging from zero to one, making any wage ranging from θ_L to θ_H admissible.[18] The level of education e_n makes the low-productivity type indifferent between choosing no education for $w = \theta_L$ and choosing e_n for $w = \theta_H$ (see Figure 2).

[18]For the sake of simplicity, we consider a family of PBEs that covers all levels of education on which the workers can pool and all possible wages paid for $e \geq e_n$. Note that the family does not cover all admissible wages for $e \leq e_n$, such that $e \neq e^*$ (e.g., for each e such that $e^* \leq e < e_n$, $w(e)$ such that $\mathbb{E}(\theta) \leq w(e) \leq \theta_L$ is admissible).

Note that for each Pooling PBE, there is an out-of-equilibrium message that only the high-productivity type could be better off than his equilibrium payoff. If the employer believes that only θ_H chooses such a message, she will offer the highest wage and then type θ_H will indeed deviate from the equilibrium. For this reason, each Pooling PBE fails the Intuitive Criterion in this game.

In the next section, we derive the set of education levels that can be supported by a Pooling Rational HTE. Moreover, we will show that the "cardinality" of this set depends on the prior, p.

5.2 Pooling Rational HTE

Let us first elucidate the reason why a Pooling PBE may fail to be a Rational HTE. Consider an off-path education level $e \in \{e_n, \ldots, e_N\}$.[19] Note that the low-productivity type has no incentive to choose such an e, even if the highest wage, $w(e) = \theta_H$, is paid. For each $e \in \{e_n, \ldots, e_N\}$, the payoff of the low-productivity type is lower than his payoff for no education at the minimum wage $w(e) = \theta_L$. Thus, strategies where type θ_L chooses $e \in \{e_n, \ldots, e_N\}$ are never-best responses.[20] Since there is no rational hypothesis according to which θ_L chooses $e \in \{e_n, \ldots, e_N\}$ with a strictly positive probability, each Pooling PBE with $\mu(\theta_L|e) > 0$ fails to be a Rational HTE.

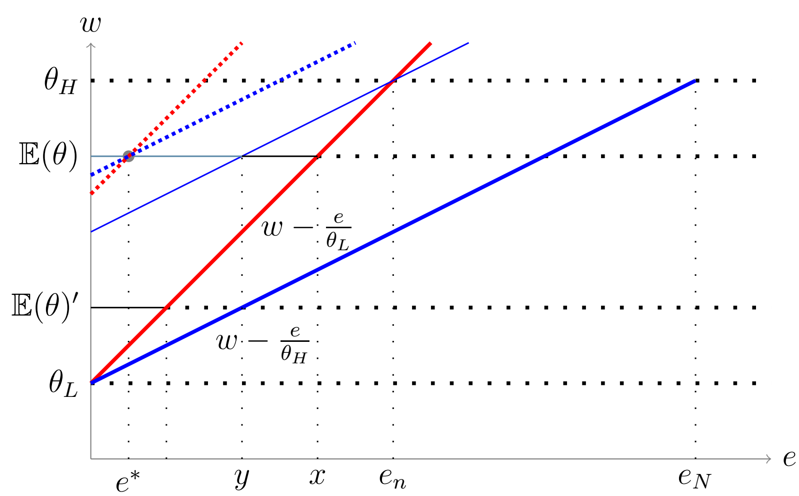

Figure 3: Refinement of Pooling PBE at $\mathbb{E}(\theta)$ but no refinement at $\mathbb{E}(\theta)'$.

What levels of education can be explained by a Rational HTE? Let us consider simple hypotheses. Since only the high-productivity type can choose $e \in \{e_n, \ldots, e_N\}$ as a best response, $\mu(\theta_H|e) = 1$ is the only off-path belief that can be justified by a rational hypothesis. This means that the payoff for type θ_H from choosing a (pooling) message e^* at $w(e^*) = \mathbb{E}(\theta)$ must be greater

[19] The lower and upper bounds of this set are $e_n := \theta_L(\theta_H - \theta_L)$ and $e_N := \theta_H(\theta_H - \theta_L)$.

[20] Even under wage $w(e)$, such that $w(e) = \theta_H$ for $e > e_n$ and $w(e) = \theta_L$ for $e \leq e_n$, θ_L will not choose $e > e_n$.

than the payoff from choosing $e \in \{e_n, \ldots, e_N\}$ at the highest wage θ_H. That is, e^* must satisfy

$$\mathbb{E}(\theta) - \frac{e^*}{\theta_H} \geq \theta_H - \frac{e_n}{\theta_H}, \quad \text{or equivalently,} \quad e^* \leq \underbrace{(\theta_H - \theta_L)(\theta_L - (1-\alpha)\theta_H)}_{:=y}, \tag{19}$$

which specifies an upper bound y for education that can be supported by a Pooling Rational HTE. Note that a Rational HTE exists if and only if $y \geq 0$, or equivalently, $\alpha \geq \frac{\theta_H - \theta_L}{\theta_H}$. In other words, the number of Rational HTE is a function of α, the fraction of high-productivity types in the market. The larger α is, the more Rational HTEs exist. If $\alpha < \frac{\theta_H - \theta_L}{\theta_H}$, none of the Pooling PBEs can be justified by a Rational HTE (see Figure 3).

The next proposition derives a family of Pooling Rational HTEs.

Proposition 2 *For each education level e^* such that $0 \leq e^* \leq y := (\theta_H - \theta_L)\big(\theta_L - (1-\alpha)\theta_H\big)$, the strategy profile (b_S^*, w_R^*) and beliefs $\mu^* := (\mu^*(\cdot|e))_{e \in \mathcal{M}}$ such that*

(i) $b_S^*(e^*|\theta_L) = b_S^*(e^*|\theta_H) = 1$,

$$(ii) \;\; w_R^*(e) = \begin{cases} \theta_L & \text{if } e < e^*, \\ \mathbb{E}(\theta) & \text{if } e^* \leq e \leq e_n, \\ \theta_H & \text{if } e_n < e \leq e_N. \end{cases} \quad \textit{and} \quad (iii) \;\; \mu^*(\theta_H|e) = \begin{cases} 0 & \text{if } e < e^*, \\ \alpha & \text{if } e^* \leq e \leq e_n, \\ 1 & \text{if } e_n < e \leq e_N, \end{cases}$$

constitute a Pooling PBE that is Rational HTE.

In Rational HTE, the employer believes that any education level below e^* is chosen by the low-productivity type, and pays the minimum wage. For any education level between e^* and e_n, the employer cannot infer the type, and pays the average productivity. Finally, the employer pays the highest wage for any education level above e_n, which is chosen by the high-productivity type,

Note that Rational HTE refines Pooling PBEs in two dimensions. The first dimension is the level of education.[21] The second dimension is off-path belief (equivalently, the wage).

Finally, let us briefly remark on the role of players' rationality for the Spence game. To this end, assume that hypotheses reflect first-order rationality as in Ortoleva (2012) (see Remark 3). That is, hypotheses are about strategies for the worker that best respond to some (not-necessarily rational) strategy of the employer. For each education level $e \in \{e_n, \ldots, e_N\}$, there is a strategy according to which each type best responds with e (e.g., when the employer pays the wage equal to $\theta_L + \varepsilon + \frac{e}{\theta_L}$ for e, where $\varepsilon > 0$, and θ_L otherwise). Hence, for each $e \in \{e_n, \ldots, e_N\}$, we can construct a hypothesis that justifies the off-path belief $\mu^*(\theta_H|e) = \alpha$, allowing us to explain each Pooling PBE (see Observation 1) in which the average productivity, $\mathbb{E}(\theta)$, is paid off the path.

[21] Interestingly enough, there is experimental evidence in favor of such equilibria. Especially, Kübler, Müller, and Normann (2008) found pooling behavior at lower education levels in the framework of the Spence model.

Paying average productivity is, however, inconsistent with mutual knowledge of rationality. The rational employer offers the expected productivity (i.e., $w(e) = \mathbb{E}(\cdot|e)$ for each e). Knowing that the employer is rational, the worker will never play a strategy according to which the low-productivity worker signals $e \in \{e_n, \ldots, e_N\}$. On the other hand, knowing that the worker is rational, in the sense of never playing a dominated strategy, the employer must know that each $e \in \{e_n, \ldots, e_N\}$ can only be chosen by the high-productivity type. Consequently, the rational employer will offer the highest wage $w(e) = \theta_H$ for e, and the rational worker knows this fact.

Under mutual knowledge of rationality, the Pooling PBEs that fail to be Rational HTEs are not robust. Consider such a Pooling PBE with e^*, such that $y < e^* \leq x$, and the average productivity $w(e) = \mathbb{E}(\theta)$ paid for each $e \in \{e_n, \ldots, e_N\}$ (see Figure 2). Knowing that the employer is rational, the payoff of the high-productivity worker from e_n at wage θ_H is larger than his equilibrium payoff, providing him an incentive to signal e_n instead of e^*. Rational HTE rules out such incentives to deviate, making the equilibrium robust due to its consistency with mutual knowledge of rationality.

In sum, our requirement to justify beliefs by second-order rational strategies significantly reduces the number of Pooling PBEs in the signaling game of Spence (1973). A Rational HTE, provided it exists, is one in which the employer offers the highest wage for each out-of-equilibrium-message that only the high-productivity type has an incentive to choose (i.e., each $e \in \{e_n, \ldots, e_N\}$).

6 Experimental Findingsand Rational HTE

In this section, we show that Rational HTE is consistent with the experimental findings in Brandts and Holt (1992) (henceforth, BH) on equilibrium behavior in signaling games of Figures 1 and 4. For each game, BH observed a different type-dependent behavior, one being inconsistent with the Intuitive Criterion. However, Rational HTE can account for the reported behavior.

Recall, the game in Figure 1 has two equilibria, whereby only the equilibrium with pooling on E is the (intuitive) Rational HTE-1 (see Example). BH reported that a significant majority of subjects behaved consistently with this equilibrium.[22] More interestingly, BH found some evidence for the type-dependent behavior that underlies the new hypothesis π_1' of the Rational HTE-1; 84 out of 84 high-skilled subjects played E, while 24 out of 44 low-skilled subjects played the out-of-equilibrium message N that is consistent with π_1'. As Brandts and Holt (1992, p.1357) remark:

> "*This type-dependence is consistent with the out-of-equilibrium beliefs that support the intuitive [...] equilibrium*".

A majority of Receivers seemed to believe that N is sent by low-skilled types in accordance with π_1'. Then, 17 out of 24 Receivers who observed N responded with the equilibrium action m. Hence,

[22] 102 out of 128 decisions matched with Rational HTE-1, while only 7 matched with the (unintuitive) HTE-2.

the reported type-dependence and subjects' replies to N suggest that π_1' is a reasonable hypothesis.

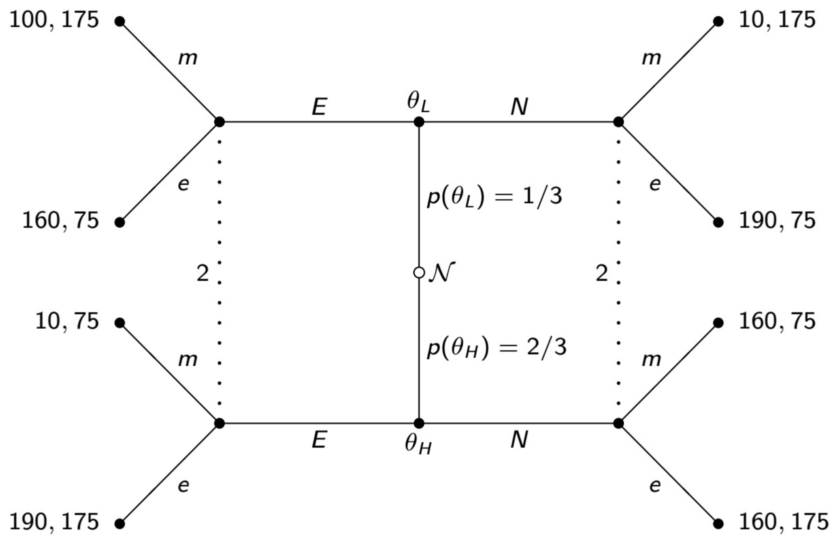

Figure 4: Labor-Market Game 2 in Brandts and Holt (1992)

Now, consider the game in Figure 4. For this game, our predictions differ from the Intuitive-Criterion predictions. There are two Rational HTEs but only one passes the Intuitive Criterion. **HTE-3:** In this equilibrium, both types pool on E; i.e,

$$b_S^*(E|\theta_L) = b_S^*(E|\theta_H) = 1,\ \ b_R^*(e|E) = b_R^*(m|N) = 1,$$
$$supp(\rho) = \{\pi_3, \pi_3'\} \text{ such that } \rho(\pi_3') < \rho(\pi_3),\ \ \mu_\rho^*(\theta_L|E) = 1/3 \text{ and } \mu_\rho^*(\theta_L|N) = 1,\ \text{ where}$$
$$\pi_3 := \{\pi_3(E, \theta_L) = 1/3, \pi_3(E, \theta_H) = 2/3\} \text{ where } \bar{\beta}_R = \left(\bar{\beta}_R(E|\theta_L) = 1,\ \bar{\beta}_R(E|\theta_H) = 1\right),$$
$$\pi_3' := \{\pi_3'(N, \theta_L) = 1/3, \pi_3'(E, \theta_H) = 2/3\} \text{ where } \bar{\beta}_R' = \left(\bar{\beta}_R'(N|\theta_L) = 1,\ \bar{\beta}_R'(E|\theta_H) = 1\right).\text{[23]}$$

HTE-4: In this equilibrium, both types pool on N; i.e.,

$$b_S^*(N|\theta_L) = b_S^*(N|\theta_H) = 1,\ \ b_R^*(m|E) = b_R^*(e|N) = 1,$$
$$supp(\rho) = \{\pi_4, \pi_4'\} \text{ such that } \rho(\pi_4') < \rho(\pi_4),\ \mu_\rho^*(\theta_L|E) = 1 \text{ and } (\theta_L|N) = 1/3,\ \text{ where}$$
$$\pi_4 := \{\pi_4(N, \theta_L) = 1/3,\ \pi_4(N, \theta_H) = 2/3\} \text{ given } \bar{\beta}_R := (\bar{\beta}_R(N|\theta_L) = 1,\ \bar{\beta}_R(N|\theta_H) = 1),$$
$$\pi_4' := \{\pi_4'(E, \theta_L) = 1/3,\ \pi_4'(N, \theta_H) = 2/3\} \text{ given } \bar{\beta}_R' := (\bar{\beta}_R'(E|\theta_L) = 1,\ \bar{\beta}_R'(N|\theta_H) = 1).\text{[24]}$$

Note that HTE-3 passes the Intuitive Criterion but not HTE-4.

Interestingly, a majority of subjects behaved consistently with Rational HTE-4.[25] As in the previous case, BH found some evidence for the new hypothesis π_4'. According to π_4', Senders

[23] $\bar{\beta}_R$ and $\bar{\beta}_R'$ best respond to $b_R(e|E) = b_R(m|N) = 1$ and $b_R'(e|E) = b_R'(e|N) = 1$, respectively.
[24] $\bar{\beta}_R$ and $\bar{\beta}_R'$ best respond to $b_R'(m|E) = b_R'(m|N) = 1$ and $b_R(m|E) = b_R(e|N) = 1$, respectively.
[25] 23 out of 144 decisions matched with Rational HTE-3, while 84 out of 144 matched with Rational HTE-4.

"reversely" separate, i.e., the low-skilled type chooses E, while the high-skilled type chooses N. BH found that 72 out of 99 high-skilled Senders played N while 20 out of 45 low-skilled Senders played E. Notably, a significant number of Receivers believed that Senders do "reversely" separate. Then, as BH reported, 24 out of 47 Receivers responded with m to E.

In sum, the Rational HTE can explain the experimental results of Brandts and Holt (1992) better than the Intuitive Criterion, making our solution concept empirically relevant.[26]

7 Conclusion

In this paper, we have suggested a solution concept for signaling games that admits belief updating at information sets with zero probability via the Hypothesis Testing model of Ortoleva (2012). In our Rational Hypothesis Testing Equilibrium, beliefs are derived from hypotheses about rational behavior of the Sender, together with the prior information about types. We have argued that this equilibrium has two desirable properties: First, beliefs are structurally consistency in the spirit of Kreps and Wilson (1982). Second, beliefs are consistent with mutual knowledge of rationality.

Our equilibrium notion provides an alternative tool for equilibrium selection. On the one hand, the Rational Hypothesis Testing Equilibrium refines PBE for which the Intuitive Criterion admits arbitrary beliefs. On the other hand, our equilibrium notion is more general than the Intuitive Criterion, allowing us explain empirical findings reporting behavior that is inconsistent with Intuitive PBE. Thus, we believe that our equilibrium notion is worth further exploration and application.

So far, we have studied the Rational Hypothesis Testing Equilibrium in signaling games. One direction for future research is to extend the solution concept to general extensive-form games. To this end, however, one first needs to solve the puzzling result by Kreps and Ramey (1987) and characterize the class of extensive-form games that admit structural consistency. Later, one could explore the implications of rational hypotheses on equilibrium selection. We leave this for future research.

A Proofs

Proof of Theorem 1. Let (b_S^*, b_R^*, μ^*) be a PBE and $\mathcal{M}'$ be the set of out-of-equilibrium messages.

Consider the equilibrium strategy b_S^*. Let β_R be the Receiver's belief such that $\beta_R = b_S^*$. This belief together with the prior p on Θ defines the initial hypothesis: for each $(m, \theta) \in \mathcal{M} \times \Theta$,

[26]Other studies have tested the Intuitive Criterion. For instance, Banks, Camerer, and Porter (1994) found evidence in favor of intuitive PBE. However, implementing similar games as Banks, Camerer, and Porter (1994), Brandts and Holt (1993) could not find unequivocal support for Intuitive PBEs. Instead, Brandts and Holt (1993) replicated similar patterns of equilibrium behaviors as the ones reported in Brandts and Holt (1992). Over a series of treatments, a majority of subjects behaved consistently with an Unintuitive PBE that is in fact consistent with Rational HTE.

$\pi^*(m,\theta) = b_S^*(m|\theta)p(\theta)$. By Condition (iii) in Definition 2, for any $m \in \mathcal{M}$ with $\pi^*(m,\Theta) > 0$,

$$\mu^*(\theta|m) = \mu_\rho(\theta|m) = \frac{\pi^*(m,\theta)}{\pi^*(m,\Theta)} \quad \text{for each } \theta \in \Theta, \tag{20}$$

showing that π^* induces the equilibrium beliefs on the path.

Now, fix $m' \in \mathcal{M}'$. Consider the following strategy b_S for the Sender:

$$b_S(m|\theta) = \begin{cases} \left(\frac{\mu^*(\theta|m')}{p(\theta)}\right)\frac{1}{X}, & \text{for } m = m', \\ 1 - \left(\frac{\mu^*(\theta|m')}{p(\theta)}\right)\frac{1}{X}, & \text{for some } m \in (\mathcal{M} \setminus \{m'\}), \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $X := \sum_{\theta \in \Theta} \frac{\mu^*(\theta|m')}{p(\theta)}$. Since $X \geq \frac{\mu^*(\theta|m')}{p(\theta)}$ and $\sum_{m \in \mathcal{M}} b_S(m|\theta) = 1$ for any $\theta \in \Theta$, b_S is well-defined. According to this strategy, only the types in the support of $\mu^*(\cdot|m')$ play m' with a strictly positive probability (i.e., $b_S(m'|\theta) > 0$ for each $\theta \in \Theta$, such that $\mu^*(\theta|m') > 0$).

Thus, $\beta_R = b_S$ and the prior p define the alternative hypothesis $\pi_{m'}$: for each $(m,\theta) \in \mathcal{M} \times \Theta$,

$$\pi_{m'}(m,\theta) = \beta_R(m|\theta)p(\theta). \tag{22}$$

By updating $\pi_{m'}$ conditional on m', by (21) and (22), we have

$$\mu_\rho(\theta|m') = \frac{\pi_{m'}(m',\theta)}{\pi_{m'}(m',\Theta)} = \frac{\frac{\mu^*(\theta|m')}{\sum_{\theta \in \Theta} \frac{\mu^*(\theta|m')}{p(\theta)}}}{\frac{1}{\sum_{\theta \in \Theta} \frac{\mu^*(\theta|m')}{p(\theta)}}} \quad \text{for every } \theta \in \Theta, \tag{23}$$

yielding the the PBE belief off the path; i.e., $\mu_\rho(\theta|m') = \mu^*(\theta|m')$ for each $\theta \in \Theta$.

Since m' was chosen arbitrarily, for each out-of-equilibrium message $m' \in \mathcal{M}'$, there is a hypothesis $\pi_{m'}$ that will induce $\mu^*(\cdot|m')$. Let $\{\pi_{m'}\}_{m' \in \mathcal{M}'}$ be the family of such hypotheses.

Finally, we can choose a strict partial order ρ with $supp(\rho) = \{\pi^*, \pi^{**}_{m'}\}_{m' \in \mathcal{M}'}$, such that

$$\{\pi^*\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi) \text{ and } \{\pi^{**}_{m'}\} := \arg\max_{\pi \in supp(\rho)} \rho_{m'}(\pi) \quad \text{for each } m' \in \mathcal{M}', \tag{24}$$

Hence, there exists a Focused HTE $(b_S^*, b_R^*, \rho, \mu_\rho^*)$ supporting the PBE (b_S^*, b_R^*, μ^*). ■

Proof of Proposition 1. We consider two cases.

Case 1. Consider the following family of Pooling PBEs for the game in Figure 4.

$$b_S^*(N|\theta_L) = b_S^*(N|\theta_H) = 1,\ b_R^*(m|E) = b_R^*(e|N) = 1, \tag{PBE-1}$$
$$\mu^*(\theta_L|N) = 1/3 \text{ and } \mu^*(\theta_L|E) \geq 1/2.$$

In Section 6, we have shown that this equilibrium is supported by Rational HTE-4.

Now, we argue that PBE-1 fails the Intuitive Criterion. According to the Intuitive Criterion, θ_H can be better off than his equilibrium payoff if he plays the out-of-equilibrium message E. That is, $I(E) = \{\theta_H\}$. This induces the out-of-equilibrium belief $\mu(\theta_H|E) = 1$. However, if the Receiver learns that E was chosen by θ_H, she will play e instead of m. Given that the Receiver responds e against E, type θ_H will indeed choose E, showing that PBE-1 fails the Intuitive Criterion.

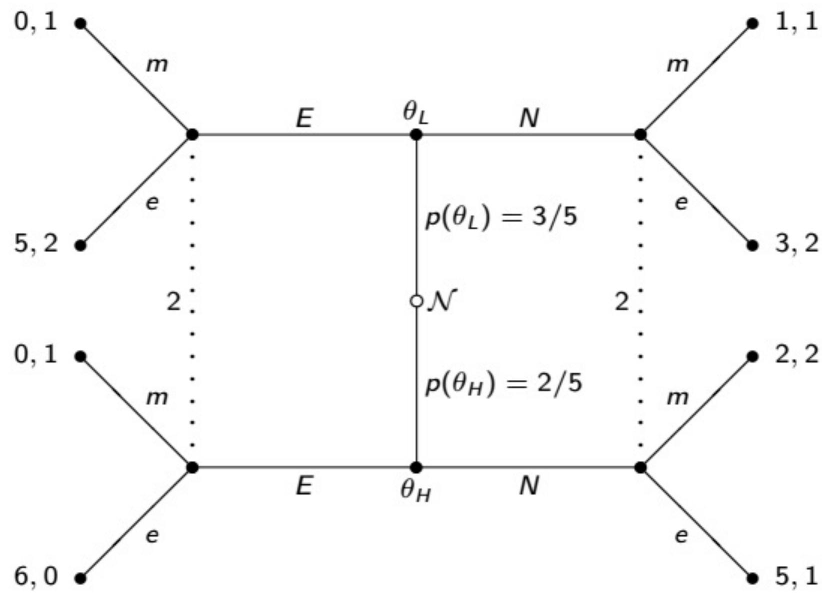

Figure 5: An intuitive PBE that fails the Rational HT refinement

Case 2. Consider the game in Figure 5. It has the following family of Pooling PBEs:

$$b_S^*(N|\theta_L) = b_S^*(N|\theta_H) = 1,\ b_R^*(m|E) = b_R^*(e|N) = 1, \qquad \text{(PBE-2)}$$
$$\mu^*(\theta_L|N) = 3/5 \text{ and } \mu^*(\theta_L|E) \leq 1/2.$$

First, we show that there does not exist any Rational HTE that supports any of the PBEs. Note that any strategy $b_R = (b_R(\cdot|E), b_R(\cdot|N))$ is rational. Thus, $\mathcal{B}_R = \mathcal{B}_R^\bullet$. To support the Receiver's off-the-equilibrium behavior $b_R^*(m|E)$, we need to determine the strategies for the Sender that best respond against $b_R \in \mathcal{B}_R^\bullet$. Denote by $x := b_R(m|E)$ and $y := b_R(m|N)$ the probabilities that the Receiver plays m in response to E and in response to N, respectively. Then, θ_L will choose E if

$$2y - 5x \geq -2 \iff y \geq \frac{5}{2}x - 1. \qquad (25)$$

Similarly, θ_H will choose E if

$$3y - 6x \geq -1 \;\;\Leftrightarrow\;\; y \geq 2x - \frac{1}{3}. \tag{26}$$

Note that $2x - \frac{1}{3} > \frac{5}{2}x - 1$ for any $x \in [0,1]$. Hence, for each $(x,y) \in [0,1] \times [0,1]$ that satisfies (26), (25) is satisfied with strict inequality. This means that type θ_L strictly prefers E to N whenever type θ_H weakly prefers E. That is, for any $b_R \in \mathcal{B}_R^\bullet$, whenever

$$\sum_{a \in \mathcal{A}} u_S(\theta_H, E, a) b_R(a|E) \geq \sum_{a \in \mathcal{A}} u_S(\theta_H, N, a) b_R(a|N), \tag{27}$$

we have

$$\sum_{a \in \mathcal{A}} u_S(\theta_L, E, a) b_R(a|E) > \sum_{a \in \mathcal{A}} u_S(\theta_L, N, a) b_R(a|N). \tag{28}$$

Hence, there is no rational strategy $b_S \in \mathcal{B}_S^\bullet$ such that $b_S(E|\theta_H) > b_S(E|\theta_L)$. Thus, by updating any hypothesis based on a system of beliefs $\bar{\beta}_R \in \mathcal{B}_S^\bullet$, we will have $\mu_\rho(\theta_L|E) \geq 3/5$. However, given such posteriors, the Receiver will choose e instead of m. Hence, there does not exist any Rational HTE supporting any of PBE-2.

Now, we show that each Pooling PBE-2 passes the Intuitive Criterion. According to the Intuitive Criterion, both types θ_L and θ_H could be better off than their equilibrium payoff by choosing the out-of-equilibrium message E. That is, $I(E) = \{\theta_L, \theta_H\}$. In this case, the Intuitive Criterion admits all beliefs over $I(E) = \{\theta_L, \theta_H\}$, including any belief, such that $\mu(\theta_L|E) \leq 1/2$. We know that no player has an incentive to deviate from the equilibrium strategy as long as $\mu(\theta_L|E) \leq 1/2$. Therefore, the whole family of PBE-2 passes the Intuitive Criterion. ■

Proof of Lemma 1. Let (b_S^*, b_R^*, μ^*) be a PBE. Fix an out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$. Since $I(m^\circ) \neq \emptyset$, there is an action $a^\circ \in BR(\Theta, m^\circ)$ such that

$$u_S^*(\theta) \leq u_S(\theta, m^\circ, a^\circ) \;\text{ for some } \theta \in I(m^\circ). \tag{29}$$

Denote by $I(m^\circ; a^\circ)$ the set of types that satisfy (29). We define a rational strategy b_R° as follows: For $m \in \mathcal{M} \setminus \{m^\circ\}$,

$$b_R^\circ(a|m) = b_R^*(a|m) \;\text{ for each } a \in \mathcal{A}, \tag{30}$$

and for $m = m^\circ$,

$$b_R^\circ(a|m) = \begin{cases} 0, & \text{for } a \in \mathcal{A} \setminus \{a^\circ\}, \\ 1, & \text{for } a = a^\circ. \end{cases} \tag{31}$$

Note that b_R° is identical to the equilibrium strategy b_R^* except when the Receiver observes m°. Since $b_R^\circ(a^\circ|m^\circ) = 1$ is a best response to some belief over Θ, b_R° is rational; i.e, $b_R^\circ \in \mathcal{B}_R^\bullet$.

For the Sender, let b_S° be a best-response to b_R° that generates m°. That is, if $\theta \in \Theta \setminus I(m^\circ; a^\circ)$,

$$b_S^\circ(m|\theta) = b_S^*(m|\theta) \text{ for each } m \in \mathcal{M}, \tag{32}$$

and if $\theta \in I(m^\circ; a^\circ)$,

$$b_S^\circ(m|\theta) = \begin{cases} 0, & \text{for } m \in \mathcal{M} \setminus \{m^\circ\}, \\ 1, & \text{for } m = m^\circ. \end{cases} \tag{33}$$

The Receiver's belief $\bar{\beta}_R = \beta_S^\circ$ together with the prior p defines the following rational hypothesis π_{m°:

$$\pi_{m^\circ}(m, \theta) := \begin{cases} p(\theta), & \text{if } (m, \theta) \in (m^\circ, I(m^\circ; a^\circ)), \\ b_S^*(m|\theta)p(\theta), & \text{if } (m, \theta) \in (\mathcal{M}, \Theta \setminus I(m^\circ; a^\circ)). \end{cases} \tag{34}$$

By updating π_{m° given m°, we get $\mu_\rho(\cdot|m^\circ)$ that satisfies $\mu_\rho(\theta|m^\circ) > 0$ for each $\theta \in I(m^\circ; a^\circ)$. Proceeding in this way for each $m^\circ \in \mathcal{M}^\circ$, we generate a rational hypothesis π_{m° consistent with m°. ■

Proof of Theorem 2. Let (b_S^*, b_R^*, μ^*) be an Intuitive PBE. First, we show that for each off-path message $m^\circ \in \mathcal{M}^\circ$, there exists a rational hypothesis π_{m° that induces the PBE belief in (13). Second, we construct a rational hypothesis that induces the equilibrium beliefs on the path.

Step 1. Fix an out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$. By assumption, there is a response a° to m° such that (i) $I(m^\circ; a^\circ) \neq \emptyset$ and (ii) the PBE belief conditional on m° (i.e., (13)) is given by

$$\mu^*(\theta|m^\circ) = \begin{cases} \frac{p(\theta)}{\sum\limits_{\theta' \in I(m^\circ; a^\circ)} p(\theta')} & \text{if } \theta \in I(m^\circ; a^\circ), \\ 0 & \text{if } \theta \notin I(m^\circ; a^\circ). \end{cases}$$

Then, we apply the arguments presented in the proof of Lemma 1 to construct (i) a rational strategy b_R° for the Receiver (as in (30) and (31)), (ii) a rational strategy b_S° for the Sender (as in (32) and (33)), and (iii) a rational hypothesis π_{m° that is consistent with m° (as in (34)), which is given by

$$\pi_{m^\circ}(m, \theta) := \begin{cases} p(\theta), & \text{if } (m, \theta) \in \{m^\circ\} \times I(m^\circ; a^\circ), \\ b_S^*(m|\theta)p(\theta), & \text{if } (m, \theta) \in \mathcal{M} \times \Theta \setminus I(m^\circ; a^\circ). \end{cases} \tag{35}$$

Since $\pi_{m^\circ}(m^\circ, \Theta) > 0$ and $b_S^*(m^\circ|\theta) = 0$ for all $\theta \notin I(m^\circ; a^\circ)$, by updating π_{m° given m°, we get $\mu_\rho(\theta|m^\circ)$ for any $\theta \in \Theta$ given by

$$\mu_\rho(\theta|m^\circ) = \frac{\pi_{m^\circ}(m^\circ, \theta)}{\sum_{\theta' \in \Theta} \pi_{m^\circ}(m^\circ, \theta')} = \frac{\pi_{m^\circ}(m^\circ, \theta)}{\sum_{\theta' \in I(m^\circ; a^\circ)} \pi_{m^\circ}(m^\circ, \theta')} = \frac{\pi_{m^\circ}(m^\circ, \theta)}{\sum_{\theta' \in I(m^\circ; a^\circ)} p(\theta')}.$$

Thus, we get the assumed PBE belief; i.e.,

$$\mu_\rho^*(\theta|m^\circ) = \begin{cases} \frac{p(\theta)}{\sum\limits_{\theta' \in I(m^\circ;a^\circ)} p(\theta')} & \text{if } \theta \in I(m^\circ; a^\circ), \\ 0 & \text{if } \theta \notin I(m^\circ; a^\circ). \end{cases} \tag{36}$$

Step 2. Consider b_S^*. Since $b_R^* \in \mathcal{B}_R^\bullet$ and b_S^* best responds to b_R^*, b_S^* is rational (i.e., $b_S^* \in \mathcal{B}_S^\bullet$). Hence, $\bar{\beta}_R = b_S^*$ and p define the rational hypothesis π^*; i.e., for each $(m, \theta) \in \mathcal{M} \times \Theta$,

$$\pi^*(m, \theta) = b_S^*(m|\theta)p(\theta). \tag{37}$$

Finally, we can choose a second-order prior ρ with $supp(\rho) = \{\pi^*, \pi_{m^\circ}^{**}\}_{m^\circ \in \mathcal{M}^\circ}$, such that

$$\{\pi^*\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi) \text{ and } \{\pi_{m^\circ}^{**}\} := \arg\max_{\pi \in supp(\rho)} \rho_{m^\circ}(\pi) \text{ for each } m^\circ \in \mathcal{M}^\circ, \tag{38}$$

showing that there exists a Rational HTE $(b_S^*, b_R^*, \rho, \mu_\rho^*)$ supporting the Intuitive PBE. ■

Proof of Corollary 1. Consider an Intuitive PBE (b_S^*, b_R^*, μ^*). Fix an out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$. By assumption, $I(m^\circ) = \{\theta_{m^\circ}\}$. Thus, the PBE belief satisfies $\mu^*(\theta_{m^\circ}|m^\circ) = 1$.

Since $I(m^\circ) = \{\theta_{m^\circ}\}$, there exists $a^\circ \in BR(\Theta, m^\circ)$ such that $I(m^\circ) = I(m^\circ; a^\circ) \neq \emptyset$. That is,

$$u_S^*(\theta_{m^\circ}) \leq u_S(\theta_{m^\circ}, m^\circ, a^\circ).$$

Moreover, given $a^\circ \in BR(\Theta, m^\circ)$, $\mu^*(\theta_{m^\circ}|m^\circ) = 1$ satisfies Condition (13) in Theorem 2. Thus, by Theorem 2, there exists a Rational HTE that explains the intuitive PBE. ■

Proof of Proposition 2. We show that the following PBEs are supported by Rational HTEs:

(i) $b_S^*(e^*|\theta_L) = b_S^*(e^*|\theta_H) = 1$ such that $e_0 \leq e^* \leq y := (\theta_H - \theta_L)\big(\theta_L - (1-\alpha)\theta_H\big)$.

$$(ii)\ w^*(e) = \begin{cases} \theta_L & \text{if } e < e^*, \\ \mathbb{E}(\theta) & \text{if } e^* \leq e \leq e_n, \\ \theta_H & \text{if } e_n < e \leq e_N. \end{cases} \qquad (iii)\ \mu^*(\theta_L|e) = \begin{cases} 1 & \text{if } e < e^*, \\ 1-\alpha & \text{if } e^* \leq e \leq e_n, \\ 0 & \text{if } e_n < e \leq e_N. \end{cases}$$

Fix a pooling message e_i^* such that $e_0 \leq e_i^* \leq y$. We first construct a rational hypothesis π_0^* that justifies the posterior on the path, $\mu^*(\theta|e_i^*)$. The equilibrium strategy b_S^* is rational, as it best responds to $w^*(e)$. Hence, $\bar{\beta}_R = b_S^*$, together with p, induces the following rational hypothesis:

$$\pi_0^*(e, \theta) = b_S^*(e|\theta)p(\theta) \text{ for each } (e, \theta) \in \mathcal{M} \times \Theta. \tag{39}$$

By updating π_0^* conditional on e_i^*, we obtain $\mu(\theta_L|e_i^*) = p(\theta_L) = 1 - \alpha$.

Now, for each $e \in \mathcal{M}^\circ = \mathcal{M} \setminus \{e_i^*\}$, we construct a rational hypothesis that is consistent with e. W.l.o.g., we limit our attention to the following partition of $\mathcal{M}^\circ$:

$$\mathcal{P}(\mathcal{M}^\circ) = \Big\{ \underbrace{\{e_0, \dots, e_{i-1}\}}_{\text{Case 1}}, \underbrace{\{e_{i+1}, \dots, e_n\}}_{\text{Case 2}}, \underbrace{\{e_{n+1}, \dots, e_N\}}_{\text{Case 3}} \Big\}, \tag{40}$$

where $e_n := \theta_L(\theta_H - \theta_L)$ and $e_N := \theta_H(\theta_H - \theta_L)$.

We consider the three cases.

Case 1. Fix $e' \in \{e_0, \dots, e_{i-1}\}$. Consider the following strategy $w_1(e)$ for the employer, together with the posterior that rationalizes it:

$$w_1(e) = \begin{cases} w' & \text{if } e = e', \\ \theta_H, & \text{if } e = e_n, \\ \theta_L, & \text{elsewhere,} \end{cases} \quad \text{and} \quad \mu(\theta_L|e) = \begin{cases} \frac{\theta_H - w'}{\theta_H - \theta_L}, & \text{if } e = e', \\ 0, & \text{if } e = e_n \\ 1, & \text{elsewhere.} \end{cases} \tag{41}$$

Note that w' must satisfy the following conditions. First, w' for e' has to make the low-productivity type better off than his payoff for offering education level 0 at the lowest wage θ_L; i.e,

$$w' - \frac{e'}{\theta_L} > \theta_L - \frac{0}{\theta_L}, \text{ or equivalently, } w' > \theta_L + \frac{e'}{\theta_L}. \tag{42}$$

Second, w' for e' has to make the high-productivity type worse off than his payoff for offering e_n at the highest wage θ_H; i.e.,

$$w' - \frac{e'}{\theta_H} < \theta_H - \frac{e_n}{\theta_H}, \text{ or equivalently, } w' < \theta_H + \frac{e'}{\theta_H} - \frac{e_n}{\theta_H}. \tag{43}$$

By (42) and (43), $\mu(\theta_L|e')$ is defined by

$$\frac{e_n - e'}{\theta_H(\theta_H - \theta_L)} < \mu(\theta_L|e') := \frac{\theta_H - w'}{\theta_H - \theta_L} < 1 - \frac{e'.}{\theta_L(\theta_H - \theta_L)}. \tag{44}$$

The worker's strategy $b_S := (b_S(e'|\theta_L) = 1, b_S(e_n|\theta_H) = 1)$ best responds to $w_1(e)$. Hence, it is rational. Therefore, $\bar{\beta}_R = b_S$, together with the prior p, induces the simple-rational hypothesis $\pi_1(e')$, yielding the PBE belief $\mu(\theta_L|e')^* = 1$ for each $e' \in \{e_0, \dots, e_{i-1}\}$.

Case 2. Fix $e' \in \{e_{i+1}, \dots, e_n\}$. Consider the following strategy $w_2(e)$ for the employer, together with the posterior that rationalizes it:

$$w_2(e) = \begin{cases} \theta_H, & \text{if } e = e', \\ \theta_L, & \text{elsewhere,} \end{cases} \quad \text{and} \quad \mu(\theta_L|e) = \begin{cases} 0, & \text{if } e = e', \\ 1, & \text{elsewhere.} \end{cases} \tag{45}$$

The strategy $b'_S := (b'_S(e'|\theta_L) = 1, b'_S(e'|\theta_H) = 1)$ best responds to $w_2(e)$. Hence, it is rational. Therefore, $\bar{\beta}'_R = b'_S$, together with the prior p, induces the simple-rational hypothesis $\pi_2(e')$, yielding the PBE belief $\mu^*(\theta_L|e') = p(\theta_L) = 1 - \alpha$ for each $e' \in \{e_{i+1}, \ldots, e_n\}$.

Case 3. Fix $e' \in \{e_{n+1}, \ldots, e_N\}$. Consider the following strategy $w_3(e)$ for the employer, together with the posterior that rationalizes it:

$$w_3(e) = \begin{cases} \theta_H, & \text{if } e = e', \\ \theta_L, & \text{elsewhere}, \end{cases} \quad \text{and} \quad \mu(\theta_L|e) = \begin{cases} 0, & \text{if } e = e', \\ 1, & \text{elsewhere}. \end{cases} \tag{46}$$

The worker's strategy $b''_S := (b''_S(e_0|\theta_L) = 1, b''_S(e'|\theta_H) = 1)$ best responds to $w_3(e)$. Hence, it is rational. Therefore, $\bar{\beta}''_R = b''_S$, together with the prior p, induces the simple-rational hypothesis $\pi_3(e')$, yielding the PBE belief $\mu^*(\theta_L|e') = 0$ for each $e' \in \{e_{n+1}, \ldots, e_N\}$.

Finally, we can suitably choose a second-order prior ρ such that

$$supp(\rho) = \{\pi_0, \pi_1(e)_{e \in \{e_0, \ldots, e_{i-1}\}}, \pi_2(e)_{e \in \{e_{i+1}, \ldots, e_n\}}, \pi_3(e)_{e \in \{e_{n+1}, \ldots, e_N\}}\},$$

$$\{\pi_0\} := \underset{\pi \in supp(\rho)}{\arg\max}\, \rho(\pi) \ \text{ and } \ \{\pi^{**}(e)\} := \underset{\pi \in supp(\rho)}{\arg\max}\, \rho_e(\pi) \ \text{ for each } e \in \mathcal{M}^\circ.$$

Thus, there exists a Rational HTE supporting the Pooling PBE with e_i^*, such that $e_0 \le e_i^* \le y$. ■

B Existence of Rational HTE

In this Appendix, we provide sufficiency conditions for existence of a Rational HTE. We consider finite monotone signaling games that resemble the properties of monotone signaling games with continuous spaces (e.g., see Mailath, 1987; Cho and Sobel, 1990; Kreps and Sobel, 1994).

We assume that Θ, $\mathcal{M}$ and $\mathcal{A}$ are finite, partially ordered sets of real numbers. That is,

$$\begin{aligned} \Theta &= \{\theta_1, \theta_2, \ldots, \theta_T\} \quad \text{where } \theta_t \in \mathbb{R} \text{ for } t = 1, \ldots, T; \\ \mathcal{M} &= \{m_1, m_2, \ldots, m_L\} \quad \text{where } m_l \in \mathbb{R} \text{ for } l = 1, \ldots, L; \\ \mathcal{A} &= \{a_1, a_2, \ldots, a_K\} \quad \text{where } a_k \in \mathbb{R} \text{ for } k = 1, \ldots, K. \end{aligned}$$

For the Sender, we assume that u_S satisfies Monotonicity and Single-Crossing Property.

(i) (Monotonicity) $u_S(\theta, m, a)$ is strictly decreasing in m and strictly increasing in a for any θ.

(ii) (Single-Crossing Property) For each $a \in A$, all $\theta, \theta' \in \Theta$ and $m, m' \in \mathcal{M}$, such that $\theta' > \theta$ and $m' > m$, $u_S(\theta, m, a) \le u_S(\theta, m', a')$ implies $u_S(\theta', m, a) < u_S(\theta', m', a')$.

For the Receiver, we assume that her best-reply correspondence is message-independent, single-valued, and increasing in θ. Moreover, the "highest" type θ_T has an incentive to signal m_L.

(iii) For each $m \in \mathcal{M}$ and $\mu := \mu(\cdot \mid m) \in \Delta(\Theta)$, $BR(\mu, m) = BR(\mu)$. Moreover, $BR(\mu(\theta) = 1)$ is increasing in θ, and $BR(\mu(\theta) = 1)$ is single-valued for each $\theta \in \Theta$.

(iv) For m_1, m_L and θ_T, $u_S(\theta_T, m_L, BR(\mu(\theta_T) = 1)) \geq u_S(\theta_T, m_1, BR(\mu(\theta_1) = 1))$.

Denote by $\mathcal{G}_M$ the family of signaling games that satisfy Conditions (i) through (iv).

We show that under another mild condition imposed on the prior, p, a Pooling Rational HTE exists. Denote by $\overline{m}$ the "highest" message that makes type θ_1 better off than choosing the "lowest" message m_1 if the Receiver believes that $\overline{m}$ is chosen by θ_H while m_1 is chosen by θ_1; i.e.,

$$\overline{m} := \max \{m \in \mathcal{M} : u_S(\theta_1, m_1, BR(\mu(\theta_1) = 1)) \leq u_S(\theta_1, m, BR(\mu(\theta_T) = 1))\}. \tag{47}$$

Notice that any message $m' \leq \overline{m}$ is rational for type θ_1 in the sense that θ_1 best replies by choosing m' if the Receiver behaves according to the following strategy:

$$b_R(BR(\mu(\theta_T) = 1)|m') = 1 \text{ and } b_R(BR(\mu(\theta_1) = 1)|m) = 1 \text{ for any } m \neq m'.$$

For a given $\overline{m}$, denote by $\underline{a}$ the "lowest" response that satisfies

$$\underline{a} := \min \{a \in \mathcal{A} : u_S(\theta_T, \overline{m}, BR(\mu(\theta_T) = 1)) \leq u_S(\theta_T, m_1, a)\}. \tag{48}$$

Since $\underline{a} \leq BR(\mu(\theta_T) = 1)$, $\underline{a}$ is well-defined. Note that for any $a \geq \underline{a}$, type θ_T prefers m_1 to $\overline{m}_1$. Moreover, Monotonicity and Single-Crossing Property imply that this is true for all types.

Lemma 2 *If the Receiver's best reply to m_1 is $a \geq \underline{a}$, then any $\theta \in \Theta$ prefers m_1 over $m > \overline{m}$.*

Proof. Let $\{a\} = BR(\mu(\theta_T) = 1)$ be the Receiver's best response against m_1. If $a \geq \underline{a}$, type θ_T prefers choosing m_1 over $\overline{m}$ by Definition (48). By Single-Crossing Property, this is true for all the other types $\theta_t < \theta_T$. Since $BR(\mu(\theta_T) = 1)$ is the highest response by the rational Receiver, Monotonicity implies that m_1 is preferred to any message $m > \overline{m}$ for all $\theta \in \Theta$. ■

For our existence result, we need to ensure that all types have potentially an incentive to pool on some message. To guarantee this, we assume that the prior probability distribution p is "skewed" towards "higher" types so that the Receiver's best response with respect to p is "higher" than $\underline{a}$.[27]

(v) (Skewness) The prior probability distribution $p \in \Delta(\Theta)$ is such that $BR(p) \geq \underline{a}$.

[27]For instance, the version of the signaling game of Spence (1973) satisfies Conditions (i)-(v) (see Section 5).

We can now prove existence of a Pooling Rational HTE for each signaling games in $\mathcal{G}_M$.

Proposition 3 *Under Conditions* (i)-(v)*, there exists a Pooling Rational HTE.*

Proof. Consider a strategy profile (b_S^*, b_R^*), such that $b_S^*(m_1|\theta) = 1$ for any $\theta \in \Theta$ and

$$b_R^*(BR(p)|m') = 1,\ b_R^*(BR(\mu(\theta_T) = 1)|m'') = 1, \tag{49}$$

for any $m', m'' \in \mathcal{M}$, such that $m_1 \leq m' \leq \overline{m}$ and $\overline{m} < m'' \leq m_L$, where $\overline{m}$ is defined as in Equation (47). The Receiver's strategy is optimal with respect to $\mu^* := \{\mu^*(\cdot|m)\}_{m\in\mathcal{M}}$, where

$$\mu^*(\theta_t|m') = p(\theta_t) \text{ for any } \theta_t \in \Theta \text{ and } m_1 \leq m' \leq \overline{m}, \tag{50}$$

$$\mu^*(\theta_T|m'') = 1 \text{ for any } \overline{m} < m'' \leq m_L. \tag{51}$$

By Condition (iii) and Lemma 1, (b_S^*, b_R^*, μ^*) is a PBE. Hence, it remains to show that μ^* can be justified by a set of rational hypotheses.

We distinguish two cases. In Case 1, $\overline{m} = m_L$ and in Case 2, $\overline{m} < m_L$.

Case 1. Consider $\overline{m} = m_L$. In this case, type θ_1 can choose any message m in $\mathcal{M}$ as a best response. Specifically, consider a rational strategy b'_R of the Receiver, such that

$$b'_R(BR(\theta_T) = 1|m') = 1 \text{ and } b'_R(BR(\theta_1) = 1|m) = 1 \text{ for any } m \neq m',$$

where $m_1 \leq m' \leq m_L$. By Monotonicity, type θ_1 will choose m' as a best response. Moreover, Single-Crossing Property implies that this is true for all types. Thus, against b'_R, each $\theta_t \in \Theta$ chooses m' as a best response. Let b'_S be such a best response strategy (i.e., $b'_S(m'|\theta_t) = p(\theta_t)$ for each $\theta_t \in \Theta$). The Receiver's belief $\bar{\beta}_R = b'_S$ and p induce the following rational hypothesis $\pi_{m'}$:

$$\pi_{m'}(m, \theta) = b'_S(m|\theta)p(\theta) \text{ for any } (m, \theta) \in \mathcal{M} \times \Theta. \tag{52}$$

By updating $\pi_{m'}$ on m', we thus obtain $\mu_\rho^*(\theta_t|m_1) = p(\theta_t)$ for any $\theta_t \in \Theta$. Thus, $\mu_\rho^*(\theta_t|m') = p(\theta_t)$ for any $\theta_t \in \Theta$ and m', such that $m_1 \leq m' \leq m_L$.

Case 2. Consider $\overline{m} < m_L$. For any m', such that $m_1 \leq m' \leq \overline{m}$, we apply Case 1 to construct the rational hypothesis $\pi_{m'}$ that is consistent with m' as defined in Equation (52).

Now, we construct a rational hypothesis consistent with m'', where $\overline{m} < m'' \leq m_L$. Condition (iv) implies that type θ_T chooses m_L as a best response to the following rational strategy b_R: $b_R := (b_R(BR(\mu(\theta_T) = 1)|m_L) = 1$ and $b_R(BR(\mu(\theta_1) = 1)|m) = 1)$ for any $m \neq m_L$.

Consider another rational strategy b''_R for the Receiver,

$$b''_R(BR(\mu(\theta_T) = 1)|m'') = 1 \text{ and } b''_R(\cdot|m) \in \Delta(\mathcal{A}) \text{ for any } m \neq m'', \tag{53}$$

such that

$$u_S(\theta_T, m'', BR(\mu(\theta_T) = 1)) = \sum_a u_S(\theta_T, m_1, a) b''_R(a|m). \tag{54}$$

By construction of $\underline{a}$, we have $u_S(\theta_T, \overline{m}, BR(\mu(\theta_T) = 1)) \leq u_S(\theta_T, m_1, \underline{a})$. Since Monotonicity implies $u_S(\theta_T, m'', BR(\mu(\theta_T) = 1)) < u_S(\theta_T, \overline{m}, BR(\mu(\theta_T) = 1))$ for $m'' > \overline{m}$, it is true that

$$\sum_a u_S(\theta_T, m_1, a) b''_R(a|m) < u_S(\theta_T, m_1, \underline{a}).$$

By Condition (iii), $\sum_a a b''_R(a|m) \in (a_1, a_T)$, where $\{a_1\} = BR(\mu(\theta_1) = 1)$ and $\{a_T\} = BR(\mu(\theta_T) = 1)$. Therefore, b''_R exists.

Since θ_T is indifferent between choosing m'' and m_1 to b''_R, by Single-Crossing Property, any type θ_t such that $\theta_t < \theta_T$ plays m_1 against b''_R. Thus, the rational strategy b''_S for the Sender,

$$b''_S := (b_S(m_1|\theta_t) = 1 \ , \ b_S(m''|\theta_T) = 1) \ \text{ for } \theta_t \neq \theta_T, \tag{55}$$

best responds to b''_R. Hence, $\bar{\beta}_R = b''_S$ and p induce the following rational hypothesis $\pi_{m''}$:

$$\pi_{m''}(m, \theta) = b''_S(m|\theta) p(\theta) \ \text{ for each } (m, \theta) \in \mathcal{M} \times \Theta. \tag{56}$$

By updating $\pi_{m''}$ on m'', we obtain $\mu^*_\rho(\theta_T|m'') = 1$. Thus, $\mu^*_\rho(\theta_T|m'') = 1$ for any $\theta_t \in \Theta$ and m'', such that $\overline{m} < m'' \leq m_L$. For both cases 1 and 2, we can suitably choose a second-order prior ρ, such that $supp(\rho) = \{\pi_{m_1}, \pi_{m^\circ}\}_{m \in \mathcal{M}^\circ}$, where

$$\{\pi^*_{m_1}\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi) \ \text{ and } \ \{\pi_{m^\circ}\} := \arg\max_{\pi \in supp(\rho)} \rho_{m^\circ}(\pi) \ \text{ for each } m^\circ \in \mathcal{M}^\circ, \tag{57}$$

showing that there exists a Rational HTE $(b^*_S, b^*_R, \rho, \mu^*_\rho)$ supporting the PBE, (b^*_S, b^*_R, μ^*). ■

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