Selective Attribute Rules*

Dongwoo Lee† Hans Haller‡

April 6, 2022

Abstract

We develop an extension of Luce's (1959) model to apply multiattributes to stochastic choice. We consider an agent who focuses on selective (salient) attributes standing out in a choice set. These attributes are endogenously determined according to the criterion of Just Noticeable Difference. The criterion selects attributes whose variation impacts the agent enough so that she cares about each of them. When such selective attributes vary with the choice set, we adjust the choice probabilities for the elements of the choice set. We find that all the violations of Luce's axioms can be attributed to a change in selective attributes. When the global choice set is of the product form, selective attributes rules are characterized. In particular, the distinguished case of (non-selective) attribute rules can be characterized by two axioms, Independence from Irrelevant Alternatives and Separability.

Keywords: Stochastic choice, Luce model, Multiattributes, Selective Attributes.

JEL Classification: D90, D91

*We would like to thank the referees for beneficial critique and suggestions. We are grateful to Eric Bahel, Adam Dominiak, Matthew Kovach, Sudipta Sarangi, and Alec Smith for their comments.

†China Center for Behavioral Economics and Finance, Southwestern University of Finance and Economics, Chengdu 610074, Sichuan, China, dwlee05@gmail.com.

‡Corresponding Author. Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA, haller@vt.edu.

1 Introduction

In this paper, we pioneer the analysis of attribute selection in stochastic choice. We study a boundedly rational agent who evaluates alternatives based on multiattributes. The main idea captures the fact that people often focus on a particular subset of attributes, which stand out distinctively among all the other attributes. For example, a person comparing Coke and Sprite may only focus on their taste, since their other traits seem similar. However, once Perrier (carbonated water) becomes available, then she may focus not only on the taste, but also on the healthiness of beverages. Schkade and Kahneman (1998) provide evidence of people focusing on distinctive features of alternatives in a choice set in order to reach a decision.

Stochastic Choice and the Luce Axioms

Since the seminal work of McFadden (1978, 1981), the random utility model is widely applied to study stochastic choice. In this kind of models, one specifies a probability distribution over alternatives in a choice set and alternatives are chosen in accordance with that distribution. The concept of stochastic choice is compelling because we may be unable to learn the preferences, but can observe choice frequency in the data. Luce's (1959) model offers one such description of individual choice behavior. The following two fundamental axioms form the basis of his model: (i) Independence of Irrelevant Alternatives (IIA), requiring the ratio of the probability of choosing one alternative to the probability of choosing another should be maintained regardless of the choice set, and (ii) Regularity, requiring that adding an alternative decreases the probability of choosing any given alternative belonging to the original choice set. Regarding the choice axioms of Luce (1959), a number of researchers have questioned their practical significance (Debreu, 1960; Huber et al., 1982; Huber and Puto, 1983).

Multiattributes

In our model, an agent perceives multiattributes for every alternative. In particular, she goes through a finite set of attributes, which may include price, quality, and so on. A probability distribution over alternatives is determined by the sum of values from the intensities of those multiattributes. Invoking multiattributes is not a novel idea in the economics literature. In individual decision making, Lancaster (1966) and Raiffa and Keeney (1976) consider a finite number of attributes in order to evaluate every alternative. In strategic decision making, multidimensional product differentiation is studied in the stage games by Economides (1989) and Irmen and Thisse (1998), among others. Status quo status in Masatlioglu and Ok (2005) can be viewed as an additional binary attribute.

Selective Attribute Rules

We further consider that an agent may utilize only a subset of attributes. These show greater differences between alternatives than "Just Noticeable Difference". In experimental psychology, Just Noticeable Difference refers to the phenomenon of imperfect response sensitivity to small changes in attribute intensities. Luce (1956) adopts the concept to explain intransitivity of preferences.[1] We interpret the subset of attributes that an agent utilizes as salient attributes. Salience is well established in choice theory and takes into account the fact that typically not all attributes are considered in decision making (Bordalo et al., 2012, 2013; Kőszegi and Szeidl, 2013).[2] We call a random choice rule based on attributes selected by Just Noticeable Differences a Selective Attribute Rule (SAR).

Prima facie, ignoring some attributes goes against the aim for more flexibility and quest for more information, which is inherent in many economic models and often plausible: A very short restaurant menu may not list some of our favorites. A larger menu tends to offer more flexibility. It may contain more favorites, but also more interesting new items. In a similar vein, a more detailed description of each item helps differentiate between alternatives and arrive at a satisfactory decision. However, a larger choice set and more information on product attributes can become a curse. It may be too time-consuming to plough through a thousand item menu. It may prove impossible to keep the elaborate descriptions of a large number of products in mind. One may cope with an abundance of alternatives in many ways, by focusing on one food group, by considering one salient item from every food group or by employing other types of heuristics. One can deal with a multitude of attributes by disregarding some attributes. Hensher (2006) performs stated choice experiments to investigate what influences a person's decision to ignore specific attributes. Among other things, he makes two observations:

[1] Luce (1956) and Luce (1959) have pioneered two different strands of literature. For recent developments in the tradition of Luce (1956), see Manzini and Mariotti (2012). Horan (2021) puts forward a theory that unifies Luce's two theories.

[2] Herweg and Müller (2021) show that original regret theory conceived by Loomes and Sugden (1982) is a special case of salience theory à la Bordalo et al. (2012), which in turn is a special case of generalized regret theory of Loomes and Sugden (1987).

Observation 1. *"As we increase the 'number of alternatives' to evaluate, ceteris paribus, the importance of considering more attributes increases, as a way of making it easier to differentiate between the alternatives."*

Observation 2. *"That is, respondents ignore more attributes when the difference between attribute levels is small."*

These two observations also obtain when attributes are selected according to an SAR. Insofar, Hensher's findings lend empirical support to our random choice model.

Results

In Section 3, we provide a characterization of (non-selective) attribute rules when the global choice set is of product form. These rules are axiomatized by Independence from Irrelevant Alternatives (IIA) and Separability. In Section 5, we provide an axiomatic characterization of selective attribute rules when the global choice set is of the product form. One can infer from the data whether a random choice rule is an SAR. Moreover, suitable model parameters can be identified if the random choice rule proves to be an SAR. Finally, once the parameters of an SAR are identified, only instances where the set of salient attributes changes have to be checked for violations of the Luce axioms. To be precise, a violation of the Luce axioms can occur only when the set of salient attributes is augmented as a choice set is expanded.

In our model, a rational agent — who takes into account all the attributes in decision-making — satisfies the choice axioms of Luce (1959). However, for a boundedly rational agent, who considers selective attributes in decision making, there exist exactly two possible cases. If the set of selective attributes has not been changed by adding an alternative, then the choice axioms still hold in the selective attributes model. Otherwise, violations of those axioms can occur. In particular, this happens when a new stand-out attribute yields high enough values. For instance, once people begin to care about healthiness for some decision problems, they tend to place enormous value on it. We find, by means of illustrative examples, that an SAR can lead to a violation of Luce's and other axioms. For instance, Example 1 exhibits violations of both Regularity and IIA.

Related Literature

The model of Gul et al. (2014) is formally most closely related to ours in that they also consider multiattributes. They first determine a separate Luce-type random choice rule for each of the attributes and then form a weighted sum of these rules. In contrast, we first aggregate across the attributes selected by the Just Noticeable Difference criterion and then apply a Luce-type formula. We call the rules introduced by Gul et al. (2014) Generalized Attribute rules (GARs) to distinguish them from SARs. By construction, a GAR satisfies Regularity whereas an SAR can violate Regularity, as illustrated by Example 1. In Appendix A, we elaborate on the fact that Luce rules are both GARs and SARs, but not all GARs are SARs and vice versa. In particular, we show that SARs satisfying Regularity need not be GARs.

The approach of Echenique et al. (2018) is most closely related to ours on an intuitive level. Incorporating perception, they also adopt a concept from psychology and behavioral economics. They presume a weak order on the set of alternatives, reflecting whether an alternative is perceived earlier or later than others. If an alternative is most preferred in a choice set, then it is assigned its Luce probability. Otherwise, the probability of an alternative is its Luce probability multiplied by a term involving the probabilities of not choosing the alternatives perceived before. This means that the choice probabilities over the elements of a choice set can sum up to less than 1. Thus their Perception-Adjusted Luce Model (PALM) obviously differs from ours, since the choice probabilities always sum up to 1 for an SAR. In a PALM, the residual probability is assigned to an outside option or no choice, which allows to explain choice overload.

PALMs can further exhibit violations of IIA, Regularity and Stochastic Transitivity — as can SARs. But that does not mean that PALMs and SARs explain the same violations. Even if one disregards normalization of choice probabilities, there remain marked differences between the two approaches. For a given number of attributes, there can exist a PALM specification that causes more violations of IIA than any SAR. Conversely, there exist SARs that violate Moral Hazard IIA, one of the axioms characterizing PALMs. Again, the Luce model is contained in both model types, the PALMs and the SARs. But none of the two encompasses the other. We shall elaborate on their differences in Appendix B.

Brady and Rehbeck (2016) consider random choice rules where the decision maker has strict preferences on the global choice set. Random choice results from randomness of choice sets, in contrast to random utility models and models following Luce (1959).

Outline

The concept of random choice rule and the Luce model are introduced in the next section. In Section 3, (non-selective) attribute rules are defined and characterized. The concept of selective attribute rule is developed in Section 4. In Section 5, we obtain an axiomatic characterization of selective attribute rules when the global choice set is of the product form. Section 6 contains comments on attributes and economic relevance. Section 7 consists of three illustrative examples. Section 8 concludes. In Appendix A, we establish by means of examples that GARs do not encompass SARs and vice versa. In Appendix B, we further explore the differences between PALMs and SARs.

2 General Model: Primitives and Luce's Model

In the sequel, $\subset$ stands for strict set inclusion, $\subseteq$ for weak set inclusion, and $\supset$ for strict set containment. $|S|$ denotes the cardinality of a set S. Consider a nonempty finite set $X \subset \mathbb{R}^n$ of n-dimensional attribute vectors, where each dimension represents an attribute. Elements $x \in X$ are also viewed as alternatives, since a decision maker will be indifferent between two alternatives with identical attribute vectors and, thus, in fact her choice can be confined to attribute vectors. We assume that $|X| \geq 3$. Let $\mathcal{X}$ be the family of subsets C of X with $|C| \geq 2$. Each $C \in \mathcal{X}$ is called a **choice set**.[3] In particular, X is the **global choice set**. We aim to model an agent who makes a stochastic choice on a choice set $C \in \mathcal{X}$. A probability distribution on C is denoted by $\rho(\cdot, C)$. $\rho(x, C)$ stands for the probability that alternative x is chosen in choice set C.

Definition 1 (Random Choice Rule) *A function $\rho : X \times \mathcal{X} \to [0,1]$ is called a* **random choice rule** *if*

$$\sum_{x \in C} \rho(x, C) = 1,$$

for all $C \in \mathcal{X}$ and $\rho(x, C) = 0$ for all $x \notin C$.

[3] In some of the literature, "menu" serves as synonym for "choice set". Some economic authors distinguish between choice sets and consideration sets. E.g., Masatlioglu et al. (2012) and Manzini and Mariotti (2014) assume that under limited attention, an agent considers only a subset of alternatives in a choice set. In contrast, we assume that an agent considers all alternatives in a choice set.

Luce (1959) introduces a random choice rule as follows. Each alternative y in X has numerical value $v(y)$ (called Luce value) such that for every $x \in C$, the probability of choosing x from the choice set $C \in \mathcal{X}$ is determined as

$$\rho(x, C) = \frac{v(x)}{\sum\limits_{y \in C} v(y)} \quad \text{for any } x \in C, \tag{1}$$

where $v : X \to \mathbb{R}_{++}$. A special case is a logit rule with $v(y) = \exp y$ for $n = 1$. Equation (1) measures choice frequencies depending on the proportion of an alternative's value compared to the sum of values over all alternatives in a choice set. Luce's model is characterized by the standard axiom of independence from irrelevant alternatives.

Axiom 1 (IIA) *A random choice rule ρ satisfies independence from irrelevant alternatives (IIA) if*

$$\frac{\rho(x, \{x, y\})}{\rho(y, \{x, y\})} = \frac{\rho(x, C)}{\rho(y, C)} \quad \textit{for any choice set } C \in \mathcal{X} \textit{ such that } x, y \in C$$

for $\rho > 0$. Obviously, (1) implies IIA. Conversely, if $\rho > 0$ satisfies IIA, set $v(x) = \rho(x, X)$ for $x \in X$. Then for $C \in \mathcal{X}, x \in C$: $1 = \sum_{y \in C} \rho(y, C) = \dfrac{\rho(x, C)}{v(x)} \cdot \sum_{y \in C} v(y)$ and, consequently (1). Moreover, any function v' that satisfies the analog of (1) is of the form $v' = \lambda v$ for some $\lambda > 0$. Namely, let $\lambda = v'(x)/v(x)$ for some $x \in X$. If $y \in X, y \neq x$, application of (1) to $C = \{x, y\}$ and both v and v' yields $v(x)/((v(x) + v(y)) = \lambda v(x)/(\lambda v(x) + v'(y))$ which implies $v'(y) = \lambda v(y)$.

IIA means that comparison of two alternatives x and y does not depend on the choice sets to which both alternatives belong. That is, the ratio of the frequency with which alternative x is chosen and of the frequency with which alternative y is chosen should be constant regardless of choice set composition. A further immediate implication of (1) and of IIA is regularity, also called monotonicity.

Axiom 2 (Regularity) *A random choice rule ρ satisfies Regularity if*

$$C \subseteq C' \;\Rightarrow \rho(x, C) \geq \rho(x, C') \quad \textit{for any } x \in C,$$

where C and C' are choice sets in $\mathcal{X}$.

Regularity requires that the probability of choosing a particular alternative is decreasing as the number of alternatives in the respective choice set is increasing.

3 Attribute Rules

In our setting, each alternative is associated with n indices of value so that an agent evaluates every alternative by means of an n-dimensional attribute intensity. Recall that we assume existence of a finite set of attributes denoted $A = \{1, \ldots, n\}$. An alternative $x \in X$ is identified by its intensity of each attribute. Thus, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We further assume a vector of attribute values (attribute value functions) $w = (w_1, \ldots, w_n)$ that assigns a positive value to each attribute intensity of an alternative. For instance, $w_i(x_i)$ is the value assigned to the i-th component (or i-th attribute intensity) of alternative x.

We develop a modification of Luce's model in the sense that a Luce value of an alternative is measured by the sum of attribute values of the alternative.[4] Specifically, under multiattributes, a Luce value of alternative $y = (y_1, \ldots, y_n) \in X$ is represented as

$$v(y) = \sum_{i \in A} w_i(y_i). \tag{2}$$

In this sense, we offer a reason why an agent has a value regarding each alternative: She enjoys each alternative via its n-dimensional intrinsic properties. Having defined an attribute value, we next introduce an attribute rule where an agent takes into account all the attributes in A to evaluate alternatives in a choice set.

Definition 2 (Attribute Rule) *A random choice rule $\rho_{A,w}$ satisfies the* **attribute rule** *based on the pair (A, w) if*

$$\rho_{A,w}(x, C) = \sum_{i \in A} \frac{w_i(x_i)}{w_A(C)} \quad \text{for each } x \in C,$$

where $w_A(C)$ is the total value of all alternatives in a choice set $C \in \mathcal{X}$, that is,

$$w_A(C) = \sum_{y \in C} \sum_{i \in A} w_i(y_i).$$

When adhering to an attribute rule, our agent chooses alternatives based on an evaluation of multiattributes. Since the attribute rule is a special form of the Luce model, it immediately follows that an attribute rule satisfies both IIA and Regularity. Therefore, if an agent is rational in the sense that she takes into account all attributes, then our model does not predict any violation of Luce (1959). This fact provides the first part of a two-part **Characterization of Attribute Rules.** First, as a special case of the Luce model, an attribute

[4] This additive form is a special case of the linear form of Lancaster (1966) and Raiffa and Keeney (1976).

rule satisfies IIA. Second, the attribute rule satisfies the system of $|X|$ linear equations (2) in the variables $w_i(x_i)$, with all $w_i(x_i) > 0$. Conversely, suppose that a random choice rule ρ satisfies IIA with Luce values $v(x), x \in X$. Suppose further that the system of equations (2) has a solution with all $w_i(x_i) > 0$. Then ρ is an attribute rule. Hence

Lemma 1 *A random choice rule is an attribute rule if and only if*

(a) *the rule satisfies IIA with Luce values $v(x), x \in X$;*

(b) *there exists a solution of the system of equations* (2) *with all $w_i(x_i) > 0$.*

Recall that we can set $v(x) = \rho(x, X)$ for $x \in X$ if ρ satisfies IIA. Therefore, in the latter case, condition (b) and subsequent conditions on v correspond to conditions on ρ where $v(x)$ is replaced by $\rho(x, X)$.

An axiomatic characterization of attribute rules can be given if the global choice set is of the product form $X = X_1 \times \ldots \times X_n$ with all $X_i \subset \mathbb{R}_+$. In case that $|A| = n > 1$ and X is of the product form, let us introduce some useful notation. For $i \in A$, let $X_{-i} = \prod_{j \neq i} X_j$. For $x = (x_1, \ldots, x_n) \in X$ and $i \in A$, it proves convenient to use the notation $x = (x_i, x_{-i})$ where $x_{-i} \in X_{-i}$ consists of the components $x_j, j \neq i$.

It turns out that if the global choice set X is of the product form, then condition (b) is equivalent to

Axiom 3 (Separability) *The differences $v(x_i', x_{-i}) - v(x_i, x_{-i})$ for $x_i, x_i' \in X_i, x_{-i} \in X_{-i}$ are independent of x_{-i} for each $i \in A$.*

The form of equations (2) is reminiscent of separable utility functions. We obtain

Proposition 1 *Suppose that the global choice set is of the product form. Then a random choice rule is an attribute rule if and only if it satisfies IIA and Separability.*

Proof. Let X be of of the product form. If a random choice rule violates IIA, it cannot be an attribute rule. Suppose now that the random choice rule ρ satisfies IIA with Luce values $v(x), x \in X$. Then ρ is an attribute rule if and only if the system of equations (2) has a solution with all $w_i(x_i) > 0$.

Consider the differences $v(x_i', x_{-i}) - v(x_i, x_{-i})$ for $x_i, x_i' \in X_i, x_{-i} \in X_{-i}$. To solve (2), those differences have to be independent of x_{-i} for each $i \in A$. If this is not the case, that is if Separability is violated, then ρ is not an attribute rule.

Next suppose that IIA and Separability hold with Luce values $v(x), x \in X$. Denote

(i) $\delta_i(x_i, x_i') = v(x_i', \cdot) - v(x_i, \cdot)$ for $x_i, x_i' \in X_i, i \in A$;

(ii) $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ such that $v(x_{i1}, \cdot) \leq v(x_{i2}, \cdot) \leq \ldots \leq v(x_{in_i}, \cdot)$ for $i \in A$.

For each $i \in A$, set $w_i(x_{i1}) = \frac{1}{n} v(x_{11}, x_{21}, \ldots, x_{n1})$ and $w_i(x_{ij}) = w_i(x_{i1}) + \delta_i(x_{i1}, x_{ij})$ for $j = 2, \ldots, n_i$. Then the values $w_i(x_i), x_i \in X_i, i \in A$ are all positive and satisfy the system (2). Namely, take any $x = (x_1, \ldots, x_n) \in X$. Define recursively

$$
\begin{aligned}
&x^0 = (x_{11}, \ldots, x_{n1}),\\
&x^1 = (x_1, x^0_{-1}),\\
&x^2 = (x_2, x^1_{-2}),\\
&\vdots\\
&x^n = (x_n, x^{n-1}_{-n}) = x.
\end{aligned}
$$

Then $\sum_i w_i(x_i^0) = \sum_i w_i(x_{i1}) = v(x^0)$ and induction yields

$$
\begin{aligned}
&\textstyle\sum_i w_i(x_i^1) = \sum_i w_i(x_i^0) + \delta_1(x_{11}, x_1) = v(x^0) + \delta_1(x_{11}, x_1) = v(x^1),\\
&\textstyle\sum_i w_i(x_i^2) = \sum_i w_i(x_i^1) + \delta_2(x_{21}, x_2) = v(x^1) + \delta_2(x_{21}, x_2) = v(x^2),\\
&\vdots\\
&\textstyle\sum_i w_i(x_i) = \sum_i w_i(x_i^n) = \sum_i w_i(x_i^{n-1}) + \delta_n(x_{n1}, x_n) = v(x^{n-1}) + \delta_n(x_{n1}, x_n)\\
&= v(x^n) = v(x),
\end{aligned}
$$

that is $v(x) = \sum_i w_i(x_i)$. Since x was an arbitrary choice, this shows (2). Hence ρ is an attribute rule. □

4 Selective Attribute Rules

We next consider the possibility that an agent utilizes only a subset of attributes to evaluate an alternative. The idea is similar to a model of salience in choice theory. Taylor and Thompson (1982) write:

> "Salience refers to the phenomenon that when one's attention is differently directed to one portion of the environment rather than to others, the information contained in that portion will receive disproportionate weighting in subsequent judgments."

Here we assume that a salient attribute is determined by an intensity difference of the attribute across alternatives. To select a subset of attributes, we adopt "Just Noticeable Difference," a well-established concept from psychology.[5] The notion is plausible in the sense

[5] In experimental psychology, the just noticeable difference (JND), also known as the difference threshold, is the minimum change of sensation (related to touch, taste, smell, hearing, sight, etc.) that a person can

that people often have imperfect response sensitivity to small changes. A Just Noticeable Difference level is given by a vector

$$k = (k_1, \ldots, k_n) \in \mathbb{R}^n_+.$$

Regarding an attribute $i \in A$, intensity differences have to be at least k_i for the agent to care about the attribute. This means that our agent ignores attribute i — intentionally or unintentionally — if the respective intensity difference between alternatives is not distinctive enough. Formally:

Definition 3 (Selective Attributes) *Given a just noticeable difference level $k \in \mathbb{R}^n_+$ and a choice set $C \in \mathcal{X}$, an agent considers a subset of attributes $A_k(C) \subseteq A$ given as*

$$A_k(C) \equiv \left\{ i \in A : \max_{x,y \in C} |x_i - y_i| \geq k_i \right\}.$$

The elements of $A_k(C)$ are called **selective attributes** *in C (given k). We also say that $i \in A_k(C)$ is* **salient** *in C.*

If $k = (k_1, \ldots, k_n)$ is the zero vector, then a set of selective attributes A_k is identical to the set of attributes A. In this case, an agent uses all the attributes to choose alternatives regardless of choice sets. An attribute rule obtains. Otherwise, an agent considers only a subset of attributes in $A_k(C)$ standing out from a choice set C. In this regard, a set of selective attributes $A_k(\cdot)$ is specified by a choice set. Here and in Definition 4, we allow for $A_k(C) = \emptyset$.

This selective attribute idea is illustrated in Figure 1. When an agent makes a decision in a choice set $\{x, y\}$, she may not consider attribute 2, since it is not distinctive enough between x and y. However, when a new alternative z becomes available, she may realize that she has to consider attribute 2 as well as attribute 1, since now attribute 2 stands out enough. In this way, selective attributes can vary according to a choice set.[6]

We are now ready to introduce a selective attribute rule $\rho_{k,w}$ that yields a probability distribution over a choice set C when an agent resorts to a subset of attributes $A_k(C)$ to evaluate alternatives.

detect 50 percent of the time. For a comprehensive overview of the literature, see Laming (1997), Algom (2001), Johnson (2004).

[6]One could argue that the agent should not use attribute 2 for comparison between x and y in $\{x, y, z\}$, since she already decided not to use attribute 2 for the comparison in $\{x, y\}$. However, we presume that our agent evaluates all alternatives using an attribute once the attribute becomes salient.

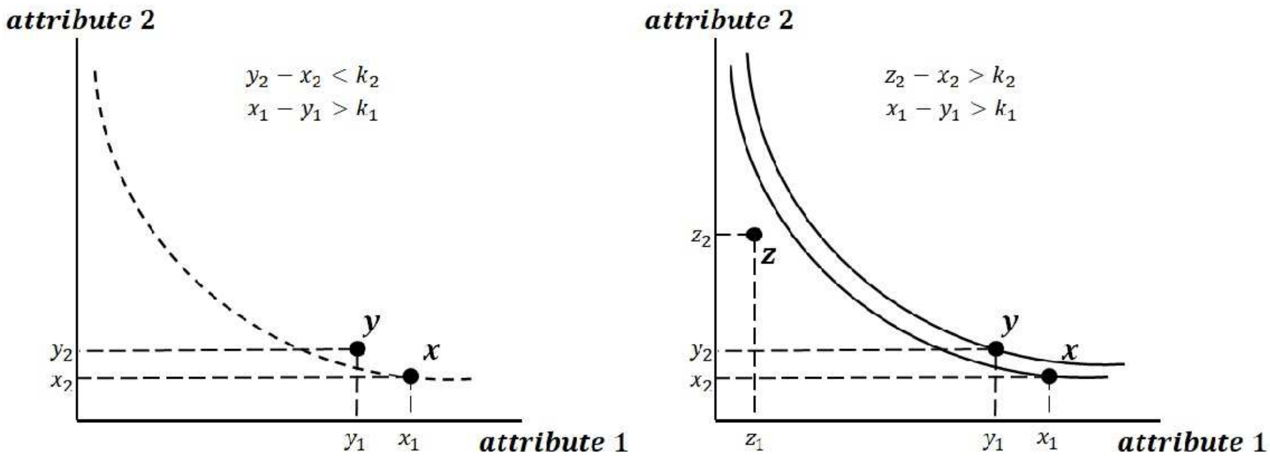

Figure 1: Selective Attribute

Definition 4 (SAR) *For a pair* (A_k, w)*, the corresponding* **selective attribute rule** *is the random choice rule* $\rho_{k,w}$ *given by*

$$\rho_{k,w}(x, C) = \begin{cases} \sum_{i \in A_k(C)} \frac{w_i(x_i)}{w_{A_k}(C)} & \text{if } A_k(C) \neq \emptyset, \\ \frac{1}{|C|} & \text{otherwise,} \end{cases}$$

where $w_{A_k}(C)$ *is the total value of a choice set* C *derived from attributes in* $A_k(C) \neq \emptyset$*, that is,*

$$w_{A_k}(C) = \sum_{y \in C} \sum_{i \in A_k(C)} w_i(y_i).$$

Notice that for $k = (k_1, \ldots, k_n) \in \mathbb{R}^n_+$, positive value functions w, $C \in \mathcal{X}$, $x \in C$, the selective attribute rule satisfies $\rho_{k,w} > 0$. If no attribute stands out in a choice set, then the second case of the definition requires that the decision maker adheres to the principle of insufficient reason. Recall that A_k is identical to A for any choice set when the Just Noticeable Difference level k is zero. Hence, an SAR $\rho_{k,w}$ with $k = 0$ is identical to an attribute rule $\rho_{A,w}$.

5 Characterization of SARs on Product Sets

A characterization of SARs is available if the global choice set is of the product form $X = X_1 \times \ldots \times X_n$ with all $X_i \subset \mathbb{R}_+$. In this section, we briefly describe random choice data sets which we study in this paper, and then provide a characterization of SARs. The key idea involves two steps. First, restrict the analysis to a finite set K of just noticeable difference levels. Second, for the random choice rule ρ at hand and each $k \in K$, aim to elicit a vector of attribute values $w[k,\rho]$ from the data. If all attempts fail, then ρ cannot be an SAR. In case of a successful attempt for a particular k, ρ is an SAR if and only if $\rho = \rho_{k,w[k,\rho]}$.

Before we succeed, let us remember useful notation in case $|A| = n > 1$. For $i \in A$, let $X_{-i} = \prod_{j \neq i} X_j$. For $x = (x_1, \ldots, x_n) \in X$ and $i \in A$, recall the notation $x = (x_i, x_{-i})$ where $x_{-i} \in X_{-i}$ consists of the components $x_j, j \neq i$. Finally, let us denote

$$C_i(x_{-i}) = X_i \times \{x_{-i}\} = \{(x_i, x_{-i}) \,|\, x_i \in X_i\}$$

for $i \in A$ and $x_{-i} \in X_{-i}$. Alternatives in $C_i(x_{-i})$ differ only in attribute i. Furthermore, define the random choice rule $\overline{\rho}$ by $\overline{\rho}(x, C) = 1/|C|$ for all $x \in C \in \mathcal{X}$.

5.1 Data Sets

Our analysis presumes (a) a given finite non-empty set of attributes A and (b) a finite non-empty list of attribute intensities X_i for each attribute $i \in A$. In laboratory experiments, both (a) and (b) tend to be part of the design, hence known to the analyst and the participants. Moreover, X frequently is of the product form. Outside the laboratory, the product form may not always hold while still a possibility. The data set reports the choice probability $\rho(x, C)$ for every pair (x, C) where C is a choice set and x is an alternative belonging to C, that is $C \in \mathcal{X}$ and $x \in C$. In short, the data set at hand is a family $(\rho(\cdot, C))_{C \in \mathcal{X}}$.

5.2 Just Noticeable Difference Level

Notice that it is impossible to recover the just noticeable difference level k from the data generated by an SAR $\rho_{k,w}$. Namely, let $K_i^* = \{|x_i - y_i| \,:\, x_i, y_i \in X_i, x_i \neq y_i\}$, $\overline{k}_i = \max K_i^*$ and $K_i = K_i^* \cup \{0, \overline{k}_i + 1\}$ for $i \in A$. Label the elements of K_i by $k_i(l), l = 0, 1, \ldots, |K_i^*| + 1$ such that $0 = k_i(0) < k_i(1) < \ldots < k_i(|K_i^*| + 1) = \overline{k}_i + 1$. If $k_i > \overline{k}_i$, then attribute i is never salient and k_i can be replaced by $\overline{k}_i + 1$ without altering the SAR. If $k_i(l) < k_i \leq k_i(l+1)$ for some $l \in \{0, \ldots, |K_i^*|\}$, then k_i can be replaced by $k_i(l+1)$ without altering the SAR. The reason is that each absolute difference $|x_i - y_i|$ belongs to K_i. On the one hand, this implies that infinitely many k give rise to the same SAR unless $k = (0, \ldots, 0)$. On the other hand, it implies that the analysis can be restricted to $k \in K = K_1 \times \ldots \times K_n$.

Definition 5 *For $k \in K$, a random choice rule ρ is a k-SAR if there exists a vector of attribute values w such that $\rho = \rho_{k,w}$.*

As an immediate consequence, a random choice rule is an SAR if and only if it is a k-SAR for some $k \in K$. Hence it suffices to check for each $k \in K$ whether the random choice rule is a k-SAR.

Given a just noticeable difference level $k = (k_1, \ldots, k_n) \in K$, we can divide the attribute set A into three disjoint categories:

$I_k = \{i \in A : \; k_i = 0\}$,

$J_k = \{i \in A : \; 0 < k_i < \overline{k} + 1\}$,

$L_k = \{i \in A : \; k_i = \overline{k} + 1\}$.

The distinguished cases $I_k = A$, dealt with in Section 3, and $J_k = A, \rho \neq \overline{\rho}$ dealt with in Subsection 5.4 form the corner stones on which the complete characterization in 5.5 - 5.7 rests.

5.3 Special Case: $\rho = \overline{\rho}$

To begin with, let us dispose of the special case $\rho = \overline{\rho}$. Whether that case prevails, is obviously revealed by the data. If it prevails, then ρ is an SAR. $\rho = \overline{\rho}$ obtains if $L_k = A$ or none of the attributes matters where we say that **attribute** $i \in A$ **matters** if there exist $x_i, y_i \in X_i$ such that $w_i(x_i) > w_i(y_i)$.[7]

5.4 Central Case: $\rho \neq \overline{\rho}$ and $J_k = A$

So far, we have focused on the cases $\rho = \overline{\rho}$ and $I_k = A$ (attribute rules). Now we shall deal with the case $\rho \neq \overline{\rho}$ and $J_k = A$, which proves central for a complete characterization. To this end, consider $k \in K$ with $J_k = A$ and SAR $\rho_{k,w} \neq \overline{\rho}$. First, we are going to show that the vector of value functions w can be retrieved from the data, up to a positive multiplicative constant.

5.4.1 Retrieving the SAR's w from the Data

We derive a number of conditions that allow to identify w up to a positive multiplicative constant. The same conditions constitute the basis for testing for k-SAR later on. Notice that all attributes are salient in X when $J_k = A$.

Define $v_i = \sum_{x_i \in X_i} w_i(x_i)$ for $i \in A$. If $n = 1$, then

$$w_1(x_1)/v_1 = \rho_{k,w}(x_1, X) \text{ for all } x_1 \in X_1 = X. \tag{3}$$

[7]Note that if $n_i = |X_i|$ and $X_i = \{x_{i1}, \ldots, x_{in_i}\}$, then w_i is given by the vector $(w_i(x_{i1}), \ldots, w_i(x_{in_i})) \in \mathbb{R}^{n_i}_{++}$. For generic w_i, $w_i(x_i) \neq w_i(y_i)$ when $x_i \neq y_i$. In particular, $w_i(x_i) \neq w_i(y_i)$ holds for $x_i \neq y_i$ if w_i is strictly increasing or strictly decreasing.

Consequently, for all $x_1, y_1 \in X$:

$$\frac{w_1(x_1)}{w_1(y_1)} = \frac{\rho_{k,w}(x_1, X)}{\rho_{k,w}(y_1, X)}. \tag{4}$$

And if $n = 1$, then the system of equations (4) suffices to recover w, up to multiplication with a positive constant.

If $n > 1$, then for each $i \in A$,

$$w_i(x_i)/v_i = \rho_{k,w}((x_i, x_{-i}), C_i(x_{-i})) \text{ for all } x_i \in X_i, x_{-i} \in X_{-i}, \tag{5}$$

since only attribute i is salient in $C_i(x_{-i})$.
Therefore, for each $i \in A$, and $x_i, y_i \in X_i, x_{-i} \in X_{-i}$:

$$\frac{w_i(x_i)}{w_i(y_i)} = \frac{\rho_{k,w}((x_i, x_{-i}), C_i(x_{-i}))}{\rho_{k,w}((y_i, x_{-i}), C_i(x_{-i}))}. \tag{6}$$

It remains to relate w_i and w_j for $i \neq j$. Since $\rho \neq \overline{\rho}$, some attribute matters. For such an attribute, the ratio in one of the corresponding equations (6) is different from 1. Let j be an attribute that matters and i be a different attribute.

If $n \geq 3$, let $X_{-ij} = \prod_{\ell \neq i,j} X_\ell$. Fix $x'_j, y'_j \in X_j$ that satisfy $w_j(x'_j) > w_j(y'_j)$. Then for every $x_{-ij} \in X_{-ij}, x_i \in X_i$:[8]

$$\begin{aligned} & (w_j(x'_j) - w_j(y'_j))/(v_i + (n_i - 1)w_j(x'_j) + v_j + (n_j - 1)w_i(x_i)) \\ = \; & \rho_{k,w}((x_{-ij}, x_i, x'_j), C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)) \\ - \; & \rho_{k,w}((x_{-ij}, x_i, y'_j), C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)), \end{aligned}$$

$$\begin{aligned} & (w_i(x_i) + w_j(x'_j))/(v_i + (n_i - 1)w_j(x'_j) + v_j + (n_j - 1)w_i(x_i)) \\ = \; & \rho_{k,w}((x_{-ij}, x_i, x'_j), C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)), \end{aligned}$$

where $n_i = |X_i|$ and $n_j = |X_j|$. We obtain

$$\left(\frac{w_i(x_i)}{w_j(x'_j)} + 1\right) \Big/ \left(1 - \frac{w_j(y'_j)}{w_j(x'_j)}\right) = (w_i(x_i) + w_j(x'_j))/(w_j(x'_j) - w_j(y'_j)) = \frac{W_{ij}}{\Delta_j} \text{ or}$$

[8] Note that only attributes i and j are salient in a choice set $C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)$.

$$\frac{w_i(x_i)}{w_j(x'_j)} = \left(1 - \frac{w_j(y'_j)}{w_j(x'_j)}\right) \cdot \frac{W_{ij}}{\Delta_j} - 1 \tag{7}$$

where

$$\begin{aligned} W_{ij} &\equiv \rho_{k,w}((x_{-ij}, x_i, x'_j), C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)); \\ \Delta_j &\equiv \rho_{k,w}((x_{-ij}, x_i, x'_j), C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)) \\ &- \rho_{k,w}((x_{-ij}, x_i, y'_j), C_i(x_{-ij}, x'_j) \cup C_j(x_{-ij}, x_i)). \end{aligned}$$

If $n = 2$, the analog of (7) holds with x_{-ij} omitted.

Now we are prepared to address the question to what extent w can be recovered from the values of $\rho_{k,w}$. Observe that if $\lambda > 0$ and $\lambda w = (\lambda w_1, \ldots, \lambda w_n)$, then $\rho_{k,\lambda w} = \rho_{k,w}$. Therefore, at best w can be recovered up to multiplication with a positive constant.

Proposition 2 *Suppose $\rho_{k,w} \neq \overline{\rho}$ and $J_k = A$. Then up to multiplication with a positive constant, w can be recovered from the values of $\rho_{k,w}$.*

Proof. If $n = 1$, set $w_1(x_1) = 1$ for some $x_1 \in X_1$. Then the system of equations (4) determines values $w_1(y_1)$ for $y_1 \in X_1$. The resulting attribute values coincide with the original ones up to multiplication with a positive constant.

If $n > 1$, then there exists an attribute j that matters. Such a j can be found by inspecting the right-hand sides of the system of equations (6): There exist $x_j, y_j \in X_j$ such that the right-hand side of the corresponding equation is different from 1. We can and shall choose $x'_j, y'_j \in X_j$ such that the corresponding right-hand side is greater than 1. Set $w_j(x'_j) = 1$. Then the system of equations (6) and (7) determines values $w_i(x_i)$ for all $x_i \in X_i, i \in A$. The obtained attribute values coincide with the original ones up to multiplication with a positive constant. □

5.4.2 Characterization

Now let ρ be any random choice rule different from $\overline{\rho}$. Let again $k = (k_1, \ldots, k_n) \in K$ with $J_k = A$. If $n = 1$, then the system of equations (4) determines an attribute value function $w[k, \rho]$, up to multiplication with a positive constant. We then say that $w[k, \rho]$ **can be constructed from the data**.

In case $n > 1$, the system of equations (6) and (7) can be used to construct an attribute value function $w[k, \rho]$ from the data, provided that

- at least one attribute matters, which amounts to the ratio in one of the corresponding equations (6) being different from 1;

- all right-hand sides in equations (7) are positive.

If the first condition does not hold, $\overline{\rho}$ is the only SAR possibly supported by the data, which we ruled out. If the second condition does not hold, then the system of equations (6) and (7) cannot define an attribute value function.

If both conditions hold, we say that $w[k, \rho]$ **can be constructed from the data**. Again, $w[k, \rho]$ is determined up to multiplication with a positive constant. When $\rho = \rho_{k,w}$ and $\rho \neq \overline{\rho}$, the attribute value function w can be reconstructed from the data, up to multiplication with a positive constant. This suggests a first axiom.

Axiom 4 (Constructability) *An attribute value function can be constructed from the data.*

When $w[k, \rho]$ can be constructed from the data, the k-SAR property automatically holds for the choice sets used in the construction. For ρ to qualify an a k-SAR, the property has to hold for all choice sets. This suggests a further axiom.

Axiom 5 (k-SAR Consistency) $\rho = \rho_{k,w[k,\rho]}$ *for an attribute value function* $w[k, \rho]$ *constructed from the data.*

Proposition 3 *Suppose that* $\rho \neq \overline{\rho}$ *is a random choice rule and* $k \in K$ *is a just noticeable difference level with* $J_k = A$*. Then* ρ *is a* k*-SAR if and only if it satisfies Constructability and* k*-SAR Consistency.*

Proof. If ρ satisfies Constructability and k-SAR Consistency, then an attribute value function $w[k, \rho]$ can be constructed from the data and $\rho = \rho_{k,w[k,\rho]}$. Hence ρ is a k-SAR. Conversely, suppose that ρ is a k-SAR, that is $\rho = \rho_{k,w}$ for some attribute value function. By Proposition 2, up to multiplication with a positive constant, w can be recovered from the values of $\rho_{k,w}$. Hence Constructability and k-SAR Consistency hold. □

5.5 All Cases

By now, we have analyzed the special case $\rho = \overline{\rho}$ and the case $\rho \neq \overline{\rho}$ with $I_k = A$, $J_k = A$, or $L_k = A$. It remains to examine a number of mixed cases.

First, consider CASE I: $I_k \neq \emptyset, J_k \neq \emptyset, L_k = \emptyset$, hence $n > 1$. Denote $X_{I_k} = \prod_{j \in I_k} X_j$ and $X_{J_k} = \prod_{j \in J_k} X_j$. A k-SAR $\rho_{k,w}$ induces an attribute value on each of the sets of alternatives $X_{I_k}(x_J) = X_{I_k} \times \{x_J\}, x_J \in X_{J_k}$ which is independent of x_J. Denote that attribute value ρ_{I_k} and v_{I_k} a corresponding Luce value function. Given k and any random choice rule ρ, one can first check whether the induced choice rule on $X_{I_k}(x_J)$, is independent of x_J.

If the test is positive, one can further check whether ρ_{I_k} meets the respective criteria of Proposition 1. If these criteria are also met, one can fix suitable attribute values $w_i, i \in I_k$. Regarding attributes $j \in J_k$, modify the systems (4)-(7) by taking into account the already fixed $w_i, i \in I_k$.

Second, consider CASE II: $L_k \neq \emptyset \wedge L_k \neq A$. In this case, the analysis can be restricted to the reduced set of attributes $A \setminus L_k$.

5.6 Summary procedure of retrieving attribute value functions

For $k \in K$ with $J_k \neq \emptyset$ and a random choice rule ρ, let us restrict the system of equations (4), (6) and (7) to attributes $i, j \in A_k(X)$ in accordance with CASE II.

(a) If $I_k = \emptyset$, check whether the right-hand sides of the pertinent equations (7) are positive: If so, derive a vector of attribute value functions $w[k, \rho]$ as before, which is unique up to multiplication with a positive constant. Thus, using part of the data, one can construct $w[k, \rho]$.

(b) If $I_k \neq \emptyset$, proceed according to CASE I: First, check whether the induced choice rule on the set $X_{I_k}(x_J)$, is independent of x_J. If the test is positive and the induced choice rule ρ_{I_k} meets the respective criteria of Proposition 1, fix suitable attribute values $w_i, i \in I_k$. As regards attributes $j \in J_k$, follow step (a), taking into account the already fixed $w_i, i \in I_k$. That yields a attribute value functions $w[k, \rho]_i, i \in A_k(X)$. $w[k, \rho]_i$ can be chosen arbitrarily for $i \in L_k$.

(c) If k and ρ fail one of the tests in (a) or (b), respectively, then ρ cannot be an SAR with just noticeable difference level k.

5.7 Complete Characterization

Following the procedure in the previous subsection, one obtains a characterization of k-SAR for each $k \in K$, as a consequence of Proposition 1, Proposition 3, both propositions combined, and the fact that $\overline{\rho}$ is an SAR. The essential cases are the special case $\rho = \overline{\rho}$ and the case $\rho \neq \overline{\rho}$ with $I_k = A$ or $J_k = A$ that have been investigated in detail above and are the corner stones of our characterization. The remaining mixed cases can be reduced to those. Since an SAR is a k-SAR for some $k \in K$, a complete characterization of SARs obtains.

Incidentally, the analysis elicits a just noticeable difference level k and a vector of values $w[k, \rho]$ that determine the random choice rule if the rule proves to be an SAR. One may ask whether this insight does not reflect a mere tautology, since by definition, ρ is an SAR if there exists (k, w) such that $\rho = \rho_{k,w}$. The very fact that $w[k, \rho]$ is not assumed but elicited from the data means that the characterization is not a mere tautology. Last but not least note that whether a random choice rule is an SAR can be determined by means of finitely many tests, since K is finite. There is an upper bound on the number of steps (computations and comparisons) it would take an algorithm to come up with a Yes-or-No answer.

5.8 Extension

Global choice sets may be of the product form. In particular, the product form may be imposed by experimental design. Yet field data need not conform with this form. In case $n = 1$, the question of product form is obsolete. If $|A| = n > 1$, then a global choice set $X \subset \mathbb{R}^n_+$ can be embedded in the finite product set $\prod_{i \in A} X_i$ where $X_i = \{x_i : (x_i, x_{-i}) \in X \text{ for some } x_{-i}\}$. Under certain conditions, the approach taken in the current section is still applicable. For instance, if $n = 2$ and for each $i \in A$, there exist sufficiently many triples (x_{-i}, x_i, y_i) with $(x_i, x_{-i}) \in X$, $(y_i, x_{-i}) \in X$ and $x_i \neq y_i$, then the counter-part of system (6) can be set up. More generally, the data must be "sufficiently rich" for the current approach to be applicable.

6 Discussion

Here we discuss the importance of some of our modeling choices and, briefly, the economic relevance of distinctive features of an SAR.

6.1 Attributes

Our analysis rests on (a) a given set of attributes A and (b) on attribute intensities x_i for alternatives $x \in X$ and attributes $i \in A$. In laboratory experiments, both (a) and (b) tend to be part of the design, hence known to the experimenter and the participants. What is not directly observable by the experimenter and other outsiders are the valuations $w_i(x_i)$. In the case of global choice sets of the product form, which are prevalent in experiments, the $w_i(x_i)$ can be inferred from behavior if an SAR is found to drive random choices.[9] In some field experiments, the data may be rich enough to overcome gaps in the product structure. In others, they may not.

[9] In a similar vein, the just noticeable difference level k may not be directly observable by outsiders, but can be inferred when the global choice set is of the product form.

Conceivably, a decision maker could apply "just noticeable difference" to the $w_i(x_i)$ rather than the x_i. If so, an outsider who only observes the x_i can come to the conclusion that the decision maker is not following an SAR even though the decision maker perceives it differently.

Attribute intensities presume a numerical representation that corresponds to a measurable entity. Such numerical scales are readily available for many attributes such as sound volume, brightness, length, weight, duration, price, etc. It may prove problematic for other attributes like color. One could distinguish spectral colors by their wave lengths. However, the latter might not mean much to the decision maker. Moreover, non-spectral colors like purple, brown and white cannot be identified by a single wave length. This constitutes a limitation of models based on attribute intensities.

Outside the laboratory, it is possible that there is a "hidden attribute" which the analyst has not thought of or was not aware of. The analyst may not account for attributes that are non-obvious or deemed irrelevant, but which are relevant to the decision maker. An example would be the dosage of food ingredients that cause certain rare allergies. Another example could be a product's country of origin while the product's material quality is not affected by its country of origin. In case such hidden attributes exist, a violation may be inferred even though the decision maker follows an SAR based on all attributes.

6.2 Economic Relevance

In addition to different axiomatic characterizations, there are qualitative differences between models that can be relevant in specific economic or business situations. An interesting question is whether and when regularity is violated. A GAR à la Gul et al. (2014) always satisfies regularity. An SAR may but need not satisfy regularity. There are situations in which regularity is satisfied, in other words a violation of regularity can be ruled out. Take a choice set C and an alternative $z \notin C$. For each attribute i, define $\min_i(C) = \min\{x_i | x \in C\}$ and $\max_i(C) = \max\{x_i | x \in C\}$. If $z_i \in [\min_i(C), \max_i(C)]$ for all i, then $A_k(C \cup \{z\}) = A_k(C)$ and $\rho_{k,w}(x, C) > \rho_{k,w}(x, C \cup \{z\})$ for all $x \in C$. In the same situation, a PALM may exhibit violation of regularity. See Proposition 1 in Echenique et al. (2018).

Now consider the situation where a company introduces a new product z that competes with the already existing products $x \in C$. Suppose x is one of the company's existing products and regularity prevails. Then x is now less frequently chosen than before introduction of the new product, an instance of what is known of "self-cannibalization" or "corporate cannibalization". Then the concern is whether the profits from the new product exceed the loss from self-cannibalization. On the other hand, violation of regularity amounts to absence of self-cannibalization. Issues of self-cannibalization arise in many industries where frequently new products, brands or packages are introduced such as car manufacturing, consumer electronics, air traffic, health insurance, automobile insurance. For instance, an automobile producer may hesitate to push electric cars for fear of reduced sales of traditional cars. The launch of new movies may be delayed not to jeopardize the revenue from current movie theater hits. In contrast, firms like Apple release new versions that cut into sales of the older models, which may still be popular. But attracting new buyers from other brands more than offsets this loss. Thus risking self-cannibalization can prove beneficial. With similar prices, this may occur when $\rho_{k,w}(x, C) - \rho_{k,w}(x, C \cup \{z\})$ is rather small or $\rho_{k,w}(z, C \cup \{z\})$ is relatively large.

Introduction of a new product or brand may have detrimental effects beyond self-cannibalization. Consider a company that has two brands, x and y, handled by its x-division and its y-division, respectively. Now suppose that the company introduces the new brand z and establishes a third division, the z-division in charge of the new brand. Then possibly, both brand x and brand y suffer from self-cannibalization, but x much more than y to the extent that $\rho_{k,w}(x, \{x, y, z\}) < \rho_{k,w}(y, \{x, y, z\})$ whereas $\rho_{k,w}(x, \{x, y\}) > \rho_{k,w}(y, \{x, y\})$. This constitutes a violation of IIA. It obtains in Example 2 of Section 7 where regularity prevails. In the example, the x-division loses its position as top sales performer to the y-division, although neither division is doing anything differently. Such a reversal in ranking can cause additional animosities and misgivings within the company.

7 Some Examples

The following examples illustrate the possibilities regarding the validity or failure of the Luce axioms. To begin with, note that (non-selective) attribute rules satisfy both IIA and Regularity. Examples 1 and 3 exhibit failure of both IIA and Regularity. In Example 2, IIA is violated while Regularity is satisfied. Furthermore, Example 2 demonstrates the "Compromise Effect" as well as the rank reversal described in subsection 6.2. Example 3 exhibits the "Attraction Effect".

Example 1

Let $n = 2$, $X = \{x, y, z\}$ with x=(1,3), y=(3,2), z=(0.1,1), k=(2,2), $w(u) = (u_1, u_2)$ for $u = (u_1, u_2) \in X$. Then we obtain:

C	$A_k(C)$	$\rho(x, C)$	$\rho(y, C)$	$\rho(z, C)$
X	$\{1, 2\}$	4/10.1	5/10.1	1.1/10.1
$\{x, y\}$	$\{1\}$	1/4	3/4	
$\{x, z\}$	$\{2\}$	3/4		1/4
$\{y, z\}$	$\{1\}$		3/3.1	0.1/3.1

Obviously, this ρ fails to satisfy IIA and Regularity. It satisfies two properties weaker than IIA:

Weak IIA. $\rho(x, \{x, y\}) \geq \rho(y, \{x, y\}) \Leftrightarrow \rho(x, C) \geq \rho(y, C)$ *for any two* $x, y \in X$ *and any choice set* $C \in \mathcal{X}$ *such that* $x, y \in C$.

Stochastic Transitivity. *If* $\rho(x, \{x, y\}) > 1/2$ *and* $\rho(y, \{y, z\}) > 1/2$*, then* $\rho(x, \{x, z\}) > 1/2$ *for any* $x, y, z \in X$.

One can show that Weak IIA implies Stochastic Transitivity. ■

Example 2 (Compromise Effect)

One well-known behavioral phenomenon that violates Weak IIA is the "Compromise Effect" where an agent selects more of a compromise alternative and avoids extreme alternatives in attributes if she is uncertain which of the attributes is more important. See, e.g., Simonson (1989), Herne (1997) and Echenique et al. (2018). In fact, Figure 1 depicts a compromise effect. This example illustrates the effect numerically.

Let $n = 2$, $X = \{x, y, z\}$ with x=(7,2), y=(4,4), z=(2,7), k=(3,3), $w(u) = (\ln u_1, \ln u_2)$ for $u = (u_1, u_2) \in X$. We obtain:

C	$A_k(C)$	$\rho(x,C)$	$\rho(y,C)$	$\rho(z,C)$
X	$\{1,2\}$	$\dfrac{\ln 14}{2\ln 56}$	$\dfrac{\ln 16}{2\ln 56}$	$\dfrac{\ln 14}{2\ln 56}$
$\{x,y\}$	$\{1\}$	$\dfrac{\ln 7}{\ln 28}$	$\dfrac{\ln 4}{\ln 28}$	
$\{x,z\}$	$\{1,2\}$	$1/2$		$1/2$
$\{y,z\}$	$\{2\}$		$\dfrac{\ln 4}{\ln 28}$	$\dfrac{\ln 7}{\ln 28}$

Since $\ln 56 > \ln 28$, we find $\dfrac{\ln 4}{\ln 28} > \dfrac{\ln 4}{\ln 56} = \dfrac{2\ln 4}{2\ln 56} = \dfrac{\ln 16}{2\ln 56}$ and consequently $\dfrac{\ln 7}{\ln 28} > \dfrac{\ln 14}{2\ln 56}$. Moreover, $1/2 > \dfrac{\ln 14}{2\ln 56}$. This shows that ρ is regular.

We observe that $\rho(y,\{y,z\}) < \rho(z,\{y,z\})$ whereas $\rho(y,X) > \rho(z,X)$. Hence ρ violates Weak IIA and exhibits the reversal of ranking described in subsection 6.2. Moreover, we observe the compromise effect with y as compromise between x and z. ■

Example 3 (Attraction Effect)
The most prominent phenomenon that violates Weak IIA is the "Attraction Effect" where in the absence of alternative y, z is more frequently chosen than x and in the presence of y, x is chosen more than half of the time. The effect occurs when x strictly dominates y whereas y is better than z in at least one dimension. See Natenzon (2019) and references therein. This example exhibits the attraction effect.

Let $n = 3$, $X = \{x,y,z\}$ with $x = (10,30,60), y = (1,11,1), z = (31,10,20)$, $k = (15,15,50)$, $w(u) = (u_1,u_2,u_3)$ for $u = (u_1,u_2,u_3) \in X$. Then we obtain:

C	$A_k(C)$	$\rho(x,C)$	$\rho(y,C)$	$\rho(z,C)$
X	A	$100/174$	$13/174$	$61/174$
$\{x,y\}$	$\{2,3\}$	$90/102$	$12/102$	
$\{x,z\}$	$\{1,2\}$	$40/81$		$41/81$
$\{y,z\}$	$\{1\}$		$1/32$	$31/32$

This ρ violates WIIA and Regularity. In addition, the attraction effect occurs: $\rho(z, \{x, z\}) > 1/2$ while $\rho(x, \{x, y, z\}) > 1/2$. Hence adding the less attractive alternative y favors x. With regard to attribute 2, alternative y is slightly better than alternative z. ■

8 Conclusion

We have presented a new model of stochastic choice that builds on Luce's (1959) model, incorporating multiattributes in a particular way. On the one hand, our model constitutes a special Luce model when an agent is fully rational. On the other hand, we ascribe violations of IIA and Regularity to bounded rationality, in the sense that an agent does not take into account all the attributes for the purpose of decision making. When the global choice set is of the product form, we provide an axiomatic characterization of SARs.

We have investigated how selective attributes vary with the choice set and how such changes alter choice probabilities. Yet varying choice sets are not the only reason why choice probabilities evolve. It may be that difference thresholds change. If a person's sensual abilities like touch, taste, smell, hearing, or sight diminish, just noticeable differences in these dimensions may become unnoticeable. Perception of the attributes of some product may change as well. For instance, people may have a misperception of the frequency of accidents related to various means of transportation. Experience and learning could lead them to a more accurate assessment. Conversely, an advertiser might successfully accentuate a particular attribute of a product. Last but not least, individuals may simply undergo a change in taste via a change of value function.

References

Algom, Daniel. "Psychophysics", pp. 800-805 in Nadel, Lynn (ed.) *Encyclopedia of Cognitive Science*, Nature Publishing Group (Macmillan): London, 2001.

Bordalo, Pedro, Nicola Gennaioli and Andrei Shleifer. "Salience Theory of Choice under Risk", *Quarterly Journal of Economics 127* (2012), 1243-1285.

Bordalo, Pedro, Nicola Gennaioli and Andrei Shleifer. "Salience and Consumer Choice", *Journal of Political Economy 121* (2013), 803-843.

Brady, Richard L. and John Rehbeck. "Menu-Dependent Stochastic Feasibility", *Econometrica 84* (2016), 1203-1223.

Debreu, Gerard. Review of R. D. Luce, "Individual Choice Behavior: A Theoretical Analysis", *American Economic Review 50* (1960), 186-188.

Echenique, Federico, Kota Saito and Gerelt Tserenjigmid. "The Perception-adjusted Luce Model", *Mathematical Social Sciences 93* (2018), 67-76.

Economides, Nicholas. "Quality Variations and Maximal Variety Differentiation", *Regional Science and Urban Economics 19* (1989), 21-29.

Gul, Faruk, Paulo Natenzon and Wolfgang Pesendorfer. "Random Choice as Behavioral Optimization", *Econometrica 82* (2014), 1873-1912.

Hensher, David A. "How Do Respondents Process Stated Choice Experiments? Attribute Consideration under Varying Information Load", *Journal of Applied Econometrics 21* (2006), 861-878.

Herne, Kaisa. "Decoy Alternatives in Policy Choices: Asymmetric Domination and Compromise Effects", *European Journal of Political Economy 13* (1997), 575-589.

Herweg, Fabian and Daniel Müller. "A Comparison of Regret Theory and Salience Theory for Decisions under Risk", *Journal of Economic Theory 193* (2021), Art.-No. 105226.

Horan, Sean. "Stochastic Semi-orders", *Journal of Economic Theory 192* (2021), 105171.

Huber, Joel, John W. Payne and Christopher Puto. "Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis", *Journal of Consumer Research 9* (1982), 90-98.

Huber, Joel and Christopher Puto. “Market Boundaries and Product Choice: Illustrating Attraction and Substitution Effects”, *Journal of Consumer Research 10* (1983), 31-44.

Irmen, Andreas and Jacques-François Thisse. “Competition in Multi-characteristics Spaces: Hotelling was Almost Right”, *Journal of Economic Theory 78* (1998), 76-102.

Johnson, James H. (2004). “Just Noticeable Difference (JND)”, pp. 502-503 in Craighead, W. Edward and Charles B. Nemeroff (eds.). *Corsini's Concise Encyclopedia of Psychology and Neuroscience*, John Wiley & Sons: New York, 2004.

Kőszegi, Botond and Adam Szeidl. “A Model of Focusing in Economic Choice”, *Quarterly Journal of Economics 128* (2013), 53-104.

Laming, Donald. *The Measurement of Sensation*, Oxford University Press: Oxford, 1997.

Lancaster, Kelvin J. “A New Approach to Consumer Theory”, *Journal of Political Economy 74* (1966), 132-157.

Loomes, Graham and Robert Sugden. “Regret Theory: An Alternative Theory of Rational Choice under Uncertainty,” *The Economic Journal 92(368)* (1982), 805-824.

Loomes, Graham and Robert Sugden. Some Implications of a more General Form of Regret Theory,” *Journal of Economic Theory 41(2)* (1987), 270-287.

Luce, R. Duncan. “Semiorders and a Theory of Utility Discrimination”, *Econometrica 24* (1956), 178-191.

Luce, R. Duncan. *Individual Choice Behavior: A Theoretical Analysis*, Wiley: New York, 1959.

Manzini, Paola and Marco Mariotti. “Choice by lexicographic semiorders”, *Theoretical Economics 7* (2012), 1-23.

Manzini, Paola and Marco Mariotti. “Stochastic Choice and Consideration Sets”, *Econometrica 82* (2014), 1153-1176.

Masatlioglu, Yusufcan, Daisuke Nakajima and Erkut Y. Ozbay. “Revealed Attention”, *American Economic Review 102* (2012), 2183-2205.

Masatlioglu, Yusufcan and Efe A. Ok. “Rational Choice with Status quo Bias”, *Journal of Economic Theory 121* (2005), 1-29.

McFadden, Daniel. “Modeling the Choice of Residential Location”, pp. 75-96 in A. Karlqvist, L. Lundqvist, F. Snickars, and J. Weibull (eds.). *Spatial Interaction Theory and Planning Models*, North-Holland: Amsterdam, 1978.

McFadden, Daniel. “Econometric Models of Probabilistic Choice”, Ch. 5 in Charles F. Manski, C.F and D. McFadden (eds.). *Structural Analysis of Discrete Data with Econometric Applications*, MIT Press: Cambridge, MA, 1981.

Natenzon, Paulo. “Random Choice and Learning”, *Journal of Political Economy 127* (2019), 419-457.

Raiffa, Howard and Ralph L. Keeney. *Decisions with Multiple Objectives: Preferences and Value Trade-Offs*, Cambridge University Press: Cambridge, UK, 1976.

Schkade, David A. and Daniel Kahneman, “Does Living in California Make People Happy? A Focusing Illusion in Judgments of Life Satisfaction”, *Psychological Science 9* (1998), 340-346.

Simonson, Itamar. “Choice Based on Reasons: The Case of Attraction and Compromise Effects”, *Journal of Consumer Research 16* (1989), 158-174.

Taylor, Shelley E. and Suzanne C. Thompson. “Stalking the Elusive “Vividness” Effect”, *Psychological Review 89* (1982), 155-181.

Appendices

A Generalized Attribute Rules of Gul, Natenzon and Pesendorfer

Gul et al. (2014) consider another generalization of Luce's random choice model which they call "attribute rule". Since we already use this term with a different meaning, we shall call their concept a **Generalized Attribute Rule (GAR)**. Their analysis is based on a "rich" and consequently infinite set of alternatives whereas ours assumes a finite set. But even in the case of finitely many alternatives, not all selective attribute rules are generalized attribute rules and vice versa. We are going to use the notation $\widehat{\rho}$ for generalized attribute rules and ρ for selective attribute rules.

Let $X \subseteq \mathbb{R}^n_{++}$ with n-dimensional attributes $A = \{1, \ldots, n\}$.[10] For an attribute $i \in A$ and a choice set C, define $w_i(C) = \sum_{y \in C} w_i(y_i)$. For i, C and $x \in C$, define $\sigma_i(x, C) = w_i(x_i)/w_i(C)$. $\sigma_i(x, C)$ is the probability that x is chosen from C if only attribute i is used in a Luce-type formula. A generalized attribute rule $\widehat{\rho}(x, C)$ is obtained as the weighted sum of the $\sigma_i(x, C)$ as follows: There exists a vector $q = (q_1, \ldots, q_n) \in \mathbb{R}^n_{++}$ such that $\sum_{i \in A} q_i = 1$ and $\widehat{\rho}(x, C) = \sum_{i \in A} q_i \sigma_i(x, C)$ for x and C with $x \in C$. In short, we can write $\widehat{\rho}(x, C) = q \cdot \sigma(x, C)$ where $\sigma(x, C) = (\sigma_1(x, C), \ldots, \sigma_n(x, C)) \in \mathbb{R}^n_{++}$.

Both generalized attribute rules and selective attribute rules include traditional Luce rules. But neither set contains the other.

First, we show that not every selective attribute rule can be obtained as generalized attribute rule. To show this, let us consider $n = 2$ and the SAR ρ given by $X = \{x, y, z\}$ with $x = (10, 30), y = (10, 10), z = (30, 10)$, $k = (15, 15)$, $w(u) = (u_1, u_2)$ for $u = (u_1, u_2) \in X$. Then $x = (x_1, x_2)$, $y = (x_1, z_2), z = (z_1, z_2)$ and $z_1 > x_1 > 0, x_2 > z_2 > 0$. For an arbitrary attribute value function $\widehat{w}$ and weight vector $q = (q_1, q_1)$, we obtain

$$\begin{aligned} &\sigma_1(x, \{x, y\}) = 1/2; && \sigma_2(x, \{x, y\}) = \widehat{w}_2(x_2)/(\widehat{w}_2(x_2) + \widehat{w}_2(y_2)); \\ &\sigma_1(z, \{y, z\}) = \widehat{w}_1(z_1)/(\widehat{w}_1(y_1) + \widehat{w}_1(z_1)); && \sigma_2(z, \{y, z\}) = 1/2. \end{aligned}$$

$\widehat{\rho} = \rho$ requires $\widehat{\rho}(x, \{x, y\}) = \rho(x, \{x, y\})$ and $\widehat{\rho}(z, \{y, z\}) = \rho(z, \{y, z\})$, that is

[10] For the sake of easy comparison, we consider attribute densities $x_i \in \mathbb{R}_{++}$ while Gul et al. (2014) consider $x_i \in \mathbb{N} \cup \{0\}$. The following counter-examples still work with the latter provision.

$$q_1 \cdot \frac{1}{2} + q_2 \cdot \frac{\widehat{w}_2(x_2)}{\widehat{w}_2(x_2) + \widehat{w}_2(y_2)} = \frac{3}{4}; \quad q_1 \cdot \frac{\widehat{w}_1(z_1)}{\widehat{w}_1(y_1) + \widehat{w}_1(z_1)} + q_2 \cdot \frac{1}{2} = \frac{3}{4}.$$

Adding the two equations yields

$$q_1 \cdot \frac{\widehat{w}_1(z_1)}{\widehat{w}_1(y_1) + \widehat{w}_1(z_1)} + q_2 \cdot \frac{\widehat{w}_2(x_2)}{\widehat{w}_2(x_2) + \widehat{w}_2(y_2)} = 1,$$

which is a contradiction because of $\frac{\widehat{w}_1(z_1)}{\widehat{w}_1(y_1) + \widehat{w}_1(z_1)} < 1$ and $\frac{\widehat{w}_2(x_2)}{\widehat{w}_2(x_2) + \widehat{w}_2(y_2)} < 1$. Hence $\widehat{\rho} \neq \rho$ has to hold. This shows that ρ cannot be obtained as generalized attribute rule.

Second, we show that not every generalized attribute rule can be obtained as selective attribute rule. To see this, let $n = 2$, $q = (1/2, 1/2)$, $X = \{x, y, z\}$ with $x = (10, 30), y = (10, 10), z = (30, 10)$, and attribute values $\widehat{w}(u) = (u_1, u_2)$ for $u = (u_1, u_2) \in X$. Then

C	$\sigma_1(x, C)$ $\sigma_2(x, C)$	$\widehat{\rho}(x, C)$	$\sigma_1(y, C)$ $\sigma_2(y, C)$	$\widehat{\rho}(y, C)$	$\sigma_1(z, C)$ $\sigma_2(z, C)$	$\widehat{\rho}(z, C)$
X	1/5 3/5	2/5	1/5 1/5	1/5	3/5 1/5	2/5
$\{x, y\}$	1/2 3/4	5/8	1/2 1/4	3/8		
$\{x, z\}$	1/4 3/4	1/2			3/4 1/4	1/2
$\{y, z\}$			1/4 1/2	3/8	3/4 1/2	5/8

Now consider any attribute value function w and some $k = (k_1, k_2) \geq 0$. Then $x = (x_1, x_2)$, $y = (x_1, z_2), z = (z_1, z_2)$ and $z_1 > x_1 > 0, x_2 > z_2 > 0$. Suppose that the resulting selective attribute rule ρ satisfies $\rho = \widehat{\rho}$. Then $A_k(X) \neq \emptyset$.

• If $k \gg (0, 0)$, then three potential cases result:

CASE 1: $A_k(X) = \{1\}$. In that case, $A_k(\{x, y\}) = \{1\}$ has to hold as well whereas $x_1 = y_1$ implies $A_k(\{x, y\}) \neq \{1\}$.

CASE 2: $A_k(X) = \{2\}$. In that case, $A_k(\{y, z\}) = \{2\}$ has to hold as well whereas $y_2 = z_2$ implies $A_k(\{y, z\}) \neq \{2\}$.

CASE 3: $A_k(X) = \{1, 2\}$. Then further $A_k(\{x, y\}) = \{2\}$ and $A_k(\{y, z\}) = \{1\}$. From $\rho(x, \{x, y\}) = \widehat{\rho}(x, \{x, y\})$ and $\rho(y, \{y, z\}) = \widehat{\rho}(y, \{y, z\})$, we have

$w_2(x_2) = (5/3)w_2(y_2) = (5/3)w_2(z_2)$ and $w_1(z_1) = (5/3)w_1(y_1) = (5/3)w_1(x_1)$, which is equivalent to

$w_2(y_2) = w_2(z_2) = (3/5)w_2(x_2)$ and $w_1(x_1) = w_1(y_1) = (3/5)w_1(z_1)$. It follows that

$$\rho(y, X) = \frac{(3/5)w_1(z_1) + (3/5)w_2(x_2)}{(11/5)w_1(z_1) + (11/5)w_2(x_2)} = \frac{3}{11},$$

whereas $\widehat{\rho}(y, X) = 1/5$.

• If $k_1 = 0, k_2 > 0$, then two more potential cases result:

CASE 4: $A_k(X) = \{1\}$. Then $A_k(C) = \{1\}$ for all C and ρ is a Luce rule and satisfies IIA whereas $\widehat{\rho}$ violates IIA.

CASE 5: $A_k(X) = \{1, 2\}$. Then $A_k(\{x, y\}) = \{1, 2\}$ holds as well so that IIA prevails for ρ when alternative z is added to the choice set $\{x, y\}$, which is not the case for $\widehat{\rho}$.

• If $k_1 > 0, k_2 = 0$, the resulting potential cases are similar to CASE 4 and CASE 5.

• If $k = (0, 0)$, then ρ is an attribute rule and satisfies IIA whereas $\widehat{\rho}$ violates IIA.

Thus in all cases, $\rho = \widehat{\rho}$ cannot hold. Therefore, $\widehat{\rho}$ cannot be obtained as selective attribute rule.

B PALMs versus SARs

PALMs and SARs simply differ because an SAR satisfies $\sum_{x \in C} \rho(x, C) = 1$ for all $C \in \mathcal{X}$ whereas a PALM can yield $\sum_{x \in C} \rho(x, C) < 1$ for some $C \in \mathcal{X}$, because of the outside option.[11]

Regardless of the normalization of probabilities, one can find specific differences in specific examples.

Let us first consider an extended Luce model in the sense of Echenique et al. (2018). Let $X = \{x^1, x^2, x^3, x^4\}$ and $x^0 \notin X$ be the "outside option". Let $v(x) = 1$ for $x \in X$ and $v(x^0) = 0$. We obtain "extended Luce values" given as

$$\mu(C, x) = \frac{v(x)}{\sum_{z \in C} v(z) + v(x^0)} = \frac{1}{|C|}$$

for $C \in \mathcal{X}$, $x \in C$. Moreover, let $\succ$ be s strict order on X such that $x^1 \succ x^2 \succ x^3 \succ x^4$. Then formula (2) in Echenique et al. (2018) yields

$$\rho(x, C) = \mu(x, C) \cdot \prod_{z \in C, z \succ x} (1 - \mu(z, C)).$$

[11] Echenique et al. (2018) show that a version without outside option can still explain violations of Regularity, but do not demonstrate further properties.

In particular, we get

$\rho(x^1, \{x^1, x^4\}) = \mu(x^1, \{x^1, x^4\}) = \frac{1}{2}$,
$\rho(x^4, \{x^1, x^4\} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$,
$\rho(x^1, \{x^1, x^3, x^4\}) = \mu(x^1, \{x^1, x^3, x^4\}) = \frac{1}{3}$,
$\rho(x^4, \{x^1, x^3, x^4\}) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{27}$,
$\rho(x^1, X) = \mu(x^1, X) = \frac{1}{4}$,
$\rho(x^4, X) = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{256}$.

Hence when the choice set is expanded first from $\{x^1, x^4\}$ to $\{x^1, x^3, x^4\}$ and then to X, two violations of IIA occur whereas $n = 1$ allows only one increase in the number of salient attributes.

Next reconsider the SAR ρ of Example 1. Then there do not exist a finite set of alternatives X' and a PALM specified by $(v, \succeq)$ with the set of alternatives X' and outside option $x_0 \notin X'$ such that

(a) $X \subset X'$,

(b) $\succeq$ is "rich",

(c) the stochastic choice rule $\rho_{(v,\succeq)}$ defined by $(v, \succeq)$ coincides with ρ on $\mathcal{X}$.

For suppose there exist a finite set of alternatives X' and a PALM $(v, \succeq)$ with set of alternatives X' and outside option $x_0 \notin X'$ such that (a) – (c) hold. By Theorem 2 of Echenique et al. (2018), $\succeq^* = \succeq$ holds for the revealed preference relation $\succeq^*$ and IIA holds for the hazard rate function q given by $\rho_{(v,\succeq)}$ and $\succeq^* = \succeq$. In particular,

$$\frac{q(x, \{x, y\})}{q(y, \{x, y\})} = \frac{q(x, X)}{q(y, X)}$$

would hold. Because of (b) and (c), the pertinent values of q can be calculated using ρ and the weak order induced by $\succeq$ on X. There are thirteen potential weak orders on X. In each case, the above IIA identity is violated; see table below. Hence a finite set of alternatives X' and a PALM $(v, \succeq)$ with set of alternatives X' and outside option $x_0 \notin X'$ such that (a) – (c) hold does not exist. On the other hand, the SAR can be extended to an arbitrary finite set $X' \supset X$.

$\succsim$	$q(x,\{x,y\})$	$q(y,\{x,y\})$	$\boldsymbol{\frac{q(x,\{x,y\})}{q(y,\{x,y\})}}$	$q(x,X)$	$q(y,X)$	$\boldsymbol{\frac{q(x,X)}{q(y,X)}}$
$z \succ x \succ y$	1/4	1	**1/4**	4/9	1	**4/9**
$z \sim x \succ y$	1/4	1	**1/4**	4/10.1	1	**4/10.1**
$x \succ y \succ z$	1/4	1	**1/4**	4/10.1	5/6.1	**24.4/50.5**
$x \succ y \sim z$	1/4	1	**1/4**	4/10.1	5/6.1	**24.4/50.5**
$x \succ z \succ y$	1/4	1	**1/4**	4/10.1	1	**4/10.1**
$z \succ x \sim y$	1/4	3/4	**1/3**	4/9	5/9	**4/5**
$z \sim x \sim y$	1/4	3/4	**1/3**	4/10.1	5/10.1	**4/5**
$x \sim y \succ z$	1/4	3/4	**1/3**	4/10.1	5/10.1	**4/5**
$z \succ y \succ x$	1	3/4	**4/3**	1	5/9	**9/5**
$z \sim y \succ x$	1	3/4	**4/3**	1	5/10.1	**10.1/5**
$y \succ x \succ z$	1	3/4	**4/3**	4/5.1	5/10.1	**40.4/25.5**
$y \succ x \sim z$	1	3/4	**4/3**	4/5.1	5/10.1	**40.4/25.5**
$y \succ z \succ x$	1	3/4	**4/3**	1	5/10.1	**10.1/5**