

ML-Index Methods

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This write-up is meant to explain the methods in the data-driven indexing approach.

I am very interested in feedback on the methods. I feel that there is large room for improvement in the Miller index assignment and unit cell optimization aspects of this algorithm.

There has been no refinement whatsoever. No hyper parameter training, equal amounts of under and over fitting.

Text in *italics* is commentary. For example, my hunches, opinions, acknowledgments of errors and bugs in the code, or features and approaches I would like to take, and features that are implemented but not used. Normal case text describes currently implemented methods.

Coelho’s 2003 algorithm that this is based on is called SVD-Index, I’m thinking of calling this algorithm ML-Index. It sounds like Windex.

My understand of reading through powder diffraction indexing methods papers is that existing algorithms work really well on good data. The SVD-Index index algorithm showed 100% accuracy in their paper with peak lists generated with random unit cells. The biggest issue is the generalization of these algorithms to real experimental data, especially low quality data. So the benchmark for this project is near perfect results on calculated data, or demonstrating superior generality to experimental data.

1 Introduction

Powder diffraction indexing is the process of using a 1-dimensional projection of the 3-dimensional diffraction to infer the symmetries that exist within a material. This is a problem as old as crystallography (Runge 1917) and has been the subject of many analytical efforts. Current trends in machine learning and artificial intelligence have produced new knowledge and tools for data-driven analytics. This project merges recent advancements in analytical methods with knowledge from existing indexing algorithms to produce a new approach to powder diffraction indexing.

This algorithm is developed with general use in mind, but the foremost use case is the small molecule serial crystallography community (Schriber, et. al. 2022). This software is developed as an open-source Python program, in line with the Computational Crystallography Initiative (CITE). Serial femtosecond crystallography data analysis is performed exclusively on super-computers and computational clusters. ML-Index is developed specifically for mass-distributed computing.

1.1 Inspirational Quotes

- de Wolff 1957: The ‘indexing problem’ is essentially a puzzle: it cannot be stated in rigorous terms, or be solved in a general way by clear-cut methods, because of the unpredictable distribution of unobserved lines in a powder pattern. It would be quite an easy puzzle if errors of measurement did not exist. This added inconvenience, however, is enough to raise some doubt as to whether the determination of the unit cell of an arbitrary polycrystalline phase is possible in, say, an average of half a day’s work or less.

Le Bail (2004) provides some quotes:

- ‘Indexing is more an art than a science,’
- ‘It is entirely the users responsibility to decide whether any of the suggested unit cells is the correct cell’ (ITO manual)
- ‘Powder indexing works beautifully on good data, but with poor data it will usually not work at all’ (Shirley, 1980)
- ‘DICVOL proposes solutions, the user disposes of them’ (DICVOL manual)

- Users now want solutions fast, without thinking too much, ignoring that ‘part of the beauty of structure determination by powder diffraction does consist in its complexity, i.e., in the lack of complete automatism as well as in the necessity of a careful and sagacious human interpretation of the experimental data,’ (anonymous reviewer)

1.2 Powder diffraction indexing

Diffraction is produced in 3-dimensional spherical coordinates - reciprocal space. In the powder diffraction method, the 3d diffraction pattern is recorded in the radial dimension, the distance of the measured diffraction from the center of the spherical coordinate system. The recorded pattern is a series of peaks with the location of the peaks representing the spacing of Miller planes through the crystal's unit cell. In the indexing problem, the objective is to determine the unit cell level symmetries of the crystal that produced this pattern. When the diffraction is in coordinates of $q = \frac{2 \sin(\theta)}{\lambda}$, where θ is the diffraction angle and λ is the wavelength, the crystal's unit cell can be related to the observed peak locations with through the general equation:

$$q_{hkl}^2 = \frac{1}{V} (S_{hh}h^2 + S_{kk}k^2 + S_{ll}l^2 + S_{hk}hk + S_{hl}hl + S_{kl}kl), \quad (1)$$

$$\begin{aligned} S_{hh} &= b^2 c^2 \sin^2(\alpha), \\ S_{kk} &= a^2 c^2 \sin^2(\beta), \\ S_{ll} &= a^2 b^2 \sin^2(\gamma), \\ S_{hk} &= abc^2 [\cos(\alpha) \cos(\beta) - \cos(\gamma)], \\ S_{hl} &= a^2 bc [\cos(\beta) \cos(\gamma) - \cos(\alpha)], \\ S_{kl} &= ab^2 c [\cos(\alpha) \cos(\gamma) - \cos(\beta)], \\ V(\text{Volume}) &= abc \sqrt{1 - \cos^2(\alpha) - \cos^2(\beta) - \cos^2(\gamma) + 2 \cos(\alpha) \cos(\beta) \cos(\gamma)}. \end{aligned}$$

This general equation assumes no constraints among the crystal lattice lengths, a, b, c , and angles, α, β, γ . It is simplified by the lattice parameters constraints defining the seven lattice systems,

triclinic : $a \neq b \neq c, \alpha \neq \beta \neq \gamma \neq 90^\circ$

cubic : $a = b = c, \alpha = \beta = \gamma = 90^\circ$

$$q_{hkl}^2 = \frac{h^2 + k^2 + l^2}{a^2}, \quad (2)$$

tetragonal : $a = b \neq c, \alpha = \beta = \gamma = 90^\circ$

$$q_{hkl}^2 = \frac{h^2 + k^2}{a^2} + \frac{l^2}{c^2} \quad (3)$$

hexagonal : $a = b \neq c, \alpha = \beta = 90^\circ, \gamma = 120^\circ$

$$q_{hkl}^2 = \frac{4}{3} \frac{h^2 + hk + k^2}{a^2} + \frac{l^2}{c^2} \quad (4)$$

rhombohedral : $a = b = c, \alpha = \beta = \gamma \neq 90^\circ$

$$q_{hkl}^2 = \frac{(h^2 + k^2 + l^2) \sin^2(\alpha) + 2(hk + kl + hl) \cos^2(\alpha) - \cos(\alpha)}{a^2(1 - 3 \cos^2(\alpha) + 2 \cos^3(\alpha))} \quad (5)$$

orthorhombic : $a \neq b \neq c, \alpha = \beta = \gamma = 90^\circ$

$$q_{hkl}^2 = \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} \quad (6)$$

monoclinic : $a \neq b \neq c, \alpha = \gamma = 90^\circ, \beta \neq 90^\circ$

$$q_{hkl}^2 = \frac{h^2}{a^2 \sin^2(\beta)} + \frac{k^2}{b^2} + \frac{l^2}{c^2 \sin^2(\beta)} - \frac{2hl \cos(\beta)}{ac \sin^2(\beta)} \quad (7)$$

The triclinic lattice system imposes no unit cell parameter constraints and is known as the triclinic lattice system. The general case gives the peak spacing formula.

These equations can be rewritten in the reciprocal space unit cell parameters, $a^*, b^*, c^*, \alpha^*, \beta^*, \gamma^*$, which are related to the real-space unit cell parameters through the inverse of the metric tensor,

$$S = \begin{pmatrix} a^2 & ab \cos(\gamma) & ac \cos(\beta) \\ ab \cos(\gamma) & b^2 & bc \cos(\gamma) \\ ac \cos(\beta) & bc \cos(\gamma) & c^2 \end{pmatrix} = \begin{pmatrix} a^{*2} & a^* b^* \cos(\gamma^*) & a^* c^* \cos(\beta^*) \\ a^* b^* \cos(\gamma^*) & b^{*2} & b^* c^* \cos(\alpha^*) \\ a^* c^* \cos(\beta^*) & b^* c^* \cos(\alpha^*) & c^{*2} \end{pmatrix}^{-1} \quad (8)$$

Eq. 1-7 can be rewritten using reciprocal space unit cells,
triclinic : $a^* \neq b^* \neq c^*$, $\alpha^* \neq \beta^* \neq \gamma^* \neq 90^\circ$

$$q_{hkl}^2 = a^{*2}h^2 + b^{*2}k^2 + c^{*2}l^2 + 2a^*b^* \cos(\gamma^*)hk + 2a^*c^* \cos(\beta^*)hl + 2b^*c^* \cos(\alpha^*)kl \quad (9)$$

cubic : $a^* = b^* = c^*$, $\alpha^* = \beta^* = \gamma^* = 90^\circ$

$$q_{hkl}^2 = (h^2 + k^2 + l^2) a^{*2} \quad (10)$$

tetragonal : $a^* = b^* \neq c^*$, $\alpha^* = \beta^* = \gamma^* = 90^\circ$

$$q_{hkl}^2 = (h^2 + k^2) a^{*2} + l^2 c^{*2} \quad (11)$$

hexagonal : $a^* = b^* \neq c^*$, $\alpha^* = \beta^* = 90^\circ$, $\gamma^* = 120^\circ$

$$q_{hkl}^2 = (h^2 + hk + k^2) a^{*2} + l^2 c^{*2} \quad (12)$$

rhombohedral : $a^* = b^* = c^*$, $\alpha^* = \beta^* = \gamma^* \neq 90^\circ$

$$q_{hkl}^2 = (h^2 + k^2 + l^2) a^{*2} + 2(hk + kl + hl) a^{*2} \cos^2(\alpha^*) \quad (13)$$

orthorhombic : $a^* \neq b^* \neq c^*$, $\alpha^* = \beta^* = \gamma^* = 90^\circ$

$$q_{hkl}^2 = h^2 a^{*2} + k^2 b^{*2} + l^2 c^{*2} \quad (14)$$

monoclinic : $a^* \neq b^* \neq c^*$, $\alpha^* = \gamma^* = 90^\circ$, $\beta^* \neq 90^\circ$

$$q_{hkl}^2 = h^2 a^{*2} + k^2 b^{*2} + l^2 c^{*2} + 2a^*c^* \cos(\beta^*)hl \quad (15)$$

An advantage of working in reciprocal space is that the Eq. 8-15 can be rewritten as a linear function of reciprocal unit cell parameters and Miller indices,

$$q_{hkl}^2 = x_{hh}h^2 + x_{kk}k^2 + x_{ll}l^2 + x_{hk}hk + x_{hl}hl + x_{kl}kl \quad (16)$$

$$\vec{q}_{hkl}^2 = H \cdot \vec{x}. \quad (17)$$

1.2.1 Difficulties with perfect data

In practice, obtaining the solution to the inverse problem of Eq. 1-7 is hindered by numerous obstacles which can be separated into two types of obstacles; those that will occur with "perfect data" and those that occur due to measurement errors. In the case of perfect data, such as powder diffraction patterns calculated from known crystal structures, existing algorithms exist which have claimed to be

The "unit cell" in these relations is an arbitrary mathematical representation of the repeating unit within a crystal as defined by convention. The IUCR defines the conventional unit cell as the smallest volume unit cell with a right-hand basis and edges along the lattice's symmetry directions. This is ambiguous, and there are multiple "settings" that describe the same unit cell. For example, the orthorhombic unit cell can be placed with any permutation of the unit cell lattices. However, convention dictates that **WHAT IS THIS CONVENTION**. In the monoclinic lattice system, convention dictates that the unit cell be so the monoclinic angle is β . The monoclinic unit cell can be defined just as well with the monoclinic angle at α or γ . Further, there exist two permutations of the monoclinic unit cell axes that retain the monoclinic angle at β , forming a six-fold ambiguity for the monoclinic lattice system.

There are additional ambiguities when considering the centered Bravais lattices which exist for all lattice systems except triclinic. In these cases, the minimum volume unit cell that describes the crystal's repetition is a triclinic (OR XXX???) lattice system. However, a larger volume unit cell also describes the crystal at a higher degree of symmetry, lending easier mathematical analysis. Convention dictates the use of the centered Bravais lattices

There are other definitions, such as the reduced unit cell (Buerger 1957) defined as INSERT DEFINITION, or the $s6$ (CITE) and $g6$ spaces (Andrews & Bernstein 1988). These definitions are unambiguous, however, they lose the mathematical convenience afforded by the centered, conventional unit cell representations and their simple relations to the experimentally observed diffraction.

Unit cells are also not unique. There are special cases where multiple, conventional unit cells can produce the same diffraction peak spacings. These unique cells are not simply different settings or representations of the same unit cell, but completely different unit cells. These cases have been tabulated, allowing their resolution (Oishi-Tomiyasu, 2016).

The powder diffraction indexing problem usually starts with a set of 20 observed peak positions $\{q_{obs}\}_{hkl}$. The lattice system equations relate each peak in the set to the same six lattice parameters. The relationships in Eq. 1 - 7 are relatively simple, and as shown in Eq. 1, can be formulated as a set of linear equations. However, each of the 20 equations is parameterized by a unique set of Miller indices $\{hkl\}$. If n_k peaks are chosen, there will be between $3n_k + 1$ and $3n_k + 6$ unknown parameters, there will always be three times more unknown parameters than equations. This is an under-determined inverse problem - more parameters being predicted than constraints that can be imposed. Determining the unknown parameters h , k , & l has long been recognized as the primary obstacle to powder diffraction indexing (de Wolff, 1961; de Wolff, 1968).

For cubic lattice systems, the single free lattice parameter, a can be found rather simply by pencil and paper, each peak occurs at an integer value divided by a^2 . For tetragonal, hexagonal, and rhombohedral with two free lattice parameters, solutions can be reliably found without computation aids through graphical methods (Hull & Davey, 1921). The difficulty begins with orthorhombic cases of three free lattice parameters, however, computational methods reliably identify the correct unit cell (CITE). Monoclinic and triclinic lattice systems are known to be difficult for computational algorithms.

1.3 Overview of existing powder diffraction indexing algorithms

Shirley (2003) gives a good overview of the history of powder diffraction indexing.

1.3.1 Dichotomy method

ADD DISCUSSION ON DICVOL.

1.3.2 Zone Indexing

The zone indexing method, first described by Runge (1917), rediscovered by Ito (1949), refined by de Wolff (1957, 1958), and implemented computationally by Visser (1969) demonstrates a methodological approach to determining the angles of a triclinic lattice. For the monoclinic case, the method starts with the assumption that the reciprocal lattice vectors a^* and ca^* have been determined by the position of the low angle reflections. If the monoclinic angle, β^* is 90° , there will be a diffraction peak at $q_{h0l}^2 = h^2a^{*2} + l^2c^{*2}$. When the monoclinic angle is not 90° , this peak will be split, with two peaks occurring at $q_{h0l}^2 = h^2a^{*2} + l^2c^{*2} + 2hla^*c^*\cos(\beta^*)$ and $q_{h0-l}^2 = h^2a^{*2} + l^2c^{*2} - 2hla^*c^*\cos(\beta^*)$. The monoclinic angle is obtained from difference of these two peaks, $\cos(\beta^*) = \frac{q_{h0l}^2 - q_{h0-l}^2}{4hla^*c^*}$.

This method demonstrates the difficulty in determining lattice angles from powder diffraction. First, an assumption is made that the lattice parameter lengths have been accurately determined. Then, a both sides of the split peak must be observed and identified. Lastly, the angle is determined from a combination of two peak positions and two lattice parameters. In practice, this is performed through a guess and check method and requires powder diffraction patterns determined at high accuracy.

This method should struggle with poor data. Missing low angle peaks will preclude lattice parameter determination. Contaminant peaks will add confusion to lattice parameter determination and identification of peak pairs. Error in peak positions will compound to a larger error in angle.

Machine learning methods for unit cell determination have not been demonstrated to determine unit cell lengths to the precision necessary to use in the zone indexing algorithm.

1.3.3 SVD-Index

The method presented here draws inspiration heavily from the SVD-Index indexing presented by Coehlo (2003). This algorithm takes a generate-and-test approach to the problem, it starts with random guesses of the lattice parameters, then iteratively assigns Miller indices and optimizes the lattice parameters to explain the observed diffraction spacing's. This is a complex algorithm and an abridged overview is given here to support the development of ML-Index. SVD-Index consists of three primary components: 1) Initial unit cell generation; 2) Miller index assignment; and 3) Unit cell optimization.

SVD-Index generates initial unit cell candidates by randomly generating numbers for the unit cell parameters, then rescaling the parameters to match a given unit cell volume. The algorithm incrementally increases the unit cell volume that is used for rescaling until a solution is found. This is a brute-force grid search of the entire parameter space. While inefficient, SVD-Index compensates with speed. Candidate entries are rapidly tested, with nearly 100,000 candidates tested in a matter of seconds. This is a very powerful approach, no assumptions are made regarding how the input diffraction maps onto unit cell parameters. These assumptions could easily lead to algorithm omitting the neighborhood of the true unit cell parameters from the search space in the presence of contaminate diffraction or heavily missing peaks, causing it to fail to find the correct unit cell parameters.

Miller index assignment starts by establishing a reference set of Miller indices. This is done by calculating all possible Miller indices up to a certain resolution cutoff given by the unit cell volume. For each reference Miller index and candidate unit cell, the forward models (Eq. 1-7), are used to calculate reference peak spacings, $\{q_{ref}\}_{hkl}$. Observed peaks are assigned the Miller index associated with the closest reference peak.

Unit cell parameters are optimized to match the agreement between the observed, $\{q_{obs}\}_{hkl}$, and calculated $\{q_{cal}\}_{hkl}$ peak spacings. This is performed using weighted least squares solved by singular value decomposition (SVD). The forward models (Eq. 1-7) are formulated as a linear function of Miller indices, H , and a transformation of the unit cell parameters, \vec{x} , $\vec{q}_{cal} = H\vec{x}$. The least squares solution to \vec{x} to $w_{hkl}\vec{q}_{obs} = w_{hkl}H\vec{x}$, where w is a set of weights for each peak, is then found using SVD (CITE). The weight function is

$$w_{hkl} = d_{obs,hkl}^m |2\theta_{obs,hkl} - 2\theta_{cal,hkl}| I_{obs}. \quad (18)$$

Here, $d_{obs,hkl}$ and $2\theta_{obs,hkl}$ are the observed peak spacings in resolution (d-spacing: $d = 1/q$) and angular units. $2\theta_{cal,hkl}$ is the calculated peak spacing in angular units. I_{obs} are the peak's integrated intensity. The ideal exponent, m , was noted to be 4, however using a random number generated between 0 and 4 was found to be the superior choice. This randomness is important to note, it is suggestive that a greedy algorithm, one that always picks the most probable next step, will not be able to find the correct unit cells.

This algorithm works by assuming a lattice system, and exhaustively searching for a solution within each lattice system.

1.4 Data-driven indexing approach

ML-Index seeks to improve the three primary components of SVD-Index using machine learning and Monte-Carlo inspired randomization.

Machine learning models are used to create random unit cell generators to reduce the initial search space. These are either derived by random sampling from neural network predictions of unit cell distributions, or individual trees in random forest models. For each lattice system, an ensemble of these models is trained on data grouped by unresolvable unit cell ambiguities or expected differences in peak spacing "distributions".

For the orthorhombic lattice system, this is an ambiguity about which unit cell axis is the shortest and which is the longest. This ambiguity is resolved by reindexing each entry in the training data so axis a is the shortest and c is the longest. The same reindexing is performed for monoclinic, but this leads to an unresolvable ambiguity for which angle is not set at 90° . Systematic absences of peaks are caused by symmetry elements at a scale finer than the lattice system. Within a lattice system, crystals with different reflection conditions result in different "distributions" of observed peak spacings.

A Neural network classification model predicts a probability distribution over the possible Miller index assignments. This improves the accuracy of Miller index assignments compared to picking the closest peak calculated from the set of reference Miller indices. It also provides confidence on the assignments and other probable assignments. This is performed by creating an array of pairwise differences between the observed peak positions and those calculated from the reference Miller indices, $q_{obs,hkl}^2 - q_{ref,hkl}^2$. This array is used as input to a neural network classifier.

Optimization of the candidate unit cells given assigned Miller indices is performed using a statistical optimization approach within the Markov Chain Monte Carlo (MCMC) framework. It seeks the most probable unit cell parameters and Miller index assignments given the observed diffraction. Randomization is applied to the optimization to encourage a full exploration of parameter space until the unit cells can describe the observed diffraction. New unit cell parameters are accepted or rejected based on the ratio of the next and current likelihoods, as performed in the Metropolis-Hasting algorithm.

2 Methods

2.1 Data

Crystallographic data for training the machine learning models are sourced from the Cambridge Structural Database (CSD) and the Crystallographic Open Database (COD). These databases do not contain experimental data, they contain structural information about materials that were determined experimentally. Powder diffraction patterns are calculated from entries in these databases.

True experimental data, as in experimentally measured powder diffraction patterns are sourced from the RRUFF database (Lafuente, et. al., 2015).

The generalization to true experimental data is at the very bottom of my to-do list. I have downloaded the RRUFF database and done some initial exploration. It is very clear that there are serious curation issues that need to be resolved to use this database.

2.1.1 Preparation

The first step of the dataset generation process is an initial quality control of entries in the CSD and COD. The following criteria must be met by an entry for its inclusion in either training or validating the machine learning models.

1. Chemical formula must be listed.
2. Spacegroup number must be listed and between 1 and 230, inclusively.
3. The Hermann–Mauguin symbol must be obtainable from the listed spacegroup symbol.
4. The crystal system must be listed and compatible with the given conventional unit cell parameters. Crystal systems are the same as lattice systems presented in Eq. 1-7, except with hexagonal and rhombohedral grouped.
5. A Niggli reduction, conversion from the conventional to reduced setting, must be successful.
6. Unit cell volume calculated from the given conventional and reduced unit cell parameters must match the given volume.
7. Unit cell volume must be consistent with the number of atoms in the unit cell deduced from the chemical formula. The given volume in \AA^3 must be within an order of magnitude of 18 times the number of atoms in the unit cell (CITE).

The next step is a duplicate removal step performed on the CSD and COD independently. A duplicate is defined as the following:

1. Belongs to the same Bravais lattice and,
2. The unit cell has the same chemical composition deduced from the chemical formula & the unit cell volume is within 5%.
3. or, The unit cell volume is exactly the same.

In the case of duplicate entries, the entry with the lowest reported r-factor is chosen. Databases are merged by first taking all the unique entries from the CSD, then only adding entries from the COD if they are not duplicated with an existing CSD entry. For the chemical composition of the unit cell, hydrogen and deuterium atoms, and atoms that comprise less than 5% of the total number of atoms in the unit cell are not taken into consideration. Many studies test the effects of changing one atom in a material. These entries could have nearly the same unit cell and very similar diffraction patterns. Omitting atoms that make up a small portion of the total number of atoms in the unit cell should consider these entries as duplicates.

I get the best performance when I use wide shallow neural networks, this was also observed by Vanessa. My understanding of wide shallow neural networks is they tend to work more by memorization as opposed to generalization. A strict duplication removal procedure could prevent the models memorizing identical patterns as opposed to generalization.

Datasets are generated from the unique entries in both the CSD and COD using the CCDI API. The following information is extracted from each entry:

- Diffraction pattern
- Database source
- Identifier (If from CSD)
- Local cif file name (If from COD)
- Spacegroup number
- Bravais Lattice
- Lattice system
- Hermann–Mauguin spacegroup symbol
- Unit cell parameters
- Unit cell volume

- Peak lists

The diffraction pattern is calculated between $2\theta = 0^\circ - > 60^\circ$ in steps of 0.02° with peak breadths of 0.1° full width at half max. It is calculated assuming the Cu K- α wavelength of 1.54 \AA . It is used to determine the peak list and is retained, although not used after dataset generation.

Two peak lists are included, these are 1) the first 60 non-systematically absent peaks and 2) the first 60 peaks that could be realistically observed in a diffraction pattern. For a diffraction peak to be included in peak list 2, each of the following conditions must be met:

1. Not systematically absent.
2. The diffraction pattern is normalized such that $\sum I^2 = 1$. The intensity at the peak position of the normalized diffraction pattern must be larger than 0.005.
3. The second derivative of the diffraction pattern at the peak position must be less than -1.
4. The peak must not be within $0.1^\circ / 1.5$ of another peak. If so, the peak with the largest intensity is retained.

The 0.1 FWHM figure is arbitrary. My impression is that a good diffraction pattern will be narrower. de Wolff 1957 provides 0.05 (FWHM) as their instrumental broadening resolution. The 1.5 was chosen by adjusting to a point where peaks were discarded when they were too close I feel like there should be a simple rule of thumb for this...

Three different representations of the unit cell are included, the conventional unit cell is taken directly from the the database. Next is a reindexed unit cell. For orthorhombic, monoclinic, and triclinic, the conventional unit cell is reindexed so the unit cell lengths, a , b , c are in increasing lengths. This reindex changes the Hermann–Mauguin spacegroup symbol, unit cell parameters, and Miller indices. Last is the Niggli reduced unit cell, which results in changes to the unit cell parameters and volume.

2.1.2 Grouping

Ensembles of machine learning models are created by training multiple models on different groupings of data. These groupings are performed according to the hierarchy:

1. Lattice system, then
2. Bravais lattice, then
3. Patterns in reflection conditions.

Grouping hierarchies 1 and 2 are well defined, and hierarchy 3 is subjective. This hierarchy is roughly based on the powder extinction class. In the original paper proposing using artificial neural networks to predict unit cell parameters (Habershon 2004), data was grouped into different powder extinction classes, and models were trained specifically for these groups.

To be clear what is meant by patterns in reflection conditions, consider the primitive orthorhombic space groups P 2 2 2 and P 21 21 21. P 2 2 2 has no systematic absences. The P 21 21 21 spacegroup has screw axis symmetry along the a , b , and c axes, leading to the reflection conditions that Miller index h must be even when $k = l = 0$; the $h00$ reflection must have an even h . The same condition is imposed on k and l for $0k0$ and $00l$ respectively. In the case that a mathematical model that determines the orthorhombic unit cell parameters, and does so by placing all of its emphasis on the low angle peaks which are more likely to have Miller indices 100, 010, and 001, It would first need to identify the difference between the P 2 2 2 and P 21 21 21 entries before predicting the unit cell parameters. If it cannot distinguish these cases, it should have great difficulty in predicting the unit cell parameters.

It should be noted that Hierarchy 3 splits up entries with the same space group. Consider spacegroup 18 which has two screw axes and one axis without a symmetry element. This one spacegroup contains spacegroup symbols P 21 21 2, P 21 2 21, and P 2 21 21. The P 21 21 2 spacegroup will have two reflection conditions $h00; h = 2n$ and $0k0; k = 2n$, and no reflection condition for l . The other two space symbols represent different permutations of the axes and reflection conditions. Again, orthorhombic crystals are reindexed so $a < b < c$, so all three settings will be contained in the training/testing datasets. Spacegroup 18 is split into three different groups, where they will be grouped with entries from other spacegroups with similar reflection conditions. On the other hand, consider spacegroup 17, with one screw axis and two axes without a symmetry element. This spacegroup contains symbols P 21 2 2, P 2 21 2, and P 2 2 21. Ideally, they would be split into different groups like Spacegroup 18, however, spacegroup 17, and other spacegroups with similar patterns in reflection conditions, have very few entries, so they are grouped into a single group due to limited data.

Specifications for the different groups can be found at "data/GroupSpec_lattice_system.xlsx"

I have a hunch that this grouping will help with generalization. Absent peaks, for whatever reason, should look like systematic absent patterns in a different group, so there should always be a random unit cell generator that matches the observed patterns in missing peaks, regardless if they are from systematic absences or other reasons.

2.1.3 Training / Validation split

An 80 / 20% split is made for the training and validation sets. This split occurs at the level of the the Hermann–Mauguin spacegroup symbol. So the separation of training and validation data is performed in groups of common Hermann–Mauguin spacegroup symbol.

2.1.4 Parameter scaling

Neural networks perform best with inputs and outputs that are on the scale of one. So there is typically a rescaling of inputs and outputs so they look like a Normal distribution, or fit between 0 and 1. Training occurs considering each lattice system at a time. For a given lattice system, the observed peak spacing as \bar{q}_{obs}^2 , is scaled using a standard scaler. Meaning the mean and standard deviation of \bar{q}_{obs}^2 for all entries in the training set are calculated. Then the mean is subtracted from each peak spacing and the difference is divided by the standard deviation. All of the peaks are scaled with the same mean and standard deviation regardless of the order in the peak list.

Unit cell length parameters are also scaled using a standard scaler. For a given lattice system, the mean and standard deviation of all the unit cell lengths of the training set entries are calculated and used for scaling.

Unit cell angle parameters are converted to radians and subtracted by $\pi/2$. This puts right angles (90°) at zero. Then the standard deviation of all angles, which were not originally 90° , is calculated in radians, and the zero-centered unit cell angles are divided by this scale.

Figure 1 shows the distribution of unit cell parameter and observed peak spacing for cubic. The top row shows the original values and the bottom row shows the values after scaling. The original entries from the databases are in the blue histogram, the augmented entries are in the orange histogram.

parameters a , b , and c don't look exactly the same, especially for the augmented entries. FIGURE 1

Figure 2 shows the distribution of unit cell parameter and observed peak spacing for orthorhombic. FIGURE 2

2.1.5 Augmentation

The grouping approach requires a broad diversity of data, as opposed to just simply more data. A broader range of data allows for more groups. To increase the breadth of data, augmentation is performed, preferentially augmenting entries associated with Hermann–Mauguin spacegroup symbols with fewer entries. Augmentation happens after splitting the dataset into training and validation sets. If the original entry is in the training set, all entries augmented from this entry will remain in the training set and visa-versa for the validation set. A single entry is augmented at most 10 times.

Two augmentations need to be performed for each entry. First, the unit cell parameters are randomly perturbed. Second, the observed peaks are resampled from the non-systemically absent peaks. There are three implemented methods to randomly perturb the unit cell parameters, all of these perturbations occur in the scaled unit cell parameter space:

1. Random perturbation. A unit cell parameter for an entry is perturbed by randomly sampled from a Normal distribution with the original entry's unit cell parameter as a mean and 0.2 as the standard deviation.
2. Covariance perturbation. The covariance matrix of the scaled unit cell parameters is calculated over all the training entries within a lattice system. Scaled unit cell parameters for an entry are perturbed by sampling from a multivariate normal distribution centered at the unperturbed scaled unit cell parameters and with a covariance matrix equal to 0.2^2 times the covariance matrix.
3. SVD perturbations. An SVD is performed on the scaled unit cell parameters from the training set within a lattice system. The training set's unit cell parameters are then transformed into the new orthogonal basis from the SVD and the standard deviation of each component is calculated. An entry's unit cells are perturbed by transforming the scaled unit cell parameters using the SVD, then sampling new components from a Normal distribution centered at the entry's transformed component values and with a standard deviation equal to 0.2 times the standard deviation of that component calculated over the training set.

The random perturbation method is used only for the cubic lattice system because a covariance or SVD is not possible with one parameter. All lattice systems other than cubic use the SVD perturbation method. The covariance and SVD methods were implemented so any correlation between unit cell parameters observed in the training data would be retained in the augmented entries. The covariance method is implemented in the code, but not used.

The peak list is augmented by resampling peaks from the list of non-systematically absent peaks. This resampling method was developed so the distribution of peaks in the augmented entries matched the distribution of the unaugmented entries.

There are two peak lists available at this time; Peak list 1, the first 60 non-systematically absent peaks. Peak list 2, the first 60 peaks that could be realistically observed in a diffraction pattern. Again, only using the training data to set up this augmentation. The position of the first observed peak in the list of all non-systematically absent peaks is recorded. An empirical distribution is made from a histogram of this position list. For the augmented entries, the empirical distribution is used to randomly select a first peak from the list of all non-systematically absent peaks.

After the first peak is picked, the list of non-systematically absent peaks is stepped through and the next peak is picked based on its distance from its neighboring peaks. Selecting the next peak based only on its separation is chosen to make observability only based on whether or not the peak will overlap another peak. The intensity of the peak is not a consideration. This approach is set up by calculating the minimum separation, Δq^2 in units of q^2 , between each peak in Peak list 1 and its neighbors. The fraction of neighboring peaks in Peak List 2 that are also neighboring in Peak List 1 are calculated in binned units of Δq^2 . This binning is in 50 bins log spaced between 10^{-3} to 10^{-2} for cubic and 10^{-4} to 10^{-2} otherwise. A function of the form,

$$\rho(\Delta q^2, q^2) = c(q^2) \left[1 - r_0^{-r_1 \Delta q^2} \right]. \quad (19)$$

is fit to the binned fraction and is used as an acceptance probability. This form was chosen because when $\Delta q^2 = 0$ the probability of acceptance is zero and as $\Delta q^2 \Rightarrow \infty$, the probability of acceptance approaches one. $c(q^2)$ is a linear function of q^2 . $c(q^2)$ should be bounded above by 1. It should be noted that $\rho(\Delta q^2, q^2)$ remains less than one in the observed range of Δq^2 so a well separated peak can be rejected. The probability increases monotonically between these two extremes.

For cubic, Figure 3 shows the probability distribution of the position of the first peak in the left column. The bottom right displays the distribution of Δq^2 and the top left shows $\rho(\Delta q^2)$ as calculated from the data (blue) and the fitting curve (orange). Figure 4 shows the equivalent plot for orthorhombic.

After the first peak is selected, there are three different cases for peaks:

1. Peak is within $0.1^\circ / 1.5$ of the preceding selected peak. Reject peak.
2. Peak is further than $0.1^\circ / 1.5$ of the preceding selected peak and the next peak in Peak list 1. Accept peak with probability $\rho(\Delta q^2)$.
3. Peak is further than $0.1^\circ / 1.5$ of the preceding selected peak and with $0.1^\circ / 1.5$ of the next peak in Peak list 1. Accept peak with 50% probability. If rejected, select the next peak. This assumes that if two peaks overlap, only one will be observed. Which peak is completely random.

The peak resampling method is a bit complicated. I arrived at it by trying to ensure the distribution of peaks in the augmented set matched the distribution in the unaugmented set.

For the cubic face centered Bravais lattice cF, the distribution of the Miller index label, position of the peak in the reference peak list, for each peak in the 10 peak list is shown in Figure 5. This is for unaugmented data.

2.2 Regression models

Regression models are developed to use their outputs to randomly generate unit cell candidates based on a set of peaks. For all lattice systems except cubic, 20 peaks from peak list 2 are used. For cubic lattice systems, 10 peaks are used because there tend to be fewer observable peaks in the diffraction patterns and 10 peaks are sufficient for one free parameter.

2.2.1 Neural Networks

Regression neural networks are set up as variance networks (Nix & Weigend, 1994). They output a mean and variance and are trained using a negative log-likelihood target function, which assumes a Normal likelihood. The approach presented here is based heavily on recent research on Variance networks (Detlefsen, et. al., 2019; Seitzer, et. al., 2022; Stirn & Knowles, 2021; Stirn et. al., 2022). Features implemented from this work include:

1. A distribution is predicted for variance as opposed to a point estimate. The Inverse-Gamma distribution is the posterior distribution for the unknown variance of a Normal distribution and is parameterized by α_{σ^2} and β_{σ^2} . The neural network outputs three values for each unit cell parameter, mean= μ , α_{σ^2} and β_{σ^2} .
2. Each model contains three parallel neural networks outputting μ , α_{σ^2} and β_{σ^2} separately. These networks start independently from the input peak spacings and have no connections with each other.
3. The training occurs cyclically. First, the weights of the μ network are optimized while the weights of the α_{σ^2} and β_{σ^2} networks are held fixed. Then the weights of the μ network are fixed while the weights of the α_{σ^2} and β_{σ^2} networks are optimized. Generally, each cycle is run for 10 epochs, and five cycles are performed.
4. A β -NLL target function is used when training the μ network and an NLL target function is used when training the α_{σ^2} and β_{σ^2} networks. The NLL target function is defined as:

$$L_{NLL} = \sum_{i=0}^{n_{uc}} \frac{1}{2} \ln(2\pi) - \alpha_{\sigma^2,i} \ln(\beta_{\sigma^2,i}) - \ln(\Gamma(\alpha_{\sigma^2,i} + 1/2)) + \ln(\Gamma(\alpha_{\sigma^2,i})) + (\alpha_{\sigma^2,i} + 1/2) \log(\beta_{\sigma^2,i} + \frac{1}{2}(u_{c_i} - \hat{u}_{c_i})^2). \quad (20)$$

Here, the summation occurs over the unit cell parameters, u_{c_i} are the true unit cell parameters, and \hat{u}_{c_i} is the estimated mean. The β -NLL target function multiplies the NLL target function by the variance estimate raised to the power β_L . This multiplicative term is excluded from the gradients calculations, as denoted by the stop-gradient operator $[\cdot]$,

$$L_{\beta-NLL} = \sum_{i=0}^{n_{uc}} \left[\left(\frac{\beta_{\sigma^2,i}}{\alpha_{\sigma^2,i} - 1} \right)^{\beta_L} \right] \left[\frac{1}{2} \ln(2\pi) - \alpha_{\sigma^2,i} \ln(\beta_{\sigma^2,i}) - \ln(\Gamma(\alpha_{\sigma^2,i} + 1/2)) + \ln(\Gamma(\alpha_{\sigma^2,i})) + (\alpha_{\sigma^2,i} + 1/2) \log(\beta_{\sigma^2,i} + \frac{1}{2}(u_{c_i} - \hat{u}_{c_i})^2) \right] \quad (21)$$

In the Inverse-Gamma distribution estimate of variance, the variance is estimated as $\frac{\beta_{\sigma^2}}{\alpha_{\sigma^2} - 1}$ where $\alpha_{\sigma^2} > 1$. The β -NLL target function raises this value to the power of $\beta_L = 1/2$ and multiplies the standard NLL target function.

I do not know which of the four of these points are important or not. There are still a few recommendations in the four papers referenced to improve the variance estimates that could be implemented.

There is an insidious memory leak in Tensorflow. I need to recompile the models between each cycle as the fixed weights and target function changes. Every time the model recompiles, a new "graph" is created in the backend of Tensorflow and the old graph is not released from memory. This is extremely bad when I am training 10 - 20 models per lattice system and I need to recompile the model 10 times during the training. I have spent some time trying to fix it without resolution. This is not the first time I have struggled with Tensorflow memory leaks and feel like jumping ship to Pytorch might be justified to solve this issue.

Each of the networks separately predicting μ , α_{σ^2} and β_{σ^2} are dense networks (MLP, Feed-forward). I do have an LSTM model that I used with orthorhombic. However, I am omitting this from the discussion.

The default mean network uses three hidden layers of 200, 100, and 60 units. These hidden layers proceed in the following order, 1) Dense layer, 2) Layer Normalization, 3) GELU activation, 4) Dropout at a rate of 50%. The final layer is a Dense layer that outputs with linear activation.

The default α_{σ^2} and β_{σ^2} networks use two hidden layers of 100 and 60 units and are otherwise similar to the mean network. The final layer of the α_{σ^2} and β_{σ^2} networks uses a softplus activation to enforce a positive constraint on β and one is added to the α_{σ^2} network after softplus activation to enforce an $\alpha_{\sigma^2} > 1$ constraint.

All training occurs with the Adam optimizer using a learning rate of 0.0001 and a batch size of 64. The Neural networks are implemented in TensorFlow.

I have not done any hyperparameter optimization. There is a lot of over and underfitting.

2.2.2 Random Forest

In addition to the neural networks described in Section 2.2.2, random forest regression models are fit for each group of data. These are implemented using SKlearn with the parameters: 80 decision trees each trained with a 10% subsample of the training set and a minimum of 10 samples per leaf. Otherwise, the parameters were the defaults of `sklearn.ensemble.RandomForestRegressor`.

I have not done the evaluation, but I would like to know how similar or different the random forest and neural network models are if the predictions overlap 100%, but the neural networks are more accurate, the random forest models should be dropped. If the predictions occasionally produce very different estimates, it would make sense to keep both models.

2.3 Miller Index Assignment Model

The second primary component of an indexing algorithm is the assignment of Miller indices given a unit cell and observed peak spacings. This is performed by establishing a reference set of Miller indices. These are all the possible Miller indices for a given lattice system without consideration for systematic absences. The forward diffraction models, (Eq. 1-7) are used to calculate a reference set of peak spacings from this reference set of Miller indices and a given unit cell. A pairwise difference array is then calculated between the observed and reference set of peak spacings. This array is then used as inputs to a neural network which produces a probability distribution, for each observed peak, for the correct Miller index assignment over the reference set of Miller indices.

2.3.1 Reference Miller Indices

1. Cubic lattice system

For each Miller index in the large reference array, $h^2 + k^2 + l^2$ is calculated. For each unique value of $h^2 + k^2 + l^2$, one set of Miller indices is retained. These are then sorted in order of increasing $h^2 + k^2 + l^2$.

2. Tetragonal lattice system

$h^2 + k^2$, and l^2 are calculated. For each unique set of $h^2 + k^2$ and l^2 , one set of Miller indices is retained. These are then sorted in order of increasing $h^2 + k^2 + l^2$.

3. Hexagonal lattice system $h^2 + hk + k^2$, and l^2 are calculated. For each unique set of $h^2 + hk + k^2$, and l^2 , one set of Miller indices is retained. These are then sorted in order of increasing $\frac{4}{3}(h^2 + hk + k^2) + l^2$.

4. Rhombohedral lattice system $h^2 + k^2 + l^2$ and $hk + kl + hl$ are calculated. For each unique set of $h^2 + k^2 + l^2$ and $hk + kl + hl$, one set of Miller indices is retained. These are then sorted in order of increasing $h^2 + k^2 + l^2$.

5. Orthorhombic lattice system For each Miller index in the "all possible set", h^2 , k^2 , and l^2 are calculated. Each unique set of h^2 , k^2 , and l^2 results in a unique peak. These are then sorted in order of increasing $h^2 + k^2 + l^2$.

6. Monoclinic lattice system h^2 , k^2 , l^2 , and hl are calculated. For each unique set of h^2 , k^2 , l^2 , and hl , one set of Miller indices is retained. These are then sorted in order of increasing $h^2 + k^2 + l^2$.

7. Triclinic lattice system h^2 , k^2 , l^2 , hk , kl and hl are calculated. For each unique set of h^2 , k^2 , l^2 , hk , kl and hl , one set of Miller indices is retained. These are then sorted in order of increasing $h^2 + k^2 + l^2$.

One reference Miller index set is created for each lattice system. The last step to establish the reference set of Miller indices is a final sorting. The peak spacing is calculated for each of peak in the set of unaugmented training data given the reference Miller index. The average peak spacings are then averaged over all the entries. Then the reference Miller index set is sorted so these averaged peak spacings increase monotonically. For cubic, 100 Miller indices are used in the reference set, for all other lattice systems 500 are used. The Miller indices taken from the peak list are then assigned a nominal label corresponding to the position of the equivalent Miller index in the sorted reference list. If a peak's Miller index is not found in the reference list, which occurs because the equivalent Miller index is beyond the 100 or 500 length cutoff, it is assigned the last nominal label which is given a Miller index of (000).

2.3.2 Neural Network

The inputs of the assignment neural network are scaled unit cell parameters and scaled \bar{q}_{obs}^2 . Pairwise differences are calculated by first calculating the scaled peak spacing for each reference Miller index given the input unit cell parameter, \bar{q}_{ref}^2 . This is subtracted from the observed peak spacing in a pairwise manner, $\Delta_{\bar{q}^2} = \bar{q}_{obs}^2 - \bar{q}_{ref}^2$. If there are 20 input peaks and a reference peak list of 500, this will result in a 20 x 500 array. If the pairwise differences are used to predict peak positions, the correct assignment will occur at small values of $\Delta_{\bar{q}^2}$. Additionally, incorrect assignments will occur at large absolute values of $\Delta_{\bar{q}^2}$. $\Delta_{\bar{q}^2}$ is transformed so it will span the range of 0 to 1, where the correct assignment will occur at larger values,

$$\Delta_{\bar{q}^2}^* = \frac{\epsilon_{\Delta}}{|\Delta_{\bar{q}^2}| + \epsilon_{\Delta}}. \quad (22)$$

$\Delta_{\bar{q}^2}^*$ is used as the input to a dense neural network. The default network uses four hidden layers of 200, 200, 200, and 100 units. These hidden layers proceed in the following order, 1) Dense layer, 2) Layer Normalization, 3) GELU activation, and 4) Dropout at a rate of 25%. These hidden layers operate on two-dimensional arrays of shape (# peaks, # reference Miller indices) for the first hidden layer and (# peaks, # previous layer units) for the second hidden layer. The Dense layers in Tensorflow operate on these 2D inputs as follows.

1. Create a kernel with shape (# previous layer units, # current layer units).
2. For each peak dimension, operate on the input array with the same kernel from step 1).

In other words, the same kernel operates on each peak dimension in the hidden layer independently. I believe the implications of this are 1) information isn't being shared between peak dimensions, and 2) Each peak dimension is considered equivalently.

The final layer is a dense layer that outputs with softmax activation. Instead of one dense layer output, the network is branched for each peak and a separate dense layer - meaning a separate kernel - acts on the one-dimensional input for the given peak.

An important aspect of this NN is fast inference. I would like to find the smallest possible NN that provides reasonable results. Some testing shows that I can move to a one hidden layer NN with minimal loss of accuracy. It is tempting to try to use a convolutional neural network here. I did see improved accuracy with cubic. In that case, the reference set of Miller indices increases with q^2 . There is a well defined spatial relationship in the pairwise difference array. This is lost in the other lattice systems.

An important question is, what data should these networks be trained with? One option is the true unit cell parameters, this results in 100% accurate Miller index assignments (and it does). On the other hand, they could be trained using the predicted unit cell parameters from the regression networks. This should represent a worst-case scenario for unit cells. There should be a few randomly generated unit cells better than the expected value. After several rounds of optimization, the best candidate unit cells should be much better than their initial value. Ideally, the Miller index assignment networks would be trained on unit cell parameters that differ from the true values by amounts consistent with the current best candidate unit cells during optimization. To these means, a series of assignment networks are trained with input unit cell parameters randomly perturbed from their true values. This is performed by adding Gaussian noise to the scaled unit cell parameters before they are used to calculate pairwise differences. This is set up in a way that each entry receives a different perturbation for each training epoch for additional regularization.

Training of the assignment neural networks is performed with all the unaugmented entries for a given lattice system. Augmented entries are omitted because there is plenty of unaugmented data to train the networks - and I have a bug somewhere in the augmentation algorithm regarding the Miller index assignments. The sparse categorical cross-entropy target function is chosen and the Adam optimizer is used with a learning rate of 0.002 and a batch size of 256. Networks are trained with Gaussian noise levels of 0.10, 0.05, 0.025, and 0.01. For monoclinic, the associated ϵ_Δ values were 0.01, 0.0075, 0.01, and 0.005.

2.4 Optimization

The third component of the indexing algorithm is the optimization of unit cell parameters given initial unit cell candidates and a mechanism to assign Miller indices. Currently, the program only considers one lattice system at a time. So if a crystal is known to be monoclinic, it will be optimized with that as a known fact. After getting this working with all lattice systems, I will work on the case where the lattice system is unknown.

2.4.1 Target function

The optimization seeks the most probable Miller index, H , and unit cell, \vec{c} , given a set of observed peak spacings, \vec{q}_{obs} . This can be expanded using Baye's formula.

$$\rho(H, \vec{c} | \vec{q}_{obs}) = \frac{\rho(\vec{q}_{obs} | H, \vec{c}) \rho(H, \vec{c})}{\rho(\vec{q}_{obs})}. \quad (23)$$

The denominator can be dropped because it will be constant for all unit cell parameters and Miller index assignments,

$$\rho(H, \vec{c} | \vec{q}_{obs}) \propto \rho(\vec{q}_{obs} | H, \vec{c}) \rho(H, \vec{c}). \quad (24)$$

Miller index assignments are conditional on the unit cell parameters,

$$\rho(H, \vec{c} | \vec{q}_{obs}) \propto \rho(\vec{q}_{obs} | H, \vec{c}) \rho(H | \vec{c}) \rho(\vec{c}). \quad (25)$$

This can be written as a product of probabilities over each observed peak spacing $q_{obs,k}$, assuming there are n_k peaks and independence between peaks.

$$\rho(H, \vec{c} | \vec{q}_{obs}) \propto \prod_{k=1}^{n_k} [\rho(q_{obs,k} | H_k, \vec{c}) \rho(H_k | \vec{c})] \rho(\vec{c}). \quad (26)$$

This equation can be further expanded as a summation over all n_h reference Miller indices for each of the n_k peaks,

$$\rho(H, \vec{c}|\vec{q}_{obs}) \propto \prod_{k=1}^{n_k} [\sum_{h=0}^{n_h} \rho(q_{obs,k}|H_{k,h}, \vec{c}) \rho(H_{k,h}|\vec{c})] \rho(\vec{c}). \quad (27)$$

From here, this likelihood function is simplified. The probability distribution of the unit cell lengths is assumed to be constant and bounded between 2 and 500 Å. The probability distribution of the unit cell angles is assumed to be constant and bounded between 0° and 180°. The term $\rho(\vec{c})$ is removed from the likelihood function and out-of-bounds unit cells are removed from consideration,

$$\rho(H, \vec{c}|\vec{q}_{obs}) \propto \prod_{k=1}^{n_k} [\sum_{h=0}^{n_h} \rho(q_{obs,k}|H_{k,h}, \vec{c}) \rho(H_{k,h}|\vec{c})]. \quad (28)$$

On my to-do list is to optimize this term directly. It is entirely possible with the current set of infrastructure. The Miller index neural networks are differentiable. I can optimize the unit cell parameters considering all possible Miller Index assignments. Or put it into a Tensorflow Probability MCMC method. Until then, I am only considering one at a time for simplification.

The likelihood equation is again simplified by only considering the j^{th} Miller index assignment at a time. The most obvious choice is the most probable Miller index assignment, but this is not necessarily the case.

$$\rho(H, \vec{c}|\vec{q}_{obs}) \propto \prod_{k=1}^{n_k} \rho(q_{obs,k}|H_{k,j}, \vec{c}) \rho(H_{k,j}|\vec{c}). \quad (29)$$

The term $\rho(H_{k,j}|\vec{c})$ is the softmax output from the assignment neural network. The term $\rho(q_{obs,k}|H_{k,j}, \vec{c})$ is a normal distribution,

$$\rho(q_{obs,k}|H_{k,j}, \vec{c}) = \frac{1}{\sqrt{2\pi}\sigma_{q^2}} \exp\left(-\frac{1}{2} \frac{(q_{obs,k}^2 - q_{cal,kj}^2)^2}{\sigma_{q^2}^2}\right). \quad (30)$$

Eq. 1-7 are used to calculate $q_{cal,kj}^2$. The standard deviation parameter, $\sigma_{q^2}^2$, is defined like the weighting by SVD-Index,

$$\sigma_{q^2}^2 = q_{obs,k}^2 |q_{obs,k}^2 - q_{cal,kj}^2 + \epsilon_{q^2}|. \quad (31)$$

Notably, the $\sigma_{q^2}^2$ presented here does not include the randomness in the same way as the weighting in SVD-Index. Randomness is important to the algorithm and is included by other means.

2.4.2 Random Unit Cell Generation

Random unit cells are generated from the neural networks. For each entry, a set of mean and variance estimates are taken from each neural network trained on the different groups. Each set of mean and variance estimates are used as parameters for a Normal distribution and n_{nn} random unit cell parameters are generated. n_{rf} random unit cells are also selected from individual decision trees in the random forest model.

Once random unit cells are generated, there will be $n_g(n_{nn} + n_{rf})$ candidate unit cells where n_g is the number of groups.

2.4.3 Unit Cell Optimization

An algorithm within the Markov Chain Monte Carlo (MCMC) framework is used to perform statistical optimization of the unit cell parameters. MCMC algorithms roughly follow these steps

1. Start with an initial set of parameters and a posterior or likelihood function.
2. Select a new set of parameters according to a well-defined protocol that includes a randomness aspect.
3. Either accept or reject the new set of parameters based on the ratio of the likelihoods a detailed balance condition calculated from the new set and original set of parameters.

Then iteratively repeat steps 2) and 3), until a predefined stopping point or criteria. Applying an MCMC algorithm to find the unit cell parameters that maximize Eq. 29 faces several challenges. One challenge is the discrete nature of this problem. Essentially, we are trying to find unit cell parameters that calculate a set of grid points, \vec{q}_{calc} that match the known grid, \vec{q}_{obs} . Gradient based MCMC algorithms generally provide faster convergence, but are difficult to adapt to discrete problems (Rhodes & Gutmann 2022).

A novel MCMC algorithm is developed using the gradient based Metropolis Adjusted Langevin Algorithm (MALA) that uses random subsampling or resampling of the Miller indices to apply randomness through the discrete aspect of this optimization, with the gradient of the Eq. 29 to move the continuous unit cells into the direction higher probability. A typical MALA implementation would select the next set of unit cell parameters, \vec{c}_{t+1} from the current set of unit cell parameters, \vec{c}_t according to the formula,

$$\vec{c}_{t+1} = \vec{c}_t + \tau A \nabla \ln[\rho(\vec{q}_{obs}|H_t, \vec{c}_t)] + \sqrt{2\tau A} \epsilon. \quad (32)$$

In this equation, τ is an adjustable parameter, A is a preconditioning matrix, ϵ is drawn from a standard Normal distribution, and $\nabla \ln[\rho(\vec{q}_{obs}|H_t, \vec{c}_t)]$ is the gradient of the target distribution. The preconditioning matrix is an optional addition to help when the target distribution varies differently with each parameter, or are unnormalized so the $\sqrt{2\tau}\epsilon$ term is on a different scale than the gradient term. Conveniently, the second derivatives of q_{cal}^2 are zero with respect to the optimizable parameters, easing the calculation of the Hessian of $\ln[\rho(\vec{q}_{obs}|H_t, \vec{c}_t)]$. We can choose the preconditioning matrix to be the inverse of the Hessian, $A = \mathcal{H}^{-1} = (\nabla^2 \ln[\rho(\vec{q}_{obs}|H_t, \vec{c}_t)])^{-1}$. (READ AND CITE WORK THAT DOES THIS, TITSIAS 2023 WILL HAVE THE REFERENCES).

This is entirely speculation. Due to the discrete nature of this problem, I expect $\rho(\vec{q}_{obs}|H_t, \vec{c}_t)$ to have sharp maximums that the term $\sqrt{2\tau}\mathcal{H}^{-1}\epsilon$ might not be to move the unit cell parameters out of. For this reason, this term is removed and randomness is added to the MALA step by random subsampling or resampling of the assigned Miller indices. The standard MALA method uses random perturbations to allow the MALA trajectory to escape the local maximums in a fixed target probability distribution. If the maximums are sharp, this may not work. Random resampling or subsampling of the peaks will instead randomly perturb the target probability distribution's landscape, allowing the gradient term to effectively explore the unit cell distribution.

In a way, this was the successful approach taken by SVD-Index. Their weighting term for least-squares was Eq. 18, $w_{hkl} = d_{obs,hkl}^m |2\theta_{obs,hkl} - 2\theta_{cal,hkl}| I_{obs}$. In this equation, m is randomly selected between 0 and 4. When m is zero, or otherwise very small, this term will receive very little weight. It will be dropped out in a manner similar to subsampling the peaks.

The complete MALA inspired algorithm is

1. Perform inference using the assignment Neural networks to obtain a set of initial unit cells.
2. Assign Miller indices. There are four different assignment neural networks that were trained on data with varying degrees of random perturbation to the unit cell parameters. One of these neural networks is randomly selected and used for inference for all candidate unit cells.
 - a) Resampling: Miller indices are randomly assigned with probabilities equal to their softmaxes calculated at the current set of unit cell parameters, $\rho(H_t|\vec{c}_t)$.
 - b) Subsampling: The most probable Miller index is assigned H_{max} . However, only a fraction of peaks are retained with probabilities taken from their softmaxes calculated at the current set of unit cell parameters, $\rho(H_{max,t}|\vec{c}_t)$ (after a renormalization step).
3. Calculate $\sigma_{q^2}^2$ based on the current unit cell and Miller index assignment
4. Step the unit cell parameters in the direction of the gradient of the negative log-likelihood, given by Eq. 8. In this method, the next unit cell parameters, \vec{c}_{t+1} are calculated from the current unit cell parameters, \vec{c}_t using the Gauss-Newton step.

$$\vec{c}_{t+1} = \vec{c}_t + \mathcal{H}_t^{-1} \nabla \ln[\rho(\vec{q}_{obs}|H_t, \vec{c}_t)] \quad (33)$$

5. Accept the new set of unit cell parameters with probability given by,

$$a = \max \left\{ 1, \frac{\rho(\vec{q}_{obs}|H_{t+1}, \vec{c}_{t+1})}{\rho(\vec{q}_{obs}|H_t, \vec{c}_t)} \right\}. \quad (34)$$

These probabilities are calculated using the full set of the most probable Miller index assignments. Detailed balance is ignored, adding this is on my to-do list.

Coelho (2003) shows that 20 peaks will result in 12% of triclinic unit cells will be unsolvable. 9% for monoclinic. These numbers are much larger at 10 peaks. For these lattice systems, subsampling peaks is likely inappropriate. However, subsampling is faster and appropriate for higher symmetry lattice systems

2.4.4 Candidate Removal

Candidate unit cells are removed from consideration if they meet the following criteria:

1. The standard deviation and median of all the negative log likelihoods are calculated. If a candidate unit cell's likelihood is four standard deviations larger than the median, it is removed.
2. If one unit cell axes length is less than 2 Å or greater than 500 Å.

3. If one unit cell angle is less than 0° or greater than 180° .
4. If a candidate unit cell is found to "explain" the observed diffraction, they are placed in a list of "explainers".

The criteria for explaining the observed diffraction is based on the numerical value of the negative log-likelihood,

$$\begin{aligned}
-\ln [\Pi_{k=1}^{n_k} \rho(q_{obs,k} | H_j, \vec{c})] &= -\sum_{k=1}^{n_k} \ln \left[\frac{1}{\sqrt{2\pi}\sigma_{q^2}} \exp \left(-\frac{1}{2} \frac{(q_{obs,k}^2 - q_{cal,k}^2)^2}{\sigma_{q^2}^2} \right) \right], \\
&= -\sum_{k=1}^{n_k} \ln \left[\frac{1}{\sqrt{2\pi}\sigma_{q^2}} \right] + \ln \left[\exp \left(-\frac{1}{2} \frac{(q_{obs,k}^2 - q_{cal,k}^2)^2}{\sigma_{q^2}^2} \right) \right], \\
&= -\sum_{k=1}^{n_k} -\ln [\sqrt{2\pi}\sigma_{q^2}] - \frac{1}{2} \frac{(q_{obs,k}^2 - q_{cal,k}^2)^2}{\sigma_{q^2}^2}, \\
&= \sum_{k=1}^{n_k} \ln [\sqrt{2\pi}] + \ln [\sigma_{q^2}] + \frac{1}{2} \frac{(q_{obs,k}^2 - q_{cal,k}^2)^2}{\sigma_{q^2}^2}, \\
&= n_k \ln [\sqrt{2\pi}] + \sum_{k=1}^{n_k} \frac{1}{2} \ln [\sigma_{q^2}^2] + \frac{1}{2} \frac{(q_{obs,k}^2 - q_{cal,k}^2)^2}{\sigma_{q^2}^2}.
\end{aligned}$$

Now given $\sigma_{q^2}^2 = q_{obs,k}^2 | q_{obs,k}^2 - q_{cal,k}^2 + \epsilon_{q^2} | \approx q_{obs,k}^2 \epsilon_{q^2}$ when $q_{cal,k}^2 \approx q_{obs,k}^2$, this likelihood simplifies to

$$\begin{aligned}
&= n_k \ln [\sqrt{2\pi}] + \sum_{k=1}^{n_k} \frac{1}{2} \ln [q_{obs,k}^2] + \frac{1}{2} \ln [\epsilon_{q^2}], \\
&= n_k [\ln [\sqrt{2\pi}] + \frac{1}{2} \ln [\epsilon_{q^2}]] + \sum_{k=1}^{n_k} \frac{1}{2} \ln [q_{obs,k}^2].
\end{aligned}$$

As implemented, the number $\epsilon_{q^2} = 10^{-10}$ is a small value used to prevent a division by zero error. This gives the result that

$$= n_k [\ln [\sqrt{2\pi}] - 10 \frac{1}{2} \ln [10]] + \sum_{k=1}^{n_k} \frac{1}{2} \ln [q_{obs,k}^2].$$

The $-10n_k \frac{1}{2} \ln [10]$ term is the dominant term in this expression and provides an approximate value for the likelihood when the unit cell parameters can explain the observed diffraction to numerical error. This will be approximately $-10n_k$. When a candidate unit cell produces a likelihood less than this value, it is assumed to fully explain the diffraction and is removed from the list of unit cell candidates being optimized.

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