Delayed Differential equations Through Epidemiological models

Shwetabh Singh

Thesis Presentation

15 May, 2021

Recap

You are probably how we ended up here, well its a long story. Well for that we need to go back, way back.

What is a DDE ?

A Delay Differential equation* is simply a differential equation* in which time derivative of a function at the current time depends on its value or its derivatives at some previous time, mathematically

$$\begin{cases} \dot{x} = F(t, x(t), x(t-\tau_1)...x(t-\tau_n), \dot{x}(t-\sigma_1)...\dot{x}(t-\sigma_m)) & t \leq t_0 \\ x(t) = \phi(t) & t \leq t_0 \end{cases}$$

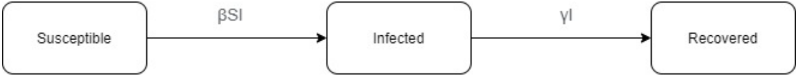
Here τ_i and σ_j are the time delays

Compartments

Before moving forward, lets describe the compartments in the model
Susceptible - People who haven't either been infected with the disease yet, or are simply just susceptible to it.
Infected - People who catch the disease and now act as carriers and further spreaders of the disease to the susceptible individuals.
Recovered - People who have recovered from the disease. In the most basic SIR model, recoveries are assumed to be permanent and if the disease is fatal might include fatalities too.

SIR Model

One of the most basic compartmental epidemiological model.



SIR model

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta}{N}S(t)I(t)$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}S(t)I(t) - \gamma I(t)$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I(t)$$

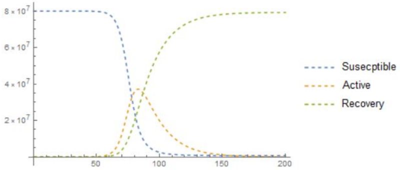


Figure : SIR model

Introducing a Delay

We introduced a delay of 14 days (assuming COVID takes 14 days to recover).

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta}{N}S(t)I(t)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}S(t)I(t) - \frac{\beta}{N}S(t-\tau)I(t-\tau)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\beta}{N}S(t-\tau)I(t-\tau)$$

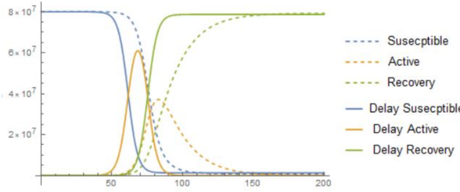


Figure : Comparison between a vanilla SIR model and a delayed SIR model

An upgrade - SIRD model

An upgrade to the model would be to introduce fatality (as observed in COVID too), to separate the recovered from the dead. Now this doesn't make a lot of difference from the last model, but is necessary.

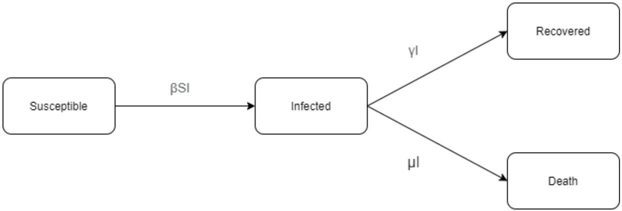


Figure : A block model of SIRD Model

Comparison

Mathematically, it becomes

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta}{N}S(t)I(t)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}S(t)I(t) - \frac{\beta}{N}S(t-\tau)I(t-\tau)e^{-\mu\tau} - \mu I(t)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\beta}{N}S(t-\tau)I(t-\tau)e^{-\mu\tau}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \mu I(t)$$

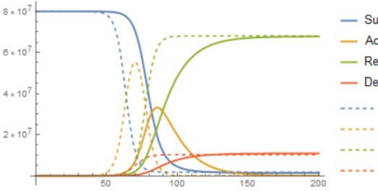
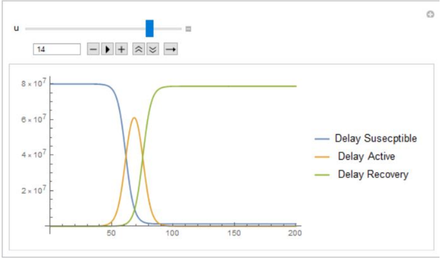


Figure : Comparison between a vanilla SIRD model and a delayed SIRD model

Varying the delay

As the focus of our study is effect of delay on the model, we can try and vary the delay, seeing how it effects the model



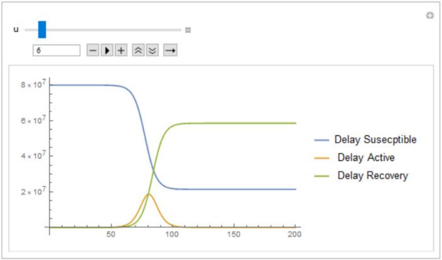


Figure : Comparison of Delayed SIR models with $\tau_1 = 14$ units and $\tau_2 = 6$ days

Varying the delay

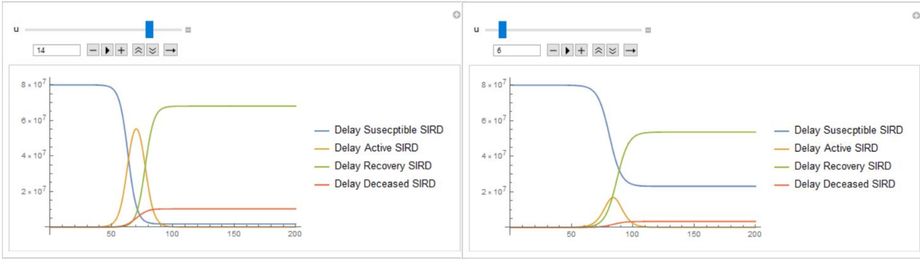


Figure : Comparison of Delayed SIRD models with $\tau_1 = 14$ units and $\tau_2 = 6$ days

Phase Space Diagrams

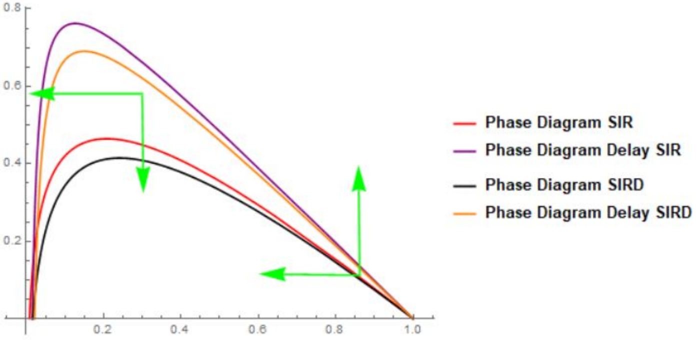


Figure : Phase Space Diagrams of Susceptible vs Infected

Phase Space Diagrams

Phase space diagram in the model helps us understand two important aspects of an epidemic, Overshoot and Herd Immunity.

Herd Immunity is when a significant fraction of people who are immune to a disease, outbreaks generally die out.

Overshoot is the difference between Herd Immunity threshold and final amount of infected people.

We can vary the delay to observe the change in Overshoot and Herd immunity threshold.

SIRS Model

An assumption of the SIR model is that, once recovered the individual becomes permanently immune to the disease, and hence the epidemic dies down. Hence, the idea of **temporary immunity** comes up when discussing epidemics. It means that a recovered individual is only immune to the disease for a short time, and after a fixed average duration becomes re-susceptible to the disease.

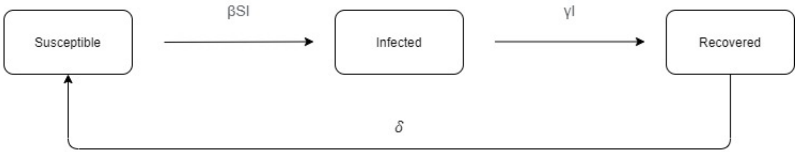


Figure : Block Diagram of SIRS model

SIRS Model

The model can be written as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta}{N}S(t)I(t) + t_i R(t)$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}S(t)I(t) - \gamma I(t)$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I(t) - t_i R(t)$$

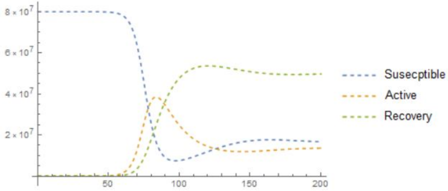


Figure :A vanilla SIRS model

Where the temporary immunity factor t_i is defined as $\frac{1}{t_0}$, where t_0 is the time in days by when the immunity from the disease nullifies. In the models under consideration the value of t_0 is taken as 60 days.

Delayed SIRS Model

The model can be written as

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\frac{\beta}{N}S(t)I(t) + t_i R(t) \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta}{N}S(t)I(t) - \frac{\beta}{N}S(t-\tau)I(t-\tau) \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \frac{\beta}{N}S(t-\tau)I(t-\tau) - t_i R(t)\end{aligned}$$

SIRS Model

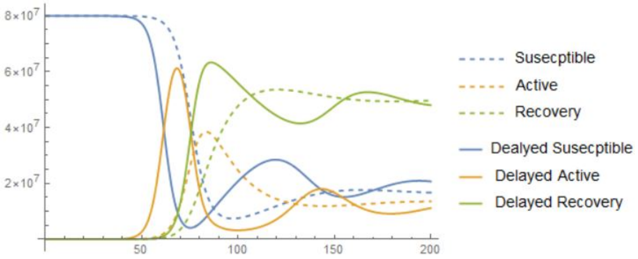


Figure : Delayed SIRS model

SIRS Model

Two questions arise -

1. What does the phase space look like?
2. Are the oscillations dependent on the delay factor?

SIRS Model

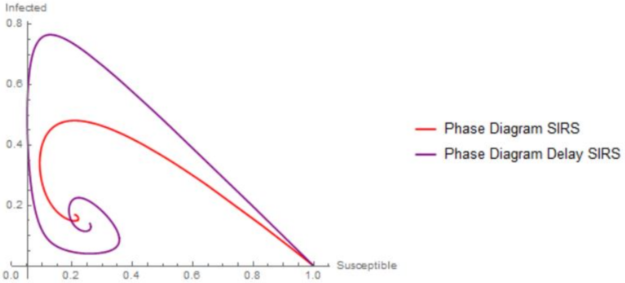


Figure : Phase space diagram of Delayed SIRS model superimposed to the vanilla SIRS model

SIRS Model

Varying the delay

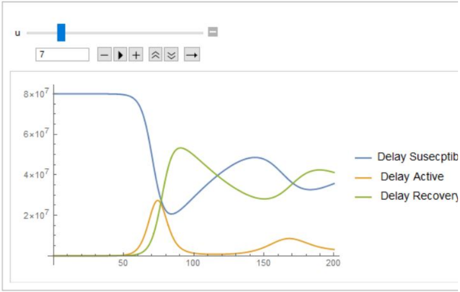


Figure : SIRS model with variable delay, delay factor u = 7 days

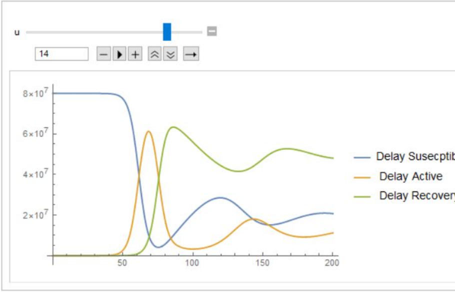


Figure : SIRS model with variable delay, delay factor u = 14 days

SIRS Model

Varying the delay and its impact on phase space

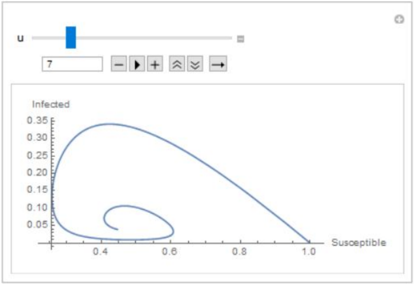


Figure : Parametric Phase plot of SIRS model with variable delay, delay factor u = 7 days

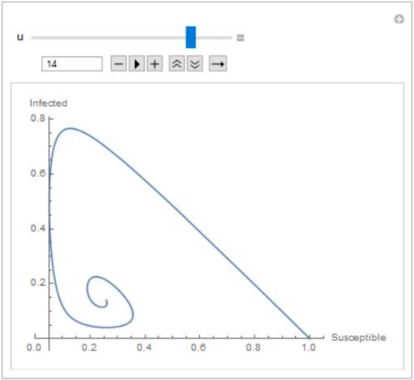


Figure : Parametric Phase Plot of SIRS model with variable delay, delay factor u = 14 days

Coupled SIR Model

In the next step we introduce a **coupling**. The model involved taking 2 populations with two different sets of the same disease with their own characteristic coefficients, which depends on external factors like masking, severity of social distancing etc and the intrinsic factors of disease like the delay factor due to mutation etc, modeling as two different geographical and then coupled.

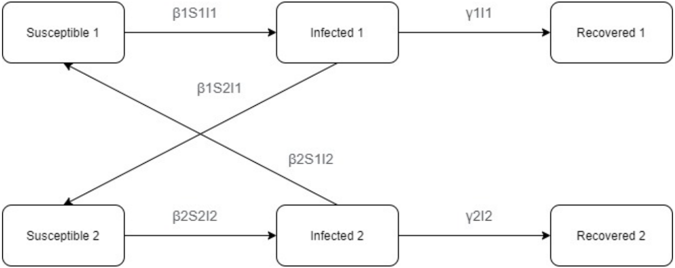


Figure : Block Diagram of SIRS model

Coupled SIR Model

The model can be written as

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= -S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t)) \\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= -S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t)) \\
\frac{\mathrm{d}I_1}{\mathrm{d}t} &= S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t)) - \gamma I_1(t) \\
\frac{\mathrm{d}I_2}{\mathrm{d}t} &= S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t)) - \gamma I_2(t) \\
\frac{\mathrm{d}R_1}{\mathrm{d}t} &= \gamma_1 I_1(t) \\
\frac{\mathrm{d}R_2}{\mathrm{d}t} &= \gamma_2 I_1(t)
\end{aligned}$$

Coupled SIR Model

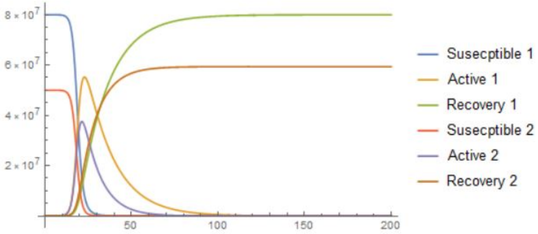


Figure : Coupled SIR model

Setting the values of $\beta_1 = 0.3$, $N_1 = 80 \times 10^6$ $\gamma^{-1} = \mathrm{T} = 16$ days, $\beta_2 = 0.5$, $N_2 = 50 \times 10^6$, $\gamma_2^{-1} = 10$ days, we can finally solve the coupled model

Coupled SIR Model

Introducing the Delay, we get

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = -S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t))$$

$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = -S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t))$$

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t)) - S_1(t-\tau_1)\frac{\beta_1}{N_1}I_1(t-\tau_1) - S_1(t-\tau_2)\frac{\beta_2}{N_1}I_2(t-\tau_2)$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t)) - S_2(t-\tau_1)\frac{\beta_1}{N_2}I_1(t-\tau_1) - S_2(t-\tau_2)\frac{\beta_2}{N_2}I_2(t-\tau_2)$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = S_1(t-\tau_1)\frac{\beta_1}{N_1}I_1(t-\tau_1) + S_1(t-\tau_2)\frac{\beta_2}{N_1}I_2(t-\tau_2)$$

$$\frac{\mathrm{d}R_2}{\mathrm{d}t} = S_2(t-\tau_1)\frac{\beta_1}{N_2}I_1(t-\tau_1) + S_2(t-\tau_2)\frac{\beta_2}{N_2}I_2(t-\tau_2)$$

Coupled SIR Model

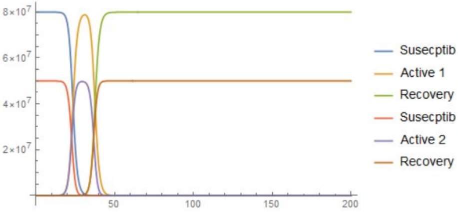


Figure : Coupled Delayed SIR model with same delay, $\tau_1 = \tau_2 = 14$ days

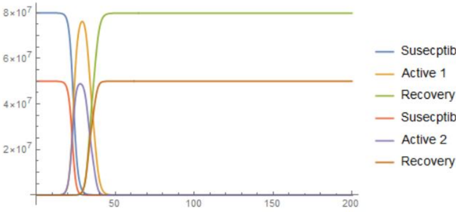


Figure : Coupled Delayed SIR model with different delays, $\tau_1 = 14$ days and $\tau_2 = 10$ days

Coupled SIR Model

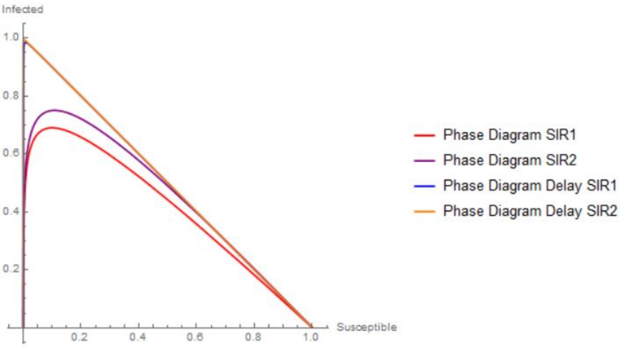


Figure : Phase Plot of Coupled Delayed SIR model with different delays, $\tau_1 =$ 14 days and $\tau_2 = 10$ days

Coupled SIRS Model

Combining the Coupled SIR and SIRS models we get

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = -S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t)) + t_{i1}R_1(t)$$

$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = -S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t)) + t_{i2}R_2(t)$$

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t)) - \gamma I_1(t)$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t)) - \gamma I_2(t)$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = \gamma_1 I_1(t) - t_{i1}R_1(t)$$

$$\frac{\mathrm{d}R_2}{\mathrm{d}t} = \gamma_2 I_1(t) - t_{i2}R_2(t)$$

Coupled SIR Model

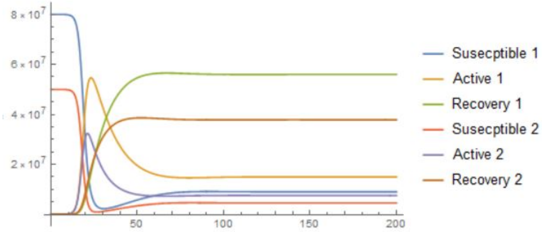


Figure : Coupled SIRS model with $t_{01} = 60$ days and $t_{02} = 50$ days

Coupled SIR Model

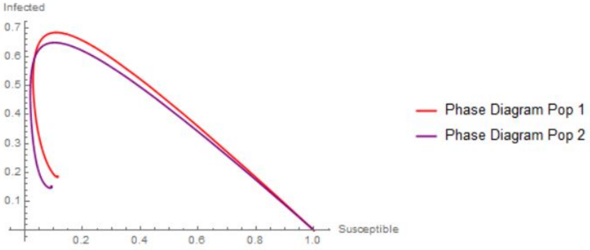


Figure : Phase Plot of Coupled SIRS model with $t_{01} = 60$ days and $t_{02} = 50$ days

Coupled SIRS Model

Combining last two models and introducing a delay

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = -S_1(t)(\frac{\beta_1}{N_1}I_1(t) + \frac{\beta_2}{N_1}I_2(t)) + t_{i1}R_1(t)$$

$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = -S_2(t)(\frac{\beta_1}{N_2}I_1(t) + \frac{\beta_2}{N_2}I_2(t)) + t_{i2}R_2(t)$$

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = S_1(t)(\frac{\beta_1}{N_1}I_1(t)+\frac{\beta_2}{N_1}I_2(t)) - S_1(t-\tau_1)\frac{\beta_1}{N_1}I_1(t-\tau_1) - S_1(t-\tau_2)\frac{\beta_2}{N_1}I_2(t-\tau_2)$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = S_2(t)(\frac{\beta_1}{N_2}I_1(t)+\frac{\beta_2}{N_2}I_2(t)) - S_2(t-\tau_1)\frac{\beta_1}{N_2}I_1(t-\tau_1) - S_2(t-\tau_2)\frac{\beta_2}{N_2}I_2(t-\tau_2)$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = S_1(t-\tau_1)\frac{\beta_1}{N_1}I_1(t-\tau_1) + S_1(t-\tau_2)\frac{\beta_2}{N_1}I_2(t-\tau_2) - t_{i1}R_1(t)$$

$$\frac{\mathrm{d}R_2}{\mathrm{d}t} = S_2(t-\tau_1)\frac{\beta_1}{N_2}I_1(t-\tau_1) + S_2(t-\tau_2)\frac{\beta_2}{N_2}I_2(t-\tau_2) - t_{i2}R_2(t)$$

Coupled SIRS Model

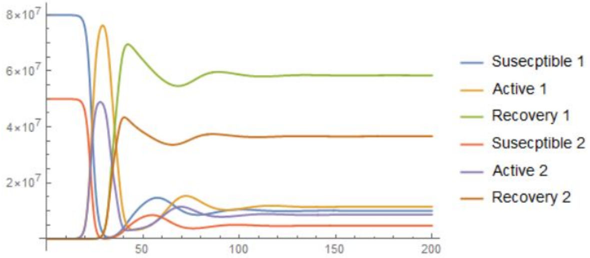


Figure : Coupled Delayed SIRS model with different delays, $\tau_1 =$ 14 days and τ_2 = 10 days and t_{01} = 60 days and t_{02} = 50 days

Coupled SIRS Model

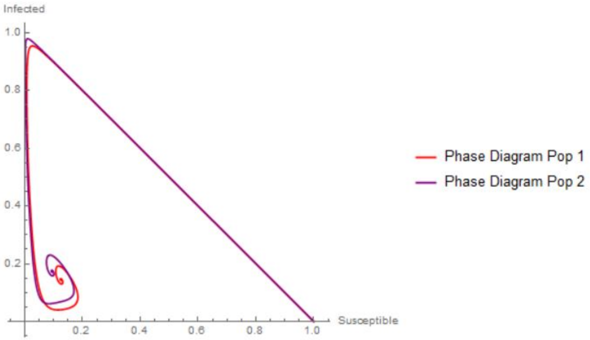


Figure : Phase space plot of Coupled Delayed SIRS model with different delays, $\tau_1 = 14$ days and $\tau_2 = 10$ days and $t_{01} = 60$ days and $t_{02} = 50$ days

Lockdown and SIR Model

In the light of pandemic, with vaccines far away, one of the primary steps taken by countries worldwide was to introduce lockdown, of varying degrees, in an attempt to gain the upper hand and bide some time. Mathematically, we can model it by assuming a variable spread factor, that depends on the severity of lockdowns.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\tilde{\beta}}{N} S(t) I(t)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\tilde{\beta}}{N} S(t) I(t) - \frac{\tilde{\beta}}{N} S(t-\tau) I(t-\tau)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\tilde{\beta}}{N} S(t-\tau) I(t-\tau)$$

Lockdown and SIR Model

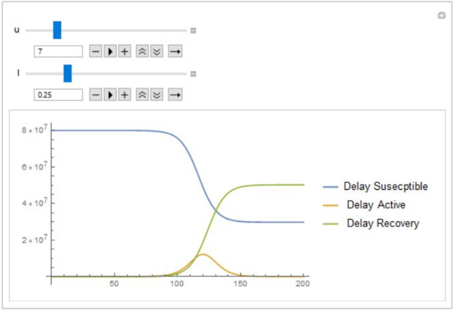


Figure : Lockdown delayed SIR model, with u = 7 days and a lockdown of severity $l_t = 0.25$

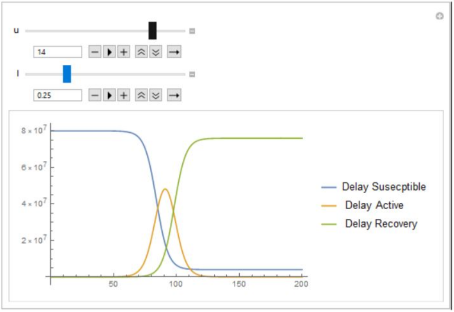


Figure : Lockdown delayed SIR model, with u = 14 days and a lockdown of severity $l_t = 0.25$

Lockdown and SIR Model

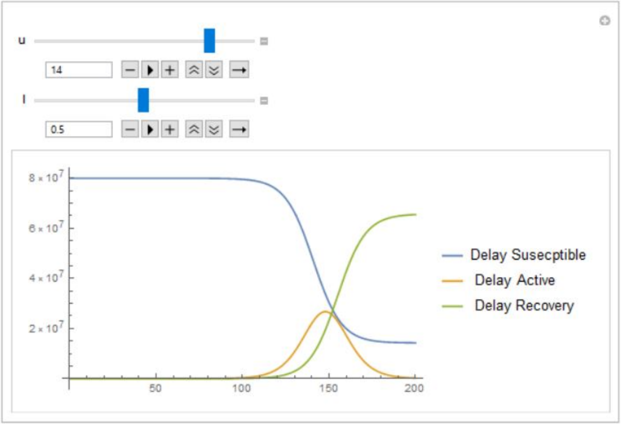


Figure : Lockdown delayed SIR model, with u = 7 days and a lockdown of severity $l_t = 0.5$

Lockdown and SIRS Model

On similar lines, we can model the SIRS model with a lockdown

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\tilde{\beta}}{N}S(t)I(t) + t_i R(t)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\tilde{\beta}}{N}S(t)I(t) - \frac{\tilde{\beta}}{N}S(t-\tau)I(t-\tau)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\tilde{\beta}}{N}S(t-\tau)I(t-\tau) - t_i R(t)$$

Lockdown and SIRS Model

Figure : Lockdown delayed SIRS model, with u = 7 days, r = 0.05 and a lockdown of severity $l_t = 0.25$

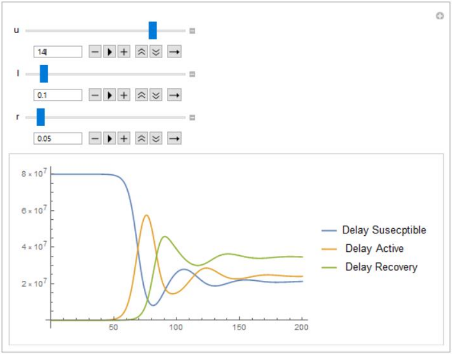


Figure : Lockdown delayed SIRS model, with u = 14 days, r = 0.05 and a lockdown of severity $l_t = 0.25$

Lockdown and SIRS Model

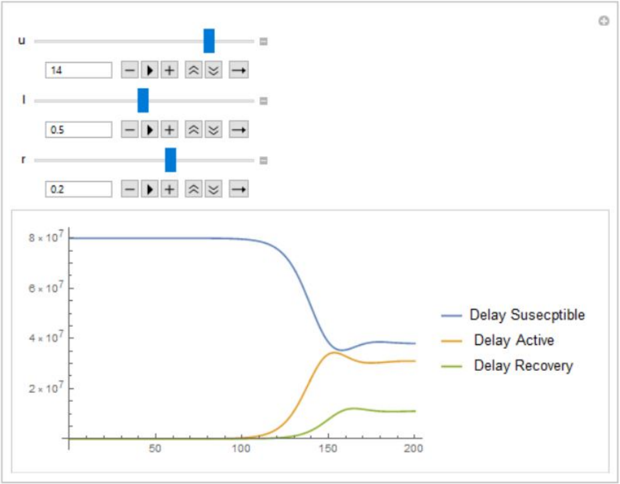


Figure : Lockdown delayed SIRS model, with u = 7 days, r = 0.2 and a lockdown of severity $l_t = 0.5$

Future avenues

Multi Strain Model and Vaccinations By the end of 2020, we saw the emergence of **newer strains** of virus. The emergence of a new strain of the virus can come about from mutation of the virus, antigenic drift/shift or even through introduction of a newer strain from an external source.
Hence, one of the developments of the Coupled model, which assumes two sets of coefficients for two geographical locations, can be assumption of two different strains of virus itself. This model can be applied either by introduction of two strains in a singular population, with either both of them attacking simultaneously or one leading the other by a certain amount of time.

Future avenues

Multi Strain Model and Vaccinations The other upgrade could be to introduce vaccination in the SIR model, and in the multi strain model. Both the cases of blanket immunity from all strains by the vaccines and selective immunity from original strain but not from newer strain can be considered.

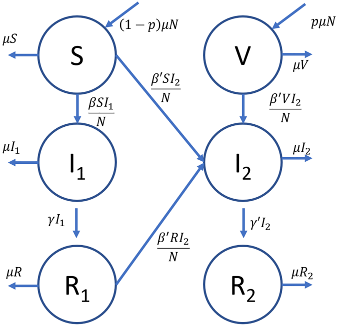


Figure : Compartment diagram for the emerging disease model

Future avenues

Multi Vectors Aside from studying different strains, different spatial and other couplings, one could even try to study impact of different vectors of the same disease on SIR model. As COVID was earlier thought to spread not only from close contact with the infected person, and transmitting through air, but also small droplets suspended on various surfaces, and thus bringing haptic aspect to the transmission. The delayed model there could be a further development of the continuous type delay.

Different types of Lockdowns The upgradation of the lockdown model can start with the changing of the function representing the lockdown in the model. These can further be applied in the coupled model, mimicking the situations when one country or state had lifted lockdown but the other hadn't and based on the severity of lockdown, a small transfer of population still took place. One could also focus on finding a lockdown function that minimises the total deaths. One could also working on finding the lockdown function which effectively minimises the total deaths.

Future avenues

Changing the Nature of Delay itself For the entirety of the investigation, we have assumed the delay to be constant in nature, *ie* τ assumes a constant value, but that isn't necessarily true. One could consider a case of disease such as malaria, where not only the susceptible individual gets the disease from an infected vector(mosquito), but also an uninfected vector getting the infection from an infected individual. This situation, when coupled with the assumption that the disease might show retardation like COVID, we arrive at a situation where the delay isn't constant, but rather can be modeled as a distribution.

Thank You