PROJECT REPORT

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Submitted to-
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Introduction

Duffing Oscillators are a class of oscillators having linear damping and a cubic non linearity. Written as

$$\ddot{x} + +\delta\dot{x} + \alpha x + \beta x^3 = F\sin\omega t$$

where δ ,α , β F and ω are given constants. This equation, due to its non linearity, is capable and often used to display chaotic behaviour. In the case of undamped and undriven, an exact solution in form of Jacobi's Elliptic functions can be obtained.

Duffing Oscillator with non linearity ($\beta \neq 0$)exhibits jumps in amplitude of solutions. This is due to the fact that for certain parametric values the response may no longer be a singular valued function. And one of the values of the solution may not be retained for long, and hence jumps occur.

The Undamped and Unforced Oscillator

The equation of Forced Duffing Oscillator is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + bx^3 = F_0\sin\omega t$$

Taking $\gamma = 0 = F_0$, the equation becomes

$$\ddot{x} + \omega_0^2 x + bx^3 = 0$$

The equation is then solved in Mathematica using the **DSolve** command, which gave solution is form of elliptical functions. Applying the initial conditions, $x(0) = 1$, $\dot{x}(0) = 0$, the solution $x(t)$ is plotted against t.

Below is the plot of solution $x(t)$ for some arbitary parametric values.

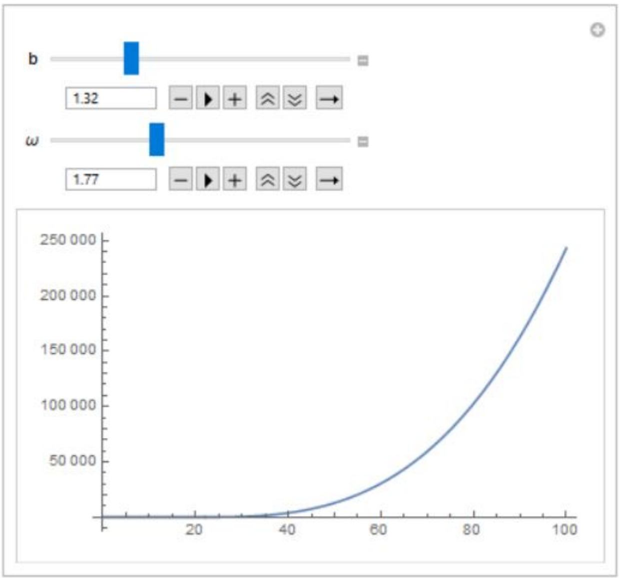

Figure 1 : Solution to the undamped, unforced non linear oscillator equation at some arbitrary parameters

The solution plot in this case, is highly sensitive to the parameters' values and changes values radically based on them.

Fully Damped and Driven Oscillator

Now consider the case for fully damped, driven oscillator, we look back at the original equation

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x + b x^3 = F_0 \sin \omega t$$

With $\omega = 2$ and $\omega_0 = 1$, the above mentioned equation is then solved in Mathematica, using the **NDSolve** command, which gives the numerical solution, as opposed to the analytic solution by the **DSolve**.

One then observe, that one gets a Interpolating Function as a solution. The solution is then plotted. A phase plot is plotted. While tuning the parameters, namely F, γ,b, one can observe that at certain combination of the values, the amplitude of solution suddenly jumps to a higher value. I decided to keep b and γ constant at 0.7 and 0.5 respectively, while increasing F. Before the point, amplitude of steady state solution keeps increasing slowly, implying a proportional relation with driving force's value. At $F = 2.71$, the amplitude suddenly jumps from ≈ 1 to ≈ 2.4. After the point, it continues to grow slowly like before. This signifies that there are two steady state solutions to the equation, with different domains.

Below are $x(t)$ vs t graph, depicting that jump in amplitudes.

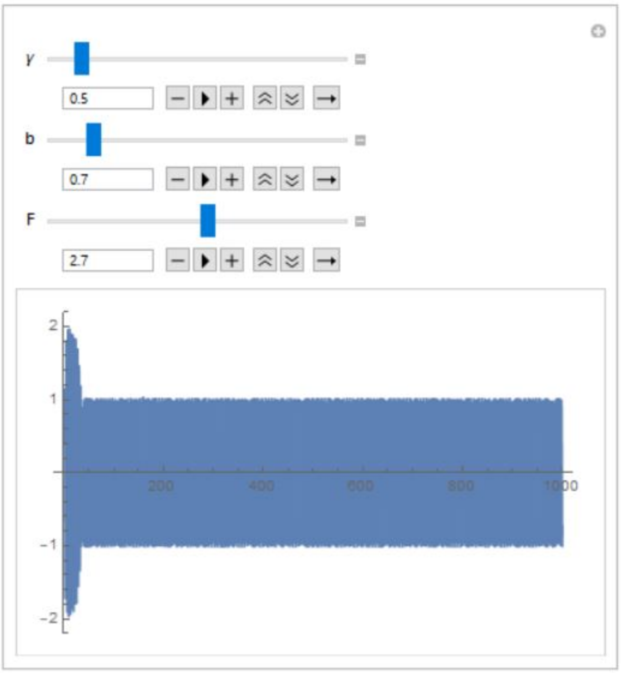

Figure 2 : The steady solution at point $F = 2.70$

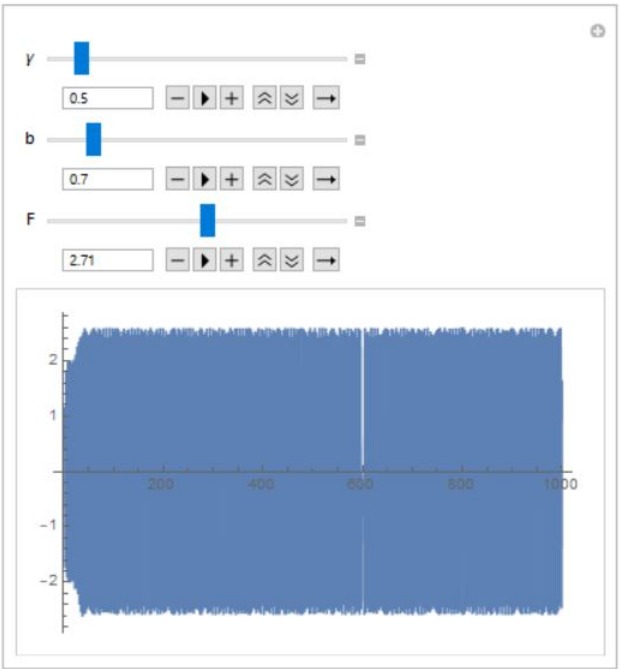

Figure 3 : The steady solution at point $F = 2.71$

Below are phase plots for the respective cases

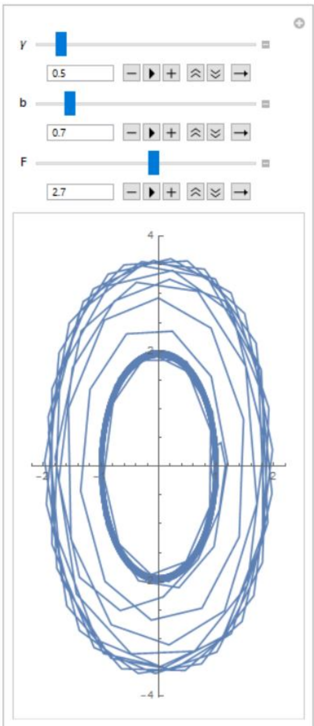

Figure 4 : Phase Plot of solution at the point $F = 2.70$

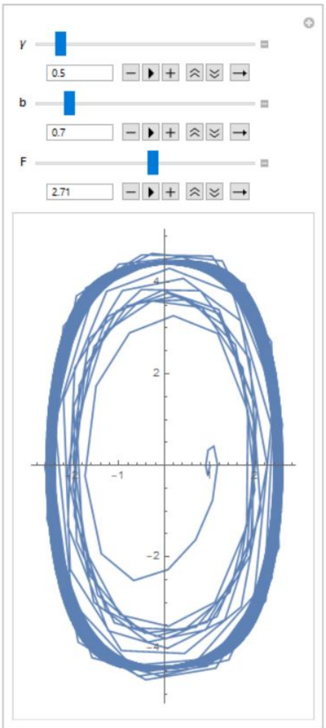

Figure 5 : Phase Plot of solution at point $F = 2.70$

Looking at the phase plots, we can see two limit cycles (approximately) in both cases, with the inner cycle having higher density for the case of $F = 2.7$ and outer one for $F = 2.71$. With a significant change in density of curves in the limit cycle, we can conclude a different steady state behaviour, coming from the jump. This confirms our observation of 2.7 being the point of change in behaviour of steady state.

If we plot a section of the solution, for both cases, we can observe the frequencies for them. Below are zoomed in solutions for both cases, plotted between the points {t,500,550}

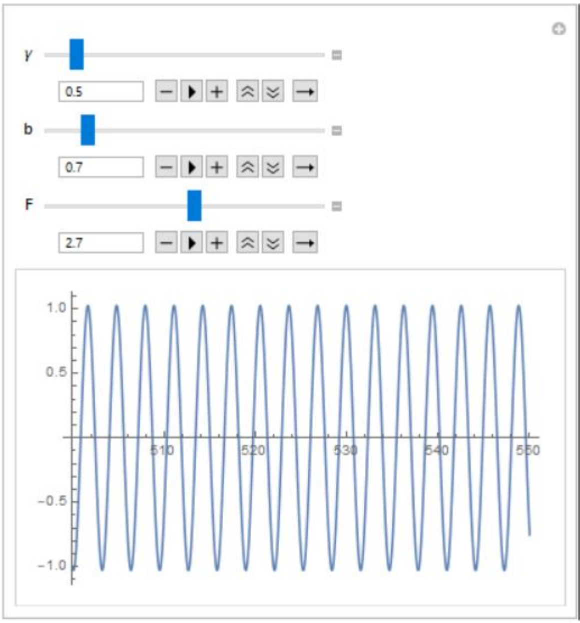

Figure 6 : Zoomed in solution at $F = 2.7$

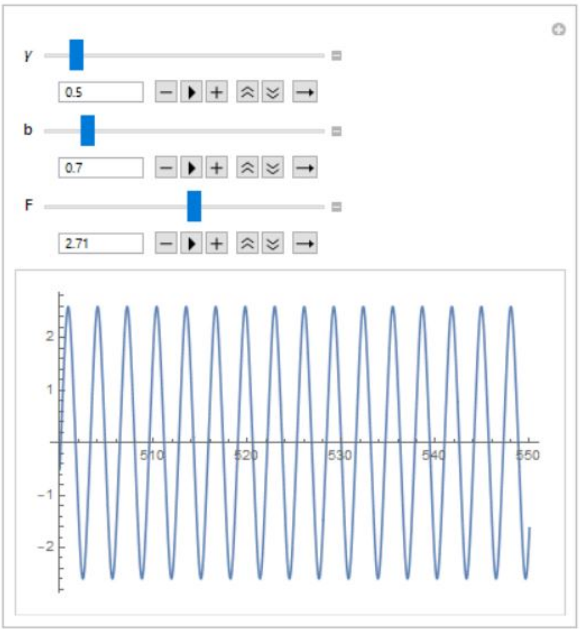

Figure 7 : Zoomed in solution at $F = 2.71$

One can observe from them, that frequency of both solutions remains approximately the same, yet the amplitude changes drastically.

To observe the index of the fixed points and fixed points themselves, we break the 2nd order differential equation in 2 first order coupled equations.

$$\dot{x} = y$$
$$\dot{y} = F\sin 2t - bx^3 - x - \gamma y$$

As there is a t dependence in the system, the system's fixed points and vector field would evolve with time making any conclusion difficult, thus, I take F =0, making the system

$$\dot{x} = y$$
$$\dot{y} = -bx^3 - x - \gamma y$$

For this system, we can find upto 3 fixed points depending on the values of parameters, who also define their index. For example

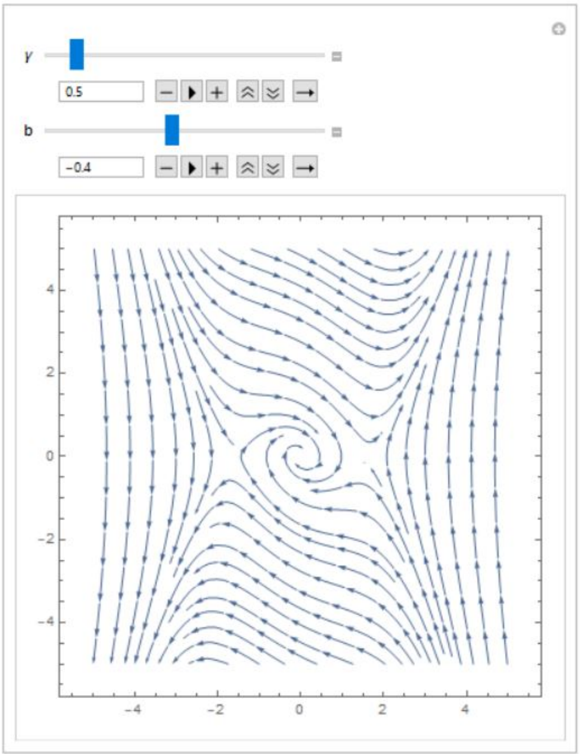

Figure 8 : Stream plot of the reduced system

For b = -0.4 and γ = 0.5, we get 3 points, two of which are saddle points with Index = -1 and one of them is a center with Index = 1, making the total index of the curve surrounding all 3 points to be I = -1.

For different parameters, we get different fixed points and different indices.

Quadratic Damped Oscillator

The non linear oscillator with non linear damping has the equation. Following the above mentioned method for the Driven Duffing oscillator, I try to plot this equation's solution $x(t)$ and its phase plot.

$$\ddot{x} + \eta \dot{x}|x| + x = F \cos 2t$$

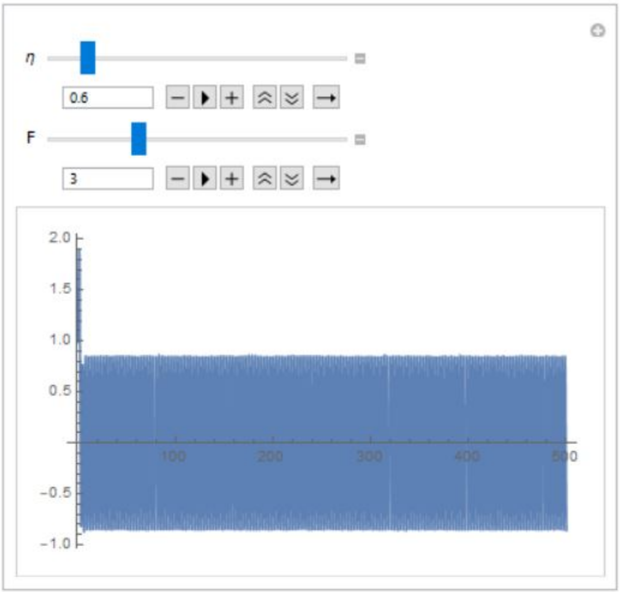

Figure 9: The steady solution at point

To observe limit cycles in the system, we look at phase plots below

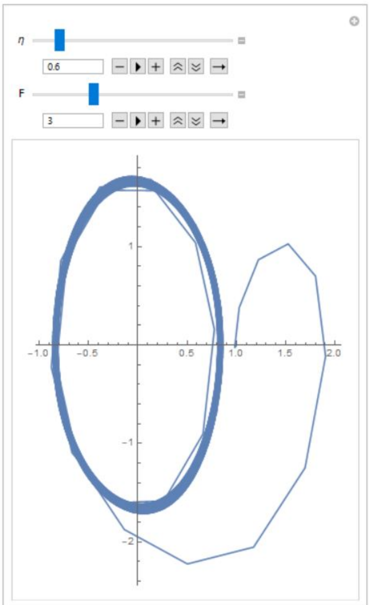

Figure 10: Phase plot of solution at point

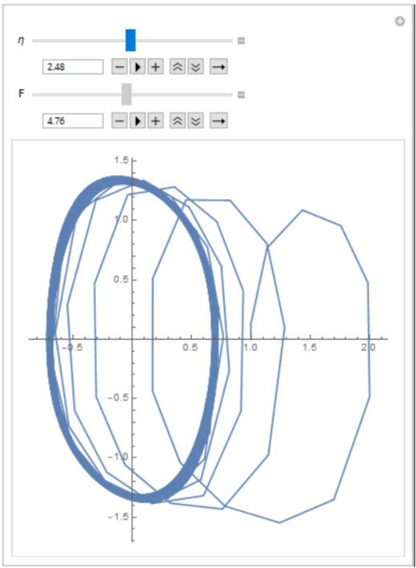

Figure 11: Phase plot of solution at point

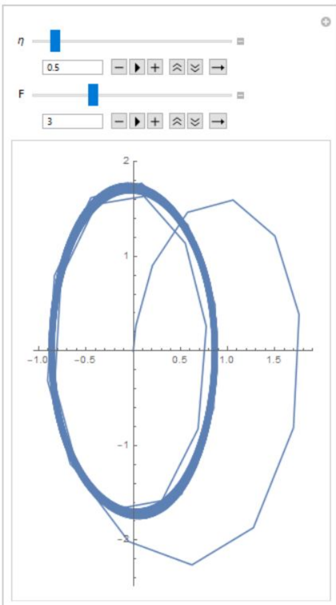

Figure 12: Phase plot of solution at point

From the above plots, one can observe the existence of limit cycles for different parametric values as well as initial conditions, for this oscillator.

The steady state behaviours are being judged by relative the change in amplitude with change in parameters. At $\eta = 1$, there was fast growing of amplitude about a point and after it growth slowed down, thus possible existence of 2 steady states, similar result was obtained for $\eta = 1.5$, but with more drastic dip in evolution of amplitude, suggesting possibly two solutions. For most values of η, changing F doesn't produce a change in steady state solution.

Bonus

Doing a Fourier transform can also reveal things about the behaviour of the oscillator.

To undertake a Fourier transform, we first create a table of values to solution, $x(t)$. Then we do a discrete Fourier transform on the data and plot it. Taking the above case, I do a Fourier transform around the point of change of amplitude,

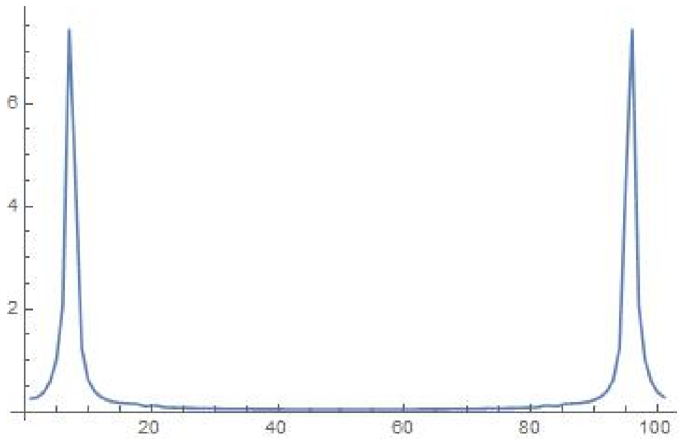

Figure 13 : Fourier transform with same parameters at F = 2.7

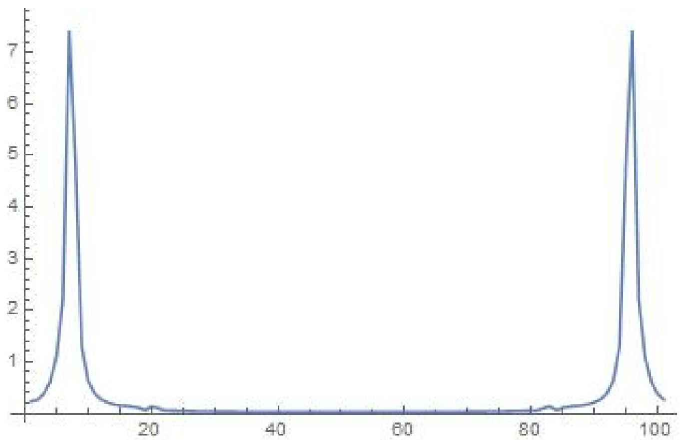

Figure 14 : Fourier transform with same parameters at F = 2.71

We get two peaks, latter of which is just complex conjugate reflection of the first one. Looking at the first peak, one can observe it to be around 7 Hz in both graphs, which could be seen from the zoomed in plot from the second part of the report too, confirming that frequencies would remain same. This can be explained by looking at the fact that steady state solution are defined by usually Driving term, hence even though there are two steady states, the external term's frequency isn't changing and consequently the steady state frequency isn't changing.

Conclusion and Discussion

I would like to discuss my choice to set the forcing term to 0, trying to find the index. While writing the system as two coupled first order equations, in the $\dot{y}$ equation, we get explicit time dependence in the equation. While doing stability analysis, we try to find the fixed points by setting the derivatives, representing flow, to zero. We then simultaneously solve the $f(x, y)$ and $g(x, y)$ to find the spatial points where the time derivative is zero. When there is explicit time dependence, the value and direction of flow starts to evolve with time. Hence, we would get different fixed points for different time values, which isn't desirable. I also thought about treating time as another coordinate and making the plot 3D, but couldn't get about writing the third coupled equation.

In the Fourier transform, we get twin peaks but only consider the first one, as the second one is just first's complex conjugate image. While calculating the transform, the value of r was chosen through trial and error to be higher than a certain value, which has certain consequences which I am not so clear on.

On the steady state solutions of the quadratic damping oscillator, the steady state solution was judged was again by the method employed with Duffing Oscillator. The amplitude growth of solution was looked with respect to varying parameters. For the two mentioned values, the average growth of amplitude of x for integral values of F was observed,in case of $\eta = 1.5$, the growth significantly dropped around $F = 6$, signifying a slower growing steady state solution, while for a smaller value of η no such dip was observed. For many different values of η no substantial change rate of change of amplitude was observed. Points were observed that suggest two solutions, but for a certain value more solutions might exist.