Qualitative and quantitative modelling of dynamic systems: how do they relate?

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Abstract

In this paper, we focus on how the qualitative vocabulary of Dynalearn, which is used for describing dynamic systems, corresponds to the mathematical equations used in quantitative modeling. Then, we demonstrate the translation of a qualitative model into a quantitative model, using the example of an object falling with air resistance.

1 Introduction

Understanding the behaviour of dynamic systems (e.g., climate change, economic growth and recession, population dynamics) is an important goal in secondary education. Educational developments that strive for future-oriented curricula emphasize this and consider practices such as causal reasoning and modelling as important skills.

Modelling is widely recommended as a way to provide learners with a deeper understanding of dynamic systems [1]. Modelling of a dynamic system on the computer can be done both qualitatively and quantitatively. Both forms can and are used in education [2, 3], but largely independent of each other. Both forms of modelling have their unique ways of representing and reasoning about system behaviour. As learning tools, each has its own pedagogical approach and offers distinct advantages and downfalls for understanding systems [4, 5, 6, 7]. Quantitative modelling allows for precise predictions and is closely aligned with the content of various school subjects such as gravitational acceleration (physics), predator-prey relationships (biology), the pig cycle (economics), and global warming (geography). Qualitative modelling, on the other hand, aligns more closely with the human reasoning about systems and emphasizes causality and the potential states of a system [8]. It also allows for automated support [2].

Education will benefit from a software solution and corresponding pedagogical approach that supports the strengths of both modelling forms. The software should integrate qualitative and quantitative representations of a system. If learners construct a qualitative model, the software can assist in translating it into a quantitative model, which learners often find challenging. Conversely, moving from quantitative to qualitative helps to verify whether the constructed quantitative model assumes plausible causal relationships. This approach also aligns with recommendations from the scientific community [9]. It is important to note that such software does not yet exist, and that the potential impact of this innovation could extend to many other sectors in society.

In this paper, we focus on how qualitative representations of dynamic systems in Dynalearn [10] relate to mathematical equations. Chapter 2 begins by outlining the qualitative vocabulary of Dynalearn. We then discuss in Chapter 3 how dynamic systems can be quantitatively described using mathematical equations. A considerable portion of this paper, Chapter 4, is dedicated to examining the relationships between the qualitative vocabulary of Dynalearn and the corresponding mathematical equations. Following this, we use the dynamics of an object falling with air resistance as a case study in Chapter 5 to demonstrate the translation of a qualitative model into a quantitative model. The paper finalizes with a conclusion and discussion in Chapter 6.

2 Qualitative modelling

Qualitative representations provide a framework for modelling dynamic systems without relying on numerical data. The Dynalearn software facilitates the construction of these models at five distinct levels of complexity, each introducing new ingredients to accommodate a more nuanced description of system dynamics. In this paper we focus on level 4. Hence, this section discusses the ingredients of the Dynalearn software at that level.

Entities are either physical objects or abstract concepts, characterized by one or more *quantities*—changeable features of entities, such as temperature or speed. Each quantity has a *derivative*, denoted as δ, indicating its direction of change: decreasing, constant, or increasing. *Quantity spaces* define the possible states of the system by determining the range of possible *values* for each quantity, represented as alternating *point* and *interval* values. *Correspondences* (C) can be added to co-occurring values to further determine the possible states of the system. The relationships between quantities are described by two types of causal relationships:

influence and *proportionality*. A causal relationship is of type influence (I) when an active process, indicated by a quantity, is the primary cause of a change in another quantity. This relationship can be either positive (I+) or negative (I−), depending on the directionality of the effect initiated by the process. When the relationship is of type positive, a positive value of the process results in an increase of the related quantity, while a negative value results in a decrease. In cases of a negative influence, a positive value of the process causes a decrease in the related quantity, and a negative value causes an increase. Causal relationships of type proportionality (P) describe how changes in one quantity lead to corresponding changes in another quantity, either in the same direction (P+) or in opposite directions (P−). Exogenous influences are external factors that have a continuous effect on the change of a quantity. In the present paper we restrict to exogenous influences that are either decreasing, constant, or increasing. The behaviour of the system can be further described by (in)equalities, which set ordinal relationships between quantities (<, ≤, =, ≥, >). Calculi allow the execution of qualitative operations such as addition and subtraction.

Simulation within Dynalearn starts with a scenario: the initial settings that define the starting conditions of the model. From these settings, a state graph is generated, visually representing the possible states and transitions of the system. Learners can use this graph to explore and understand the behaviour of the system by navigating through different states. Simulation preferences can be adjusted so that the underlying Garp 3 reasoning engine [11] accounts for possible changes in the first derivative of a quantity (i.e., the second derivative), potentially leading to new states or transitions. Value and inequality history offer an overview of the changes, values and (in)equality of quantities throughout the simulation.

3 Quantitative modelling

In the case of quantitative modelling in secondary education, mathematical equations are used to describe and analyse how systems evolve over time. These models typically use differential equations, linear equations, and nonlinear equations to describe system dynamics.

The differential equation $y(t + \Delta t) = y(t) + m \cdot x(t) \cdot \Delta t$ describes how the value of a function y at time $t + \Delta t$ is derived from its value at a previous time t by adding an increment that depends on the constant m, the value of $x(t)$, and the time step Δt. This formulation uses Euler's method, a finite difference approach commonly used in simulations to approximate the solutions of differential equations. Note, that $x(t)$ itself is a function of time, and its behavior directly influences the behavior of $y(t)$. For example, if $x(t)$ is constant, then $y(t)$ increases linearly. Conversely, if $x(t)$ increases linearly (e.g., $x(t) = mt$), then $y(t)$ exhibits quadratic growth as each increment added to $y(t)$ increases over time. We use Euler here for keeping things simple, though other numerical methods like the Runge-Kutta 4 (RK4) are also commonly employed for more accuracy and stability. Numerical analysis for solving differential equations is crucial when analytical solutions are not feasible.

Relationships between quantities in a dynamic system can often be described using linear equations, such as $y(t) = m \cdot x(t) \pm b$, where m represents the slope and b is the y-intercept, indicating the value of y when $x = 0$. Here, $x(t)$ denotes the value of x at time t. It's important to note that $y(t)$ exhibits linear behavior relative to $x(t)$; however, the overall behavior of $y(t)$ in terms of time depends on the behavior of $x(t)$. Specifically, $y(t)$ will only show constant behaviour if $x'(t) = 0$ (i.e., if $x(t)$ is constant over time). For example, in modelling a dynamic system that describes the behaviour of gases, the relationship between temperature and pressure is typically linear under constant conditions.

Non-linear equations describe scenarios in dynamic systems where quantities appear as exponents, products, or other non-linear combinations. For example, the non-linear equation $y(t) = -m \cdot x(t)^2 + b$ illustrates how the intensity of light, $y(t)$, diminishes with the square of the distance, $x(t)$, from a point source as an object moves away over time.

After defining the equations of the dynamic system, a simulation can be initiated. Initial values for the variables must be set, along with the duration of the simulation and the size of each time step. The values of each quantity are then calculated for each time step using an integration method, such as Euler's method.

4 Qualitative vocabulary and mathematical equations

In this chapter, we describe how ingredient types of the qualitative vocabulary relate to mathematical equations. For clarity, when referring to quantities in qualitative representations, we use *x*, *y*, *z* without the time notation *t* and use the δ symbol to indicate their direction of change. When discussing mathematical equations, we denote these quantities as $x(t)$, $y(t)$, $z(t)$ to specify that they are functions of time, and we use the prime notation to discuss the direction of change of these quantities, for example, $x'(t)$. For discussing time steps, we use the notation Δt, and m and b are used in equations to denote the slope and intercept, respectively.

4.1 Exogenous influence, change, and quantity space

Fig. 1 presents a qualitative representation of quantity *x* with quantity space {0, +} and an increasing exogenous influence acting on it. The initial value of *x* is zero (0). The simulation result shows two consecutive states: in the first state, *x* is zero and increasing (δ*x* > 0), and in the second state (shown), *x* is positive (+) and continues to increase. The mathematical equation corresponding to the value of x is $x(t + \Delta t) = x(t) + x'(t - \Delta t) \cdot \Delta t$. The quantity space of x defines the range as $x(t) \geq 0$. Given that δ*x* is increasing linearly, $x'(t) > 0$ and remains constant. Conversely, for a constant exogenous influence, $x'(t) = 0$ and remains constant, while for a decreasing exogenous influence, $x'(t) < 0$ and remains constant.

Hence, to transition from a qualitative model to a quantitative one, if the exogenous influence on a variable is increasing or decreasing, then the numerical value of $x'(t)$ must be provided. Additionally, if the initial setting of x starts at an interval, then the initial numerical value of $x(t)$, namely $x(0)$, must also be specified. Furthermore, the value of Δt also needs to be set.

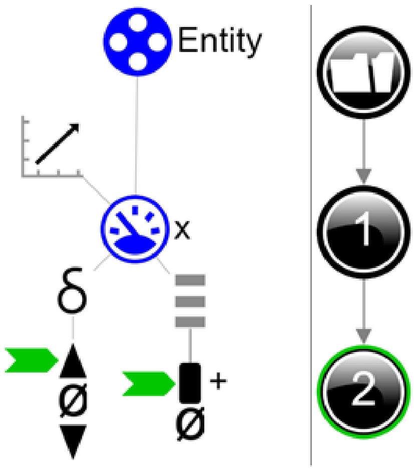

Fig. 1. An increasing exogenous influence acting on quantity x with quantity space {0, +}.The right side shows the state-graph starting with the scenario followed by two consecutive states. The left side shows the model and the simulation result of the 2nd state (in green).

4.2 Causal relationships

Fig. 2 shows a qualitative representation with a positive proportional relationship (P+) between quantities x and y, with an increasing exogenous influence acting on x. The simulation result demonstrates that as x increases, y also increases.

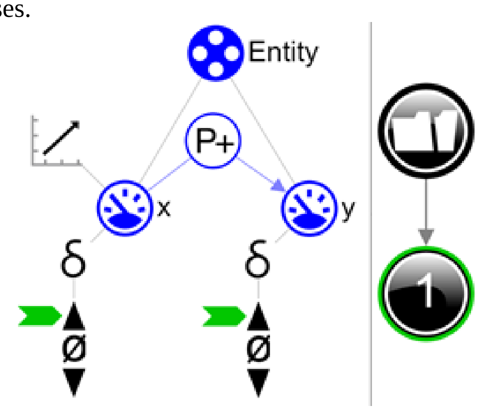

Fig. 2. Positive proportional relationship between x and y.

Assuming a linear relationship between x and y, the general mathematical expression corresponding to this is $y(t) = m \cdot x(t) \pm b$, and the derivative is $y'(t) = m \cdot x'(t)$. Given the positive proportional relationship, the value of m must be greater than 0. Conversely, for a negative proportional relationship holds $m < 0$. The value of b can be any real number ($b \in \mathbb{R}$), as there are no quantity spaces defined for x and y that dictate how the values of x and y are related. For further discussion on the latter, see paragraph 4.3.

Fig. 2 could also depict a non-linear positive proportional relationship between x and y, for example dose-response relationship of a certain drug (Fig. 3).

Fig 3. Non-linear positive proportional relationship between dose and response.

When accounting for non-linear positive proportional relationships, the qualitative representation in Fig. 2 could be described by any mathematical equation whose first derivative is always greater than zero. For example, consider the equation $y(t) = x(t)^3 + 3x(t)$. Following the chain rule, the derivative is $y'(t) = 3x(t)2 \cdot x'(t) + 3x'(t)$. If $x'(t) > 0$ and remains constant, then $y'(t) > 0$, which indicates that $y(t)$ is a strictly increasing function of $x(t)$.

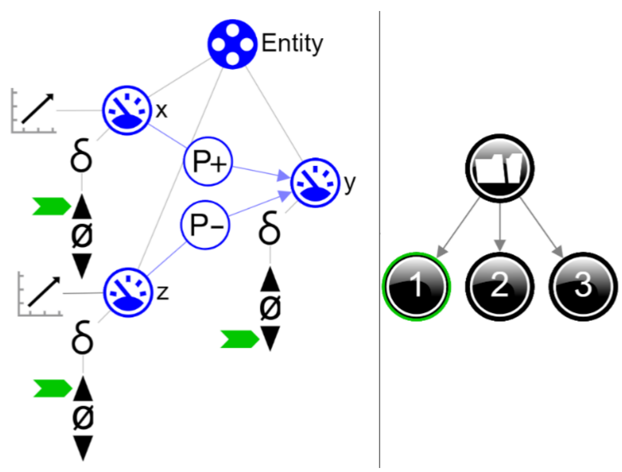

Fig. 4. Positive and negative proportional relationship. The left side shows the model and the simulation result of state 1 (in green).

Fig. 4 shows a qualitative representation where x has a positive proportional relationship with y, and z has a negative proportional relationship with y; both x and z are increasing due to an increasing exogenous influence. When simulation preferences are set to only consider first changes in the first derivative, the simulation result is ambiguous with three possible final states. In state 1, y is decreasing ($\delta y < 0$); in state 2 (not shown), y is constant; and in state 3 (not shown), y is increasing. The general mathematical equation describing the change in y, considering a linear relationship between z, x, and y, that corresponds to this representation is $y'(t) = m_1 \cdot x'(t) - m_2 \cdot z'(t)$. Note the minus sign indicates that $z(t)$ has a negative proportional relationship with $y(t)$. The ambiguous simulation result arises not only because m_1 and m_2 are unknown but also because $x'(t)$ and $z'(t)$ are not specified. For example, if $y'(t) = 3x'(t) - 4z'(t)$, and $x'(t)$ is less than 4/3 times $z'(t)$, then $y(t)$ is decreasing ($y'(t) < 0$). However, if $x'(t)$ is equal or larger than 4/3 times $z'(t)$, then $y(t)$ is constant or increases. Table 1 shows numerical examples over a time step that illustrate the impact of different ratios of $x'(t)$ and $z'(t)$ on $y'(t) = 3x'(t) - 4z'(t)$. The table demonstrates that if the ratio between $x'(t)$ and $z'(t)$ is 1, then $y'(t) < 0$; if the ratio is 4/3, then $y'(t) = 0$; and if the ratio is 2, then $y'(t) > 0$.

Table 1. The impact of different ratios of x'(t) and z'(t) on y'(t).

	$y'(t) = 3x'(t) - 4z'(t)$								
	$x'(t) = 1$ $z'(t) = 1$			$x'(t) = 1$ $z'(t) = 3/4$			$x'(t) = 2$ $z'(t) = 1$		
t	$x'(t)$	$z'(t)$	$y'(t)$	$x'(t)$	$z'(t)$	$y'(t)$	$x'(t)$	$z'(t)$	$y'(t)$
0	1	1	-1	1	-¾	0	2	1	2
1	1	1	-1	1	-¾	0	2	1	2
…	…	…	…	…	…	…	…	…	…

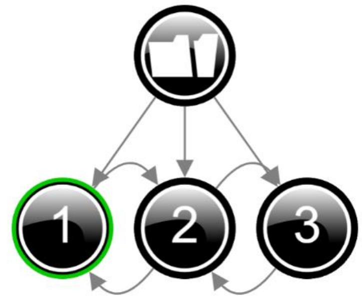

Fig. 5. Ambiguous simulation result with transitions between states 1, 2, and 3.

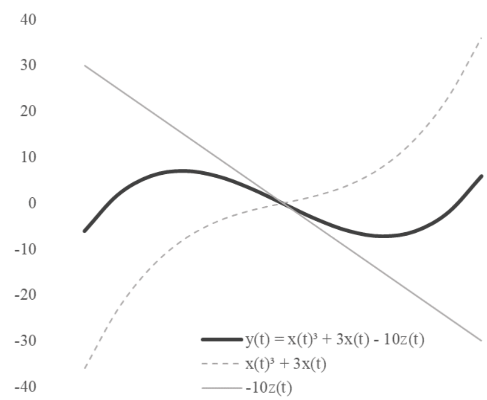

Fig 6. The combined effect of a nonlinear and linear relationship.

Fig. 5 shows the simulation result corresponding to the qualitative representation in Fig. 4, with adjustments in the simulation settings[1] to account for changes in the second-order derivative. These adjustments reveal that transitions between states 1, 2, and 3 are now feasible. Specifically, if one or both relationships of y with x and z are non-linear, the combined effect on δy may depend on specific values. For example, consider if the mathematical equation associated with the qualitative representation of Fig. 4 is $y(t) = x(t)^3 + 3x(t) - 10z(t)$. If both x(t) and z(t) increase consistently (with $x'(t) = 1$ and $z'(t) = 1$) from -3 to 3, $y(t)$ initially increases, becomes constant, decreases, becomes constant again, and finally increases (Fig. 6). This pattern corresponds to the transitions along path 3 → 2 → 1 → 2 → 3 as shown in the simulation result of Fig. 5.

Fig. 7 shows a qualitative representation of a causal relationship with a positive influence (I+) between x and y, with x having quantity space {0, +}. The simulation result indicates that x is positive and remains constant, which leads to an increase in y ($\delta y > 0$). Note that y does not have a quantity space. Assuming y increases linearly, the corresponding mathematical equation that represents this qualitative relationship is $y'(t) = m \cdot x(t)$. Given the positive influence of x on y, m > 0. Furthermore, $x(t) > 0$ and is constant.

Fig. 8 extends the qualitative representation shown in Fig. 7 by including quantity z with a negative influence (I−) on y, and now y has quantity space {−, 0, +}.

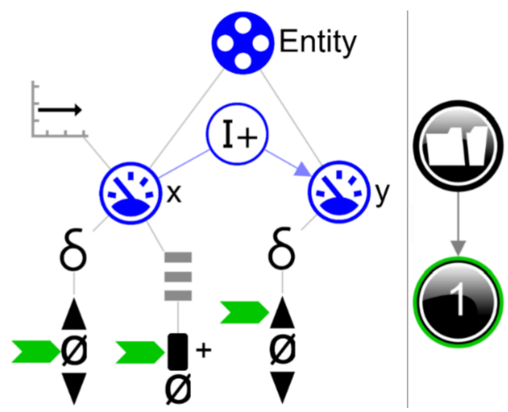

Fig 7. Causal relationship of type influence.

The initial settings are such that y is 0, while both x and z are positive (+) and constant (δx = 0 and δz = 0). These settings introduce ambiguity in the simulation result due to the opposing influences: x has a positive effect on y, while z has a negative effect, and their relative magnitudes are unknown. If the influence of z on y is greater than that of x, y will decrease and become negative (path 1 → 5); if the influences are equal, y remains at zero (state 2); and if the influence of x is greater than z, y will increase and become positive (path 3 → 4).

[1] We differentiate between initial and simulation settings. The former refers to starting values (and inequalities) when starting a simulation. The latter refers to characteristics of the reasoning engine [12].

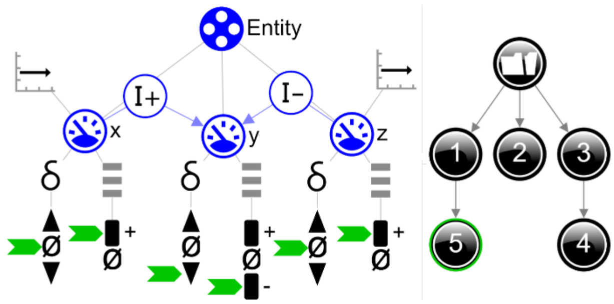

Fig. 8. A negative and positive influence acting on y.

Assuming y increases or decreases linearly, the corresponding mathematical equations are $y(t+\Delta t) = y(t) + m_1 \cdot x(t) \cdot \Delta t$ for the influence of x on y, and $y(t) = y(t-\Delta t) - m_2 \cdot z(t) \cdot \Delta t$ for z's influence, where $m_2 < 0$ indicating a negative influence. Combining these, the overall expression for $y(t)$ becomes $y(t-\Delta t) = y(t) + (m_1 \cdot x(t) - m_2 \cdot z(t)) \cdot \Delta t$. Here, if $m_1 \cdot x(t) < m_2 \cdot z(t)$, then $y(t)$ decreases; if $m_1 \cdot x(t) = m_2 \cdot z(t)$, then $y(t)$ remains steady; and if $m_1 \cdot x(t) > m_2 \cdot z(t)$, then $y(t)$ increases.

Hence, to transition from a qualitative to a quantitative model, the mathematical equations that describe the causal relationships must be specified. Additionally, the numerical values for the parameters of these equations, such as *m* and *b*, must also be provided.

4.3 Correspondence and quantity space

Fig. 9 shows a positive proportional relationship between *x* and *y*. An increasing exogenous influence is acting on *x*, and *x* has quantity space {0, +}. Because the quantity space of *x* includes no negative numbers, any equation for which *y(t)* is increasing within $x(t) \geq 0$ is valid.

For example, if we assume a linear relationship between *x* and *y*, then the general mathematical equation $y(t) = mx(t) \pm b$, with $x(t) \geq 0$ and $m > 0$, is valid. If we assume a non-linear relationship, then $y(t) = x(t)^2$, is also valid. Fig. 10 shows that these two equations are strictly increasing in the range $x(t) \geq 0$. Note that $y(t) = x(t)^2$ would not be strictly increasing if the quantity space included negative values for *x*.

Fig. 11 extends the representation shown in Fig. 9, now defining quantity spaces {0, +} for both *x* and *y*. This additional specification for *y* narrows the scope of the proportional relationship between *x* and *y*. The initial values are set with *x* at zero (0) and *y* positive (+). The simulation result depicts two consecutive states: In state 1, *x* is zero and increasing, while *y* is positive and also increasing. In state 2 (not shown), both *x* and *y* are positive and continue to increase.

These initial settings inform the mathematical relationship between $x(t)$ and $y(t)$. Given that at $x(0) = 0$, $y(0) > 0$, assuming a linear relationship, the general mathematical equation would be $y(t) = m \cdot x(t) + b$, where $x(t) \geq 0$ and $y(t) > 0$. Conversely, if the initial values were $x(0) > 0$ and $y(0) = 0$, then the equation would be $y(t) = m \cdot x(t) - b$, with $x(t) > 0$ and $y(t) \geq 0$. If the initial values were $x(0) = 0$ and $y(0) = 0$, then $y(t)$ simplifies to $y(t) = m \cdot x(t)$. Fig. 12 displays line graphs illustrating these three mathematical relationships.

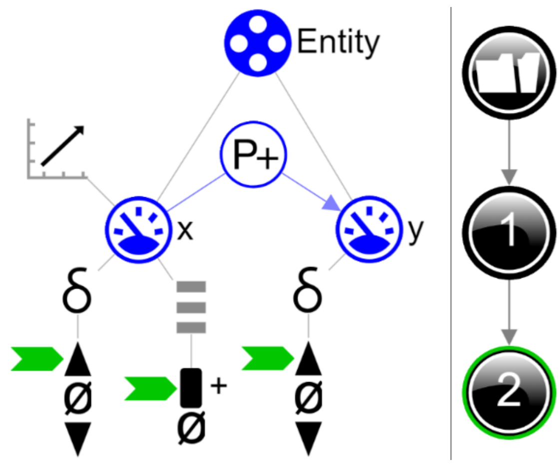

Fig 9. A positive proportional relationship between x and y, where x has quantity space {0, +}.

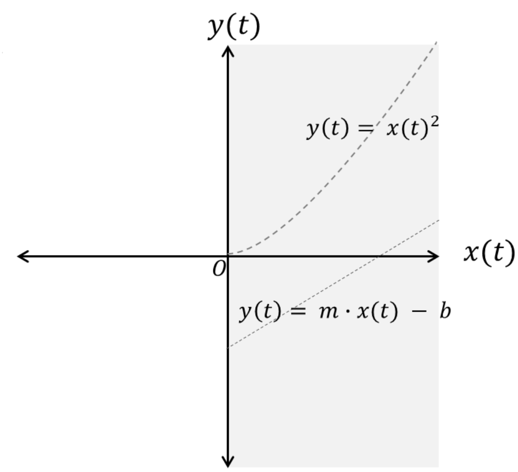

Fig. 10. Examples of linear and non-linear relationships between x(t) and y(t) in the range x(t) ≥ 0.

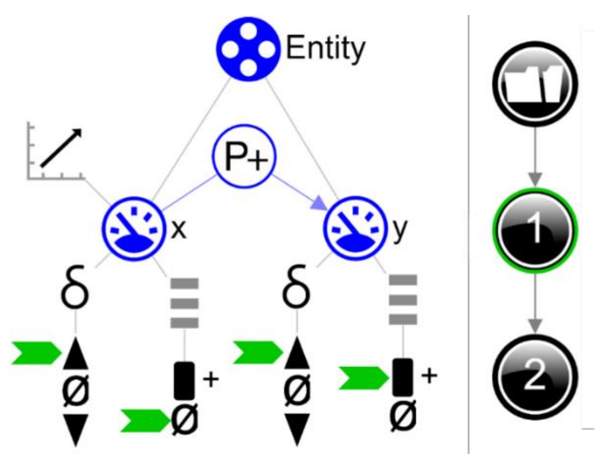

Fig 11. Both x and y have quantity space {0, +}. The left side shows the model and the simulation result of state 1 (in green).

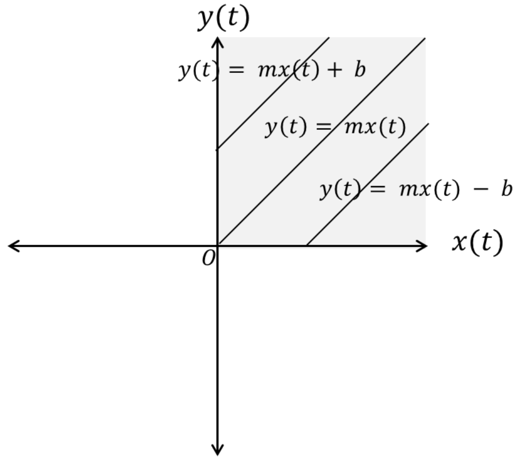

Fig. 12. Three mathematical equations corresponding to different initial settings.

Fig. 13 shows a qualitative representation where both *x* and *y* have quantity spaces {−, 0, +} and there is a bi-directional correspondence (C) between these quantity spaces This correspondence define that if *x* = − than *y* = −, if *x* = 0 than *y* = 0, and if *x* = + than *y* = +).

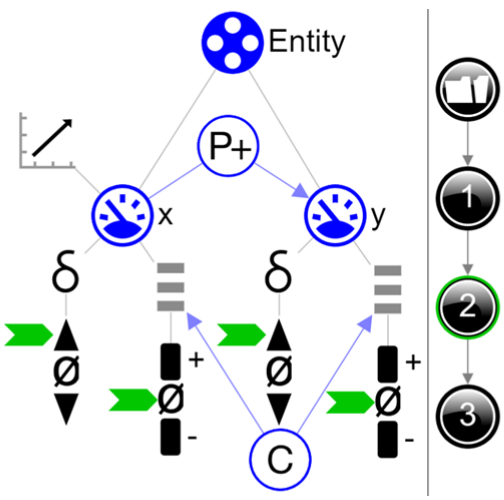

Fig. 13. Bi-directional correspondence between quantity spaces.

Mathematically any equation that goes through the origin and is strictly increasing is valid. For example, if we assume a linear relationship between *x* and *y*, then the mathematical equation $y(t) = 2x(t)$. holds. If we assume a non-linear relationship, then $y(t) = x(t)^3 + 3x(t)$ is also valid. Fig. 14 shows that these two equations are strictly increasing.

4.4 Inequality and calculus

Fig. 15 shows a qualitative representation with quantity *x* with quantity space {0, +, transition, ++} and quantity *z* with quantity space {0, low, mid, high}. Quantity *x* has a positive influence (I+) on *y* and quantity *z* has a negative influence on *y*. The initial value for *x* is '++' and for *y* the initial value is 'low'. There is an (in)equality (=) between the 'transition' point from quantity *x* and 'mid' from quantity *z*. The (in)equality provides information about the relative size of the influences on *y*. Given that the value '++' for quantity *x* is above 'transition', and the value 'low' for quantity *z* is below 'mid', the impact of *x* on *y* is greater than that of *z*. Consequently, the simulation result indicates that *y* will increase. The corresponding mathematical equation is $y(t - \Delta t) = y(t) + (m_1 \cdot x(t) - m_2 \cdot z(t)) \cdot \Delta t$, with $m_1 \cdot x(t) > m_2 \cdot z(t)$, as also discussed in the accompanying text of Figure 8.

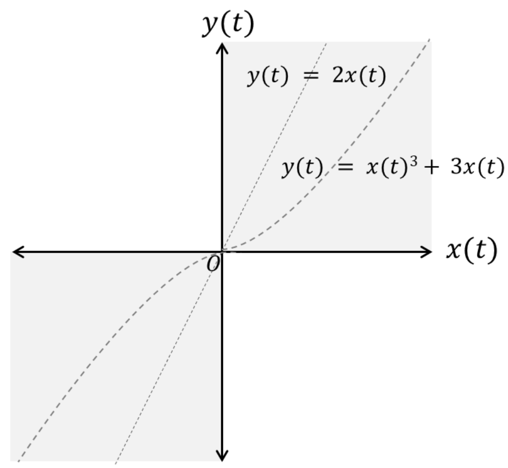

Fig. 14. Two strictly increasing equations that go through the origin.

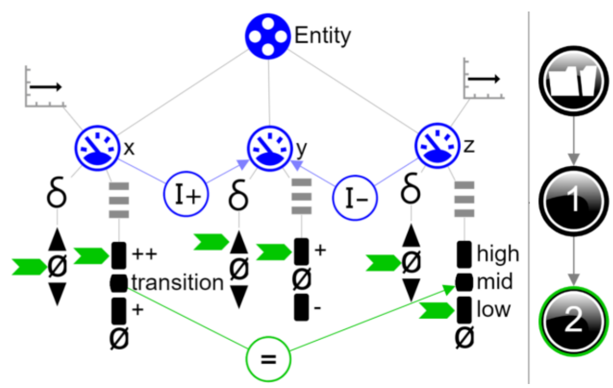

Fig. 15. An (in)equality between two points.

Fig. 16 shows a qualitative representation where *x* has a positive proportional relationship with *y* and *z* has a negative proportional relationship with *y*. Quantity *x* and *z* have quantity space {0, +} and *y* has quantity space {−, 0, +}. Quantity *x* has a decreasing exogenous influence acting on it, whereas *z* has a constant exogenous influence acting on it. There is a calculus that determines that the value of *y* is the value of *x* minus the value of *z* (*y* = *x* – *z*). Initially, both *x* and *y* are positive (+), with *x* being greater than *y* as indicated in the inequality history. The simulation result shows 4 consecutive states. In state 1, *x* is positive and decreasing, while *x* > *z*, hence y is positive and decreasing. In state 2, *x* is still positive and decreasing, *x* is now equal to *z* (*x* = *z*).

Consequently, y is zero and decreasing. In state 3, x is positive, but $x < z$, hence y is negative and decreasing. In state 4, x is zero and steady, thereby y is negative and steady.

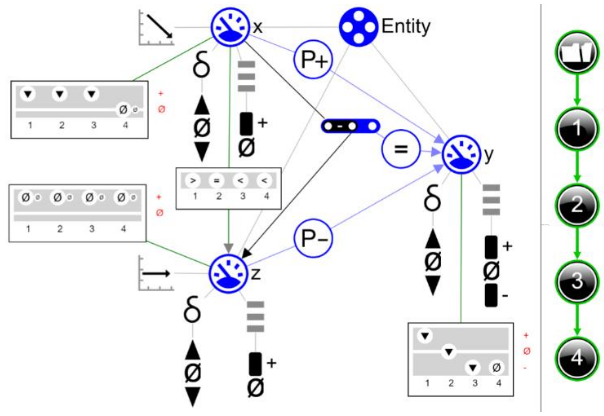

Fig. 16. A calculus specifies that the value of y is x minus z. The grey coloured rectangles show value and inequality histories. For the value history the arrows depict direction of change, the values are show on the right side, and the state numbers are listed below (e.g., x has value + and is decreasing in state 1). The inequality history depicts the relationship between two quantities (e.g., x > y in state 1, x = y in state 2, etc.).

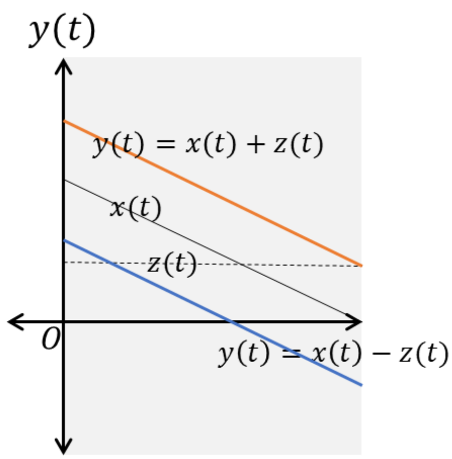

Fig. 17. Lines showing how *y(t)* changes with *x(t)* and *z(t)*, both added and subtracted.

The corresponding mathematical equation that models the calculus of the qualitative representation is $y(t) = x(t) - z(t)$, where $x(t) \geq 0$ and $z(t) \geq 0$, and $y(t)$ can be any real number. Note that δx is decreasing linearly, hence $x'(t) < 0$ and remains constant, while $y'(t) = 0$ and remains constant, implying that the rate of change of $y(t)$ is negative ($y'(t) < 0$). Conversely, if the calculus involved addition, as in $y = x + z$, then y would always be positive because z remains positive and x cannot be smaller than zero. Mathematically, if $y(t) = x(t) + z(t)$ and both $x(t)$ and $z(t)$ are non-negative, then $y(t) > 0$. Fig. 17 illustrates the lines corresponding to $x(t)$, $z(t)$, and both $y(t) = x(t) - z(t)$ and $y(t) = x(t) + z(t)$.

5 Dynamics of a falling object as an example

Fig. 18 shows a qualitative representation of the dynamics involved when an object falls and encounters air resistance. The quantities include gravitational force (*Fg*), air resistance (*Fair*), net force (*Fnet*), acceleration (*a*), velocity (*v*), and distance (*s*), each with a quantity space of {0, +}. The net force acting on the object is calculated by subtracting air resistance from gravitational force (i.e., *Fnet* = *Fg* – *Fair)*.

Gravitational force has a positive proportional relationship with net force and air resistance has a negative proportional relationship with net force. Acceleration has a positive proportional relationship with net force, and there is a directed correspondence (C) between the quantity spaces of net force and acceleration. Acceleration has a positive influence on velocity, which in turn positively influences distance. Velocity has a positive proportional relationship with air resistance. The initial settings are that gravitational force has a constant exogenous influence acting in it, velocity and distance are both zero. Acceleration and air resistance derive their values by the directed correspondences.

The simulation of this system with these initial settings shows four consecutive states. In state 1, gravitational force is positive and steady and air resistance is zero and about to increase, resulting in a positive net force (*Fnet* > 0). This positive net force results in acceleration, which in turn causes an increase in velocity (δ*v* > 0). As the velocity increases, air resistance increases (δ*Fair* > 0), which decreases the net force (δ*Fnet* < 0). In state 2, velocity is positive (+) and thereby distance increases (δs > 0) and air resistance is positive (+). In state 3, distance is positive (+) and increasing (δs > 0). In state 4, air resistance is equal to gravitational force and the net force is zero (*Fnet* = 0). Thereby acceleration is zero (0) and velocity is positive (+) and constant (δv = 0).

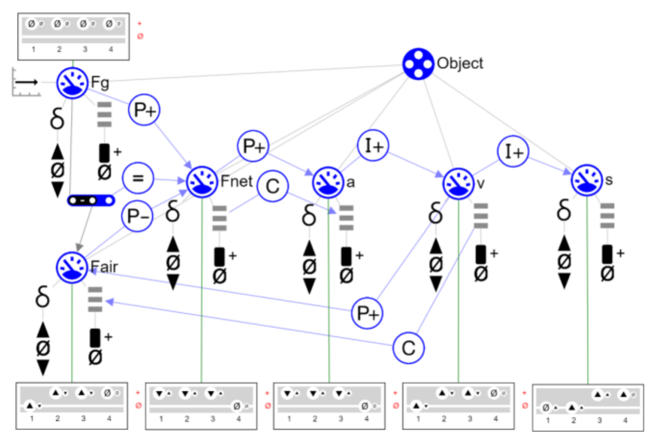

Fig. 18. Qualitative representation of the dynamics involved when an object falls and encounters air resistance. Value history shows the first and second derivative.

To transition from the qualitative representation to an accurate quantitative model, the mathematical equations, along with several initial numerical values for parameters need to be set. The mathematical equations corresponding to the qualitative representation in Fig. 18, which describe the system of a falling object that encounters air resistance, are detailed in Table 2. The differential equations for velocity $v(t)$ and distance $s(t)$ are linear. The starting values of velocity and distance can be directly taken from the qualitative representation ($v(0) = 0$ and $s(0) = 0$). The equation for gravitational force $Fg(t)$ is also treated as a differential equation. Typically, in software for numerical simulation, $Fg(t)$ would be considered a constant; however, the vocabulary of Dynalearn does not include an ingredient for constants. The numerical starting value of gravitational force must be explicitly specified, as it remains constant within a given interval and its exact value is otherwise undefined. In numerical simulations, the parameters mass (m) and the gravitational constant (g) are used to calculate the gravitational force acting on the object ($Fg = m \cdot g$). Because $Fg(t)$ is represented in the model as a differential equation but is intended to remain constant, the parameter that governs the increase over time should be set to zero ($c = 0$), ensuring that $Fg(t)$ does not change. The equation for calculating air resistance incorporates several parameters: Cd is the drag coefficient, which varies based on the object's shape and its movement through the air; ρ represents the air density; and A denotes the cross-sectional area of the object. Additionally, the value of $v(t)$ is squared within this context, reflecting its impact on air resistance as velocity increases.

Table 2. Mathematical equations of the dynamics involved when an object falls and encounters air resistance.

Equations	Initial values
$v(t) = v(t - \Delta t) + a(t) \cdot \Delta t$	$m = .1; g = 9.81$
$s(t) = s(t - \Delta t) + v(t) \cdot \Delta t$	$\rho = 1.3; A = .05; Cd = .3$
$Fg(t) = Fg\ (t - \Delta t) + c \cdot \Delta t$	$v(0) = 0$
$Fair(t) = ½ \cdot Cd \cdot \rho \cdot A \cdot v(t)^2$	$s(0) = 0$
$Fnet(t) = Fg(t) – Fair(t)$	$Fg\ (0) = m \cdot g$
$a(t) = Fnet(t) / m$	$\Delta t = .1$
	$c = 0$

Fig. 19 shows the simulation result for velocity per time, based on the equations and initial values listed in Table 2. It shows that velocity starts at zero and increases, aligning with state 1 in Fig. 18. Next, velocity is increases at a decreasing rate, corresponding to states 2 and 3, before finally stabilizing at a constant value, which corresponds to state 4 in Fig. 18.

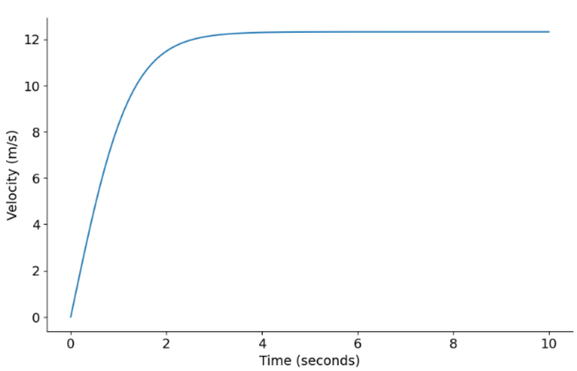

Fig. 19. Simulation result of velocity per time based on equations and initial values of Table 2.

6 Conclusion and future work

In this paper, we focus on how the qualitative vocabulary of Dynalearn, which is used for describing dynamic systems, corresponds to the mathematical equations used in quantitative modeling. We demonstrate how qualitative relationships can be mapped to linear and nonlinear general and differential equations. We also describe how quantity spaces and correspondences define the range of the mathematical equations. The initial values and inequalities set the scenarios in the qualitative representation and provide information about the starting values for parameters in the mathematical equations. Furthermore, a qualitative calculus that specifies operations such as addition or subtraction can be expressed through corresponding mathematical equations.

Dynalearn serves as a learning tool, and for the integration of quantitative modeling, a pedagogical approach should be developed to optimize learning. This approach should include support functions that assist learners in describing the mathematical equations that correspond with the behavior of the qualitative model, as learners often find this challenging [4, 13]. For instance, the software could automatically generate general equations which learners can then edit. For example, the differential equations for $v(t)$ and $s(t)$ as shown in Table 2 could be a derived from the quantitative representation in Fig. 18 and presented as the default option.

Another option is to provide feedback based on whether the behavior of the quantitative model aligns with the qualitative model. Since an analytic solution is often not feasible, analysis of whether behaviors align needs to be derived from the simulation result of both models. From the mathematical model, we know that there is no ambiguity in behavior; all values and changes are determined, and the simulation result should at least be a subset of a single path of states from the simulation result of the qualitative model. Remember, a transition in states in the qualitative simulation indicates a change in value or derivative of one or more quantities. To detect changes in the results of the quantitative simulation, it is necessary to check at each time interval whether derivatives change or certain thresholds are reached. If discrepancies are identified between the behaviors, feedback should be provided. For instance, if the results from the quantitative analysis only partially align with a path of states and a final state is not achieved, then the simulation duration may not have been sufficient to reach those subsequent states, or some parameters might need adjustment. For example, if the simulation based on the equations and initial values listed in Table 2 is run for an insufficient duration, the velocity may not stabilize at its final constant state.

With support options in place, the next step is to develop an educational approach that optimizes learning in such integrated software. For instance, a step-by-step approach alternating between qualitative and quantitative modeling, or initially constructing a complete qualitative model to understand system behavior conceptually before transitioning to a quantitative model. Further research on optimizing learning in integrated qualitative and quantitative modeling software is therefore essential.

References

[1] Jacobson, M., & Wilensky, U. Complex systems in education: Scientific and educational importance and implications for the learning sciences. *The Journal of the Learning Sciences*, 15(1), 11-34, 2006.

[2] Bredeweg, B., Kragten, M., Holt, J., Kruit, P., van Eijck, T., Pijls, M., Bouwer, A., Sprinkhuizen, M., Jaspar, E., & de Boer, M. Learning with interactive knowledge representations. *Applied Sciences*, 13(9), 5256, 2023.

[3] Fisher, D. Reflections on teaching system dynamics modeling to secondary school students for over 20 Years. *Systems*, 6(2), 12, 2018.

[4] Sins, P., Savelsbergh, E., van Joolingen, W. R. The difficult process of scientific modelling: An analysis of novices' reasoning during computer-based modelling. *International Journal of Science Education*, 27(14), 1695-1721, 2005.

[5] Kragten, M., Spitz, L., & Bredeweg, B. Learning domain knowledge and systems thinking using qualitative representations in secondary education (grades 9-10). In *Proceedings of the 34th International Workshop on Qualitative Reasoning*. Montreal, Canada, 2021.

[6] Kragten, M., Jaspar, E., & Bredeweg, B. Learning domain knowledge and systems thinking using qualitative representations in secondary education (grades 10-12). *In Proceedings of the 35th International Workshop on Qualitative Reasoning*. Vienna, Austria, 2022.

[7] Spitz, L., Kragten, M., & Bredeweg, B. Learning domain knowledge and systems thinking using qualitative representations in secondary education (grades 8-9). In *Proceedings of the 34th International Workshop on Qualitative Reasoning*. Montreal, Canada, 2021.

[8] Forbus, K. D. *Qualitative representations: How people reason and learn about the continuous world*. MIT Press, 2019.

[9] Coyle, G. Qualitative and quantitative modeling in system dynamics: some research questions. *System Dynamics Review*, 16(3), 225-244, 2000.

[10] Bredeweg, B., Liem, J., Beek, W., Linnebank, F., Gracia, J., Lozano, E., Wißner, M., Bühling, R., Salles, P., Noble, R. A., Zitek, A., Borisova, P., & Mioduser, D. (2013). DynaLearn - An intelligent learning environment for learning conceptual knowledge. *AI Magazine*, 34(4), 46-65.

[11] Bredeweg, B., Linnebank, F., Bouwer, A., & Liem, J. Garp3—Workbench for qualitative modelling and simulation. Ecological informatics, 4(5-6), 263-281, 2009.

[12] Bredeweg, B., & Linnebank, F. E. Simulation preferences - Means towards usable QR engines. In *Proceedings of the 26th International Workshop on Qualitative Reasoning*. Playa del Rey, CA, USA, 2012.

[13] Buuren, O., Heck, A., & Ellermeijer, T. Understanding of Relation Structures of Graphical Models by Lower Secondary Students. *Research in Science Education*, 46, 633 – 666, 2016.