Dynamics Of A Quadcopter

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1 Generalized Coordinates

In order to properly define the dynamics of a quadcopter, two coordinate systems will be defined, body & inertial coordinate systems. In the inertial frame the position will be defined by X_G, Y_G, Z_G with angles of rotation in the body frame defined as ψ,θ,ϕ, which allows to define the generalized coordinates q:

$$q = \begin{bmatrix} X_G & Y_G & Z_G & \psi & \theta & \phi \end{bmatrix}$$

The location of the body frame defined from the origin of the inertial frame is defined by the following:

$$\overrightarrow{r_G} = X_G\overrightarrow{I} + Y_G\overrightarrow{J} + Z_G\overrightarrow{K}$$

In the case of the quadcopter where there are four motors, there are four forces produced by the motors defined in the positive $\overrightarrow{k}$ direction of the body frame, in general the forces are defined as:

$$\overrightarrow{F_i} = \begin{bmatrix} 0 \\ 0 \\ F_i \end{bmatrix} i = 1:4$$

Which will yeild a resultant force in the body frame of:

$$\overrightarrow{F_R} = \sum_{i=1}^{4}\overrightarrow{F_i} - mg$$

These forces will also need to be defined in the inertial frame, this is acheived by the rotation matrix:

$$\overrightarrow{F_R} = R(\psi,\theta,\phi)\cdot\overrightarrow{F_R}$$

Where the rotation matrix is defined as:

$$R(\psi,\theta,\phi) = \begin{bmatrix} cos\psi cos\theta & cos\theta sin\psi & -sin\theta \\ cos\psi sin\phi sin\theta - cos\phi sin\psi & cos\phi cos\psi + sin\phi sin\psi sin\theta & cos\theta sin\phi \\ sin\phi sin\psi + cos\phi cos\psi sin\theta & cos\phi sin\psi sin\theta - cos\psi sin\phi & cos\phi cos\theta \end{bmatrix}$$

Which can be expressed in terms of the generalized coordinates q as:

$$R(q) = \begin{bmatrix} \cos(q_4)\cos(q_5) & \cos(q_5)\sin(q_4) & sin(q_5) \\ \cos(q_4)\sin(q_6)\sin(q_5) cos(q_6)\sin(q_4) & \cos(q_6)\cos(q_4) + \sin(q_6)\sin(q_4)\sin(q_5) & \cos(q_5)\sin(q_6) \\ \sin(q_6)\sin(q_4) + \cos(q_6)\cos(q_4)\sin(q_5) & \cos(q_6)\sin(q_4)\sin(q_5) - \cos(q_4)\sin(q_6) & \cos(q_6)\cos(q_5) \end{bmatrix}$$

which yields the force resultant in the inertial frame as:

$$\overrightarrow{F_R}(q) = \begin{bmatrix} -sin(q_5)\cdot(F_1 + F_2 + F_3 + F_4 - mg) \\ cos(q_5)\sin(q_6)\cdot(F_1 + F_2 + F_3 + F_4 - mg) \\ cos(q_6)\cos(q_5)\cdot(F_1 + F_2 + F_3 + F_4 - mg) \end{bmatrix}$$

The position of the forces in the body frame are defined by the position vectors $\overrightarrow{r_i}$ (i=1:4) as follows:

$$\overrightarrow{r_1} = \frac{a}{2}\overrightarrow{i} + \frac{b}{2}\overrightarrow{j} + 0\overrightarrow{k}$$

$$\overrightarrow{r_2} = -\frac{a}{2}\overrightarrow{i} + \frac{b}{2}\overrightarrow{j} + 0\overrightarrow{k}$$

$$\overrightarrow{r_3} = \frac{a}{2}\overrightarrow{i} - \frac{b}{2}\overrightarrow{j} + 0\overrightarrow{k}$$

$$\overrightarrow{r_4} = -\frac{a}{2}\overrightarrow{i} - \frac{b}{2}\overrightarrow{j} + 0\overrightarrow{k}$$

Using the forces and their position vectors the moment about origin G in the body frame is defined as:

$$\overrightarrow{M_G} = \sum_{i=1}^{4}(\overrightarrow{r_i} x \overrightarrow{F_i})$$

this will also need to be rotated into the inertial frame yielding:

$$\overrightarrow{M_G}(q) = \begin{bmatrix} cosq_4cosq_5M_1 + cosq_5sinq_4M_2 \\ -(cosq_6sinq_4 - cosq_4sinq_6sinq_5)M_1 + (cosq_6cosq_4 + sinq_6sinq_4sinq_5)M_2 \\ (sinq_6sinq_4 + cosq_6cosq_4sinq_5)M_1 - (cosq_4sinq_6 - cosq_6sinq_4sinq_5)M_2 \end{bmatrix}$$

$$M_1 = \frac{b}{2}F_1 + \frac{b}{2}F_2 - \frac{b}{2}F_3 - \frac{b}{2}F_4$$

$$M_2 = \frac{a}{2}F_1 - \frac{a}{2}F_2 + \frac{a}{2}F_3 - \frac{a}{2}F_4$$

For a full depiction of the dynamics of the system in the global frame first we start with looking at the local angular velocities as expressed in the body frame:

$$\omega_{body} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -sin\theta & 0 & 1 \\ cos\theta cos\phi & cos\phi & 0 \\ cos\phi cos\theta & -sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Expressing these in the generalized coordinates of q yields:

$$\omega(q) = R(q) \cdot \omega_{body}$$

With the local angular velocities and the moments in the body frame rotated into the inertial, the generalized forces $Q_i(q)$ can be defined by:

$$Q_j = \overrightarrow{F}_R \frac{\partial \overrightarrow{r_G}}{\partial q_j} + \overrightarrow{M}_G \frac{\partial \overrightarrow{\omega}}{\partial \dot{q_j}}$$

This allows for a full generalized force representation as:

$$Q_1 = \overrightarrow{F_R} \frac{\partial \overrightarrow{r_G}}{\partial q_1}$$

$$Q_2 = \overrightarrow{F_R} \frac{\partial \overrightarrow{r_G}}{\partial q_2}$$

$$Q_3 = \overrightarrow{F_R} \frac{\partial \overrightarrow{r_G}}{\partial q_3}$$

$$Q_4 = \overrightarrow{F_R} \frac{\partial \overrightarrow{r_G}}{\partial q_4} + \overrightarrow{M_G} \frac{\partial \omega}{\partial \dot{q_4}}$$

$$Q_5 = \overrightarrow{F_R} \frac{\partial \overrightarrow{r_G}}{\partial q_5} + \overrightarrow{M_G} \frac{\partial \omega}{\partial \dot{q_5}}$$

$$Q_6 = \overrightarrow{F_R} \frac{\partial \overrightarrow{r_G}}{\partial q_6} + \overrightarrow{M_G} \frac{\partial \omega}{\partial \dot{q_6}}$$

The Kinetic Energy and the Equations of Motion

The kinetic energy of the quadcopter is defined as:

$$T(q) = \frac{1}{2}\dot{q}^T M(q) * \dot{q}$$

where M(q) is defined as:

$$M(q) = J_v(q)^T m_{rr} J_v(q) + J_w(q)^T m_{\theta\theta} J_w(q)$$

with:

$$J_v(q) = \begin{bmatrix} \frac{\partial q_1}{\partial q_1} & \frac{\partial q_1}{\partial q_2} & \frac{\partial q_1}{\partial q_3} & 0 & 0 & 0 \\ \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial q_2} & \frac{\partial q_2}{\partial q_3} & 0 & 0 & 0 \\ \frac{\partial q_3}{\partial q_1} & \frac{\partial q_3}{\partial q_1} & \frac{\partial q_3}{\partial q_1} & 0 & 0 & 0 \end{bmatrix}$$

$$J_w(q) = \begin{bmatrix} 0 & 0 & 0 & -sinq_5 & 0 & 1 \\ 0 & 0 & 0 & cosq_5 sinq_6 & cosq_6 & 0 \\ 0 & 0 & 0 & cosq_5 cosq_6 & -sinq_6 & 0 \end{bmatrix}$$

$$m_{rr} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$m_{\theta\theta} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

which expands to:

$$T(q) = \frac{1}{2}[m\dot{q_1}^2 + m\dot{q_2}^2 + m\dot{q_3}^2 + \dot{q_4}^2(I_{zz}cos(q_5)^2cos(q_6)^2 + I_{yy}cos(q_5)^2sin(q_6)^2 + I_{xx}sin(q_5)^2) + \ldots$$

$$\ldots + \dot{q_5}^2(I_{yy}cos(q_6)^2 + I_{zz}sin(q_6)^2) + 2\dot{q_5}\dot{q_4}\sigma + \dot{q_6}^2 I_{xx} - 2\dot{q_6}\dot{q_4}I_{xx}sin(q_5)]$$

where:

$$\sigma = I_{yy}cos(q_5)cos(q_6)sin(q_6) - I_{zz}cos(q_5)cos(q_6)sin(q_6)$$

The potential energy of the drone is defined as:

$$\overrightarrow{V}(q) = mgq_3$$

which allows for the lagrangian to be:

$$\mathcal{L}(q) = T(q) - V(q)$$

with $\mathcal{L}(q)$ defined this allows us to find the corresponding Euler-Lagrange equations of motions from the following equation:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}(q)}{\partial \dot{q}_j}) - \frac{\partial \mathcal{L}(q)}{\partial q_j} = Q_j(q)$$

$$Q_1(q) = m\ddot{q_1}$$

$$Q_2(q) = m\ddot{q_2}$$

$$Q_3(q) = m\ddot{q_3} - mg$$

$$Q_4(q) = 2I_{xx}sin(q_5)cos(q_5)\dot{q_4}\dot{q_5} - I_{xx}sin(q_5)\ddot{q_6} - I_{xx}cos^2(q_5)\ddot{q_4} - I_{xx}cos(q_5)\dot{q_5}\dot{q_6}\ldots$$

$$\ldots + I_{xx}\ddot{q_4} + \frac{1}{4}I_{yy}(-(\dot{q_5} - 2\dot{q_6})cos(q_5 - 2q_6) + (\dot{q_5} + 2\dot{q_6})cos(q_5 + 2q_6))\dot{q_5} + \frac{1}{4}I_{yy}(-sin(q_5 - 2q_6) + sin(q_5 + 2q_6))\ddot{q_5}\ldots$$

$$\ldots + 2I_{yy}sin(q_5)cos(q_5)cos^2(q_6)\dot{q}_4\dot{q}_5 - 2I_{yy}sin(q_5)cos(q_5)\dot{q}_4\dot{q}_5 + 2I_{yy}sin(q_6)cos^2(q_5)cos(q_6)\dot{q}_4\dot{q}_6 \ldots$$

$$\ldots - I_{yy}cos^2(q_5)cos^2(q_6)\ddot{q}_4 + I_{yy}cos^2(q_5)\ddot{q}_4 + \frac{1}{4}I_{zz}(-(\dot{q}_5 - 2\dot{q}_6)cos(q_5 - 2q_6) + (\dot{q}_5 + 2\dot{q}_6)cos(q_5 + 2q_6))\dot{q}_5 \ldots$$

$$\ldots + \frac{1}{4}I_{zz}(-sin(q_5 - 2q_6) + sin(q_5 + 2q_6))\ddot{q}_5 - 2I_{zz}sin(q_5)cos(q_5)cos^2(q_6)\dot{q}_4\dot{q}_5 \ldots$$

$$\ldots - 2I_{zz}sin(q_6)cos^2(q_5)cos(q_6)\dot{q}_4\dot{q}_6 + I_{zz}cos^2(q_5)cos^2(q_6)\ddot{q}_4$$

$$Q_5(q) = -I_{xx}sin(q_5)cos(q_5)\dot{q}_4^2 + I_{xx}cos(q_5)\dot{q}_4\dot{q}_6 + \frac{1}{4}I_{yy}(-(\dot{q}_5 - 2\dot{q}_6)cos(q_5 - 2q_6) + (\dot{q}_5 + 2\dot{q}_6)cos(q_5 + 2q_6))\dot{q}_4 \ldots$$

$$\ldots + \frac{1}{4}I_{yy}(-sin(q_5 - 2q_6) + sin(q_5 + 2q_6))\ddot{q}_4 - \frac{1}{4}I_{yy}(-cos(q_5 - 2q_6) + cos(q_5 + 2q_6))\dot{q}_4\dot{q}_5 \ldots$$

$$\ldots - I_{yy}sin(q_5)cos(q_5)cos^2(q_6)\dot{q}_4^2 + I_{yy}sin(q_5)cos(q_5)\dot{q}_4^2 - 2.0I_{yy}sin(q_6)cos(q_6)\dot{q}_5\dot{q}_6 \ldots$$

$$\ldots + I_{yy}cos^2(q_6)\ddot{q}_5 + \frac{1}{4}I_{zz}(-(\dot{q}_5 - 2\dot{q}_6)cos(q_5 - 2q_6) + (\dot{q}_5 + 2\dot{q}_6)cos(q_5 + 2q_6))\dot{q}_4 \ldots$$

$$\ldots + \frac{1}{4}I_{zz}(-sin(q_5 - 2q_6) + sin(q_5 + 2q_6))\ddot{q}_4 - \frac{1}{4}I_{zz}(-cos(q_5 - 2q_6) + cos(q_5 + 2q_6))\dot{q}_4\dot{q}_5 \ldots$$

$$\ldots + I_{zz}sin(q_5)cos(q_5)cos^2(q_6)\dot{q}_4^2 + 2.0I_{zz}sin(q_6)cos(q_6)\dot{q}_5\dot{q}_6 - I_{zz}cos^2(q_6)\ddot{q}_5 + I_{zz}\ddot{q}_5$$

$$Q_6(q) = -I_{xx}sin(q_5)\ddot{q}_4 - I_{xx}cos(q_5)\dot{q}_4\dot{q}_5 + I_{xx}\ddot{q}_6 - \frac{1}{4}I_{yy}(2cos(q_5 - 2q_6) + 2cos(q_5 + 2q_6))\dot{q}_4\dot{q}_5 \ldots$$

$$\ldots - I_{yy}sin(q_6)cos(q_5)^2cos(q_6)\dot{q}_4^2 + I_{yy}sin(q_6)cos(q_6)\dot{q}_5^2 - \frac{1}{4}I_{zz}(2cos(q_5 - 2q_6) + 2cos(q_5 + 2q_6))\dot{q}_4\dot{q}_5 \ldots$$

$$\ldots + I_{zz}sin(q_6)cos(q_5)^2cos(q_6)\dot{q}_4^2 - I_{zz}sin(q_6)cos(q_6)\dot{q}_5^2$$

State Space Representation

To define a controller, the dynamics of the system must be first put into state space form:

$$x = [q, \dot{q}]^T$$

$$\frac{d\dot{q}_4}{dt} = \frac{-1}{((I_{yy} - I_{zz})cos^2(q_6) + I_{zz})(2I_{xx}sin^2(q_5)) + (I_{yy} - I_{zz})cos^2(q_5)cos^2(q_6)}$$