

# Control Of A Quadcopter

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April 11, 2021

## 1 Generalized Coordinates

When analyzing the dynamics of a quadcopter, two coordinate systems need to be defined, the body & inertial coordinate frames. The inertial frame is earth fixed, this can be thought of as a centralized home position that all motion is relative to, in this case with a center point defined as O. The body frame is a coordinate frame defined from the center of gravity of the quadcopter, this will be defined with a center point of G. Managing two coordinate frames in a dynamics system can be achieved efficiently by introducing a generalized coordinate q, that is comprised of the necessary variables to fully define the dynamics of the body frame in the inertial.

A vector can then be related from O to G as  $\vec{r}_G$ :

$$\vec{r}_G = \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

To relate the motion of the body in the inertial frame, three angles of rotation  $\psi, \theta, \phi$  are needed, which are known in literature as Type I Euler angles[cite greenwood]. This leads to the state vector of the system in generalized coordinates  $q \in \mathbb{R}^6$ :

$$q = [X_G \ Y_G \ Z_G \ \psi \ \theta \ \phi] = [q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6]$$

The predominant forces acting on a quadcopter are gravity and the forces created by the motors. In the case of the quadcopter, there are four forces produced by the motors defined in the positive  $\vec{k}$  direction of the body frame, in general the forces are defined as:

$$\vec{F}_i = \begin{bmatrix} 0 \\ 0 \\ F_i \end{bmatrix} \quad i = 1 : 4$$

Which will yield a resultant force in the body frame of:

$$\vec{F}_R = \sum_{i=1}^4 \vec{F}_i - mg$$

These forces will also need to be defined in the inertial frame, this is achieved by the rotation matrix:

$$\vec{F}_R = R(\psi, \theta, \phi) \cdot \vec{F}_R$$

With:

$$R(\psi, \theta, \phi) = \begin{bmatrix} \cos \psi \cos \theta & \cos \theta \sin \psi & -\sin \theta \\ \cos \psi \sin \phi \sin \theta - \cos \phi \sin \psi & \cos \phi \cos \psi + \sin \phi \sin \psi \sin \theta & \cos \theta \sin \phi \\ \sin \phi \sin \psi + \cos \phi \cos \psi \sin \theta & \cos \phi \sin \psi \sin \theta - \cos \psi \sin \phi & \cos \phi \cos \theta \end{bmatrix}$$

Or in terms of the generalized coordinates q:

$$R(q) = \begin{bmatrix} \cos(q_4) \cos(q_5) & \cos(q_5) \sin(q_4) & -\sin(q_5) \\ \cos(q_4) \sin(q_6) \sin(q_5) - \cos(q_6) \sin(q_4) & \cos(q_6) \cos(q_4) + \sin(q_6) \sin(q_4) \sin(q_5) & \cos(q_5) \sin(q_6) \\ \sin(q_6) \sin(q_4) + \cos(q_6) \cos(q_4) \sin(q_5) & \cos(q_6) \sin(q_4) \sin(q_5) - \cos(q_4) \sin(q_6) & \cos(q_6) \cos(q_5) \end{bmatrix}$$

yielding the force resultant in the inertial frame as:

$$\vec{F}_R = \begin{bmatrix} -\sin(q_5) \cdot (F_1 + F_2 + F_3 + F_4 - mg) \\ \cos(q_5) \sin(q_6) \cdot (F_1 + F_2 + F_3 + F_4 - mg) \\ \cos(q_6) \cos(q_5) \cdot (F_1 + F_2 + F_3 + F_4 - mg) \end{bmatrix}$$

The position of the forces in the body frame are defined by the position vectors  $\vec{r}_i$  (i=1:4) as follows:

$$\vec{r}_1 = \frac{a}{2}\vec{i} + \frac{b}{2}\vec{j} + 0\vec{k}$$

$$\vec{r}_2 = -\frac{a}{2}\vec{i} + \frac{b}{2}\vec{j} + 0\vec{k}$$

$$\vec{r}_3 = \frac{a}{2}\vec{i} - \frac{b}{2}\vec{j} + 0\vec{k}$$

$$\vec{r}_4 = -\frac{a}{2}\vec{i} - \frac{b}{2}\vec{j} + 0\vec{k}$$

Using the forces and their position vectors the moment about origin G in the body frame is defined as:

$$\vec{M}_G = \sum_{i=1}^4 (\vec{r}_i \times \vec{F}_i)$$

this will also need to be rotated into the inertial frame with a center point of O yielding:

$$\vec{M}_O = \vec{M}_G \cdot R(q) = \begin{bmatrix} \cos q_4 \cos q_5 M_1 + \cos q_5 \sin q_4 M_2 \\ -(\cos q_6 \sin q_4 - \cos q_4 \sin q_6 \sin q_5) M_1 + (\cos q_6 \cos q_4 + \sin q_6 \sin q_4 \sin q_5) M_2 \\ (\sin q_6 \sin q_4 + \cos q_6 \cos q_4 \sin q_5) M_1 - (\cos q_4 \sin q_6 - \cos q_6 \sin q_4 \sin q_5) M_2 \end{bmatrix}$$

$$M_1 = \frac{b}{2}F_1 + \frac{b}{2}F_2 - \frac{b}{2}F_3 - \frac{b}{2}F_4$$

$$M_2 = \frac{a}{2}F_1 - \frac{a}{2}F_2 + \frac{a}{2}F_3 - \frac{a}{2}F_4$$

For a full depiction of the dynamics of the system in the inertial frame first we start with looking at the local angular velocities as expressed in the body frame:

$$\omega_{body} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\cos\phi & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Expressing these in the generalized coordinates of q yields:

$$\omega(q) = R(q) \cdot \omega_{body}$$

With the local angular velocities and the moments in the body frame rotated into the inertial, the generalized forces  $Q_i$  can be defined by:

$$Q_i = \vec{F}_R \frac{\partial \vec{r}_G}{\partial q_i} + \vec{M}_G \frac{\partial \vec{\omega}}{\partial \dot{q}_i}, i = 1 : 6, j = 4 : 6$$

## The Kinetic Energy and the Equations of Motion

The kinetic energy of the quadcopter is defined based on Koenig's Theorem [Cite Greenwood] which states the kinetic energy is the sum of the kinetic energy due to translational velocity of the center of mass as well as the rotation:

$$T(q) = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

where M(q) is defined as:

$$M(q) = J_v(q)^T m_{rr} J_v(q) + J_w(q)^T m_{\theta\theta} J_w(q)$$

with:

$$J_v(q) = \begin{bmatrix} \frac{\partial q_1}{\partial q_1} & \frac{\partial q_1}{\partial q_2} & \frac{\partial q_1}{\partial q_3} & 0 & 0 & 0 \\ \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial q_2} & \frac{\partial q_2}{\partial q_3} & 0 & 0 & 0 \\ \frac{\partial q_3}{\partial q_1} & \frac{\partial q_3}{\partial q_2} & \frac{\partial q_3}{\partial q_3} & 0 & 0 & 0 \end{bmatrix}$$

$$J_w(q) = \begin{bmatrix} 0 & 0 & 0 & -\sin q_5 & 0 & 1 \\ 0 & 0 & 0 & \cos q_5 \sin q_6 & \cos q_6 & 0 \\ 0 & 0 & 0 & \cos q_5 \cos q_6 & -\sin q_6 & 0 \end{bmatrix}$$

$$m_{rr} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$m_{\theta\theta} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

which expands to:

$$T(q) = \frac{1}{2}(m\dot{q}_1^2 + m\dot{q}_2^2 + m\dot{q}_3^2 + \dot{q}_4^2((I_{yy} + I_{zz})\cos(q_5)^2\cos(q_6)^2 + I_{xx}\sin(q_5)^2) + \dots$$

$$\dots + \dot{q}_5^2(I_{yy}\cos(q_6)^2 + I_{zz}\sin(q_6)^2) + 2\dot{q}_5\dot{q}_4\sigma + \dot{q}_6^2 I_{xx} - 2\dot{q}_6\dot{q}_4 I_{xx}\sin(q_5))$$

where:

$$\sigma = \frac{1}{2}(I_{yy} - I_{zz})\cos(q_5)\sin(2q_6)$$

The potential energy of the drone is defined as:

$$\vec{V}(q) = mgq_3$$

which allows for the lagrangian to be:

$$\mathcal{L}(q) = T(q) - V(q)$$

yielding,

$$\mathcal{L}(q) = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2 + \frac{1}{2}m\dot{q}_3^2 +$$

$$\dots (\frac{1}{2}I_{xx}\sin^2(q_5) + \frac{1}{2}I_{yy}\cos^2(q_6) - (\frac{1}{2}I_{yy} - \frac{1}{2}I_{zz})\cos^2(q_5)\cos^2(q_6))\dot{q}_4^2 +$$

$$\dots (\frac{1}{2}I_{zz} + \frac{1}{2}(I_{yy} - I_{zz})\cos^2(q_6))\dot{q}_5^2 + \frac{1}{2}I_{xx}\dot{q}_6^2 - I_{xx}\sin(q_5)\dot{q}_4\dot{q}_6$$

$$\dots + \frac{1}{2}(I_{yy} + I_{zz})\cos(q_5)\sin(2q_6)\dot{q}_4\dot{q}_5 + gmq_3$$

with  $\mathcal{L}(q)$  defined this allows us to find the corresponding Euler-Lagrange equations of motions from the following equation, which is also known as the fundamental holonomic form of Lagrange's equation:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}(q)}{\partial \dot{q}_j}) - \frac{\partial \mathcal{L}(q)}{\partial q_j} = Q_j(q)$$

$$Q_1(q) = m\ddot{q}_1$$

$$Q_2(q) = m\ddot{q}_2$$

$$Q_3(q) = m\ddot{q}_3 - mg$$

$$Q_4(q) = -I_{xx}\sin(q_5)\ddot{q}_6 + (-I_{xx} + (I_{yy} - I_{zz})\cos^2(q_5))\sin(q_6)\cos(q_6)\dot{q}_4\dot{q}_6 +$$

$$\dots 2(I_{xx} + (I_{yy} - I_{zz})\cos^2(q_6))\sin(q_5)\cos(q_5)\dot{q}_4\dot{q}_5 + (-0.5I_{yy} + 0.5I_{zz})\cos^2(q_5)\cos^2(q_6)\ddot{q}_4 +$$

$$\dots (I_{yy} + I_{zz})\cos(q_5)\sin(2q_6)\ddot{q}_5$$

$$Q_5(q) = (I_{xx} + (I_{yy} - I_{zz})\cos^2(q_6))\sin(q_5)\cos(q_5)\dot{q}_4^2 + (I_{xx} + (I_{yy} + I_{zz})\cos(2q_6))\cos(q_5)\dot{q}_4\dot{q}_6 +$$

$$\dots (I_{yy} + I_{zz})\cos(q_5)\sin(2q_6)\ddot{q}_4 - (2I_{yy} - 2I_{zz})\sin(q_6)\cos(q_6)\dot{q}_5\dot{q}_6 + (I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6))\ddot{q}_5$$

$$Q_6(q) = -I_{xx}\sin(q_5)\ddot{q}_4 + I_{xx}\ddot{q}_6 - (I_{xx} + (I_{yy} + I_{zz})\cos(2q_6))\cos(q_5)\dot{q}_4\dot{q}_5 +$$

$$\dots (I_{yy} - I_{zz})\sin(q_6)\cos(q_6)\dot{q}_5^2 - (I_{yy} + (I_{yy} - I_{zz})\cos^2(q_5))\sin(q_6)\cos(q_6)\dot{q}_4^2$$

## State Space Representation

To define a controller, the dynamics of the system must be first put into state space form:

$$x = [q, \dot{q}]^T$$

and,

$$\dot{x} = [\frac{dq}{dt}, \frac{d\dot{q}}{dt}]^T$$

where,

$$\frac{d\dot{q}_1}{dt} = \frac{Q_1}{m}$$

$$\frac{d\dot{q}_2}{dt} = \frac{Q_2}{m}$$

$$\frac{d\dot{q}_3}{dt} = g + \frac{Q_3}{m}$$

$$\frac{d\dot{q}_4}{dt} = \frac{A_1}{(I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6))(2I_{xx}\sin^2(q_5) + (I_{yy} - I_{zz})\cos^2(q_5)\cos^2(q_6))}$$

where:

$$\begin{aligned} A_1 = & -(I_{yy} + I_{zz})((I_{xx} + (I_{yy} - I_{zz})\cos^2(q_6))\sin(q_5)\cos(q_6)\dot{q}_4^2 + \\ & \dots (I_{xx} + (I_{yy} + I_{zz})\cos(2q_6))\cos(q_5)\dot{q}_4\dot{q}_6 + \\ & \dots (-I_{yy} + I_{zz})\sin(2q_6)\dot{q}_5\dot{q}_6 - Q_5)\sin(2q_6)\cos(q_5) - \\ & \dots (2I_{zz} + 2(I_{yy} - I_{zz})\cos^2(q_6))((-I_{xx} + (I_{yy} - I_{zz})\cos^2(q_6))\sin(2q_5)\dot{q}_4\dot{q}_5 + \\ & (I_{xx} + (I_{yy} - I_{zz})\cos^2(q_5))\sin(q_6)\cos(q_6)\dot{q}_4\dot{q}_6 + Q_4)\sin(q_5) - \\ & \dots (2I_{zz} + 2(I_{yy} - I_{zz})\cos^2(q_6))((I_{xx} + (I_{yy} + I_{zz})\cos(2q_6))\cos(q_5)\dot{q}_4\dot{q}_5 + (-0.5I_{yy} + 0.5I_{zz})\sin(2q_6)\dot{q}_5^2 + \\ & (-0.125I_{yy} + 0.125I_{zz})\sin(2q_5 - 2q_6) + (0.125I_{yy} - 0.125I_{zz})\sin(2q_5 + 2q_6) + (1.5I_{yy} - 0.5I_{zz})\sin(q_5)\cos(q_5)\dot{q}_4^2) \end{aligned}$$

$$\frac{d\dot{q}_5}{dt} = \frac{A_2}{2I_{xx}\sin^2(q_5)(I_{yy}\cos^2(q_6) + I_{zz}\sin^2(q_6)) + \cos^2(q_5)\cos^2(q_6)(I_{yy}^2(\sin^2(q_6) + 1) + I_{yy}I_{zz}(6\sin^2(q_6) - 1)) + I_{zz}^2\sin^2(q_6)}$$

where,

$$\begin{aligned} A_2 = & (0.5I_{yy} + 0.5I_{zz})((-0.5I_{yy} + 0.5I_{zz})\sin(2q_6)\dot{q}_5^2 + (I_{xx}\cos(q_5) + (0.5I_{yy} + 0.5I_{zz})\cos(q_5)\cos(2q_6))\dot{q}_4\dot{q}_5 + \\ & \dots ((-0.125I_{yy} + 0.125I_{zz})\sin(2q_5 - 2q_6) + (0.125I_{yy} - 0.125I_{zz})\sin(2q_5 + 2q_6) + \\ & \dots (0.75I_{yy} - 0.25I_{zz})\sin(2q_6))\dot{q}_4^2 + Q_6)\sin(2q_5)\sin(2q_6) + (I_{yy} + I_{zz})((-I_{xx} + (-I_{yy} + I_{zz})\cos^2(q_6))\sin(2q_5)\dot{q}_4\dot{q}_5 + \\ & \dots (I_{xx} + (-0.5I_{yy} + 0.5I_{zz})\cos^2(q_5))\sin(2q_6)\dot{q}_4\dot{q}_6 + Q_4)\sin(2q_6)\cos(q_5) - \\ & \dots (2I_{xx}\sin^2(q_5) + (I_{yy} - I_{zz})\cos^2(q_5)\cos^2(q_6))((0.5I_{xx} + (0.5I_{yy} - 0.5I_{zz})\cos^2(q_6))\sin(2q_5)\dot{q}_4^2 + \\ & \dots (-I_{yy} + I_{zz})\sin(2q_6)\dot{q}_5\dot{q}_6 + (I_{xx}\cos(q_5) + (I_{yy} + I_{zz})\cos(q_5)\cos(2q_6))\dot{q}_4\dot{q}_6 - Q_6) \end{aligned}$$

$$\frac{d\dot{q}_6}{dt} = \frac{A_3}{I_{xx}(2I_{xx}((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\sin^2(q_5) + (I_{yy} - I_{zz})((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\cos^2(q_5)\cos^2(q_6) + 2(I_{yy} + I_{zz})^2\sin^2(q_6)\cos^2(q_5)\cos^2(q_6))}$$

where,

$$\begin{aligned} A_3 = & -0.25I_{xx}(I_{yy} + I_{zz})((-I_{yy} + I_{zz})\sin(2q_6)\dot{q}_5\dot{q}_6 + (0.5I_{xx}\sin(2q_5) + (I_{yy} - I_{zz})\sin(q_5)\cos(q_5)\cos^2(q_6))\dot{q}_4^2 + \\ & \dots (I_{xx}\cos(q_5) + (I_{yy} + I_{zz})\cos(q_5)\cos(2q_6))\dot{q}_4\dot{q}_6 - Q_5)\sin(2q_5)\sin(2q_6) - \\ & \dots 2I_{xx}(I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6))((-0.5I_{xx} + (-I_{yy} + I_{zz})\cos^2(q_6))\sin(2q_5)\dot{q}_4\dot{q}_5 + \\ & \dots (I_{xx} + (-0.5I_{yy} + 0.5I_{zz})\cos^2(q_5))\sin(2q_6)\dot{q}_4\dot{q}_6 + Q_4)\sin(q_5) + \end{aligned}$$

$$\begin{aligned} & \dots ((I_{yy} - I_{zz})(I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6)) + 2(I_{yy} + I_{zz})^2\sin(q_6))((I_{xx} + (I_{yy} + I_{zz})\cos(2q_6))\cos(q_5)\dot{q}_4\dot{q}_5 + \\ & \dots (-0.5I_{yy} + 0.5I_{zz})\sin(2q_6)\dot{q}_5^2 + ((-0.25I_{yy} + 0.25I_{zz})\sin(2q_5 - 2q_6) + (0.25I_{yy} + 0.25I_{zz})\sin(2q_5 + 2q_6) + \\ & \dots (0.75I_{yy} - 0.25I_{zz})\sin(2q_6))\dot{q}_4^2 + Q_6)\cos^2(q_5)\cos^2(q_6) \end{aligned}$$

To control the dynamics of the system the state space needs to be written in the following form:

$$\dot{x} = F(q, \dot{q}) + G(q, \dot{q})Q$$

where  $G(q, \dot{q})$  is a function of the states, and the generalized forces  $Q_{1x12}$  are the system inputs.

$$G(q, \dot{q}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1010} & g_{1011} & g_{1012} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1110} & g_{1111} & g_{1112} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{1210} & g_{1211} & g_{1212} \end{bmatrix}, Q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}$$

where,

$$g_{1010} = \frac{(I_{yy} - I_{zz})\cos q_6^2 + I_{zz}}{(I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6))(2I_{xx}\sin^2(q_5) + (I_{yy} - I_{zz})\cos^2(q_5)\cos^2(q_6))}$$

$$g_{1011} = \frac{(I_{yy} + I_{zz})\cos q_5 \sin 2q_6}{(I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6))(2I_{xx}\sin^2(q_5) + (I_{yy} - I_{zz})\cos^2(q_5)\cos^2(q_6))}$$

$$g_{1012} = \frac{2\sin q_5((I_{yy} - I_{zz})\cos q_6^2 + I_{zz})}{(I_{zz} + (I_{yy} - I_{zz})\cos^2(q_6))(2I_{xx}\sin^2(q_5) + (I_{yy} - I_{zz})\cos^2(q_5)\cos^2(q_6))}$$

$$g_{1110} = \frac{(I_{yy} + I_{zz})\cos(q_5)\sin(2q_6)}{2I_{xx}\sin^2(q_5)(I_{yy}\cos^2(q_6) + I_{zz}\sin^2(q_6)) + \cos^2(q_5)\cos^2(q_6)(I_{yy}^2(\sin^2(q_6) + 1) + I_{yy}I_{zz}(6\sin^2(q_6) - 1)) + I_{zz}^2\sin^2(q_6)}$$

$$g_{1111} = \frac{2I_{xx}\sin^2(q_5) + (I_{yy} - I_{zz})\cos^2(q_5)\cos^2(q_6)}{2I_{xx}\sin^2(q_5)(I_{yy}\cos^2(q_6) + I_{zz}\sin^2(q_6)) + \cos^2(q_5)\cos^2(q_6)(I_{yy}^2(\sin^2(q_6) + 1) + I_{yy}I_{zz}(6\sin^2(q_6) - 1)) + I_{zz}^2\sin^2(q_6)}$$

$$g_{1112} = \frac{\frac{1}{2}(I_{yy} + I_{zz})\sin(2q_5)\sin(2q_6)}{2I_{xx}\sin^2(q_5)(I_{yy}\cos^2(q_6) + I_{zz}\sin^2(q_6)) + \cos^2(q_5)\cos^2(q_6)(I_{yy}^2(\sin^2(q_6) + 1) + I_{yy}I_{zz}(6\sin^2(q_6) - 1)) + I_{zz}^2\sin^2(q_6)}$$

$$g_{1210} = \frac{2\sin(q_5)((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})}{I_{xx}(2I_{xx}((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\sin^2(q_5) + (I_{yy} - I_{zz})((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\cos^2(q_5)\cos^2(q_6) + 2(I_{yy} + I_{zz})^2\sin^2(q_6)\cos^2(q_5)\cos^2(q_6))}$$

$$g_{1211} = \frac{\frac{1}{2}(I_{yy} + I_{zz})\sin(2q_5)\sin(2q_6)}{I_{xx}(2I_{xx}((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\sin^2(q_5) + (I_{yy} - I_{zz})((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\cos^2(q_5)\cos^2(q_6) + 2(I_{yy} + I_{zz})^2\sin^2(q_6)\cos^2(q_5)\cos^2(q_6))}$$

$$g_{1212} = \frac{(I_{yy} - I_{zz})^2\cos^2(q_5)\cos^4(q_6) + 2(I_{yy} + I_{zz})^2\sin^2(q_6)\cos^2(q_5)\cos^2(q_6) + I_{zz}\cos^2(q_5)\cos^2(q_6)(I_{yy} - I_{zz})}{I_{xx}(2I_{xx}((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\sin^2(q_5) + (I_{yy} - I_{zz})((I_{yy} - I_{zz})\cos^2(q_6) + I_{zz})\cos^2(q_5)\cos^2(q_6) + 2(I_{yy} + I_{zz})^2\sin^2(q_6)\cos^2(q_5)\cos^2(q_6))}$$