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# Modeling of Quadrotor Helicopter Dynamics

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**Abstract**: Quadrotor Helicopter or simply quadrotor is rotorcraft that has four lift-generating propellers. Two of the propellers spin clock wise and the other two counter-clockwise. Control of the machine can be achieved by varying relative speed of the propellers. Quadrotor concept is not new, however the modern quadrotors are mostly unmanned. Advancement in miniaturized IMU technology, availability of high speed brushless motors and high power to weight ratio Li-Polymer battery technology, quadrotors can now be successfully designed and fabricated. This paper proposes a mathematical model of quadrotor dynamics. A simplified approach is adopted using momentum theory, where the gyroscopic effect and air friction on frame of machine has been neglected, resulting in a simplified model, which is useful for designing a controller to stabilize the machine in hover state. Proposed model is non-linear since the rotor dynamics are a function of square of motor inputs.

**Keywords:** quadrotor; dynamics; propeller; moment of inertia; thrust; roll; pitch

## 1. INTRODUCTION

A Quadrotor Helicopter, also called Quadrotor, is a helicopter with four lift-generating propellers mounted on motors. Two of the motors generate thrust by spinning their propellers clockwise and other two counter-clockwise. Control of the machine can be achieved by varying the relative speed of the propellers. Quadrotor concept is not new. In 1907 first quadrotor named Gyroplane No.1 was built by Breguet Brothers. In 1922, Georges de Bothezat contructed a truss structure of intersecting beams, where the propeller was located at each end of the x-shaped structure [1].

The new generation quadrotors are mostly unmanned. Due to the availability of high-speed brush-less motor, inertial measurement units based on MEMS technology and high power to weight ratio (>150W/Kg) Li-polymer batteries, unmanned quadrotors can now be designed and fabricated [2] but their control is still a challenge.

In the past few years quite a lot of research effort has been directed in this direction. Mesicopter [3] was an ambitious project, which explored the ways to fabricate centimeter-sized vehicles. Such vehicles can be used for gathering planetary atmospheric and meteorological data. The OS4 project [4] was started in March 2003, with an aim to develop devices for searching and monitoring hostile indoor environments. The X4 Flyer project of Australian National University [5], aims at developing a quadrotor for indoor and outdoor applications. The control system of the vehicle is based on classical control methodology.

## 2. CONSTRUCTION AND ASSUMPTIONS

The A quadrotor simply consists of four dc motors on which propellers are mounted. These motors are arranged at the corners of a X-shaped frame, where all the arms make an angle of 90 degrees with one another. As shown in Figure 1, two of the rotors or propellers spin in one direction and the other in the opposite direction. The motors labeled as M1 and M3 spin in the clockwise direction with velocity  and other two in the opposite direction. Each spinning propeller generates vertically upward lifting force. All the motion of machine is a consequence of this force.

The mathematical model developed in this text is based on certain basic assumptions as given below:

- Quadrotor body is rigid.
- Propellers are rigid.
- There is no air friction on quadrotor body.
- Free stream air velocity is zero.
- Drag torque td is proportional to propeller speed with D as drag constant.
- Design is symmetrical.. Please do not revise any of the current designations.

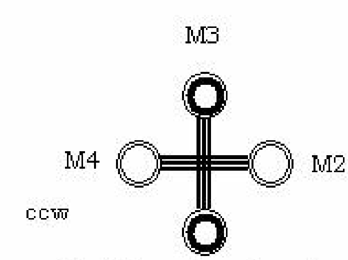


Fig 1. Top view of quadrotor.

## 3. FRAME OF REFERENCE

In order to model quadrotor dynamics a frame of reference should be defined. Let E be an earth fixed frame with mutually orthogonal axes *x, y, z* with unit vector *^i, ^j, ^k*. The center point of this frame is O. Quadrotor is placed in this frame of reference with its center of mass at O. If $l$ is the length of arms then each motor $M_1$ to $M_4$ can be located by position vectors as shown in Figure 2, and given as:

$$\vec{L}_1 = l\hat{i}$$
$$\vec{L}_2 = l\hat{j}$$
$$\vec{L}_3 = -l\hat{i}$$
$$\vec{L}_4 = -l\hat{j}$$

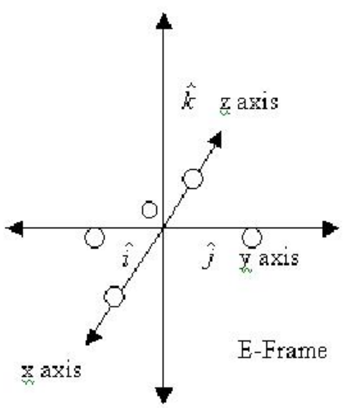


Fig 2. Frame of reference (Circles showing motor positions.)

## 4. ROLL, PITCH AND YAW ANGLE

The free body diagrams of quadrotor with reference to E are shown in Figure 3. Figure The free body diagrams of quadrotor with reference to E are shown in Figure 3. Figure 3(a) shows the yaw angular displacement which is represented by Ψ, and it is due to rotation about positive z-axis.

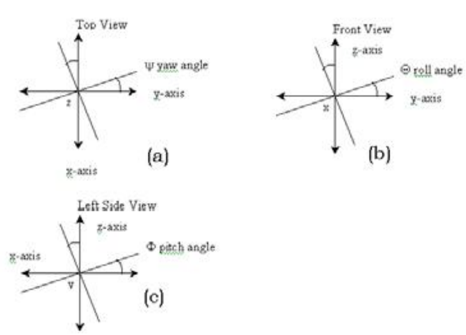


Fig 3. Roll, pitch and yaw angles.

Figure 3(b) shows the front view of quadrotor along with an axis that is perpendicular to the plane formed by the frame of machine. This angular displacement is due to the rotation about positive x-axis, and is represented by Θ. Figure 3(c) represents the pitch angular displacement, which is about positive y-axis and is represented by Φ.

## 5. FORCES ACTING ON QUADROTOR

Lifting forces generated by the spinning propeller and the weight, are responsible for all the motion of body, as the external effects such as air friction, wind pressure etc. have been neglected. Consider a detailed free body diagram of quadrotor with reference to E, as shown in Figure 4.

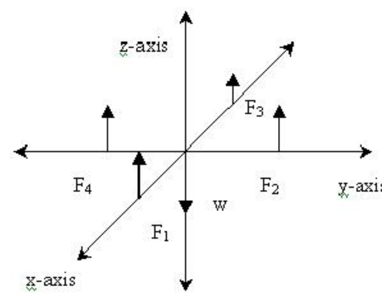


Fig 4. Detailed free body diagram of quadrotor.

All the forces shown in the figure by $F_i$ are located from the origin of E by position vectors $\vec{L}_i$ as mention in the previous section. The design of quadrotor motors and propellers is assumed to be such that these thrust forces always act perpendicular to the plane of propellers and therefore plane of quadrotor body. The center of mass of machine is located at the origin of the *E*-frame. And the weight force acts at this point. Weight force is always along negative z-axis. Propellers mounted on motors $M_1$ and $M_3$ spin in the clockwise direction at speeds $\Omega_1$ and $\Omega_3$ respectively and those mounted on motors $M_2$ and $M_4$ spin in counter-clockwise direction at speeds $\Omega_2$ and $\Omega_3$ respectively.

The imbalance of the forces $F_i$, where i = 1, 3 or 2, 4 results in moment, along a direction perpendicular to the plane formed by the force $F_i$ and the vector $\vec{L}_i$. This torque is responsible for the rotation of machine along x-axis and y-axis. The rotation about z-axis is due to imbalance of clockwise and counter-clockwise torques.

## 6. MOMENTS OF INERTIA OF QUADROTOR BODY

Determination of moment of inertia can be divided into two parts, firstly about *x* and *y*-axis and secondly about *z*-axis. It is assumed that if the machine is rolling, pitching and spinning etc. then that does not change the moment of inertia about a specific axis.

### 6.1 Moments of inertia along x and y-axis

In derivation of moment of inertia along $x$ (and $y$) axis following assumptions are made:

- Motors $M_1$ and $M_3$ are cylindrical in shape with radius $\rho$, height $h$ and mass $m$.
- Central hub or body of quadrotor is also cylindrical with radius $R$, height $H$ and mass $m_o$.

Since moment of inertia about $x$-axis is to be determined, therefore consider only the rolling motion.

The moment of inertia about $x$-axis consists of two parts:

- Due to motion of motors $M_2$ and $M_4$ about x-axis with radius of rotation $l$.
- Due to $M_1$, $M_3$ and central hub about x-axis.
-

The moment of inertia of a cylinder about an axis perpendicular to its body, as specified in [6], is

$$I = \frac{mass(radius)^2}{4} + \frac{mass(height)^2}{12} \quad (6.1)$$

The moment of inertia of two identical spheres connected together by a horizontal arm, and rotating about a vertical axis, which is passing through the center of the arm and is perpendicular to it, as specified in [6], is:

$$I = 0.5.mass.armlength^2 \quad (6.2)$$

For part(a) the moment of inertia due to motors $M_2$ and $M_4$ rotating about x-axis is approximated by:

$$inertia = 2ml^2 \quad (6.3)$$

For part(b) the moment of inertia due to $M_1$, $M_3$ and central hub about x-axis is approximated by:

$$inertia = 2\left[\frac{ml^2}{4} + \frac{mh^2}{12}\right] + \frac{m_o R^2}{4} + \frac{m_o H^2}{12}$$

Therefore the total inertia due to part (a) and part (b) is:

$$I_{xx} = \frac{m\rho^2}{2} + \frac{mh^2}{6} + 2ml^2 + \frac{m_o R^2}{4} + \frac{m_o H^2}{12} \quad (6.4)$$

Which is the inertia about x-axis. By exactly similar procedure moment of inertia about y-axis can be found.

$$I_{yy} = \frac{m\rho^2}{2} + \frac{mh^2}{6} + 2ml^2 + \frac{m_o R^2}{4} + \frac{m_o H^2}{12} \quad (6.5)$$

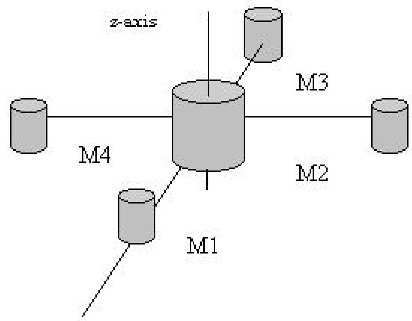


Fig 5. Moments of inertia about x, y and z-axis.

### 6.2 Momentof inertia about z-axis

Moment of inertia about z-axis, can also be calculated in two parts:

- Moment of inertia due to central hub.
- Moment of inertia due to motors $M_1$, $M_3$, $M_2$ and $M_4$.

For part (a) the moment of inertia due to central hub is:

$$inertia = \frac{m_o R^2}{2}$$

For part (b) the moment of inertia due to motors $M_1$, $M_3$, $M_2$ and $M_4$ is:

$$inertia = 4ml^2$$

Therefore the total inertia about z-axis is:

$$I_{zz} = \frac{m_o R^2}{2} + 4ml^2 \quad (6.6)$$

### 6.3 Moment of inertia about z-axis

Since the machine is assumed to be exactly symmetric therefore the products of inertia can be safely neglected.

## 7. THRUST AND VOLTAGE RELATIONSHIP

The plant (quadrotor) input is voltage Vi. And the propellers are generating thrust force, therefore a relationship between plant's input Vi and thrust force Fi is useful. This relationship is derived using momentum theory methodology.

The electrical power input into the motor as specified in [7] is :

$$P = iV \quad (7.1)$$

Suppose each motor's efficiency is η, then the mechanical power output will be:

$$P_m = \eta P = \eta i V \tag{7.2}$$

Ideally the power induced in the air when the free stream air velocity is zero and the induced velocity is $v_h$ , specified in [8] is:

$$P_h = F v_h \tag{7.3}$$

where $F$ is the thrust force of propeller. If $f$ is the figure of merit of propeller then mechanical power and the power induced in air can be related by:

$$P_h = f P_m = f \eta P = f \eta i V$$

$$P_h = f \eta i V \tag{7.4}$$

A simplified model of dc separately excited motor is shown in Figure 6, where the inductance has been neglected, as its small for dc motors [7]. The motor constants are $K_q$ and $K_e$.

$$\tau_m = K_q i, e_m = K_e \Omega \tag{7.5}$$

In (7.4) substituting (7.5) gives:

$$P_h = f \eta \left( \frac{\tau_m}{K_q} \right) V \tag{7.6}$$

The induced velocity $v_h$ is related to thrust, geometry of propeller and air density $\rho_a$ as mentioned in [9] by:

$$v_h = \sqrt{\frac{F}{2 \rho_a A}} \tag{7.7}$$

Therefore induce power in air is:

$$P_h = F \sqrt{\frac{F}{2 \rho_a A}} \tag{7.8}$$

The torque of propeller is proportional to the thrust generated by propeller [8] and constant of proportionality is $K_t$, therefore:

$$\tau_m = K_t F \tag{7.9}$$

Comparing (7.8), (7.6) and using (7.9) a relationship between thrust and voltage input to plant is obtained which is given bellow.

$$F = 2 \rho_a A \left[ \frac{f \eta K_t}{K_a} \right]^2 V^2 \tag{7.10}$$

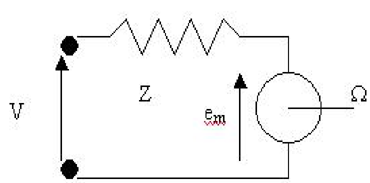


Fig. 6 Simplified schematic of dc motor.

## 8. ANGULAR VELOCITY AND VOLTAGE RELATIONSHIP

In order to relate the plant output to the plant input a sub-system is formed. This sub-system relates the plant input to the motors speed, Ω. The motor torque is given as:

$$\tau_m = \tau_d + J \dot{\Omega} \tag{8.1}$$

Using the dc motor model as specified in Figure 6, motor input voltage is:

$$V = e_m - iZ \tag{8.2}$$

Using (7.5) and (8.1) in (8.3) voltage input to subsystem is given by:

$$V = \tau \frac{Z}{K_q} + \frac{ZJ\dot{\Omega}}{K_q} + K_e \Omega \tag{8.3}$$

As specified in assumptions,

$$\tau_d = D \Omega^2 \tag{8.4}$$

Using (8.3) and (8.4) a relationship between sub-system input and output is obtained:

$$\frac{JZ\dot{\Omega}}{K_q} + K_e \Omega + \frac{ZD\Omega^2}{K_q} = V \tag{8.5}$$

## 9. ANGULAR VELOCITY AND VOLTAGE RELATIONSHIP

Under the gravity and actuator action and no other external effects the machine experiences following motions:

- Rolling about x-axis
- Pitching about y-axis
- Yaw motion about z-axis
- Vertical acceleration
- Horizontal Acceleration

Since the focus here is on the hover control of quadrotor, and the fact that horizontal motion is a consequence of rotational motion [10], therefore relations for (1) to (4) are developed.

### 9.1 Rolling motion about x-axis

As specified earlier, rolling motion is about x-axis. Rolling angular displacement as measured with respect to E-frame using right hand rule is Θ. Therefore $\ddot{\Theta}$ expresses angular acceleration about x-axis. And as found earlier moment of inertia of machine about x-axis is $I_{xx}$. Therefore the rolling torque on machine is :

$$\tau_{xx} = I_{xx}\ddot{\Theta}\hat{i} \quad (9.1)$$

Here $\hat{i}$ is the unit vector in direction of x-axis. The rolling torque is due to the thrust difference of motors $M_2$ and $M_4$. Due to the design of machine, the thrust force of each motor is assumed to be perpendicular to the plane of propeller and therefore the plane of quadrotor frame. As specified in section 3, the length of moment arm is $l$. Therefore the rolling torque is:

$$I_{xx}\ddot{\Theta}\hat{i} = l\hat{j}xF_2\hat{k} + \left(-l\hat{j}\right)xF_2\hat{k} \quad (9.2)$$

or

$$I_{xx}\ddot{\Theta}\hat{i} = l\left(F_2 - F_4\right)\hat{i} \quad (9.3)$$

Here $I_{xx}$ is as in (6.4) and unit vector $\hat{i}$ specifies the direction of torque. Dropping the unit vector gives a simpler relationship.

$$I_{xx}\ddot{\Theta} = l\left(F_2 - F_4\right) \quad (9.4)$$

Using (7.10) final equation for rolling motion is obtained as:

$$\ddot{\Theta} = \frac{2l\rho_a A}{I_{xx}}\left(\frac{f\eta K_t}{K_q}\right)^2\left[V_2^2 - V_4^2\right] \quad (9.5)$$

**9.2 Pitching motion about y-axis**

By repeating the derivation of previous section and using (6.5), with thrust forces of motors $M_1$ and $M_3$ relationship for pitching torque is obtained.

$$I_{yy}\ddot{\Phi}\hat{j} = l\left(F_3 - F_1\right)\hat{j} \quad (9.6)$$

Here the unit vector specifies the direction of torque about y-axis. It can be dropped to simplify the relationship.

$$I_{yy}\ddot{\Phi} = l\left(F_3 - F_1\right) \quad (9.7)$$

Using (7.10) final equation for pitching motion is obtained as:

$$\ddot{\Phi} = \frac{2l\rho_a A}{I_{yy}}\left(\frac{f\eta K_t}{K_q}\right)^2\left[V_3^2 - V_1^2\right] \quad (9.8)$$

**9.3 Yaw motion about z-axis**

Yaw motion is due to torque imbalance. As a propeller spins under machine torque it experiences a drag torque. This torque is exerted back on the machine as reaction torque and tends to spin the motor back. But since all the motors are connected to the frame of the machine, a net torque about z-axis is experienced. This torque is responsible for yaw motion in accordance with right hand rule.

Each motor supplies machine torque $\tau_m$, which is balanced by the drag torque so the net torque on propeller, is:

$$J\dot{\Omega} = \tau_m - \tau_d \quad (9.9)$$

$J$ is the rotational inertia of the rotor and propeller of motor. If all the clockwise and counter clockwise torques are added, sum of torques is given by:

$$\sum\tau = \tau_{m2} + \tau_{m4} - \tau_{m1} - \tau_{m3} \quad (9.10)$$

The yaw torque is along z-axis given by:

$$\tau_{zz} = I_{zz}\ddot{\Psi} \quad (9.11)$$

In (9.10) the sum of machine torques is specified and reaction to these torques is in opposite direction to it. Therefore yaw torque is:

$$I_{zz}\ddot{\Psi}\hat{k} = -\sum\tau_{zz}\hat{k} \quad (9.12)$$

Unit vector specifies the direction of yaw torque. Dropping the unit vector for simplification, yaw torque is given by:

$$I_{zz}\ddot{\Psi} = \tau_{m1} + \tau_{m3} - \tau_{m2} - \tau_{m4} \quad (9.13)$$

Substituting (9.9) in (9.13) following equation is obtained:

$$I_{zz}\ddot{\Psi} = J\left(\dot{\Omega}_1 + \dot{\Omega}_3 - \dot{\Omega}_2 - \dot{\Omega}_4\right) + \left(\tau_{d1} + \tau_{d3} - \tau_{d2} - \tau_{d4}\right) \quad (9.14)$$

Using (8.4) final equation relating the output of sub-system to the plant out puts is obtained.

$$I_{zz}\ddot{\Psi} = J(\dot{\Omega}_1 + \dot{\Omega}_3 - \dot{\Omega}_2 - \dot{\Omega}_4) + D(\Omega_1^2 + \Omega_3^2 - \Omega_2^2 - \Omega_4^2) \quad (9.15)$$

**9.4 Vertical Acceleration**

The four spin propeller, two clockwise and two counterclockwise, generate thrust force that is perpendicular to the plane of propellers. Therefore when roll and pitch angle is zero it is along z-axis. But when roll or pitch angle is not zero then a component of this force $F_{total}\cos\Theta\cos\Phi$ acts along the z-axis. Weight of the machine always acts along negative z-axis therefore this component of thrust force has to cancel it out in order to lift the machine in air.

The net force acting at the center of mass of machine is given bellow:

$$F_{net} = Ma_z = \left(F_1 + F_2 + F_3 + F_4\right)\cos\Theta\cos\Phi - Mg \quad (9.16)$$

Here M is the net mass of the machine. Using (7.10), a relationship between vertical acceleration and voltage input to system can be deduced.

$$a_z = \frac{2\rho_a A}{M}\left(\frac{f\eta K_t}{K_q}\right)^2\left(V_1^2 + V_2^2 + V_3^2 + V_4^2\right)\cos\Theta\cos\Phi - g \quad (9.17)$$

## 10. CONCLUSION

Equations (9.17), (9.8) and (9.5) relate the system inputs $V_i$ to system outputs, roll, pitch and vertical acceleration. Equation (8.5) constitutes a sub-system which relates the system input $V_i$ to the motor output $\Omega_i$ proposed mathematical model of the dynamics of quadrotor helicopter. Equation (9.15) relates the sub-system outputs the motor speeds to the system output yaw angle. All these equations describe the dynamics of quadrotor helicopter.

## 11. ACKNOWLEDGEMENT

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