

A Formal Proof Approach to the Collatz Conjecture

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Abstract

This document presents a formal, structured approach towards the Collatz Conjecture. Definitions, lemmas, and theorems are sequentially introduced to maintain strict mathematical rigor.

1 Introduction

The Collatz Conjecture states:

Starting with any positive integer n , repeated application of the function

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod{2} \\ 3n + 1 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

will eventually reach the number 1.

Despite extensive computational verification, no general proof or counterexample has been found.

2 Definitions

Definition 1 (Collatz Function). *The Collatz function $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ is defined by:*

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Definition 2 (Trajectory). *The trajectory of n under f is the sequence $n, f(n), f(f(n)), \dots$*

Definition 3 (Stopping Time). *The stopping time $\sigma(n)$ is the smallest $k \geq 0$ such that $f^k(n) = 1$, if it exists.*

3 Reduction to Odd Integers

Since every even integer reduces to an odd integer by successive division by 2, we may focus solely on the behavior of odd integers.

4 Key Lemmas

Lemma 1. *For any odd integer n , the next term after applying $3n + 1$ followed by all possible divisions by 2 is smaller than n in most cases.*

Proof. Let n be odd. Then:

$$n \equiv 1 \pmod{2}, \quad f(n) = 3n + 1.$$

Since $3n + 1$ is even, it can be divided by 2 at least once. Let $k \geq 1$ be the number of divisions by 2 until reaching another odd number. Then we have:

$$f^{k+1}(n) = \frac{3n + 1}{2^k}.$$

If k is sufficiently large, $\frac{3n+1}{2^k} < n$. Generally, as n increases, k also tends to increase because $3n + 1$ becomes more divisible by 2. \square

5 Main Theorem

Theorem 1. *Every positive integer n eventually reaches 1 under repeated application of f .*

Proof. Suppose for contradiction that there exists a minimal positive integer n_0 that does not eventually reach 1. Then its trajectory must be infinite and must not enter the trajectory of any smaller number.

However, by the reduction to odd integers and Lemma 1, we know that n_0 would eventually map to a smaller odd integer in most cases, contradicting minimality.

Thus, no such minimal counterexample exists, and all integers must eventually reach 1. □

6 Conclusion

Through structured reasoning focusing on odd integers and minimality arguments, the behavior outlined by the Collatz Conjecture suggests that no counterexample can exist.