COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING, Vol. 9, 365–374 (1993)

COMPLETED RICHARDSON EXTRAPOLATION

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SUMMARY

The Richardson extrapolation method, which produces a 4th-order-accurate solution on a subgrid by combining 2nd-order solutions on the fine grid and the subgrid, is 'completed' – in the sense that a higher-order-accurate solution is produced on all the fine grid points.

INTRODUCTION

In his classic paper in 1910, Richardson[1] presented a method for obtaining 4th-order-accurate solutions. The method, known variously as Richardson extrapolation, extrapolation to the limit, deferred approach to the limit, or iterated extrapolation takes separate 2nd-order solutions on a fine grid and on the subgrid formed of alternate points, and combines them to obtain a 4th-order solution on the subgrid. It is also the basis of Romberg integration.[3]

The usual assumptions of smoothness apply, as well as the assumption (or perhaps presumption) common to finite-difference methods that the local error is indicative of global error. The method must be used with considerable caution, since it involves additional assumptions of monotone truncation error convergence in the mesh spacing h (which may not be valid for coarse grids) and since it magnifies machine round-off errors and incomplete iteration errors.[2,4] In spite of these caveats, the method is extremely convenient to use compared to forming and solving direct 4th-order discretizations, which involve more complicated stencils, wider-bandwidth matrices, special considerations for near-boundary points and non-Dirichlet boundary conditions, additional stability analyses, etc., especially in non-orthogonal co-ordinates which generate cross-derivative terms and generally complicated equations. Such an application was given in Reference 5 by the first author. The method is in fact oblivious to the equations being discretized and to the dimensionality of the problem, and can easily be applied as a postprocessor[5] to solutions on two grids with no reference to the codes, algorithms or governing equations which produced the solutions, as long as the original solutions are indeed 2nd-order-accurate. The difference between the 2nd-order solution and the extrapolated 4th-order solution is itself a useful diagnostic tool, obviously being a 2nd-order-accurate error estimator (although it does not provide a true *bound* on the error except possibly for certain trivial problems). It was used very carefully, with an experimental determination rather than an assumption of the *local* order of convergence, by de Vahl Davis[6] in his classic benchmark study of a model free convection problem. Also, it can be applied not only to point-by-point solution values, but to solution functionals such as drag coefficient, global heat

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0748–8025/93/050365–10$10.00

Received February 1990
Revised June 1992

transfer, etc.; for example, see References 6 and 7. Blottner[8] has used the same procedure to estimate effects of 4th-order damping.

A disadvantage of Richardson extrapolation is that it is incomplete, in the sense that it only provides the 4th-order solution on a subgrid. For example, in Reference 5 the first author obtained a sequence of 2nd-order two-dimensional solutions in grids of cell size 10×10, 20×20, 40×40 and 80×80, but could obtain the 4th-order solution by Richardson extrapolation only on the 40×40 grid. (It is also theoretically possible to continue the process, obtaining a 6th-order solution on the 20×20 grid and an 8th-order solution on the 10×10 subgrid, as done in Romberg integration,[3] but we are sceptical of its practicality in multidimensional problems.)

This paper describes a method by which Richardson extrapolation is 'completed', giving a higher-order solution on the entire fine grid rather than just a subgrid. The extension is very simple, and it would not surprise the authors if it had been used by other workers, but we have not seen it published or heard it discussed, in spite of a long-standing interest in the subject.

THE METHOD

If the 4th-order solution on the coarse subgrid were interpolated by simple two-point averaging on to the skipped fine-grid points, the interpolated solution would be only 2nd-order-accurate. Higher-order interpolation can be used, but this causes inconvenience near boundaries (as noted above in relation to the use of direct 4th-order stencils) and real problems in multidimensions. (Also, note that one can always interpolate a coarse-grid solution to consistent order on to a fine grid, but this is not what one means when one claims to have a fine-grid solution; a 10×10 grid second-order solution, when interpolated by second-order interpolation formulas on to a 100×100 grid, is in some sense a second-order solution, but it is second-order in $h = 1/10$, not $h = 1/100$. This is not comparable to obtaining a second-order solution of the discretized partial differential equation on a 100×100 grid! Otherwise, why would one ever do fine-grid solutions?)

The process advocated here is to interpolate by simple two-point averaging, *not* the 4th-order *solution*, but rather the *correction* between the 2nd-order solution and the 4th-order solution. We easily demonstrate that the result is higher-order-accurate for the entire solution on the fine grid. Also, it requires no special treatment for near-boundary points, and involves no additional loss of accuracy nor significant computation time in multidimensions.

Consider the fine grid $i = 1, 2, 3 \ldots$ on which we have obtained a 2nd-order solution. We also have a separate 2nd-order solution on the subgrid of odd points $i = 1, 3, 5, \ldots$, etc. (By 'separate' solution, we mean a solution obtained by discretization over $2h$, not simply the fine-grid solution injected into the subgrid.) By applying Richardson extrapolation, we also have a 4th-order solution on the subgrid of odd points $i = 1, 3, 5, \ldots$, etc. We want to obtain a 4th-order solution on the fine-grid points which were skipped in the Richardson extrapolation process, i.e. the subgrid of even points $i = 2, 4, 6, \ldots$, etc.

Let U_i = the exact (continuum) solution at node i, let F_{2i} = the fine-grid 2nd-order solution obtained by centred differences, and let S_{2i} = the subgrid 2nd-order solution. The extrapolated 4th-order solution F_{4i} is obtained on the subgrid $i = 1, 3, 5, \ldots$ by Richardson extrapolation as

$$F_{4i} = 4/3 F_{2i} - 1/3 S_{2i} \text{ for } i = 1, 3, 5 \ldots \tag{1}$$

(The Richardson extrapolation procedure can be more general than this situation of the subgrid mesh spacing being twice the fine-grid mesh spacing,[1,2] but this is the most convenient, accurate, and commonly used arrangement.) We conveniently express this extrapolation in terms of C_i, the correction from the 2nd- to the 4th-order solution, as

$$F_{4i} = F_{2i} + C_i \text{ for } i \text{ odd} \tag{2}$$

where

$$C_i = 1/3(F_{2i} - S_{2i}) \text{ for } i \text{ odd} \tag{3}$$

(This C_i is a 2nd-order-accurate error estimator.)

By definition of (global) solution accuracy:

$$U_i = F_{2i} + A_i h^2 + \mathrm{O}(h^{3+m}) \tag{4}$$

$$U_{i+1} = F_{2i+1} + A_{i+1} h^2 + \mathrm{O}(h^{3+m}) \tag{5}$$

$$U_{i-1} = F_{2i-1} + A_{i-1} h^2 + \mathrm{O}(h^{3+m}) \tag{6}$$

where the As are the coefficients of the leading error terms, which vary spatially but become independent of h as $h \to 0$. The term $m = 1$ if centred differences have been used throughout (due to cancellation of alternate terms in the Taylor series expansion), but $m = 0$ if any one-sided 2nd-order expression has been used. For smooth solutions (already assumed when using Richardson extrapolation), we have

$$A_{i+1} = 1/2(A_i + A_{i+2}) + \mathrm{O}(h^2),\ i+1 \text{ even} \tag{7}$$

by simple two-point interpolation. (Increasing the order of this interpolation will not improve the order of the overall method, which will be limited by the 2nd error terms of $\mathrm{O}(h^{3+m})$ above.)

Evaluating A_i for i odd from (4) gives

$$A_i = 1/h^2[U_i - F_{2i} + \mathrm{O}(h^{3+m})],\ i \text{ odd} \tag{8}$$

Using the 4th-order-accurate solution,

$$U_i = F_{4i} + \mathrm{O}(h^4),\ i \text{ odd} \tag{9}$$

Substituting (9) into (8), we obtain

$$A_i = 1/h^2[F_{4i} - F_{2i} + \mathrm{O}(h^{3+m})],\ i \text{ odd} \tag{10}$$

Similarly,

$$A_{i+2} = 1/h^2[F_{4i+2} - F_{2i+2} + \mathrm{O}(h^{3+m})],\ i \text{ odd} \tag{11}$$

Using (10) and (11) in (7) gives

$$A_{i+1} = 1/(2h^2)[F_{4i} - F_{2i} + F_{4i+2} - F_{2i+2} + \mathrm{O}(h^{3+m})] \tag{12}$$

Substituting (12) into (5) gives

$$U_{i+1} = F_{2i+1} + 1/2[F_{4i} - F_{2i} + F_{4i+2} - F_{2i+2}] + \mathrm{O}(h^{3+m}) \tag{13}$$

This defines the method, but for clarity we can write the correction C_i of (2) and (3) from the 2nd to $(3+m)$th-order solutions,

$$C_i = F_{4i} - F_{2i},\ i \text{ odd} \tag{14}$$

This part, (14), is the original Richardson extrapolation. Then at the even fine-grid points $2, 4, 6, \ldots,$ not covered by the original Richardson extrapolation, we complete the extrapolation from 2nd to $(3+m)$th-order solutions by

$$F_{4i+1} = F_{2i+1} + C_{i+1}, i+1 \text{ even} \tag{15}$$

where

$$C_{i+1} = 1/2(C_i + C_{i+2}), \ i+1 \text{ even} \tag{16}$$

The 2nd error term of the 2nd-order solution, $O(h^{3+m})$ in (4), will limit the accuracy of the completed Richardson extrapolation; for centred differences with constant grid spacing, $m=1$, and the completed Richardson extrapolation is 4th-order-accurate. However, since another interpolation is involved, i.e. equation (16), it is expected that the *size* of the error on the even fine-grid points will be larger, though still 4th order.

TESTS

The original Richardson extrapolation is sensitive to round-off error and only works when the convergence rate is in the asymptotic range, i.e. when the grid is small enough. Not surprisingly, these restrictions apply even more stringently to the completed Richardson extrapolations. In original tests by the first author, 4th-order accuracy was not demonstrated even with $m=1$ (centred differences) but rather the method appeared to be 3rd-order. This later proved to be due to round-off error and lack of asymptotic error behaviour. The following results were obtained on a microVAX II computer using double precision.

The prototype elliptic test problem is the 1-D Poisson equation,

$$U''(x) = -\pi^2 \sin(\pi x), \ U(0) = U(1) = 1 \tag{17}$$

The exact solution is

$$U = \sin(\pi x) \tag{18}$$

The convergence results are displayed in Table I. The value $E_2 = \text{maximum error}/h^2$, and $E_4 = \text{maximum error}/h^4$. For 2nd (4th)-order convergence, E_2 (E_4) will become roughly constant as the grid size asymptotically approaches zero. (Similar results are obtained for local errors; the use of the maximum error norm is more demanding of the method.) The results for C_4 are the usual Richardson extrapolation, and show the well known 4th-order convergence on the coarse grid. The results for F_4 are the completed Richardson extrapolation. Both are indeed 4th-order-accurate. The (new) F_4 results have a much larger coefficient than the (original) C_4 results, as expected, owing to the additional interpolation involved. That is, the completed Richardson extrapolations (on the even fine-grid points) are not as accurate as the original Richardson extrapolations (on the odd fine-grid points). However, both are 4th-order-accurate, and the (new) F_4 results are much more accurate than the 2nd-order F_2 results.

The same pattern holds for the other test cases. Table II shows the convergence results for the 1-D Poisson equation with an exponential forcing term,

$$U''(x) = -x(3+x)e^x, \ U(0) = 1, U(1) = 0 \tag{19}$$

which has the solution

$$U(x) = x(1-x)e^x \tag{20}$$

The method readily extends to multidimensions (see *Extensions* Section below). Table III shows the convergence results for the 2D elliptic problem on the unit square,

$$\nabla^2 U = (1 - \pi^2/4)\sin((\pi/2)x)e^y \tag{21}$$

Table I. Convergence results for 1D Poisson equation with sine forcing term (equation (17))

N	Max error	E_2
	2nd-order coarse mesh (C_2)	
4	0·23370055	3·73920
8	0·05302929	3·39387
16	0·01295075	3·31539
32	0·00321896	3·29622
64	0·00080358	3·29145
128	0·00020082	3·29026
	2nd-order fine mesh (F_2)	
N	Max error	E_2
4	0·05302929	0·84847
8	0·01295075	0·82885
16	0·00321896	0·82405
32	0·00080358	0·82286
64	0·00020082	0·82257
128	0·00005020	0·82249
	4th-order coarse mesh (C_4)	
N	Max error	E_4
4	0·00719447	1·84178
8	0·00040877	1·67431
16	0·00002496	1·63597
32	0·00000155	1·62659
64	0·00000010	1·62426
128	0·00000001	1·62368
	4th-order fine mesh (F_4)	
N	Max error	E_4
4	0·00738549	1·89069
8	0·00056187	2·30143
16	0·00003665	2·40190
32	0·00000231	2·42690
64	0·00000015	2·43315
128	0·00000001	2·43470

The constancy of E_4 as the grid is refined indicates 4th-order accuracy. Coarse-mesh (C_4) results are for usual Richardson extrapolation; fine-mesh (F_4) results are for completed Richardson extrapolation.

Table II. Convergence results for 1D Poisson equation with exp. forcing term (equation (19))

N	Max error	E_2
	2nd-order coarse mesh (F_2)	
4	0·05152254	0·82436
8	0·01318477	0·84383
16	0·00333075	0·85267
32	0·00084567	0·86597
64	0·00021150	0·86628
128	0·00005288	0·86636
	2nd-order fine mesh (F_2)	
N	Max error	E_2
4	0·01318477	0·21096
8	0·00333075	0·21317
16	0·00084567	0·21649
32	0·00021150	0·21657
64	0·00005288	0·21659
128	0·00001322	0·21662
	4th-order coarse mesh (C_4)	
N	Max-error	E_4
4	0·00040551	0·10381
8	0·00002579	0·10562
16	0·00000162	0·10608
32	0·00000010	0·10760
64	0·00000001	0·10763
128	0·00000000	0·10764
	4th-order fine mesh (F_4)	
N	Max error	E_4
4	0·00498564	1·27632
8	0·00037469	1·53472
16	0·00002550	1·67120
32	0·00000166	1·74115
64	0·00000011	1·77653
128	0·00000001	1·79433

The constancy of E_4 as the grid is refined indicates 4th-order accuracy. Coarse-mesh (C_4) results are for usual Richardson extrapolation; fine-mesh (F_4) results are for completed Richardson extrapolation.

with boundary conditions

$$U(0, y) = 0,\ U(1, y) = e^y \tag{22a}$$

$$U(x, 0) = \sin((\pi/2)x),\ U(x, 1) = e\ \sin((\pi/2)x) \tag{22b}$$

which has the solution

$$U(x, y) = \sin((\pi/2)x)e^y \tag{23}$$

Table III. Convergence results for 2D Poisson equation with exp × sine forcing term (equations (21)–(22))

	2nd-order coarse mesh (C_2)	
N	Max error	E_2
4	0·01058443	0·16935
8	0·00295224	0·18894
16	0·00080010	0·20483
32	0·00020317	0·20805
64	0·00005107	0·20920
	2nd-order fine mesh (F_2)	
N	Max error	E_2
4	0·00295224	0·04724
8	0·00080010	0·05121
16	0·00020317	0·05201
32	0·00005108	0·05231
64	0·00001275	0·05223
	4th-order coarse mesh (C_4)	
N	Max error	E_4
4	0·00108699	0·27827
8	0·00009642	0·39493
16	0·00000690	0·45188
32	0·00000045	0·47316
64	0·00000003	0·49328
	4th-order fine mesh (F_4)	
N	Max error	E_4
4	0·00185288	0·47434
8	0·00016135	0·66089
16	0·00001265	0·82908
32	0·00000087	0·90943
64	0·00000006	0·92867

The constancy of E_4 as the grid is refined indicates 4th-order accuracy. Coarse-mesh (C_4) results are for usual Richardson extrapolation; fine-mesh (F_4) results are for completed Richardson extrapolation.

Table IVa. Convergence results for 1D advection–diffusion equation (24) with $R = 1$

	2nd-order coarse mesh (C_2)	
N	Max error	E_2
4	0·00254066880	0·040651
8	0·00061759188	0·039526
16	0·00015638354	0·040034
32	0·00003928711	0·040230
64	0·00000982752	0·040254
128	0·00000245794	0·040271
256	0·00000061447	0·040270
512	0·00000015363	0·040272
1024	0·00000003841	0·040272
	2nd-order fine mesh (F_2)	
N	Max error	E_2
4	0·00061759188	0·009881
8	0·00015638354	0·010009
16	0·00003928711	0·010058
32	0·00000982752	0·010063
64	0·00000245794	0·010068
128	0·00000061447	0·010067
256	0·00000015363	0·010068
512	0·00000003841	0·010068
1024	0·00000000960	0·010068
	4th-order coarse mesh (C_4)	
N	Max error	E_4
4	0·00002343377	0·005999
8	0·00000140457	0·005753
16	0·00000009079	0·005950
32	0·00000000566	0·005934
64	0·00000000035	0·005940
128	0·00000000002	0·005939
256	0·00000000000	0·005806
512	0·00000000000	0·002441
1024	0·00000000000	0·487701
	4th-order fine mesh (F_4)	
N	Max error	E_4
4	0·00022824765	0·058431
8	0·00001781814	0·072983
16	0·00000125631	0·082333
32	0·00000008359	0·087655
64	0·00000000539	0·090497
128	0·00000000034	0·091967
256	0·00000000002	0·092717
512	0·00000000000	0·093193
1024	0·00000000000	0·537369

The constancy of E_4 as the grid is refined indicates 4th-order accuracy. Coarse-mesh (C_4) results are for usual Richardson extrapolation; fine-mesh (F_4) results are for completed Richardson extrapolation.

Table IVb. Convergence results for 1D advection–diffusion equation (24) with $R = 16$

N	Max error	E_2
	2nd-order coarse mesh (C_2)	
4	1·50033535013	24·005366
8	0·36831552841	23·572194
16	0·13533518593	34·645808
32	0·03454605219	35·375157
64	0·00787942097	32·274108
128	0·00192912318	31·606754
256	0·00047982091	31·445543
512	0·00011980277	31·405578
1024	0·00002994118	31·395607
	2nd-order fine mesh (F_2)	
N	Max error	E_2
4	0·36831552841	5·893048
8	0·13533518593	8·661452
16	0·03454605219	8·843789
32	0·00787942097	8·068527
64	0·00192912318	7·901689
128	0·00047982091	7·861386
256	0·00011980277	7·851394
512	0·00002994118	7·848902
1024	0·00000748470	7·848279
	4th-order coarse mesh (C_4)	
N	Max error	E_4
4	0·63299798320	162·047484
8	0·09835113825	402·846262
16	0·01281293468	839·708487
32	0·00100945610	1058·491441
64	0·00005430942	911·160837
128	0·00000346486	930·091560
256	0·00000021462	921·783898
512	0·00000001338	919·728825
1024	0·00000000084	919·202469
	4th-order fine mesh (F_4)	
N	Max Error	E_4
4	0·63299798320	162·047484
8	0·09835113825	402·846262
16	0·01602753711	1050·380672
32	0·00208620873	2187·548407
64	0·00019995963	3354·765899
128	0·00001580555	4242·769819
256	0·00000111805	4802·000014
512	0·00000007447	5117·806417
1024	0·00000000481	5285·915802

The constancy of E_4 as the grid is refined indicates 4th-order accuracy. Coarse mesh (C_4) results are for usual Richardson extrapolation; fine mesh (F_4) results are for completed Richardson extrapolation.

Table IV. Convergence results for 1D advection–diffusion equation (24) with $R = 100$

N	Max error	E_2
	2nd-order coarse mesh (C_2)	
4	12·00000000000	192·00
8	2·91156597776	186·34
16	0·86516351087	221·48
32	0·51711924504	529·53
64	0·26344912875	1079·09
128	0·08680436961	1422·20
256	0·01963111458	1286·54
512	0·00468893880	1229·18
1024	0·00117403293	1231·06
	2nd-order fine mesh (F_2)	
N	Max error	E_2
4	2·91156597776	46·59
8	0·86516351087	55·37
16	0·51711924504	132·38
32	0·26344912875	269·77
64	0·08680436961	355·55
128	0·01963111458	321·64
256	0·00468893880	307·29
512	0·00117403293	307·77
1024	0·00029258786	306·80
	4th-order coarse mesh (C_4)	
N	Max error	E_4
4	4·56067832538	1167·53
8	1·61781954320	6626·59
16	0·64220021538	42087·23
32	0·23404661458	245415·66
64	0·04934254949	827830·61
128	0·00548121959	1471353·68
256	0·00036823584	1581560·89
512	0·00002098026	1441752·64
1024	0·00000128280	1410449·00
	4th-order fine mesh (F_4)	
N	Max error	E_4
4	4·56067832538	1167·53
8	1·61781954320	6626·59
16	0·64220021538	42087·23
32	0·23404661458	245415·66
64	0·04934254949	827830·61
128	0·00809541593	2173096·67
256	0·00093131298	3999958·79
512	0·00008238890	5661722·35
1024	0·00000621227	6830460·85

The constancy of E_4 as the grid is refined indicates 4th-order accuracy. Coarse-mesh (C_4) results are for usual Richardson extrapolation; fine-mesh (F_4) results are for completed Richardson extrapolation.

The addition of first-order advection terms affects the convergence for coarse grids. Table IV shows the results for the 1D steady linear advection diffusion equation,

$$U'' - RU' = 0,\ U(0) = 1,\ U(1) = 0 \tag{24}$$

The exact solution is

$$U(x) = (\mathrm{e}^{-Rx} - \mathrm{e}^{-R})/(1 - \mathrm{e}^{R}) \tag{25}$$

Error behaviour due to advection terms

The linear advection–diffusion equation appears to display only 3rd-order convergence until the grid is sufficiently refined. This behaviour is explained by the following analysis.

The kth derivative of $U(x)$ is

$$U^{(k)}(x) = (-1)^k R^k\, \mathrm{e}^{-Rx}/(1 - \mathrm{e}^{-R}) \tag{26}$$

In particular,

$$U^{(k)}/U^{(2)} = (-1)^k R^{k-2} \tag{27}$$

The Taylor Series for U reads:

$$\begin{aligned} U(x+h) &= \sum_{k=0}^{\infty} \frac{h^k}{k!}\, U^{(k)}(x) \\ &= [U(x) + hU'(x)] + U^{(2)}(x)\left[\frac{h^2}{2!} - \frac{h^3}{3!}\, R + \frac{h^4}{4!}\, R^2 + \cdots\right] \end{aligned} \tag{28}$$

The significance of this expression is that the higher-order error terms contain not only the grid spacing h, but also the continuum parameter R. Now, $F(x) = U(x) + hU'(x)$ acts like a 2nd-order approximation to $U(x+h)$ only if $h^2/2 \gg h^3 R/6$, etc. This translates into the requirement that $N \gg R/3$.* Therefore, if R is large, a large N (or small h) is needed to make F behave in a 2nd-order-accurate manner. If N is not sufficiently large, C_2 will not behave as 2nd-order, so F_4 cannot behave as 4th-order.

This type of behaviour is seen in Table IV for $R = 1$, 16, 100. 4th-order accuracy is not apparent for N small because the solution is not yet in the asymptotic range, but, as N becomes sufficiently large, 4th-order accuracy is approached. The effect is most noticeable for larger values of R. For $R = 16$, the asymptotically constant F_4 changes by only 2·9 per cent from $N = 256$ to 512, indicating that the problem is now indeed in the asymptotic range. (Note also that, for $R = 1$, the 4th-order results are so accurate at $N = 1024$ that the E_4 calculation becomes polluted by even double-precision round-off errors and is therefore meaningless.)

EXTENSIONS

With the notation of (14)–(16), the extension to arbitrary dimensions is also clear. At the two-dimensional fine-grid points $(i+1, j)$ for $i + 1 = 2, 4, 6 \ldots$ and $j = 1, 3, 5 \ldots$, the analogue of

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* The requirement for 2nd-order accuracy of the usual centred difference approximations to derivatives gives the same ordered estimate, and increases only slightly with higher derivatives. For the kth derivative, the requirement is $N \gg R/\sqrt{[(k+4)/(k+3)]}$. This requirement is clearly reminiscent of the well known fact that centred differences do not behave $O(h^2)$ for large cell Reynolds (or Peclet) numbers $R_c = R \times h$, and $R_c \leqslant 2$ is required for non-oscillatory solutions. However, the present analysis shows the requirement $N \gg R/3$ or $R_c \ll 3$ for 2nd-order accuracy.

(15) and (16) applies directly:

$$F_{4i+1,j} = F_{2i+1,j} + C_{i+1,j} \tag{29}$$

$$C_{i+1,j} = 1/2(C_{i,j} + C_{i+2,j}), \tag{30}$$

$$i+1 = 2,4,6\ldots \quad \text{and} \quad j = 1,3,5\ldots$$

At fine-grid points $(i, j+1)$ for $i = 1,2,3\ldots$ and $j+1 = 2,4,6\ldots$, we have

$$C_{i,j+1} = 1/2(C_{i,j} + C_{i,j+2}), \tag{31}$$

$$i = 1,2,3\ldots \text{ and } j+1 = 2,4,6\ldots$$

At the centre points,

$$C_{i+1,j+1} = 1/4\ (C_{i,j} + C_{i+2,j} + C_{i,j+2} + C_{i+2,j+2}), \tag{32}$$

$$i+1 = 2,4,6\ldots \text{ and } j+1 = 2,4,6\ldots$$

In 3D, consider the cube defined by the 27 points from (i, j, k) to $(i+2, j+2, k+2)$, where i, j, k are all odd. The eight corner points, e.g. (i, j, k), $(i+2, j, k+2)$, etc. are common to both the fine and coarse grids, so the original Richardson extrapolation formulas (2) and (3) apply. Then the 1D formulas (15) and (16) apply at the 12 mid-points of edges, e.g. $(i+1, j, k)$, $(i+2, j+1, k)$, $(i, j, k+1)$, etc. The 2D formulas (29)–(32) apply at the six mid-points of faces, e.g. $(i+1, j+1, k)$, $(i+2, j+1, k+1)$, etc. The remaining (27th) point is evaluated from

$$F_{4p} = F_{2p} + C_p,\ p = (i+1, j+1, k+1) \tag{33}$$

where

$$C_{i+1,j+1,k+1} = 1/8(C_{i,j,k} + C_{i+2,j,k} + C_{i,j+2,k} + C_{i+2,j+2,k} + C_{i,j,k+2} + C_{i+2,j,k+2} + C_{i,j+2,k+2} + C_{i+2,i+2,k+2}) \tag{34}$$

Note that, although the present notation is suggestive of finite-difference solutions, the process is equally applicable to finite-element or other discretizations, provided that the global errors are expressible in integer powers of h, and that the 'subgrid' solution is defined.

For non-orthogonal boundary fitted co-ordinates as in Reference 5, cross-derivative terms are also evaluated by centred differences, so no new problems arise. If the transformation metrics are evaluated numerically (as they should be) they can be evaluated separately for the fine and coarse grids if convenient. However, it makes more sense, and is more accurate, to evaluate metrics numerically only on the fine grid, and to inject these values into the coarse-grid calculation.

If the grid is produced by elliptic (or other) grid-generation equations, there is no value in producing two grids with the fine and coarse spacings. Only the fine grid need be generated, and the coarse grid formed by using every other point. Also, if solution-adaptive grid generation is used, this clearly should be done only for the fine grid.

ACKNOWLEDGEMENTS

This work was partially supported by contracts from the Air Force Weapons Laboratory (Dr. N. L. Rapagnani, Technical Monitor) and from Sandia National Laboratories–Albuquerque (Dr M. G. Marietta, Contract Monitor).

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