Analytical Solutions for Advection and Advection-Diffusion Equations with Spatially Variable Coefficients

By C. Zoppou,[1] Member, ASCE, and J. H. Knight[2]

Abstract: Analytical solutions are provided for the one-dimensional transport of a pollutant in an open channel with steady unpolluted lateral inflow uniformly distributed over its whole length. This practical problem can be described approximately by spatially variable coefficient advection and advection-diffusion equations with the velocity proportional to distance, and the diffusion coefficient proportional to the square of the velocity. Using a simple transformation, the governing equations can be transformed into constant coefficient problems that have known analytical solutions for general initial and boundary conditions. Analytical solutions to the spatially variable coefficient advection and advection-diffusion equations, written in conservative and nonconservative forms, are presented. The analytical solutions are simple to evaluate and can be used to validate models for solving the advection and advection-diffusion equations with spatially variable coefficients. The analytical solutions show that nonconservative forms of the equations can yield exact solutions that are not consistent with the physical problem.

INTRODUCTION

The transport of solutes by water takes place in a large variety of environmental, agricultural, and industrial processes. Accurate prediction of the transport of these pollutants is crucial to the effective management of these processes. The one-dimensional transport of a pollutant in a channel of uniform cross section that is augmented by steady, unpolluted lateral inflow uniformly distributed over its whole length in one such problem that is considered here. This practical problem can be described approximately by spatially variable coefficient advection and advection-diffusion equations.

Analytical techniques for the solution of the advection and advection-diffusion equations are generally restricted to simple problems with constant coefficients. There are very few analytical solutions to the one-dimensional advection and advection-diffusion equations with variable velocity and diffusion coefficients. Analytical solutions have been provided by Barry and Sposito (1989) and by Basha and El-Habel (1993) for time-dependent coefficients, and by Philip (1994) for variable diffusion coefficients. Many of these analytical solutions are complicated to implement, restricted to specific initial and boundary value problems, or have limited practical relevance.

Analytical solutions will be derived for the conservative and nonconservative forms of the advection and advection-diffusion equations with a particular form of spatially variable coefficients. The fluid velocity is taken to be a linear function of distance, with the diffusion coefficient proportional to the square of the velocity, and therefore proportional to the square of the distance. In conservative form the spatial derivatives can be written in the form of the divergence of a flux of solute.

For flow in a tube of circular cross section, Taylor (1953) showed that the dispersivity of a solution was approximately proportional to the square of the flow velocity. Wooding (1960) derived a corresponding dependence of the dispersivity on the square of the velocity for flow between parallel plates. The particular forms of the spatially variable coefficients considered here are consistent with the problem of the transport of pollutant in an open channel where the flow in the channel is augmented by steady, unpolluted lateral inflow distributed along the whole length of the channel, such as a steady inflow of ground water. Therefore the analytical solutions are solutions to a practical problem.

The simple expressions considered here for the spatial variation of the coefficients facilitate the process of obtaining analytical solutions to these equations. The spatially variable coefficient equations reduce to constant coefficient equations through a simple transformation. Consequently, many of the known analytical solutions to the constant coefficient equations can be used to obtain analytical solutions to the spatially variable coefficient equations.

Analytical solutions are provided for the advection of a sudden release of pollutant into the channel and for the solution of advection-diffusion equation. The advection of an initial quasi-Gaussian concentration profile in the channel is also considered. The analytical solutions are simple to evaluate and are useful for validating numerical schemes for solving the advection and advection-diffusion equation with spatially variable coefficients written in either conservative or nonconservative form (Zoppou and Knight 1994).

The conservative and nonconservative forms of the equations are valid equations describing different physical problems. The analytical solutions to the conservative and nonconservative forms of the governing equations will be used to illustrate the importance of selecting the equation relevant to the physical problem, when spatially variable coefficients are involved.

ADVECTION EQUATION

The conservative form of the advection equation can be written as

$$\frac{\partial c(x, t)}{\partial t} + \frac{\partial [c(x, t)u(x)]}{\partial x} = 0 \quad 0 < x \le \infty \quad t > 0 \tag{1}$$

in which $c(x, t)$ = concentration of contaminant; $u(x)$ = one-dimensional fluid velocity field; x = distance; and t = time.

The transformation of (1) into a constant coefficient equation begins with the selection of a suitable function form for the spatial variation of $u(x)$ so that (1) becomes an equidimensional linear equation [Hildebrand (1962) p. 13]. An equidimensional linear equation for (1) is

[1]Sr. Engr., Engrg. Div., ACT Electricity and Water Corp., GPO Box 366, Canberra, ACT 2601, Australia.

[2]Prin. Res. Sci., Ctr. for Envir. Mech., CSIRO, GPO Box 821, Canberra, ACT 2601, Australia.

Note. Discussion open until July 1, 1997. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this technical note was submitted for review and possible publication on October 27, 1995. This technical note is part of the ***Journal of Hydraulic Engineering***, Vol. 123, No. 2, February, 1997. ©ASCE, ISSN 0733-9429/97/0002-0144–0148/$4.00 + $.50 per page. Technical Note No. 11914.

$$\frac{\partial c(x, t)}{\partial t} + \frac{\partial[c(x, t)u_0x]}{\partial x} = 0 \quad 0 < x \leq \infty \quad t > 0 \tag{2}$$

The velocity field is simply a linear function of distance, such that $u(x, t) = u_0x$, where u_0 is a constant.

Eq. (2) is further simplified by making the substitution $y = \ln(x)$ [Hildebrand (1962) p. 13] so that

$$\frac{\partial[a(y, t)]}{\partial t} + u_0\frac{\partial[a(y, t)]}{\partial y} = 0 \tag{3}$$

is a constant coefficient advection equation with $a(y, t) = xc$ as the dependent variable. Known analytical solutions to the constant coefficient advection equation can be used to obtain analytical solutions to the spatially variable coefficient problem.

The analytical solution for the advection of a step boundary condition and a quasi-Gaussian initial condition in a spatially variable velocity field will now be derived.

Step Profile

The following initial and boundary condition is imposed on (2): $c(x, 0) = 0$ and $c(\infty, t) = 0$. A singularity occurs at $x = 0$, therefore the boundary condition will be imposed at $x = x_0$ where $0 < x_0 < \infty$, $c(x_0, t) = c_0$, and the initial boundary value problem is solved in the domain $x_0 \leq x \leq \infty$.

Eq. (3) is solved using the Laplace transform with respect to time, t. The evolution in time of the step concentration profile is given by

$$c(x, t) = \frac{c_0}{x} H[u_0t - \ln(x/x_0)] \tag{4}$$

where H = Heaviside function.

In a variable velocity field, the concentration front is located at $x = x_0 \exp(u_0t)$ and the profile decays as $1/x$. This is illustrated in Fig. 1, where $u_0 = 1$, $x_0 = 1$, $c_0 = 100$, and $t = 2$. This is what would be expected for this problem. The polluted water in the channel is diluted by the unpolluted lateral inflow entering the channel, and therefore the concentration of pollutant in the channel decreases with distance.

Since this is the conservative form of the advection equation, mass should be conserved. The mass entering the computational domain at $x = 1$ is given by c_0u_0t for a conservative system. Therefore

$$\int_1^\infty c(x, t)\, dx = c_0u_0t \tag{5}$$

Substituting (4) for $c(x,t)$ into (5), then

$$\int_1^\infty c(x, t)\, dx = \int_1^\infty \frac{c_0}{x} H[u_0t - \ln(x)]\, dx$$

Since $H[u_0t - \ln(x)] = 1$ when $x \leq \exp(u_0t)$ and 0 otherwise, then

$$\int_1^\infty c(x, t)\, dx = \int_1^{\exp(u_0t)} c_0/x\, dx = c_0[\ln(x)]_1^{\exp(u_0t)} = c_0u_0t$$

and mass is conserved.

The results for the advection of the step profile will now be compared with the evolution of the step profile predicted by the constant coefficient advection equation and the nonconservative advection equation.

For the constant coefficient advection equation given by

$$\frac{\partial c}{\partial t} + u_0\frac{\partial c}{\partial x} = 0$$

the solution is simply

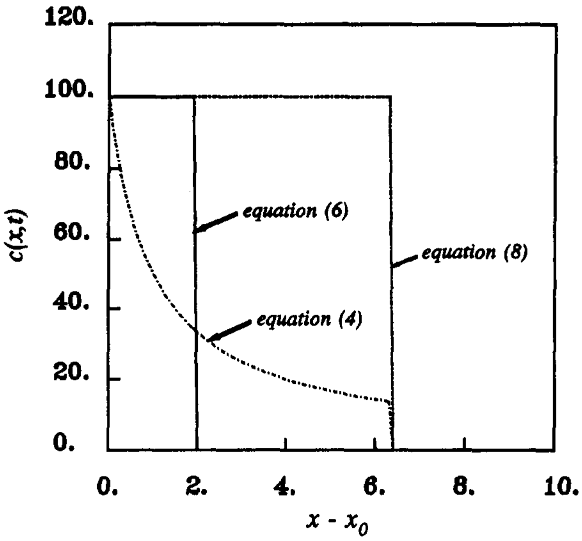

FIG. 1. Analytical Solution to Constant Coefficient, Eq. (6); Nonconservative, Eq. (8); and Conservative, Eq. (4), Advection Equation

$$c(x, t) = c_0H(u_0t - x + x_0) \tag{6}$$

The step profile is advected without changing shape at the speed u_0 and is located at $x = u_0t + x_0$ (Fig. 1). The front for the conservative equation travels much faster, and the concentration at the front is much smaller than is predicted by the constant coefficient advection equation.

Now consider the nonconservative spatially variable coefficient advection equation

$$\frac{\partial c}{\partial t} + u_0x\frac{\partial c}{\partial x} = 0 \tag{7}$$

The analytical solution to this equation is simply

$$c(x, t) = c_0H[u_0t - \ln(x/x_0)] \tag{8}$$

Comparing this equation with (4) shows that although the front is also located at $x = x_0 \exp(u_0t)$, the profile does not decay exponentially. The profile is simply advected without deformation, as shown in Fig. 1.

It is obvious from Fig. 1 that the nonconservative form of the advection equation has not conserved mass. For (8), (5) becomes

$$\int_1^\infty c(x, t)\, dx = \int_1^{\exp(u_0t)} c_0\, dx = c_0[x]_1^{\exp(u_0t)} = c_0[\exp(u_0t) - 1] \neq c_0u_0t$$

Eq. (7) is not conservative because mass increases exponentially with time. It is clear that the nonconservative form of the advection equation implies that the lateral inflow entering the channel is also polluted. This is in contrast to the physical problem being solved. This simple problem illustrates the importance of using the appropriate form of the governing equation for the problem being studied.

Quasi-Gaussian Profile

Consider a pollutant with a quasi-Guassian concentration profile. The conservative advection equation given by (2) can be transformed into the constant coefficient advection equation given by (3), which has the following analytical solution:

$$a(y, t) = a(y - u_0t, 0) \tag{9}$$

obtained using the method of characteristics, where the initial profile in the y-t plane is translated without deformation by the amount u_0t.

There is no restriction on the initial conditions that can be used. Consider the Gaussian profile

$$a(y, 0) = \frac{M_0}{\sigma\sqrt{2\,\pi}} \exp\left[\frac{-(y - y_0)^2}{2\sigma^2}\right] \tag{10}$$

in the y-t plane, where the peak concentration is located at $y_0 = \ln(x_0)$ and the profile contains a mass equal to M_0.

Recalling that $a(y, t) = xc(x, t)$, $y = \ln(x)$, and using (9), then the analytical solution in the x-t plane of (2) for the initial conditions given by (10) is

$$c(x, t) = \frac{M_0}{x\sigma\sqrt{2\pi}} \exp\left\{\frac{-[\ln(x/x_0) - u_0 t]^2}{2\sigma^2}\right\} \tag{11}$$

The mass under the profile is given by

$$\int_0^\infty c(x, t)\, dx = \int_{-\infty}^\infty xc(x, y)\, dy$$

$$= \frac{M_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left[\frac{-(y - u_0 t - y_0)^2}{2\sigma^2}\right] dy = M_0$$

therefore mass is conserved.

The peak concentration that occurs at $x_p = x_0 \exp(u_0 t)$ is given by

$$c(x_p, t) = \frac{M_0}{x_0\sigma\sqrt{2\pi}} \exp(-u_0 t)$$

Therefore the peak concentration decays exponentionally with time.

The centroid of the concentration profile is given by

$$\bar{x} = \frac{\displaystyle\int_0^\infty xc(x, t)\, dx}{\displaystyle\int_0^\infty c(x, t)\, dx}$$

where the first moment is given by

$$\int_0^\infty xc(x, t)\, dx = \frac{M_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^y \exp\left[\frac{-(y - u_0 t - y_0)^2}{2\sigma^2}\right] dy$$

$$= M_0 x_0 \exp[u_0 t + \sigma^2/2]$$

Therefore

$$\bar{x} = x_0 \exp\left[u_0 t + \frac{\sigma^2}{2}\right]$$

The peak concentration travels slower than the centroid of the profile and the profile is positively skewed in the x-t domain.

The location of the centroid and peak concentrations coincide for a Gaussian profile. In the example shown earlier, this occurs only in the y-t plane but not in the x-t plane. Therefore the profile has been called a quasi-Gaussian profile.

The analytical solution given by (11) with $u_0 = 0.1$, $\sigma = 0.2$, $c_0 = 10$, and $x_0 = 0.2$ is shown in Fig. 2 for $t = 0$–20 in time increments of 2. As predicted, the profile decays exponentially and the mass is conserved.

For the nonconservative advection equation, (7), making the substitution $y = \ln(x)$ produces the following constant coefficient equation:

$$\frac{\partial a(y, t)}{\partial t} + u_0 \frac{\partial a(y, t)}{\partial x} = 0$$

where $a(y, t) = c(x, t)$. This equation has the analytical solution given by (9). The initial profile is translated by the amount $u_0 t$. Therefore, for the initial quasi-Gaussian profile given by (10), the analytical solution of the nonconservative advection equation becomes

$$c(x, t) = \frac{M_0}{\sigma\sqrt{2\pi}} \exp\left[\frac{-[\ln(x/x_0) - u_0 t]^2}{2\sigma^2}\right] \tag{12}$$

There is no attenuation of the peak, whose concentration is located at $x_p = x_0 \exp(u_0, t)$. This is in contrast with the analytical solution to the same problem using the conservative form of the advection equation where the peak concentration travels at the same speed, but the profile attenuates.

To establish whether the nonconservative advection equation conserves mass, the mass under the profile is given by

$$\int_0^\infty c(x, t)\, \mathrm{d}x = \frac{M_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^y \exp\left[\frac{-(y - u_0 t - y_0)^2}{2\sigma^2}\right] dy$$

$$= M_0 x_0 \exp\left[u_0 t + \frac{\sigma^2}{2}\right] \neq M_0$$

Mass is not conserved even at $t = 0$, and it increases exponentially with time t.

The analytical solution given by (12) with $u_0 = 0.1$, $\sigma = 0.2$, $c_0 = 10$, and $x_0 = 0.2$, is shown in Fig. 3 for $t = 0$–20 in time

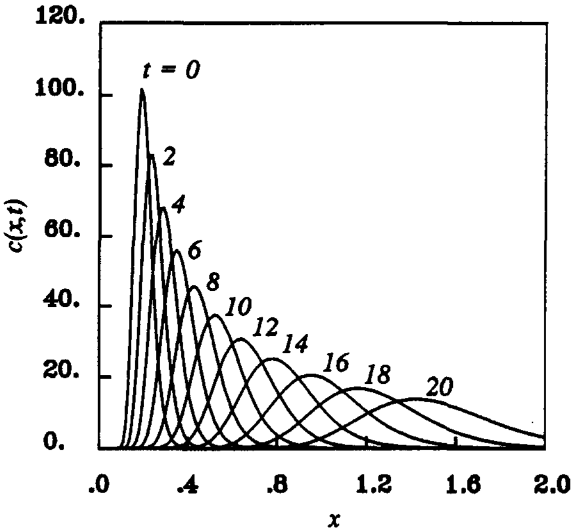

FIG. 2. Concentration Profile for Advection of a Quasi-Gaussian Profile in a Nonuniform Flow Field Using a Conservative Law

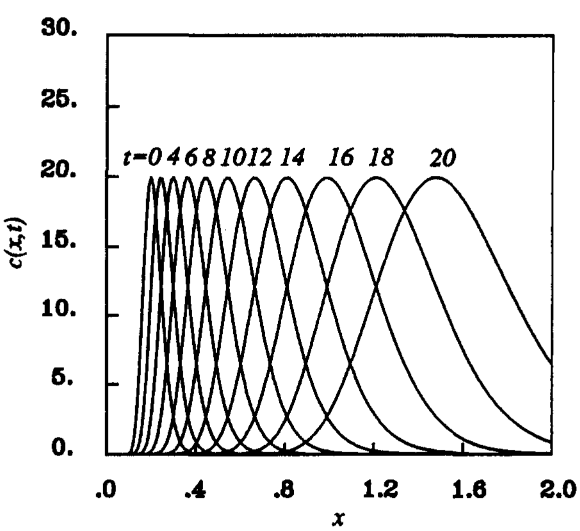

FIG. 3. Concentration Profile for Advection of a Quasi-Gaussian Profile in a Nonuniform Flow Field Using a Nonconservative Law

increments of 2. It can be seen that the profile does not attenuate and the mass increases exponentially with time.

ADVECTIVE-DIFFUSION EQUATION

Including the influence of diffusion in the transport of pollutant into the open channel is a straightforward problem. For the diffusion proportional to the square of the velocity, and the velocity being a linear function of distance, the analytical solution of the conservative and nonconservative forms of the advection-diffusion equation with spatially variable velocity and diffusion coefficients will be derived for a step initial condition.

Conservative Advection-Diffusion Equation

The advection-diffusion equation written in conservative form is given by

$$\frac{\partial c(x, t)}{\partial t} + \frac{\partial[c(x, t)u(x)]}{\partial x} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial c(x, t)}{\partial x}\right]$$

$$x_0 < x \leq \infty \quad t > 0 \tag{13}$$

in which $c(x, t)$ = concentration of a contaminant; $u(x)$ = one-dimensional fluid velocity field, and $D(x)$ = diffusion coefficient. Consider a slug of pollutant; then the following initial and boundary conditions are imposed on (13):

$$c(x, 0) = 0 \quad \text{for} \quad x > x_0, \; c(x_0, t) = c_0 \quad \text{for} \quad x \leq x_0$$

$$\text{and} \quad c(\infty, t) = 0 \tag{14}$$

If the velocity field varies linearly with distance, and the diffusion coefficient is proportional to the square of the velocity, and therefore proportional to the square of the distance, then (13) becomes

$$\frac{\partial c(x, t)}{\partial t} + \frac{\partial[c(x, t)u_0x]}{\partial x} = \frac{\partial}{\partial x}\left[D_0x^2\frac{\partial c(x, t)}{\partial x}\right] \quad x_0 < x \leq \infty \quad t > 0 \tag{15}$$

in which u_0 and D_0 are constant.

An analytical solution to (15) subject to (14) is sought. Substituting $y = \ln(x)$ into (15) and simplifying yields

$$\frac{\partial c}{\partial t} + (u_0 - D_0)\frac{\partial c}{\partial y} = D_0\frac{\partial^2 c}{\partial y^2} - cu_0 \quad y_0 < y \leq \infty \quad t > 0 \tag{16}$$

and the initial and boundary conditions become

$$c(y, 0) = 0 \quad \text{for} \quad y > y_0, \; c(y_0, t) = c_0 \quad \text{for} \quad y \leq y_0$$

$$\text{and} \quad c(\infty, t) = 0$$

which is an advection-diffusion equation with constant coefficients and a first-order decay term.

The analytical solution to (16) is given by

$$c(y, t) = \frac{c_0}{2}\left\{\exp(-y + y_0)\text{erfc}\left[\frac{y - y_0 - t(u_0 + D_0)}{2\sqrt{D_0t}}\right] + \exp\left[\frac{u_0(y - y_0)}{D_0}\right]\text{erfc}\left[\frac{y - y_0 + t(u_0 + D_0)}{2\sqrt{D_0t}}\right]\right\}$$

which was obtained using the Laplace transform and recalling the result given by Carslaw and Jaeger (1971).

Making the final substitution, $y = \ln(x)$, the analytical solution to (15) with the initial and boundary conditions given by (14) is

$$c(x, t) = \frac{c_0}{2}\left\{\frac{x_0}{x}\text{erfc}\left[\frac{\ln(x/x_0) - t(u_0 + D_0)}{2\sqrt{D_0t}}\right] + \exp\left[\frac{u_0\ln(x/x_0)}{D_0}\right]\text{erfc}\left[\frac{\ln(x/x_0) + t(u_0 + D_0)}{2\sqrt{D_0t}}\right]\right\} \tag{17}$$

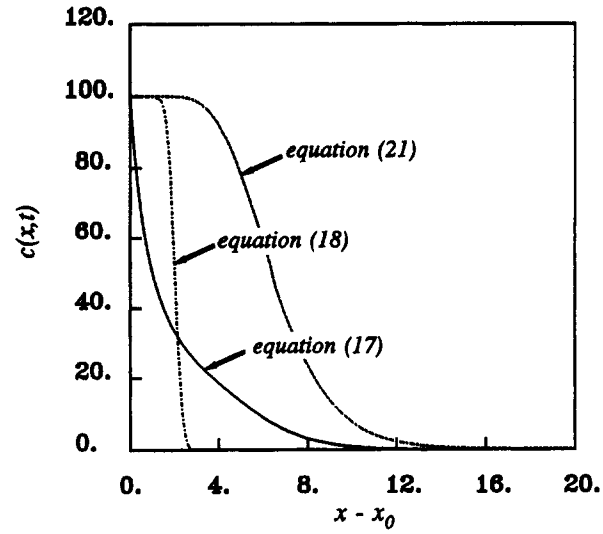

FIG. 4. Analytical Solution to Constant Coefficient, Eq. (18); Nonconservative, Eq. (21); and Conservative, Eq. (17), Advection-Diffusion Equation.

When a step function is subject to a variable velocity field and diffusion coefficient, where $u_0 = 1$, $D_0 = 0.02$, $c_0 = 100$, and $x_0 = 1$, the concentration profile predicted by (17) at $t = 2$ is illustrated in Fig. 4.

The predicted concentration profile for the constant coefficient advection-diffusion equation

$$\frac{\partial c}{\partial t} + u_0\frac{\partial c}{\partial x} = D_0\frac{\partial^2 c}{\partial x^2}$$

is also illustrated in Fig. 4. The analytical solution to the constant coefficient problem

$$c(x, t) = \frac{c_0}{2}\text{erfc}\left[\frac{x - x_0 - u_0t}{2\sqrt{D_0t}}\right] + \frac{c_0}{2}\exp\left[\frac{u_0(x - x_0)}{D_0}\right]\text{erfc}\left[\frac{x - x_0 + u_0t}{2\sqrt{D_0t}}\right] \tag{18}$$

can be found in de Marsily (1986).

The effect of variable coefficients in the advection-diffusion equation is dramatic. The concentration profile propagates at a greater speed than for the constant coefficient equation and the concentration decays exponentially.

Nonconservative Advection-Diffusion Equation

The variable coefficient nonconservative form of the advection-diffusion equation can be written as

$$\frac{\partial c(x, t)}{\partial t} + u_0x\frac{\partial c(x, t)}{\partial x} = D_0x^2\frac{\partial^2 c(x, t)}{\partial x^2} \quad 0 < x \leq \infty \quad t > 0$$

Since a singularity exists at $x = 0$, the following initial boundary value problem will be solved

$$\frac{\partial c(x, t)}{\partial t} + u_0x\frac{\partial c(x, t)}{\partial x} = D_0x^2\frac{\partial^2 c(x, t)}{\partial x^2} \quad x_0 < y \leq \infty \quad t > 0 \tag{19}$$

with $c(x_0t) = c_0$; and $x_0 > 0$.

Making use of the substitution, $y = \ln(x)$ (19) becomes

$$\frac{\partial c(y, t)}{\partial t} + u_0\frac{\partial c(y, t)}{\partial y} = D_0\frac{\partial^2 c(y, t)}{\partial y^2} \quad y_0 > y \leq \infty \quad t > 0 \tag{20}$$

with $c(y_0, t) = c_0$. This is an advection-diffusion equation with constant coefficients in the y-t plane.

The well-known analytical solution of (20) is given by (de Marsily 1986).

$$c(y, t) = \frac{c_0}{2}\,\mathrm{erfc}\left[\frac{y - y_0 - u_0 t}{2\sqrt{D_0 t}}\right] + \frac{c_0}{2}\exp\left[\frac{u_0(y - y_0)}{D_0}\right]\mathrm{erfc}\left[\frac{y - y_0 + u_0 t}{2\sqrt{D_0 t}}\right]$$

making the final substitution, $y = \ln(x)$, the analytical solution to (19) is

$$c(x, t) = \frac{c_0}{2}\,\mathrm{erfc}\left[\frac{\ln(x/x_0) - u_0 t}{2\sqrt{D_0 t}}\right] + \frac{c_0}{2}\exp\left[\frac{u_0 \ln(x/x_0)}{D_0}\right]\mathrm{erfc}\left[\frac{\ln(x/x_0) + u_0 t}{2\sqrt{D_0 t}}\right] \quad (21)$$

Eq. (21) satisfies the boundary condition at $x = x_0$, which is $c(x, t) = c_0$. The concentration profile predicted by (21) is illustrated in Fig. 4. The concentration profile moves much faster than that predicted by the constant coefficient nonconservative advection-diffusion equation. It is obvious that mass is not conserved in this case.

The foregoing examples illustrate the importance of using the correct form of the governing equation to model the transport of pollutants, as pointed out by Philip (1994). Kitanidis (1994) demonstrated the importance in particle-tracking methods of solving the conservative form of the spatially variable coefficient advection-diffusion equation instead of its nonconservative form.

CONCLUSIONS

Analytical solutions have been derived for the advection and advection-diffusion equation with a particular form of the spatially variable coefficients. The analytical solutions are simple to evaluate and implement.

The analytical solutions can be used to test numerical schemes for solving the advection and advection-diffusion equation with spatially variable coefficients in either their conservative or nonconservative forms. In addition, the analytical solution can be extended readily to multidimensional problems that will be considered in a subsequent paper.

The appropriate governing equations must be solved for variable coefficient problems. Generally this will involve the conservative form of the governing equations. Failure to do so may result in solutions that are not consistent with the physical process being modeled.

ACKNOWLEDGMENT

This research was partially supported by a grant from the Urban Water Resources Research Association, Australia.

APPENDIX I. REFERENCES

Barry, D. A., and Sposito, G. (1989). "Analytical solution to a convection-dispersion model with time-dependent transport coefficients." *Water Resour. Res.*, 25(12), 2407–2416.

Basha, H. A., and El-Habel, F. S. (1993). "Analytical solution of the one-dimensional time-dependent transport equation." *Water Resour. Res.*, 29(9), 3209–3214.

Carslaw, H. S., and Jaeger, J. C. (1971). *Conduction of heat in solids.* Oxford University Press, New York, N.Y.

de Marsily, G. (1986). *Quantitative hydrogeology: groundwater hydrology for engineers.* Academic Press, Inc., San Diego, Calif.

Hildebrand, F. B. (1962). *Advanced calculus for applications.* Prentice-Hall, Inc., Englewood Cliffs, N.J.

Kitanidis, P. K. (1994). "Particle-tracking equations for the solution of the advection-dispersion equation with variable coefficients." *Water Resour. Res.*, 30(11), 3225–3227.

Philip, J. R. (1994). "Some exact solutions of convection-diffusion and diffusion equations." *Water Resour. Res.*, 30(12), 3545–3551.

Taylor, G. I. (1953). "Dispersion of soluble matter in solvent flowing slowly through a tube." *Proc., Royal Soc.*, London, England, Ser. A, 219, 186–203.

Wooding, R. A. (1960). "Instability of a viscous liquid of variable density in a Hele-Shaw cell." *J. Fluid Mech.*, Cambridge, U.K., 7(4), 501–515.

Zoppou, C., and Knight, J. H. (1994). "Comment on 'A space time accurate method for solving solute transport problems,' by S. G. Li, F. Ruan, and D. McLaughlin." *Water Resour. Res.*, 30(11), 3233–3235.

APPENDIX II. NOTATION

The following symbols are used in this paper:

$a(y, t)$ = flux;
$c(x, t)$ = concentration;
$D(x)$ = diffusion coefficient;
D_0 = constant diffusion coefficient;
H = Heaviside function;
M_0 = constant;
t = time;
$u(x)$ = fluid velocity;
u_0 = constant fluid velocity;
x = distance;
x_p = location of the peak concentration;
x_0 = location of the boundary condition;
y = $y = \ln(x)$;
y_0 = $y_0 = \ln(x_0)$;
π = 3.1416; and
σ = standard deviation.