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# THE ACCURACY OF THE FRACTIONAL STEP METHOD*

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**Abstract.** We analyze the accuracy of the fractional step method of Kim and Moin [*J. Comput. Phys.*, 59 (1985), pp. 308–323] for the incompressible Navier–Stokes equations. We show that the boundary conditions cannot be exactly satisfied in the projection step and that this limits the accuracy of the method. We also show that the pressure in any projection method can be at best first-order accurate. Our analysis is simpler and more direct than the previous analyses of this method. We also show that there is no numerical boundary layer for velocity or pressure, but there is one for the auxiliary pressure variable.

**Key words.** incompressible Navier–Stokes, fractional step method, accuracy, boundary conditions

**AMS subject classifications.** 65M06, 76D05

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## 1. Introduction.

The fractional step method proposed by Kim and Moin [4] has been used extensively in computations of time-dependent fluid flow. Studies of its accuracy have been made by E and Liu [2], [3], Shen [9], [10], and Rannacher [8]. See also the papers by Orszag, Israeli, and Deville [6], Perot [7], and Temam [13]. In this paper we offer a rigorous analysis of the accuracy of this method using normal mode analysis, which illustrates several defects of the method. Perot [7] has argued that the pressure can be determined only to first-order accuracy in time, and we improve upon his heuristic argument and justify his claims. Furthermore, we show that this method cannot satisfy the boundary conditions on the velocity. Our argument shows that all methods based on the projection method must suffer from the same defects.

Our analysis is similar in spirit to that of E and Liu [2], [3] but is much simpler because we consider a half-plane rather than a channel. Orszag, Israeli, and Deville [6] also considered a channel. The scheme analyzed by E and Liu [2], [3] is a variant of the original Kim and Moin scheme [4] and that variant has a numerical boundary layer for the pressure. As we show in this paper, the pressure of the original Kim and Moin scheme does not have a numerical boundary layer, although the auxiliary pressure variable does have a numerical boundary layer.

The analyses of Shen [9], [10] and Rannacher [8] use Banach space methods and produce norm estimates for the errors. Here we use normal mode analysis that leads to estimates of the errors of the Laplace–Fourier transforms of the variables. Norm estimates can be obtained from our normal mode estimates, but we leave these to subsequent papers.

Our analysis here is restricted to schemes that are discretized only in time. We do not consider the discretization in the spatial variables. This restricts the analysis to implicit schemes for the time-advancement step, the explicit schemes being unstable.

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However, the conclusions about the disadvantages of projection methods are general enough that they apply to the explicit schemes as well.

There are schemes for which the pressure can be determined with high accuracy and for which the boundary conditions can be satisfied. These, however, are completely implicit; see, e.g., [12].

The unsteady incompressible Navier–Stokes equations are

$$\vec{u}_t - \frac{1}{R}\nabla^2\vec{u} + \vec{\nabla}(\vec{u}\vec{u}^T) + \vec{\nabla}p = \vec{0}, \tag{1.1}$$

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{1.2}$$

The vector function $\vec{u}$ is the velocity and the scalar function $p$ is the pressure. We consider the Navier–Stokes system holding in the right half-plane, a specific but important domain. To specify a unique solution, initial conditions and boundary conditions must be given, i.e.,

$$\vec{u}(0, x, y) = \vec{u}_0(x, y), \tag{1.3}$$

$$\vec{u}(t, 0, y) = \vec{b}(t, y) \quad \text{at} \quad x = 0. \tag{1.4}$$

Since the domain is unbounded, a few additional constraints are needed. We assume that

$$\vec{u} \to \vec{0} \quad \text{and} \quad p \to 0 \quad \text{as} \quad |x| + |y| \to \infty. \tag{1.5}$$

By considering only the half-plane we simplify the analysis as compared with E and Liu [2], [3], who considered a channel, but we do not sacrifice generality. The accuracy along any portion of a straight boundary can be of order no higher than that obtained on a half-plane. Similarly, any boundary segment can be mapped locally to a half-plane, with the introduction of source terms, and the order of accuracy for the mapped equation can be no higher than that for the simple half-plane problem. Thus the half-plane problem is the simplest problem on which to study the accuracy of boundary conditions, and the conclusions carry over to the general case.

A seemingly small, but significant, variation from the discussion of the boundary conditions by others (see [2], [3], and [7]) is the introduction of the function $\vec{b}$. In these other discussions, the boundary condition (1.4) is given as "the velocity is specified on the boundary." However, as we show, the velocity that is computed does not satisfy (1.4). That is, the computed velocity is not equal to the intended velocity on the boundary. Thus it is essential to distinguish between the velocity on the boundary, $\vec{u}$, and the intended values of the velocity on the boundary, $\vec{b}$. Kim and Moin never distinguish between $\vec{b}$ and $\vec{u}$.

In the following, we consider only the Stokes equations for simplicity. These are

$$\vec{u}_t - \nabla^2\vec{u} + \vec{\nabla}p = \vec{0}, \tag{1.6}$$

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{1.7}$$

The Stokes equations are valid in the limit of vanishing Reynolds number and are also the highest order terms in the Navier–Stokes equations. For smooth solutions of the

Navier–Stokes equations, our conclusions about the scheme for the Stokes equations apply to the Navier–Stokes equations as well.

We begin our analysis by reducing the general initial value problem (1.6) and (1.7) with data (1.3) and (1.4) to the special case with homogeneous initial data. We assume that there is an extension of the initial data (1.3) to the whole plane such that there is a solution $\vec{u}_1$ and $p_1$ to equations (1.6) and (1.7) on the whole plane. Let $\vec{u}_2 = \vec{u} - \vec{u}_1$ and $p_2 = p - p_1$. Then $\vec{u}_2$ and $p_2$ satisfy (1.6) and (1.7) on the half-plane. $\vec{u}_2$ has the initial condition $\vec{u}_2(0, x, y) = 0$ and must satisfy the boundary condition

$$\vec{u}_2(t, 0, y) = \vec{b} - \vec{u}_1(t, 0, y) \quad \text{at} \quad x = 0. \tag{1.8}$$

In the remainder of this paper, we consider $\vec{u}_2$ and $p_2$, the solution of equations (1.6) and (1.7), with initial condition (1.3) with $\vec{u}_0$ equal to 0, and boundary condition (1.4) with data given by (1.8). To simplify the discussion, we write $\vec{u}$ and $p$ instead of $\vec{u}_2$ and $p_2$, respectively.

We now consider the finite difference scheme of Kim and Moin [4] for (1.6) and (1.7). The scheme is a two-step time-advancement scheme that uses an intermediate velocity $\vec{u}^\star$ and can be written as follows:

$$\frac{\vec{u}^{\star,n+1} - \vec{u}^n}{\Delta t} = \frac{1}{2}\nabla^2(\vec{u}^{\star,n+1} + \vec{u}^n), \tag{1.9}$$

$$\frac{\vec{u}^{n+1} - \vec{u}^{\star,n+1}}{\Delta t} = -\vec{\nabla}\phi^{n+1}, \tag{1.10}$$

$$\vec{\nabla} \cdot \vec{u}^{n+1} = 0. \tag{1.11}$$

As pointed out by Kim and Moin [4], $\phi$ is not the pressure, but $p$ can be obtained from $\phi$ by the relationship

$$p = \phi - \frac{\Delta t}{2}\nabla^2\phi. \tag{1.12}$$

In the papers of E and Liu [2], [3] they identify $p$ with $\phi$ and thus, by (1.12), introduce an error of $O(\Delta t)$ in $p$.

An important issue is how to impose boundary conditions for the intermediate velocity field and the pressure in this time-splitting scheme. Kim and Moin suggest the boundary condition

$$\vec{u}^{\star,n+1}_{x=0} = \vec{b}^{n+1} + \Delta t\vec{\nabla}\phi^n_{x=0}, \tag{1.13}$$

which was developed by following the ideas of LeVeque and Oliger [5].

It is important to note that the system consisting of (1.10) and (1.11) for $\vec{u}^{n+1}$ and $\phi^{n+1}$ in terms of $\vec{u}^{\star,n+1}$ requires only one boundary condition for the solution to be well defined. This is the reason for the inability of the projection methods to satisfy the velocity boundary condition for both components.

For the solution of the system (1.10)–(1.11) we apply the one boundary condition

$$u^{n+1}_{x=0} = b^{n+1}_1 \tag{1.14}$$

for the normal component of the velocity.

We analyze the accuracy of the intermediate boundary condition (1.13) and other boundary conditions by solving (1.9), (1.10), and (1.11). In section 3 the solutions are found in the transformed variables, and in section 5 we discuss the accuracy of the fractional step method and the rate of convergence.

**2. Other projection methods.** Several other projection methods have been proposed, including the scheme of van Kan [14]. As applied to the Stokes equations, this may be seen to be equivalent to the Kim and Moin scheme by redefining the intermediate velocity.

For example, the scheme of van Kan is

$$\frac{\vec{u}^{\star,n+1} - \vec{u}^n}{\Delta t} = \frac{1}{2}\nabla^2(\vec{u}^{\star,n+1} + \vec{u}^n) - \vec{\nabla} p^n, \tag{2.1}$$

$$\frac{\vec{u}^{n+1} - \vec{u}^{\star,n+1}}{\Delta t} = -\frac{1}{2}\vec{\nabla}(p^{n+1} - p^n), \tag{2.2}$$

$$\vec{\nabla} \cdot \vec{u}^{n+1} = 0. \tag{2.3}$$

By redefining variables, we see that the scheme of van Kan in (2.1)–(2.3) is equivalent to the fractional step scheme of Kim and Moin. Letting the subscript of $km$ mark the Kim and Moin variables in (1.9)–(1.11) and the subscript $vk$ mark the variables in (2.1)–(2.3) and setting

$$\vec{u}^{\star}_{vk} = \vec{u}^{\star}_{km} - \Delta t \vec{\nabla} \phi^n_{vk}$$

and

$$p_{vk} = \phi_{vk} - \frac{\Delta t}{2}\nabla^2 \phi_{vk},$$

we see that the schemes are the same in the sense that $\vec{u}^n_{vk} = \vec{u}^n_{km}$ for all $n$.

The scheme of Bell, Colella, and Glaz [1] involves an iterative method, each step of which has a projection step similar to the scheme of Kim and Moin. Therefore, it is subject to the same difficulties as those discussed here.

E and Liu [2] discuss the relationships between other fractional step methods.

**3. The solution in the transformed variables.** By using (1.10) in (1.9) we can eliminate $\vec{u}^*$ and obtain the equations

$$\left(\frac{1}{\Delta t} - \frac{1}{2}\nabla^2\right)\vec{u}^{n+1} + \Delta t\left(\frac{1}{\Delta t} - \frac{1}{2}\nabla^2\right)\vec{\nabla}\phi^{n+1} = \left(\frac{1}{\Delta t} + \frac{1}{2}\nabla^2\right)\vec{u}^n, \tag{3.1}$$

$$\vec{\nabla} \cdot \vec{u}^{n+1} = 0.$$

We solve this system using the Fourier and Laplace transforms. Because of the homogeneous initial data, we can extend the solution back in time, taking both $\vec{u}^n$ and $p^n$ to be zero for $n$ negative. The Laplace transform of a discrete function $v^n$ on a grid with spacing $\Delta t$ is defined by

$$\tilde{v}(z) = \frac{1}{\sqrt{2\pi}}\Delta t \sum_{n=-\infty}^{\infty} z^{-n} v^n \quad \text{for} \quad |z| \geq 1$$

and the Fourier transform $\hat{u}(\omega)$ is defined by

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega y} u(y)\, dy \quad \text{for} \quad \omega \in \mathbb{R}.$$

Notice that if $w^n = v^{n+1}$, then $\tilde{w} = z\tilde{v}$. See Strikwerda [11] for more discussion of the use of transforms for analyzing difference schemes and differential equations.

We define $\bar{f}(z,\omega)$ to be both the Fourier transform and the Laplace transform of a function $f(t,y)$.

We begin our analysis of the system (3.1) by taking the Fourier transform in the variable $y$ and the discrete Laplace transform in $t$. We also now write the equations using the velocity components $u$ and $v$ for the velocity vector $\vec{u}$. By letting

$$\bar{p} = \left[1 - \frac{\Delta t}{2}(\partial_x^2 - \omega^2)\right]\bar{\phi},$$

according to (1.12) we have

$$\left[\frac{2}{\Delta t}\left(1 - \frac{1}{z}\right) + \omega^2\left(1 + \frac{1}{z}\right)\right]\bar{u} + 2\bar{p}_x = \bar{u}_{xx}\left(\frac{1}{z} + 1\right),$$

$$\left[\frac{2}{\Delta t}\left(1 - \frac{1}{z}\right) + \omega^2\left(1 + \frac{1}{z}\right)\right]\bar{v} + 2i\omega\bar{p} = \bar{v}_{xx}\left(\frac{1}{z} + 1\right), \tag{3.2}$$

$$\bar{u}_x + i\omega\bar{v} = 0.$$

We solve this system of ordinary differential equations by determining the solutions of the form

$$\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \bar{u}_0(z,\omega) \\ \bar{v}_0(z,\omega) \\ \bar{p}_0(z,\omega) \end{pmatrix} e^{-\lambda x}.$$

Substituting this expression in the system (3.2) we have the set of equations

$$-\lambda\bar{u}_0 + i\omega\bar{v}_0 = 0,$$

$$\left[\frac{2}{\Delta t}\left(1 - \frac{1}{z}\right) + (\omega^2 - \lambda^2)\left(1 + \frac{1}{z}\right)\right]\bar{u}_0 - 2\lambda\tilde{\bar{p}}_0 = 0,$$

$$\left[\frac{2}{\Delta t}\left(1 - \frac{1}{z}\right) + (\omega^2 - \lambda^2)\left(1 + \frac{1}{z}\right)\right]\bar{v}_0 + 2i\omega\tilde{\bar{p}}_0 = 0,$$

where $\bar{u}_0$, $\bar{v}_0$, and $\tilde{\bar{p}}_0$ are not all zero. There are four solutions for $\lambda$ given by

$$\lambda = \pm\sqrt{\omega^2 + \frac{2}{\Delta t}\frac{z-1}{z+1}},$$
$$\lambda = \pm|\omega|.$$

Since we are interested only in solutions that decay as $x$ increases (see (1.5)) only the two values of $\lambda$ with $\Re\lambda > 0$ are of interest. For convenience, we set

$$\kappa = \sqrt{\omega^2 + \frac{2}{\Delta t}\frac{z-1}{z+1}}.$$

Then the solution, when $|\omega| \neq \kappa$, is of the form

$$\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{p} \end{pmatrix} = A \begin{pmatrix} i\omega \\ \kappa \\ 0 \end{pmatrix} e^{-\kappa x} + B \begin{pmatrix} \Delta t \frac{z}{z-1}|\omega| \\ -\Delta t \frac{z}{z-1} i\omega \\ 1 \end{pmatrix} e^{-|\omega| x}, \tag{3.3}$$

where $A(z, \omega)$ and $B(z, \omega)$ are determined by the boundary conditions. Note that $|z| > 1$ implies that $\kappa \neq |\omega|$.

We now solve for $\bar{\phi}$. Since $\bar{p} = Be^{-|\omega| x}$ from (3.3), we obtain from (1.12)

$$Be^{-|\omega| x} = \left[1 - \frac{\Delta t}{2}\left(\partial_x^2 - \omega^2\right)\right] \bar{\phi}.$$

Hence we have

$$\bar{\phi}_{xx} - \left(\frac{2}{\Delta t} + \omega^2\right) \bar{\phi} = -\frac{2B}{\Delta t} e^{-|\omega| x}$$

and $\bar{\phi}$ is given by

$$\bar{\phi} = \phi_0 e^{-\alpha x} + Be^{-|\omega| x}, \tag{3.4}$$

where

$$\alpha = \sqrt{2/\Delta t + \omega^2},$$

and $\phi_0$ is to be determined by the boundary conditions.

From (1.10), i.e., the relationship

$$\vec{u}^\star = \vec{u} + \Delta t \vec{\nabla} \phi, \tag{3.5}$$

the transform of $\vec{u}^\star$ can be expressed by

$$\begin{pmatrix} \bar{u}^\star \\ \bar{v}^\star \end{pmatrix} = A \begin{pmatrix} i\omega \\ \kappa \end{pmatrix} e^{-\kappa x} + B \frac{\Delta t}{z-1} \begin{pmatrix} |\omega| \\ -i\omega \end{pmatrix} e^{-|\omega| x} + \Delta t \phi_0 \begin{pmatrix} -\alpha \\ i\omega \end{pmatrix} e^{-\alpha x}. \tag{3.6}$$

Note that an additional term $e^{-\alpha x}$ is introduced for $\vec{u}^\star$ and that this term with decay rate $\alpha$ is the numerical boundary layer discussed by E and Liu [2], [3]. It is of interest that this boundary layer term does not appear in the expressions for $p$ or $\vec{u}$; it appears only in $\phi$ and $\vec{u}^\star$.

**4. The solution of the partial differential equation.** The solution of the initial boundary value problem for the system given by partial differential equations (1.6) and (1.7) is

$$\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} i\omega \\ \sqrt{s + \omega^2} \\ 0 \end{pmatrix} e^{-\sqrt{s+\omega^2}\, x} + B \begin{pmatrix} \frac{|\omega|}{s} \\ -\frac{i\omega}{s} \\ 1 \end{pmatrix} e^{-|\omega| x}.$$

This can be obtained either by transforming the Stokes equations and solving the system similarly to how the difference equations were solved or, from (3.3), by taking the limit as $\Delta t$ goes to zero using the relation

$$z = e^{s\Delta t}, \tag{4.1}$$

which relates the transformed variable for the discrete Laplace transform to that of the continuous Laplace transform; see [11].

**5. The accuracy.** In this section we discuss the accuracy of the finite difference scheme and show how it depends critically on the boundary condition for the intermediate velocity $\vec{u}^\star$. Note that the elliptic system given by (1.10) and (1.11) for the second step of the scheme needs only one boundary condition for the solution to be determined. We consider several boundary conditions and show that the fractional step method cannot satisfy the intended boundary condition (1.4).

The equations (3.3)–(3.6) contain all the information in (1.9)–(1.11) except for boundary conditions. We first consider the boundary condition

$$\vec{u}^{\star,n+1} = \vec{b}^{n+1}. \tag{5.1}$$

Later we consider the more accurate boundary condition (1.13). From (3.6) and (5.1) we obtain

$$A\begin{pmatrix} i\omega \\ \kappa \end{pmatrix} + B\frac{\Delta t}{z-1}\begin{pmatrix} |\omega| \\ -i\omega \end{pmatrix} + \Delta t\phi_0\begin{pmatrix} -\alpha \\ i\omega \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}. \tag{5.2}$$

Since $u_{x=0}^{n+1} = b_1^{n+1}$ by (1.14), from (3.3) we have

$$Ai\omega + B\frac{z}{z-1}\Delta t\,|\omega| = \bar{b}_1. \tag{5.3}$$

The system composed of (5.2) and (5.3) is three equations for the three unknowns $A$, $B$, and $\phi_0$.

Using (5.3) and the first equation in (5.2) to eliminate $A$, we obtain

$$\phi_0 = -B\frac{|\omega|}{\alpha} \tag{5.4}$$

and so, from (3.4),

$$\bar{\phi} = B\left(-\frac{|\omega|}{\alpha}e^{-\alpha x} + e^{-|\omega| x}\right). \tag{5.5}$$

The form (5.5) also follows from $u_{x=0}^{n+1} = u_{x=0}^{n+1,\star}$, which is a result of (1.14) and (5.1). By (1.10), it is seen that for this boundary condition $\phi_{x,x=0}^{n+1} = 0$.

Using (5.4) in (5.2) gives

$$A\begin{pmatrix} i\omega \\ \kappa \end{pmatrix} + B\Delta t\begin{pmatrix} \frac{z}{z-1}|\omega| \\ -\left(\frac{1}{z-1} + \frac{|\omega|}{\alpha}\right)i\omega \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}. \tag{5.6}$$

Equation (5.6) is a system for $A$ and $B$ and can be solved to obtain explicit formulas for them. However, for our purposes this is unnecessary. We are interested in an expression for $v_{x=0}$, the second component of the velocity on the boundary. By (3.3), we have that

$$\bar{v}_{x=0} = A\kappa - B\Delta t\frac{z}{z-1}i\omega,$$

and, comparing this with the second equation in (5.6), we have

$$\bar{v}_{x=0} = \bar{b}_2 - i\omega\Delta t B\left(1 - \frac{|\omega|}{\alpha}\right).$$

Since $1-\dfrac{|\omega|}{\alpha}=O(1)$ and $B=O(1)$ (by (3.3) $B=\bar{p}_{x=0}$), we obtain

$$\bar{v}_{x=0}=\bar{b}_2+O(\Delta t).$$

This shows that the boundary condition (1.4) is not satisfied for the tangential component of velocity with boundary condition (5.1).

We next consider the boundary condition (1.13) suggested by Kim and Moin:

$$\vec{u}^{\star,n+1}_{x=0}=\vec{b}^{n+1}+\Delta t\vec{\nabla}\phi^n. \tag{5.7}$$

From the expression for the transform of $\vec{u}^\star$ in (3.6) and the expression for $\phi$ in (3.4), the boundary condition (5.7) is

$$A\begin{pmatrix} i\omega \\ \kappa \end{pmatrix}+B\frac{\Delta t}{z-1}\begin{pmatrix} |\omega| \\ -i\omega \end{pmatrix}+\Delta t\phi_0\begin{pmatrix} -\alpha \\ i\omega \end{pmatrix}=\begin{pmatrix} \bar{b}_1+\frac{\Delta t}{z}(-\alpha\phi_0-B|\omega|) \\ \bar{b}_2+\frac{\Delta t}{z}(\phi_0+B)i\omega \end{pmatrix}. \tag{5.8}$$

Since $u^{n+1}=b_1^{n+1}$, we again have from (3.3)

$$Ai\omega+B\frac{z}{z-1}\Delta t|\omega|=\bar{b}_1. \tag{5.9}$$

The system composed of (5.8) and (5.9) involves three equations for the three unknowns $A$, $B$, and $\phi_0$.

As with the previous boundary condition, we use (5.9) and the first equation in (5.8) to eliminate $A$ and thus obtain

$$\phi_0=-B\frac{|\omega|}{\alpha} \tag{5.10}$$

as before, and hence

$$\bar{\phi}=B\left(-\frac{|\omega|}{\alpha}e^{-\alpha x}+e^{-|\omega|x}\right). \tag{5.11}$$

Using (5.10) in (5.8) gives

$$A\begin{pmatrix} i\omega \\ \kappa \end{pmatrix}+B\Delta t\begin{pmatrix} \frac{z}{z-1}|\omega| \\ -\left(\frac{2z-1}{z(z-1)}+\frac{(z-1)|\omega|}{z\alpha}\right)i\omega \end{pmatrix}=\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix},$$

from which $A$ and $B$ can be determined. Relation (5.10) also follows from $\phi^{n+1}_{x,x=0}=\phi^n_{x,x=0}$, which is a consequence of (1.10), (1.14), and (5.7).

As with the previous case, we solve for the value of $\bar{v}$ on the boundary. From (3.5) we have

$$\bar{v}=\bar{v}^\star-i\omega\Delta t\bar{\phi}$$

and from (5.7)

$$z\bar{v}^\star_{x=0}=z\bar{b}_2+i\omega\Delta t\bar{\phi}_{x=0}.$$

Solving these last two equations for $\bar{v}_{x=0}$, we obtain from (5.11)

$$\begin{aligned} z\bar{v}_{x=0} &= z\bar{b}_2 - (z-1)\Delta t\; i\omega\bar{\phi}_{x=0} \\ &= z\bar{b}_2 - (z-1)\Delta t\; i\omega B\Big(1 - \frac{|\omega|}{\alpha}\Big) \\ &= z\bar{b}_2 + O(\Delta t^2), \end{aligned}$$

where we have used that $z - 1 = O(\Delta t)$ by (4.1). This shows that the boundary condition (5.7) gives less error than does the boundary condition (5.1), i.e., it is second-order accurate as opposed to the first-order accuracy of (5.1). But again the velocity boundary conditions are not satisfied for the tangential component of velocity. Therefore, neither of these boundary conditions gives $\vec{u}^{n+1} = \vec{b}^{n+1}$.

To be able to satisfy exactly the velocity boundary conditions using the fractional step method, it is necessary to have

$$\vec{u}^{\star,n+1}_{x=0} = \vec{b}^{n+1} + \Delta t\vec{\nabla}\phi^{n+1}_{x=0}.$$

However, it is not possible to implement this condition in an actual fractional step method since $\phi^{n+1}$ cannot be determined at the stage that $\vec{u}^{\star,n+1}$ is computed. Therefore, fractional step methods cannot satisfy the velocity boundary conditions.

Now we show that the pressure $p$ is only first-order accurate in time, not second-order, even if we impose the exact boundary condition $\vec{u}^{n+1} = \vec{b}^{n+1}$. (As we pointed out above, the method of Kim and Moin does not satisfy $\bar{v} = \bar{b}_2$.) The solution $\bar{p}_f$ to finite difference equations (1.9)–(1.11) with the exact boundary condition $\vec{u}^{n+1} = \vec{b}^{n+1}$ is given by

$$\bar{p}_f = \frac{z-1}{\Delta t z}\left(\frac{\bar{b}_1}{|\omega|} + \frac{\bar{b}_1 - \bar{b}_2 i\operatorname{sgn}(\omega)}{\sqrt{\frac{z-1}{z+1}\frac{2}{\Delta t} + \omega^2} - |\omega|}\right)e^{-|\omega|x}.$$

Let us compare the solution $\bar{p}_p$ to partial difference equations,

$$\bar{p}_p = s\left(\frac{\bar{b}_1}{|\omega|} + \frac{\bar{b}_1 - \bar{b}_2 i\operatorname{sgn}(\omega)}{\sqrt{s+\omega^2} - |\omega|}\right)e^{-|\omega|x}.$$

By considering only the coefficient of $\bar{b}_1/|\omega|$ and observing that

$$\left|s - \frac{z-1}{\Delta t z}\right| = |s|^2 O(\Delta t)$$

by (4.1), we see that the pressure is at most first-order accurate in time.

**6. Overwriting the boundary velocity.** In this section we analyze the consequences of modifying the fractional step scheme given by adding the additional step of overwriting the tangential velocity obtained in the projection step with the boundary value $b_2$. That is, after the projection step has determined the velocity components $u^{n+1}(x,y)$ and $v^{n+1}(x,y)$, we impose the condition

$$v^{n+1}(0,y) = b_2^{n+1}(y)$$

and then proceed to the next step. We presume that this is done in many codes that use similar methods, since the graphical display of the solution will appear to satisfy the correct boundary conditions. We call this scheme the fractional step scheme with overwriting.

To analyze this changed scheme, we consider the effect of the change in (1.9) where the value of $\vec{u}^n$ is replaced by $\vec{u}^n + \delta(x)(\vec{b}^n - \vec{u}^n)$. That is, we use the Dirac delta generalized function to modify the value of the function at the boundary. We then rewrite (1.9) as

$$\begin{aligned}\frac{\vec{u}^{\star,n+1} - \vec{u}^n}{\Delta t} &= \frac{1}{2}\nabla^2(\vec{u}^{\star,n+1} + [\vec{u}^n + \delta(x)(\vec{b}^n - \vec{u}^n)])\\ &= \frac{1}{2}\nabla^2([\vec{u}^{\star,n+1} + \delta(x)(\vec{b}^n - \vec{u}^n)] + \vec{u}^n).\end{aligned}$$

Notice we do not have to modify the values of $\vec{u}^{\star,n+1}$ in the time difference in the interior of the half-plane since the $\delta$ function has no effect there.

This equation shows that the effect of overwriting the value of the velocity at one step, e.g., step $n$, is equivalent to using the original fractional step scheme but changing the boundary condition to

$$\vec{u}^{*,n+1}_{x=0} = \vec{u}^{\star,n+1,0}_{x=0} + \vec{b}^n - \vec{u}^n_{x=0},$$

where the value of $\vec{u}^{\star,n+1,0}_{x=0}$ is determined by whatever boundary condition one was using for the first step of the fractional step method with overwriting. For example, with boundary condition (5.1) we have

$$\vec{u}^{*,n+1}_{x=0} = \vec{b}^{n+1} + \vec{b}^n - \vec{u}^n_{x=0}$$

and with (5.7) we have

$$\vec{u}^{*,n+1}_{x=0} = \vec{b}^{n+1} + \Delta t\vec{\nabla}\phi^n + \vec{b}^n - \vec{u}^n_{x=0}. \tag{6.1}$$

The accuracy of these boundary conditions can be analyzed in the same manner as was done in section 5. We now analyze boundary condition (6.1). Substituting the expressions (3.6) for $\vec{u}^*$ and (3.3) for $\vec{u}$, we obtain

$$\begin{aligned}&A\begin{pmatrix} i\omega \\ \kappa \end{pmatrix} + B\frac{\Delta t}{z-1}\begin{pmatrix} |\omega| \\ -i\omega \end{pmatrix} + \Delta t\phi_0\begin{pmatrix} -\alpha \\ i\omega \end{pmatrix}\\ &= \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}\left(1+\frac{1}{z}\right) + \frac{\Delta t\phi_0}{z}\begin{pmatrix} -\alpha \\ i\omega \end{pmatrix} + \frac{\Delta t B}{z}\begin{pmatrix} -|\omega| \\ i\omega \end{pmatrix}\\ &\quad - \frac{A}{z}\begin{pmatrix} i\omega \\ \kappa \end{pmatrix} - B\frac{\Delta t z}{z(z-1)}\begin{pmatrix} |\omega| \\ -i\omega \end{pmatrix}.\end{aligned}$$

After collecting terms and dividing by $(z+1)/z$, we obtain

$$A\begin{pmatrix} i\omega \\ \kappa \end{pmatrix} + B\frac{\Delta t}{z-1}\frac{3z-1}{z+1}\begin{pmatrix} |\omega| \\ -i\omega \end{pmatrix} + \Delta t\phi_0\frac{z-1}{z+1}\begin{pmatrix} -\alpha \\ i\omega \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix}.$$

Once again, we have (5.4), and the formula for $\phi$ is (5.5); hence,

$$\bar{v}_{x=0} = \bar{b}_2 - i\omega\Delta t B\frac{z-1}{z+1}\left(1 - \frac{|\omega|}{\alpha}\right),$$

and this gives

$$\bar{v}_{x=0} = \bar{b}_2 + O(\Delta t^2).$$

Thus overwriting gives the same order of accuracy for $v(t,0,y)$ before overwriting, and this gives second-order accuracy globally.

**7. Summary.** In summary, we have shown that the projection methods that rely on solving equations such as (1.10) and (1.11) cannot satisfy both boundary conditions for the projected velocity. This is inherent in the method since the system composed of (1.10) and (1.11) requires precisely one boundary condition, yet the original system requires two boundary conditions.

First, our analysis is simpler and more direct than that of E and Liu [2], [3]. Second, the analysis shows that the pressure is determined only to first order. Based on our analysis, it seems very difficult, if not impossible, to modify the scheme to get a higher-order approximation to the pressure.

We also correct the conclusion of E and Liu [2] about the boundary layer in the pressure. We show that the boundary layer is in the auxiliary pressure variable but not in the pressure itself. Our analysis of the overwritten boundary conditions is novel and fits within the general framework of our analysis.

In this paper we did not consider the spatial discretization, but the discretization cannot affect the basic conclusions we have drawn regarding the accuracy of the pressure and velocity. Norm estimates based on our analysis will be published in a subsequent paper.

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REFERENCES

[1] J. B. Bell, P. Colella, and H. M. Glaz, *A second-order projection method for the incompressible Navier–Stokes equations*, J. Comput. Phys., 85 (1989), pp. 257–283.

[2] W. E and J.-G. Liu, *Projection method* I*: Convergence and numerical boundary layers*, SIAM J. Numer. Anal., 32 (1995), pp. 1017–1057.

[3] W. E and J.-G. Liu, *Projection method* II*: Godunov–Ryabenki analysis*, SIAM J. Numer. Anal., 33 (1996), pp. 1597–1621.

[4] J. Kim and P. Moin, *Application of a fractional-step method to incompressible Navier–Stokes*, J. Comput. Phys., 59 (1985), pp. 308–323.

[5] R. L. LeVeque and J. Oliger, *Numerical methods based on additive splittings for hyperbolic partial differential equations*, Math. Comp. 40 (1983), pp. 467–497.

[6] S. A. Orszag, M. Israeli, and M. O. Deville, *Boundary conditions for incompressible flows*, J. Sci. Comput., 1 (1986), pp. 75–111.

[7] J. B. Perot, *An analysis of the fractional step method*, J. Comput. Phys., 108 (1993), pp. 51–58.

[8] R. Rannacher, *On Chorin's projection method for the incompressible Navier-Stokes equations*, in Navier-Stokes Equations II—Theory and Numerical Methods, Lecture Notes in Math. 1530, Springer-Verlag, Berlin, 1992, pp. 167–183.

[9] J. Shen, *On error estimates of projection methods for Navier–Stokes equations: First-order schemes*, SIAM J. Numer. Anal., 29 (1992), pp. 57–77.

[10] J. Shen, *On error estimates of some higher order projection and penalty-projection methods for Navier–Stokes equations: First-order schemes*, Numer. Math., 62 (1992), pp. 49–73.

[11] J. C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1989.

[12] J. C. Strikwerda, *High-order accurate schemes for incompressible viscous flow*, Internat. J. Numer. Methods Fluids, 24 (1997), pp. 715–734.

[13] R. Temam, *Remark on the pressure boundary condition for the projection method*, Theor. and Comp. Fluid Dynamics, 3 (1991), pp. 181–184.

[14] J. van Kan, *A second-order accurate pressure-correction scheme for viscous incompressible flow*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 870–891.