Decomposition Algebra

- While we could do decomposition indefinitely, we typically stop at k = 3 steps because 3 is the number of endogenous accounts within the SAM. In other words, the flow of income around the SAM undergoes 3 steps.

A_n and $A^o{}_n$

- We start by defining three matrices: A_n, $A^o{}_n$, and A*.
- A_n is the A matrix for our complete partitioned SAM

$$A_n = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & 0 & 0 \\ 0 & A_{32} & A_{33} \end{bmatrix}$$

- $A^o{}_n$ is the sub-matrix of inter-industry and inter-institutional transfers

$$A^o{}_n = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

A*

- Remember that $A^* = (I - A_n)^{-1} (A_n - A^o_n)$, where the first term is equivalent to

$$\left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right)^{-1} = \begin{bmatrix} (I - A_{11})^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I - A_{33})^{-1} \end{bmatrix}$$

and the second term is equivalent to

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & 0 & 0 \\ 0 & A_{32} & A_{33} \end{bmatrix} - \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{13} \\ A_{21} & 0 & 0 \\ 0 & A_{32} & 0 \end{bmatrix}$$

A*

- Multiplying these two terms gives

$$A^* = \begin{bmatrix} 0 & 0 & (I - A_{11})^{-1} A_{13} \\ A_{21} & 0 & 0 \\ 0 & (I - A_{33})^{-1} A_{32} & 0 \end{bmatrix}$$

Note that we can define the elements of A* as

$(I – A_{11})^{-1}A_{13} = A^*_{13}$ $A_{21} = A^*_{21}$ $(I – A_{33})^{-1}A_{32} = A^*_{32}$

$$A^* = \begin{bmatrix} 0 & 0 & A^*_{13} \\ A^*_{21} & 0 & 0 \\ 0 & A^*_{32} & 0 \end{bmatrix}$$

such that A* follows the circular income flow in the SAM.

$M_{a3}M_{a2}M_{a1}$

- With

$$y_n = (I - A^{*3})^{-1} \; (I + A^* + A^{*2}) \; (I - A^o{}_n)^{-1}x = M_a x$$

we can define the SAM multiplier M_a as the product of three matrices:

$$M_a = M_{a3}M_{a2}M_{a1}$$

where

$$M_{a1} = (I - A^o{}_n)^{-1}$$

$$M_{a2} = (I + A^* + A^{*2})$$

$$M_{a3} = (I - A^{*3})^{-1}$$

M_{a1}

- For $M_{a1} = (I - A^o{}_n)^{-1}$

  Remember that in our partitioned SAM

$$A^o{}_n = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

Thus

$$M_{a1} = \begin{bmatrix} (I - A_{11})^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I - A_{33})^{-1} \end{bmatrix}$$

M_{a1}

- From the $(I-A_{11})^{-1}$ and $(I-A_{33})^{-1}$ elements of M_{a1} you can begin to develop some intuition about how to interpret the decomposed multipliers.
- M_{a1} is typically referred to as the transfers, or direct effects, multiplier, because it captures the multiplier effects of transfers within accounts; in this case industries, i.e. $(I-A_{11})^{-1}$, and institutions, i.e. $(I-A_{33})^{-1}$.
- M_{a1} only captures within account effects; it tells us nothing about factors or institutions.

M_{a2}

- Similarly, for $M_{a2} = (I + A^* + A^{*2})$, where A^{*2} is

$$A^{*2} = \begin{bmatrix} 0 & (I - A_{11})^{-1}A_{13}(I - A_{33})^{-1}A_{32} & 0 \\ 0 & 0 & A_{21}(I - A_{11})^{-1}A_{13} \\ (I - A_{33})^{-1}A_{32}A_{21} & 0 & 0 \end{bmatrix}$$

or more simply

$$A^{*2} = \begin{bmatrix} 0 & A^*{}_{13}A^*{}_{32} & 0 \\ 0 & 0 & A^*{}_{21}A^*{}_{13} \\ A^*{}_{32}A^*{}_{21} & 0 & 0 \end{bmatrix}$$

M_{a2}

- Thus $M_{a2} = (I + A^* + A^{*2})$ is

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 & A^*{}_{13} \\ A^*{}_{21} & 0 & 0 \\ 0 & A^*{}_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & A^*{}_{13}A^*{}_{32} & 0 \\ 0 & 0 & A^*{}_{21}A^*{}_{13} \\ A^*{}_{32}A^*{}_{21} & 0 & 0 \end{bmatrix}$$

or
$$\begin{bmatrix} I & A^*{}_{13}A^*{}_{32} & A^*{}_{13} \\ A^*{}_{21} & I & A^*{}_{21}A^*{}_{13} \\ A^*{}_{32}A^*{}_{21} & A^*{}_{32} & I \end{bmatrix}$$

M_{a2}

- M_{a2} is the only matrix with off-diagonal elements, and is referred to as the cross-effects, or open-loop, multiplier.
- M_{a2} captures the effects of an injection into the system as it moves through the system without coming back to its origin (hence the name 'open-loop'). In other words, M_{a2} shows how an external injection travels from endogenous demand to income ("across" institutions), but not from income to demand.

M_{a3}

- $M_{a3} = (I - A^{*3})^{-1}$, where A^{*3} is

$$\begin{bmatrix} A^*_{13}A^*_{32}A^*_{21} & 0 & 0 \\ 0 & A^*_{21}A^*_{13}A^*_{32} & 0 \\ 0 & 0 & A^*_{32}A^*_{21}A^*_{13} \end{bmatrix}$$

and $(I - A^{*3})^{-1}$ is

$$\begin{bmatrix} (I - A^*_{13}A^*_{32}A^*_{21})^{-1} & 0 & 0 \\ 0 & (I - A^*_{21}A^*_{13}A^*_{32})^{-1} & 0 \\ 0 & 0 & (I - A^*_{32}A^*_{21}A^*_{13})^{-1} \end{bmatrix}$$

Additive Multipliers

- All three multiplier forms — aggregate, multiplicative, and additive — are related by

$$M_a = M_3 M_2 M_1 = I + T + O + C$$

  where
  - I = Identity multiplier
  - $T = (M_1 - I)$ = Net transfer multiplier
  - $O = (M_2 - I)M_1 = (M_2 M_1 - M_1)$ = Open-loop multiplier
  - $C = (M_3 - I)M_2 M_1 = (M_3 M_2 M_1 - M_2 M_1)$ = Closed-loop multiplier