Lecture 2.1
Basic Analytics of GE Modeling

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Introduction

- In this lecture, we will introduce some of the primary building blocks for constructing an economy-wide model.
- The lecture will be a bit dry since we will introduce the basic micro-economic theory that is the backbone of economic policy analysis.
- Keep in mind, that the goal is to introduce functionality to each of the individual cells in the SAM.

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1. Producer Behavior

- Because of their evolutionary role as institutions that assemble production teams, the fundamental behavioral model for enterprises is the production function.
- This mathematical specification of how factor services combine to transform resources and components into goods and services lies at the heart of the neoclassical paradigm.
- Here we review a variety of widely accepted specifications for production functions and discuss how they can be implemented empirically.

Producer Optimization

The most elementary production structure considers only two factors, ignoring other factors and intermediate goods. Assuming a given level of output, producers choose inputs in order to minimise costs. In mathematical terms the producer's decision can be formulated as:

$$\text{Min}(wL+rK) \quad \text{subject to} \quad V=F(K,L)$$

where *w* is the wage rate, *r* the rental rate of capital, *K* and *L* are the inputs to production, and *V* is the level of output. The producer chooses *K* and *L*, with *V*, *w*, and *r* given. In most GE models, the levels of output *V*, *w*, and *r* are determined not by producers but by market equilibrium conditions.

First-order Conditions

- Setting up the Lagrangian to the problem above, we have:

$$\mathcal{L} = wL + rK - P\left[V - F(K, L)\right]$$

- Setting the partial derivatives with respect to *K*, *L* and *P* equal to zero, we have the following three first order conditions:

$$\frac{\partial \mathcal{L}}{\partial L} = 0 \Rightarrow w = P\frac{\partial F}{\partial L}$$

$$\frac{\partial \mathcal{L}}{\partial P} = 0 \Rightarrow V = F(K, L)$$

$$\frac{\partial \mathcal{L}}{\partial K} = 0 \Rightarrow r = P\frac{\partial F}{\partial K}$$

CES Production

Let's explore this relation a bit further by assigning a functional form to *F*, notably the *Constant-Elasticity-of-Substitution* function, otherwise known as the CES function. The primal form of the CES function is:

$$V = F(K, L) = \left[a_l L^\rho + a_k K^\rho\right]^{1/\rho}$$

where the coefficients *al* and *ak* are called the labour and capital share parameters, respectively, and *r* is the CES exponent (which will be related to the CES substitution elasticity).

First-order CES

Differentiating this expression yields

$$\frac{\partial F}{\partial L} = \frac{1}{\rho}\left[a_l L^{\rho} + a_k K^{\rho}\right]^{1/\rho - 1} a_l \rho L^{\rho-1} = a_l \left(\frac{L}{V}\right)^{\rho-1}$$

$$\frac{\partial F}{\partial K} = a_k \left(\frac{K}{V}\right)^{\rho-1}$$

which implies

$$\begin{cases} w = P\dfrac{\partial F}{\partial L} = Pa_l\left(\dfrac{L}{V}\right)^{\rho-1} \\ \\ r = P\dfrac{\partial F}{\partial K} = Pa_k\left(\dfrac{K}{V}\right)^{\rho-1} \end{cases}$$

CES Factor Demand

Three simplifications,

$$\sigma = \frac{1}{1-\rho} \Leftrightarrow \rho = \frac{\sigma - 1}{\sigma}$$

$$\alpha_l = a_l^{1/(1-\rho)} = a_l^{\sigma}$$

$$\alpha_k = a_k^{1/(1-\rho)} = a_k^{\sigma}$$

then lead to relatively transparent derived factor demands:

$$\boxed{\begin{aligned} L &= \alpha_l \left(\frac{P}{w}\right)^{\sigma} V \\ K &= \alpha_k \left(\frac{P}{r}\right)^{\sigma} V \end{aligned}}$$

CES Unit Cost and Pricing

Consider now the total cost function:

$$PV = wL + rK$$

and substitute the reduced form expressions for L and K

$$PV = w\alpha_l\left(\frac{P}{w}\right)^{\sigma}V + r\alpha_k\left(\frac{P}{r}\right)^{\sigma}V = VP^{\sigma}\left[\alpha_l w^{1-\sigma} + \alpha_k r^{1-\sigma}\right]$$

This yields the unit price-cost equivalence from duality

$$P = \left[\alpha_l w^{1-\sigma} + \alpha_k r^{1-\sigma}\right]^{1/(1-\sigma)}$$

Generalized CES

The CES can be extended to i=2,…,n inputs as

$$\min \sum_i P_i X_i \qquad\qquad V = A\left[\sum_i a_i \left(\lambda_i X_i\right)^{\rho}\right]^{1/\rho}$$

with these reduced-from factor demands and unit costs

$$X_i = \alpha_i \left(A\lambda_i\right)^{\sigma-1}\left(\frac{P}{P_i}\right)^{\sigma} V \qquad P = \frac{1}{A}\left[\sum_i \alpha_i \left(\frac{P_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/\left(1-\sigma\right)}$$

Special Case 1: Leontief

In the case where σ=0, there is no input substitution and we have the Leontief

$$V = A \min\left(\lambda_i \frac{X_i}{\alpha_i} \right)$$

linear $\quad X_i = \dfrac{\alpha_i}{A\lambda_i} V \quad$ production technology

and the output price is simply the weighted average of input prices

$$P = \frac{1}{A} \sum_i \frac{\alpha_i}{\lambda_i} P_i$$

Special Case 2: Cobb-Douglas

In the case where [illegible]=1, we have the most widely used production specification, Cobb-Douglas

$$V = A \prod_i \left(\lambda_i X_i\right)^{\alpha_i}$$

$X_i = \alpha_i \frac{P}{P_i} V$

linear production technology

and the output price is simply the weighted average of input prices

$$P = \frac{1}{A} \prod_i \left(\frac{P_i}{\alpha_i \lambda_i}\right)^{\alpha_i}$$

Where we assure constant returns to scale with the added assumption that $\sum_i \alpha_i = 1$

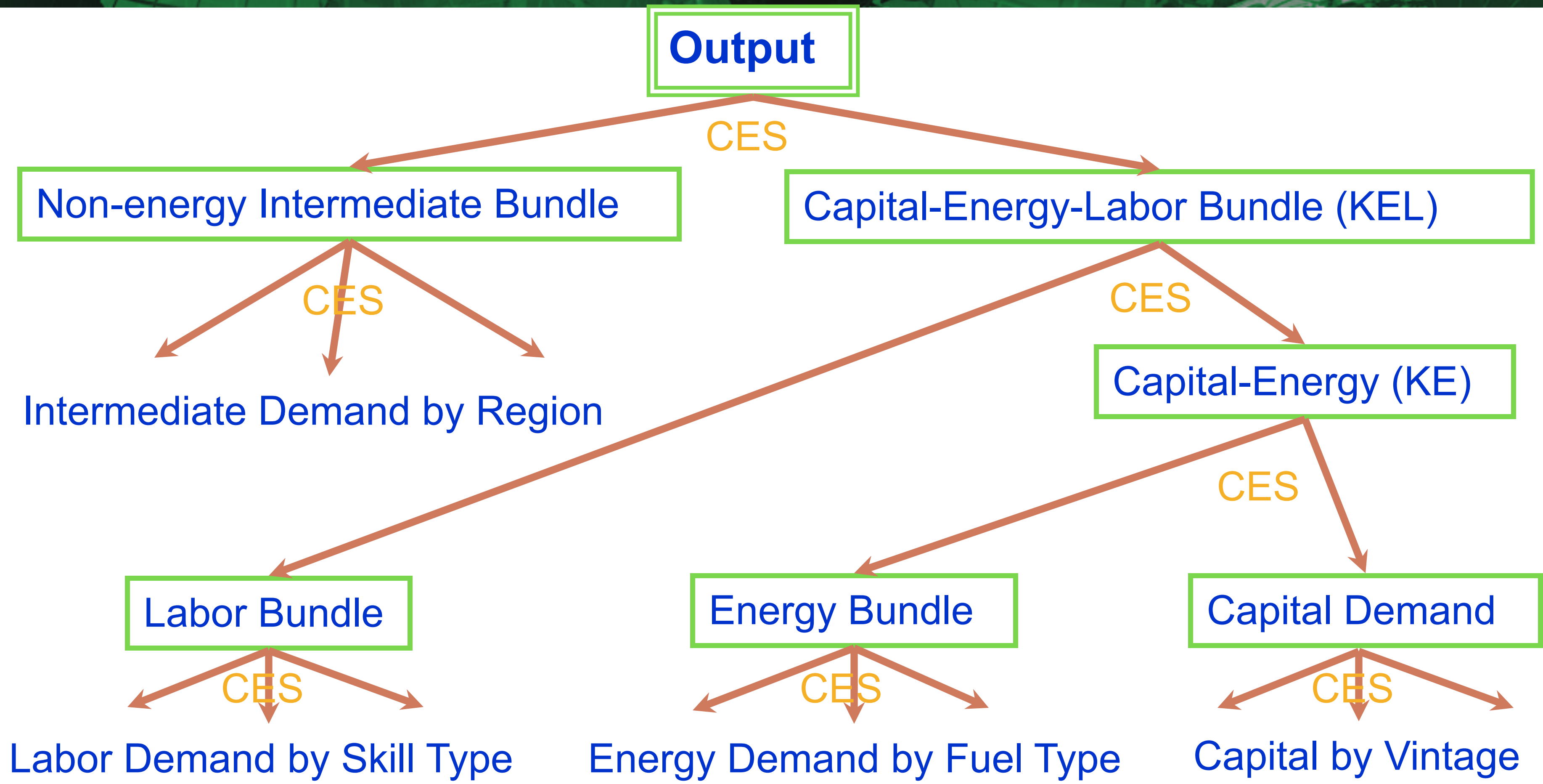
A Complication: Stratified Input Substitution

It is generally unrealistic to assume a single elasticity of substitution between all input types.

To overcome this limitation, we stratify input choice into generic sub-groups, each with their own substitution properties.

Consider an example with four types of inputs: Intermediates, Labor, Energy, and Capital.

Stratified Production Structure



Input Demand and Prices for Two Strata

Upper level (Labor, Non-labor)

$$L = \alpha_l \left(\frac{P}{w} \right)^{\sigma^1} XP$$

$$KE = \alpha_{ke} \left(\frac{P}{PKE} \right)^{\sigma^1} XP$$

$$P = \left[\alpha_l w^{1-\sigma^1} + \alpha_{ke} PKE^{1-\sigma^1} \right]^{1/\left(1-\sigma^1\right)}$$

Lower level (Capital, Energy)

$$K = \alpha_k \left(\frac{PKE}{r} \right)^{\sigma^2} KE$$

$$E = \alpha_e \left(\frac{PKE}{PE} \right)^{\sigma^2} KE$$

$$PKE = \left[\alpha_k r^{1-\sigma^2} + \alpha_e PE^{1-\sigma^2} \right]^{1/\left(1-\sigma^2\right)}$$

CES Comparison

Consider a classic comparative static example, a 10% energy tax.

	Single stratum: σ*=0.8*	*Dual strata:* σ^1*=0.8,* σ^2*=0.2*
ΔL	0.8	0.8
ΔK	0.8	-0.4
ΔE	-6.6	-2.3
$\Delta E/\Delta L$	-7.3	-3.0
$\Delta E/\Delta K$	-7.3	-1.9

The results differ in important ways. In the stratified case, energy and capital are both substitutes and complements, sharply reducing the substition away from energy.

2. Household Behavior

- Ten representative household categories, but state income tax bracket
- Income from all factors, enterprises, public and private transfers
- Consumption modeled with the Extended Linear Expenditure System
- Extensive tax and transfer mechanisms
- Demographic dynamics (population, labor force participation)

Consumer demand

- Consumer demand theory must represent the complexity of human behaviour in a mathematically tractable fashion, and is in many respects more difficult than production theory.
- Whereas, production theory essentially consists of an adequate representation of technology, and minimising the cost of production subject to the representation of technology, consumer demand theory must explain personal tastes and preferences.
- There are many subjective factors which influence consumer decisions. Economists have developed a set of axioms which are generally believed to hold in many situations:
  - *Complete*: preferences for any two goods can be ordered.
  - *Transitive*: x preferred to y, and y preferred to z means that x is preferred to z
  - *Non-satiation*: more is always better.

Linear Expenditure Systems

- Economists have devised a tool to analyse preferences which meet these axioms, it is known as a utility function.
- The utility function represents consumer preference ordering for alternative bundles of goods, subject of course to remaining within the limits of their budget. If *x* represents the bundle of goods (where *x* is an *n*-dimensional vector), then the consumer problem can be set up as follows:

Max U(x) subject to $$\sum_i p_i x_i = Y$$

where *U* is a single valued function with *n* inputs, *p* is the vector of consumer prices, and *Y* is disposable income. The constraint is of course the consumer budget constraint, which restricts expenditures to available income.

Linear Expenditure Systems

A commonly used utility function is known as the linear expenditure system (LES), also known as the Stone-Geary expenditure function. The form of the LES utility function,

$$U(x) = \sum_i \mu_i \ln\left(x_i - \theta_i\right)$$

which corresponds to the Lagrangian

$$\mathcal{L} = \sum_i \mu_i \ln\left(x_i - \theta_i\right) + \lambda\left(Y - \sum_i p_i x_i\right)$$

whose first-order conditions

$$\frac{\mu_i}{x_i - \theta_i} - \lambda p_i = 0 \qquad \text{and} \qquad Y - \sum_i p_i x_i = 0$$

yield the individual commodity demand function $x_i = \theta_i + \dfrac{\mu_i}{\lambda p_i}$

LES Derived Demand

To express demand as a reduced-form in prices, note that

$$Y = \sum_i p_i \left(\theta_i + \frac{\mu_i}{\lambda p_i} \right) = \sum_i p_i \theta_i + \sum_i p_i \left(\frac{\mu_i}{\lambda p_i} \right) = \frac{1}{\lambda} + \sum_i p_i \theta_i$$

so the Lagrange multiplier can be expressed as

$$\lambda = \frac{1}{Y - \sum_j p_j \theta_j}$$

which can be substituted back to yield the final demand equation

$$x_i = \theta_i + \frac{\mu_i}{p_i} \left(Y - \sum_j p_j \theta_j \right)$$

consisting of "subsistence minimum" demand $_i$ and a defined share of "supernumerary income," the term in parentheses.

LES Elasticities

Income elasticities can be obtained directly from

$$\frac{\partial x_i}{\partial Y} = \frac{\mu_i}{p_i} \qquad \text{and, by extension} \qquad \eta_i = \frac{\partial x_i}{\partial Y}\frac{Y}{x_i} = \frac{\mu_i}{p_i}\frac{Y}{x_i} = \frac{\mu_i}{s_i}$$

In words, LES income elasticities are the ratios of marginal (μ_i) to average (s_i) expenditure shares.

Price elasticities follow from $\frac{\partial x_i}{\partial p_i} = -\frac{\mu_i}{p_i^2}Y^* + \frac{\mu_i}{p_i}\left(-\theta_i\right) = -\frac{\mu_i}{p_i}\left(\frac{Y^*}{p_i} + \theta_i\right)$ as

$$\varepsilon_i = \frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i} = -\frac{p_i}{x_i}\frac{\mu_i}{p_i}\left(\theta_i + \frac{1}{p_i}\frac{p_i}{\mu_i}\left(x_i - \theta_i\right)\right) = -\frac{\mu_i\theta_i}{x_i} - \frac{1}{x_i}\left(x_i - \theta_i\right) = \frac{\theta_i\left(1-\mu_i\right)}{x_i} - 1$$

3. Other Final Demand

- Other final demand accounts are represented by a single demand matrix.
- Examples are
  - government current spending
  - government capital spending
  - private capital spending
  - trade and transport margins for domestic and imported goods
- All these final demand vectors are presently assumed to have fixed (linear) expenditure shares .

Government

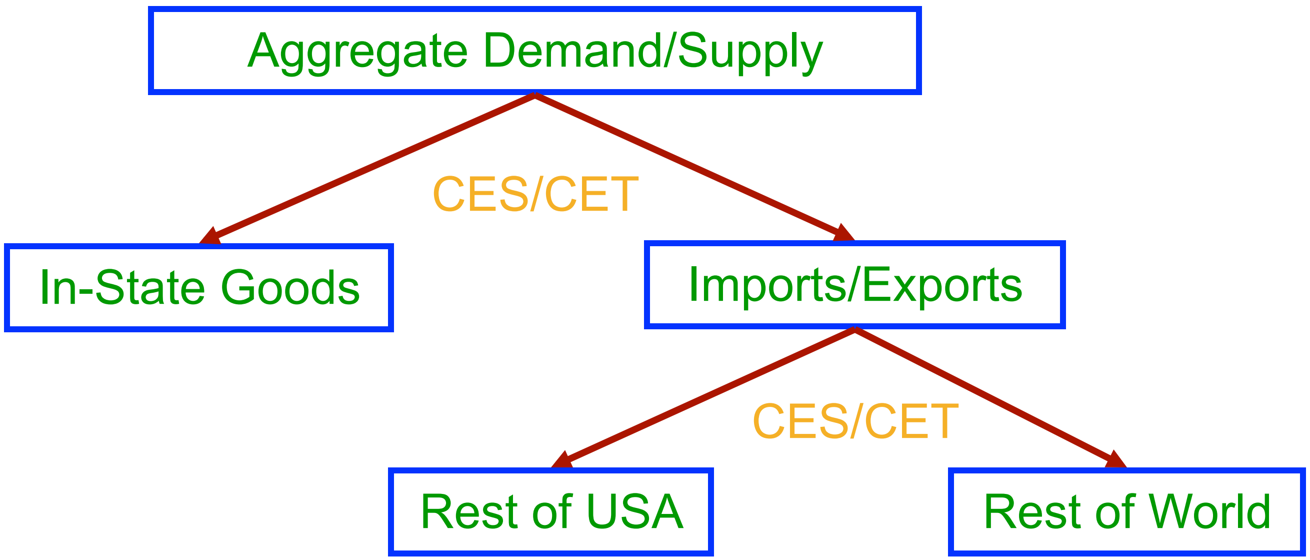
- Government is a passive actor in the baseline, adhering to established expenditure patterns and fiscal programs
- The model details extensive accounting for transfer relationships between institutions (fiscal, capital flows, remittances, etc.).
- Government behavior is a primary driver of scenarios, but this behavior remains largely exogenous (subject to fiscal closure)

4. Trade

- Trade is the final critical component of demand. In this section we will discuss exports and imports separately.
- The most widely used assumption in models of imports is that they are different from domestic goods.
- The degree of differentiation is critical. If there are no differences, i.e. imports are exactly the same as there domestic counterpart, then imports are just the residual between domestic production and domestic demand.
- If production is greater than demand, then imports will be zero, and exports will be positive. If, on the contrary, domestic demand is greater than production, then imports will be positive (i.e. the gap between demand and production), and exports will be zero.
- There are few commodities which are truly homogeneous, though some are close, e.g. cereal grains and crude oil.

Trade Stratification

- Demand is thought to combine in-state and imported goods in each product category with a nested CES aggregation



- Output is modeled symmetrically with a dual nested CET structure

Trade Analytically - Imports

Denoting domestic demand by XD and imports by XM, total demand is modeled with the CES preference function

Min(PD•XD+PM•XM) subject to $XA = \left[a_d XD^{\rho} + a_m XM^{\rho}\right]^{1/\rho}$

where PD and PM denote prices for domestic and imported goods and XA is aggregate demand. Passing over derivations from the production analytics, we have the following reduced forms

$$XD = \alpha_d \left(\frac{PA}{PD}\right)^{\sigma} XA \qquad XM = \alpha_m \left(\frac{PA}{PM}\right)^{\sigma} XA \quad \text{where} \quad \begin{cases} \alpha_d = a_d^{\sigma} \\ \alpha_m = a_m^{\sigma} \\ \rho = \dfrac{\sigma - 1}{\sigma} \end{cases}$$

and $PA = \left[\alpha_d XD^{1-\sigma} + \alpha_m XM^{1-\sigma}\right]^{1/(1-\sigma)}$ denotes the price index of XA.

Trade Analytically - Exports

Denoting domestic supply by XD and export supply by XE, total supply is modeled with the CET production frontier

Max(PD•XD+PE•XE) subject to $XP = \left[g_d XD^{\omega} + g_e XE^{\omega}\right]^{1/\omega}$

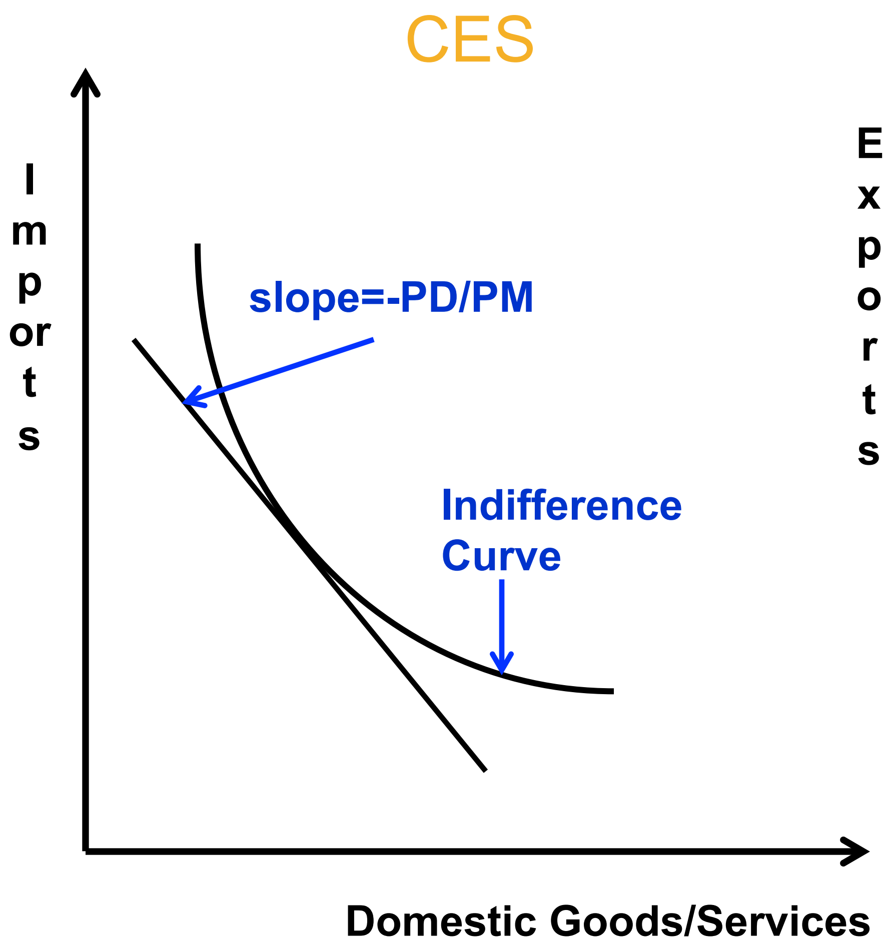
where PD and PM denote prices for domestic and imported goods and XA is aggregate demand. Passing over derivations from the production analytics, we have the following reduced forms

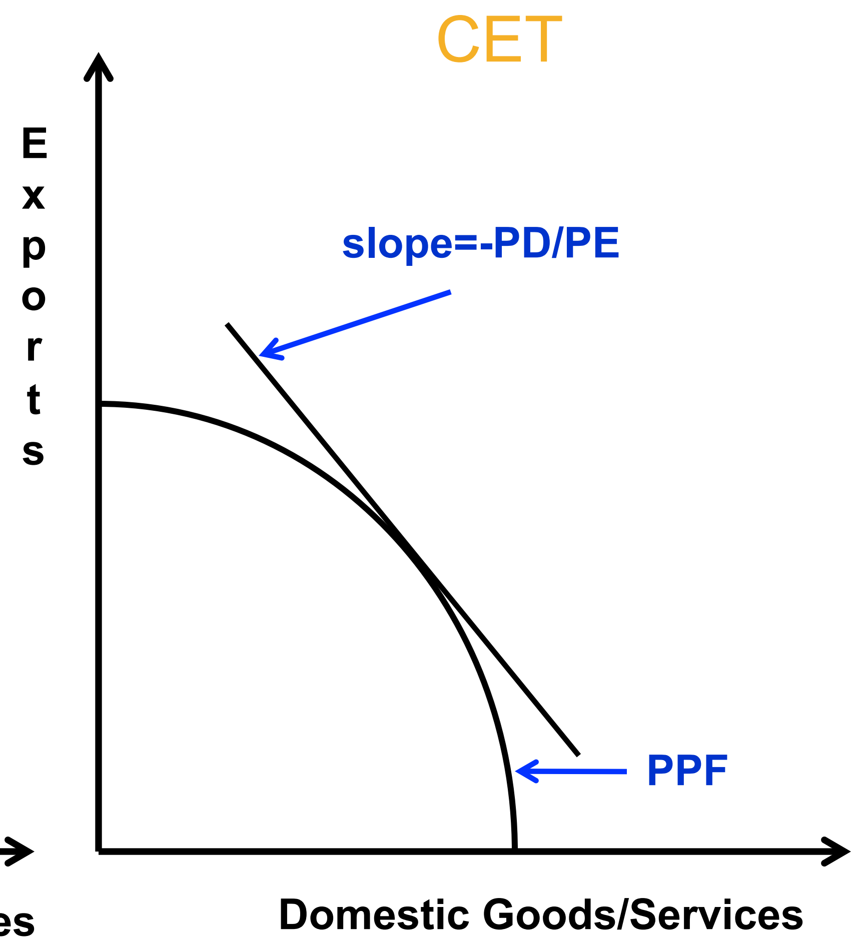
$$XD = \gamma_d \left(\frac{PD}{PP}\right)^{\nu} XP \qquad XE = \gamma_e \left(\frac{PE}{PP}\right)^{\nu} XP \quad \text{where} \quad \begin{cases} \gamma_d = g_d^{-\nu} \\ \gamma_e = g_e^{-\nu} \\ \omega = \dfrac{\nu+1}{\nu} \end{cases}$$

and $PP = \left[\gamma_d XD^{1+\nu} + \gamma_e XE^{1+\nu}\right]^{1/(1+\nu)} = \left(PD.XD + PE.XE\right) / XP$

denotes the price index of XP.

Trade Schematically





Trade Prices

- A single domestic price equilibrates demand and supply of each domestic good.
- Each trade node clears with a market-clearing price. The model thus has (nxr)(r+1) trade prices, for n goods and r trading partners.
- FOB/CIF wedges are modeled using trade and transport margins.

5. Labor

- Supplied by households in response to a labor-leisure choice
- Employed by sector and occupation, with perfect mobility between the former and none (currently) between the latter
- Labor markets are perfectly competitive
- Migration is not currently modeled

6. Income distribution

The prototype model has a rich menu of income distribution channels—factor income and intra-household, government and foreign transfers (i.e. remittances). The prototype also includes corporations used as a pass-through account for channeling operating surplus.

Factor Income

Income accrues directly to labor, capital, land, and resources:

$$LY_l = \sum_i \frac{W_{i,l} L_{i,l}^d}{1 + \tau_{i,l}^{fl}} \qquad TY_{lt} = \sum_i \frac{PT_{i,lt} T_{i,lt}^d}{1 + \tau_{i,lt}^{ft}}$$

$$KY_{kt} = \sum_i \frac{R_{i,kt} K_{i,kt}^d}{1 + \tau_{i,kt}^{fk}} \qquad RY = \sum_i \frac{PR_i R_i^d}{1 + \tau_i^{fr}}$$

Profit Distribution

Profits are distributed to enterprises (E), households (H), and to the Rest of the World (W):

$$TR^{E}_{k,kt} = \varphi^{E}_{k,kt} KY_{kt}$$

$$TR^{H}_{k,kt} = \varphi^{H}_{k,kt} KY_{kt}$$

$$TR^{W}_{k,kt} = \varphi^{W}_{k,kt} KY_{kt}$$

Enterprise Income

Enterprises have earnings that are retained (S) or distributed to households (H) or foreigners (W) :

$$CY_e = \sum_{kt} \varphi_{kt,e}^{e} TR_{k,kt}^{E} \qquad TR_{c,e}^{H} = \varphi_{c,e}^{H}\left(1-\kappa_e^c\right)CY_e$$

$$S_e^c = s_e^c\left(1-\kappa_e^c\right)CY_e \qquad TR_{c,e}^{W} = \varphi_{c,e}^{W}\left(1-\kappa_e^c\right)CY_e$$

Household Incomes

Household income comes from many sources:

$$YH_h = \underbrace{\sum_{l} \varphi^h_{l,l} LY_l}_{\text{Labor}} + \underbrace{\sum_{kt} \varphi^h_{kt,h} TR^H_{k,kt}}_{\text{Capital}} + \underbrace{\sum_{lt} \varphi^h_{lt,h} TY_{lt}}_{\text{Land}}$$

$$+ \underbrace{\varphi^h_{nr,h} RY}_{\text{Sector-specific factor}} + \underbrace{\sum_{e} \varphi^h_{e,h} TR^H_{c,e}}_{\text{Enterprise}} + \underbrace{PLEV.TR^h_{g,h}}_{\text{Transfers from government}}$$

$$+ \underbrace{\sum_{h'} TR^h_{h,h'}}_{\text{Intra-household transfers}} + \underbrace{ER \sum_{r} TR^h_{r,h}}_{\text{Foreign remittances}}$$

Household Component Accounts

Households also have several secondary accounts:

$$YD_h = \left(1 - \lambda^h \kappa_h^h\right) YH_h - TR_h^H$$

$$TR_h^H = \varphi_{h,h}^H \left(1 - \lambda^h \kappa_h^h\right) YH_h$$

$$TR_{h,h'}^h = \varphi_{h,h'}^h TR_h^H$$

$$TR_{h,r}^w = \varphi_{h,r}^w TR_h^H$$

7. Macroeconomic Closure

1. Government fiscal balance is exogenous, achieved with an endogenous direct tax schedule
2. Private investment is endogenous and is driven by available savings
3. The volume of government current and investment expenditures is exogenous
4. The volume of demand for international trade and transport services is exogenous
5. The volume of stock changes is exogenous
6. The trade balance (i.e. capital flows) is exogenous. The real exchange rate equilibrates the balance of payments.

8. Equilibrium Conditions

- Goods and Services: Combined domestic and external demand equals supply for every good and service
- Factors: Domestic factor (labor, capital, land) supply equals in-state factor demand
- Trade: Thailand's net outflow of goods and services equals its net claims on external financial assets

9. Dynamics

- Labor force and population growth are currently exogenous.
- Capital stock is driven by past investments and depreciation.
- Total factor productivity is calibrated in baseline to achieve a GDP growth target.
- Productivity is currently exogenous.

10. Model Extensions

- Health sector reform, productivity, and fiscal sustainability
- Demographic change and public health management
- More labor market structure and conduct (occupations, unemployment, migration, bargaining, rigidities, etc.)
- Trade and regional economic integration
- Emissions, Climate Change, and Public Health
- Energy and other strategic commodities
- Location/mapping

Discussion