Solutions to Ahlfors' Complex Analysis

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Contents

1 Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1. Find the values of

$$(1+2i)^3, \qquad \frac{5}{-3+4i}, \qquad \left(\frac{2+i}{3-2i}\right)^2, \qquad (1+i)^n+(1-i)^n$$

For the first problem, we have $(1+2i)^3 = (-3+4i)(1+2i) = -11-2i$. For the second problem, we should multiple by the conjugate $\bar{z} = -3-4i$.

$$\frac{5}{-3+4i}\frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \frac{-3}{5} - \frac{4}{5}i$$

For the third problem, we should first multiple by $\bar{z} = 3+2i$.

$$\frac{2+i}{3-2i}\frac{3+2i}{3+2i} = \frac{8+i}{13}$$

Now we need to just square the result.

$$\frac{1}{169}(8+i)^2 = \frac{63+16i}{169}$$

For the last problem, we will need to find the polar form of the complex numbers. Let $z_1 = 1+i$ and $z_2 = 1-i$. Then the modulus of $z_1 = \sqrt{2} = z_2$. Let ϕ_1 and ϕ_2 be the angles associated with z_1 and z_2, respectively. Then $\phi_1 = \arctan(1) = \frac{\pi}{4}$ and $\phi_2 = \arctan(-1) = \frac{-\pi}{4}$. Then $z_1 = \sqrt{2}e^{\pi i/4}$ and $z_2 = \sqrt{2}e^{-\pi i/4}$.

$$\begin{aligned} z_1^n + z_2^n &= 2^{n/2}\left[e^{n\pi i/4} + e^{-n\pi i/4}\right] \\ &= 2^{n/2+1}\left[\frac{e^{n\pi i/4} + e^{-n\pi i/4}}{2}\right] \\ &= 2^{n/2+1}\cos\left(\frac{n\pi}{4}\right) \end{aligned}$$

2. If $z = x+iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}$$

For z^4, we can use the binomial theorem since $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^n b^{n-k}$. Therefore,

$$(x+iy)^4 = \binom{4}{0}(iy)^4 + \binom{4}{1}x(iy)^3 + \binom{4}{2}x^2(iy)^2 + \binom{4}{3}x^3(iy) + \binom{4}{4}x^4 = y^4 - 4xy^3 i - 6x^2y^2 + 4x^3yi + x^4$$

Then the real and imaginary parts are

$$\begin{aligned} u(x,y) &= x^4 + y^4 - 6x^2y^2 \\ v(x,y) &= 4x^3y - 4xy^3 \end{aligned}$$

For second problem, we need to multiple by the conjugate $\bar{z}$.

$$\frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

so the real and imaginary parts are

$$u(x,y) = \frac{x}{x^2+y^2}$$
$$v(x,y) = \frac{-y}{x^2+y^2}$$

For the third problem, we have $\frac{x-1+iy}{x+1-iy}$. Then $\bar{z} = x+1+iy$.

$$\frac{x-1+iy}{x+1-iy}\frac{x+1+iy}{x+1+iy} = \frac{x^2-1+2xyi}{(x+1)^2+y^2}$$

Then real and imaginary parts are

$$u(x,y) = \frac{x^2-1}{(x+1)^2+y^2}$$
$$v(x,y) = \frac{2xy}{(x+1)^2+y^2}$$

For the last problem, we have

$$\frac{1}{z^2} = \frac{x^2-y^2-2xyi}{x^4+2x^2y^2+y^4}$$

so the real and imaginary parts are

$$u(x,y) = \frac{x^2-y^2}{x^4+2x^2y^2+y^4}$$
$$v(x,y) = \frac{-2xy}{x^4+2x^2y^2+y^4}$$

3. Show that $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3 = 1$ and $\left(\frac{\pm 1\pm i\sqrt{3}}{2}\right)^6 = 1$.

   Both problems will can be handled easily by converting to polar form. Let $z_1 = \frac{-1\pm i\sqrt{3}}{2}$. Then $|z_1| = 1$. Let ϕ_+ be the angle for the positive z_1 and ϕ_- for the negative. Then $\phi_+ = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$ and $\phi_- = \arctan(\sqrt{3}) = \frac{4\pi}{3}$. We can write $z_{1+} = e^{2i\pi/3}$ and $z_{1-} = e^{4i\pi/3}$.

$$\begin{aligned} z_{1+}^3 &= e^{2i\pi} \\ &= 1 \\ z_{1-}^3 &= e^{4i\pi} \\ &= 1 \end{aligned}$$

   Therefore, $z_1^3 = 1$. For the second problem, $\phi_{ij} = \pm\frac{\pi}{3}$ and $\pm\frac{2\pi}{3}$ for $i,j = +,-$ and the $|z_2| = 1$. When we raise z to the sixth poewr, the argument becomes $\pm 2\pi$ and $\pm 4\pi$.

$$e^{\pm 2i\pi} = e^{\pm 4i\pi} = z^6 = 1$$

1.1.2 Square Roots

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}$$

   For $\sqrt{i}$, we are looking for x and y such that

$$\begin{aligned} \sqrt{i} &= x+iy \\ i &= x^2-y^2+2xyi \\ x^2-y^2 &= 0 \qquad (1.1) \\ 2xy &= 1 \qquad (1.2) \end{aligned}$$

From equation (1.1), we see that $x^2 = y^2$ or $\pm x = \pm y$. Also, note that i is the upper half plane (UHP). That is, the angle is positive so $x = y$ and $2x^2 = 1$ from equation (1.1). Therefore, $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$. We also could have done this problem using the polar form of z. Let $z = i$. Then $z = e^{i\pi/2}$ so $\sqrt{z} = e^{i\pi/4}$ which is exactly what we obtained. For $\sqrt{-i}$, let $z = -i$. Then z in polar form is $z = e^{-i\pi/2}$ so $\sqrt{z} = e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$. For $\sqrt{1+i}$, let $z = 1+i$. Then $z = \sqrt{2}e^{i\pi/4}$ so $\sqrt{z} = 2^{1/4}e^{i\pi/8}$. Finally, for $\sqrt{\frac{1-i\sqrt{3}}{2}}$, let $z = \frac{1-i\sqrt{3}}{2}$. Then $z = e^{-i\pi/3}$ so $\sqrt{z} = e^{-i\pi/6} = \frac{1}{2}(\sqrt{3}-i)$.

2. Find the four values of $\sqrt[4]{-1}$.

   Let $z = \sqrt[4]{-1}$ so $z^4 = -1$. Let $z = re^{i\theta}$ so $r^4e^{4i\theta} = -1 = e^{i\pi(1+2k)}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{4}(1+2k) \end{aligned}$$

   where $k = 0, 1, 2, 3$. Since when $k = 4$, we have $k = 0$. Then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

$$z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$$

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

   Let $z = \sqrt[4]{i}$ and $z = re^{i\theta}$. Then $r^4e^{4i\theta} = i = e^{i\pi/2}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{8} \end{aligned}$$

   so $z = e^{i\pi/8}$. Now, let $z = \sqrt[4]{-i}$. Then $r^4e^{4i\theta} = e^{-i\pi/2}$ so $z = e^{-i\pi/8}$.

4. Solve the quadratic equation
$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

   The quadratic equation is $x = \frac{-b\pm\sqrt{b^2-ac}}{2}$. For the complex polynomial, we have

$$z = \frac{-\alpha - \beta i \pm \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}}{2}$$

   Let $a + bi = \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}$. Then

$$z = \frac{-\alpha - \beta \pm (a + bi)}{2}$$

1.1.3 Justification

1. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

   combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

1.1.4 Conjugation, Absolute Value

1. Verify by calculation the values of
$$\frac{z}{z^2+1}$$
   for $z = x + iy$ and $\bar{z} = x - iy$ are conjugate.

For z, we have that $z^2 = x^2 - y^2 + 2xyi$.

$$\begin{aligned}
\frac{z}{z^2+1} &= \frac{x+iy}{x^2-y^2+1+2xyi} \\
&= \frac{x+iy}{x^2-y^2+1+2xyi}\frac{x^2-y^2+1-2xyi}{x^2-y^2+1-2xyi} \\
&= \frac{x(x^2-y^2+1)+2xy^2+iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2}
\end{aligned} \tag{1.3}$$

For $\bar{z}$, we have that $\bar{z}^2 = x^2 - y^2 - 2xyi$.

$$\begin{aligned}
\frac{\bar{z}}{\bar{z}^2+1} &= \frac{x-iy}{x^2-y^2+1-2xyi} \\
&= \frac{x-iy}{x^2-y^2+1-2xyi}\frac{x^2-y^2+1+2xyi}{x^2-y^2+1+2xyi} \\
&= \frac{x(x^2-y^2+1)+2xy^2-iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2}
\end{aligned} \tag{1.4}$$

Therefore, we have that equations (1.3) and (1.4) are conjugates.

2. Find the absolute value (modulus) of

$$-2i(3+i)(2+4i)(1+i) \qquad \text{and} \qquad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}.$$

When we expand the first problem, we have that

$$z_1 = -2i(3+i)(2+4i)(1+i) = 32+24i$$

so

$$|z_1| = \sqrt{32^2+24^2} = 40.$$

For the second problem, we have that

$$z_2 = \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)} = 2 - \frac{3}{2}i$$

so

$$|z_2| = \sqrt{4+9/4} = \frac{5}{2}.$$

3. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Recall that $|z|^2 = z\bar{z}$.

$$\begin{aligned}
1^2 &= \left|\frac{a-b}{1-\bar{a}b}\right|^2 \\
1 &= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\overline{\frac{a-b}{1-\bar{a}b}}\right) \\
&= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\frac{\bar{a}-\bar{b}}{1-a\bar{b}}\right) \\
&= \frac{a\bar{a}-a\bar{b}-\bar{a}b+b\bar{b}}{1-\bar{a}b-a\bar{b}+a\bar{a}b\bar{b}}
\end{aligned} \tag{1.5}$$

If $|a| = 1$, then $|a|^2 = a\bar{a} = 1$ and similarly for $|b|^2 = 1$. Then equation (1.5) becomes

$$\frac{1-a\bar{b}-\bar{a}b+b\bar{b}}{1-\bar{a}b-a\bar{b}+b\bar{b}} \qquad \text{and} \qquad \frac{1-a\bar{b}-\bar{a}b+a\bar{a}}{1-\bar{a}b-a\bar{b}+a\bar{a}}$$

resepctively which is one. If $|a| = |b| = 1$, then $|a|^2 = |b|^2 = 1$ so equation (1.5) can be written as

$$\frac{2-a\bar{b}-\bar{a}b}{2-\bar{a}b-a\bar{b}}.$$

Therefore, we must have that $a\bar{b}+\bar{a}b \neq 2$.

4. Find the conditions under which the equation $az + b\bar{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

   Let $z = x + iy$. Then $az + b\bar{z} + c = a(x + iy) + b(x - iy) + c = 0$.

$$(a+b)x + c = 0 \tag{1.6a}$$
$$(a-b)y = 0 \tag{1.6b}$$

   Lets consider equation (1.6b). We either have that $a = b$ or $y = 0$. If $a = b$, then WLOG equation (1.6a) can be written as

$$x = \frac{-c}{2a}$$

   and $y \in \mathbb{R}$. For fixed a, b, c, we have infinitely many solutions when $a = b$ since $z = \frac{-c}{2a} + iy$ for $y \in \mathbb{R}$. If $y = 0$, then equation (1.6a) can be written as

$$x = \frac{-c}{a+b}.$$

   Therefore, $z = x$ and we have only one solution.

5. Prove that Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2 - \sum_{1\leqslant i\leqslant j\leqslant n}|a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

   Let's consider

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 + \sum_{1\leqslant i\leqslant j\leqslant n}|a_i\bar{b}_j - a_j\bar{b}_i|^2 = \sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2.$$

   Then we can write the lefthand side as

$$\begin{aligned}
\left|\sum_{i=1}^{n} a_i b_i\right|^2 + \sum_{1\leqslant i\leqslant j\leqslant n}|a_i\bar{b}_j - a_j\bar{b}_i|^2 &= \sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} \bar{a}_j\bar{b}_j + \sum_{1\leqslant i\leqslant j\leqslant n}(a_i\bar{b}_j - a_j\bar{b}_i)(\bar{a}_i b_j - \bar{a}_j b_i) \\
&= \sum_{i,j=1}^{n} a_i b_i \bar{a}_j \bar{b}_j + \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right) - \sum_{i\leqslant j}\left(a_i\bar{a}_j b_i\bar{b}_j + \bar{a}_i a_j \bar{b}_i b_j\right) \\
&= \sum_{i=j=1}^{n}|a_i|^2|b_i|^2 + \sum_{i\neq j} a_i b_i \bar{a}_j \bar{b}_j \begin{array}{l} + \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right) \\ - \sum_{i\leqslant j}\left(a_i\bar{a}_j b_i\bar{b}_j + \bar{a}_i a_j \bar{b}_i b_j\right) \end{array}
\end{aligned}$$

   For $i \neq j$, $\sum_{i\neq j}^{n} a_i b_i \bar{a}_j \bar{b}_j - \sum_{i<j}\left(a_i\bar{a}_j b_i\bar{b}_j + \bar{a}_i a_j \bar{b}_i b_j\right) = 0$. Thus, we now have

$$\sum_{i=1}^{n}|a_i|^2|b_i|^2 + \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right).$$

   When the indicies of both series on the right hand side coincide,

$$\sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2 = \sum_{i=1}^{n}|a_i|^2|b_i|^2. \tag{1.7}$$

   That is, both a_i and b_i index together on the left of side equation (1.7). When a_i and b_i dont index together on the left side of equation (1.7),

$$\sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2 = \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right)$$

   as was needed to be shown.

1.1.5 Inequalities

1. Prove that
$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$
if $|a| < 1$ and $|b| < 1$.

From the properties of the modulus, we have that
$$\begin{aligned}
\left|\frac{a-b}{1-\bar{a}b}\right| &= \frac{|a-b|}{|1-\bar{a}b|} \\
&= \frac{|a-b|^2}{|1-\bar{a}b|^2} \qquad (1.8)\\
&= \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})} \\
&= \frac{|a|^2+|b|^2-a\bar{b}-\bar{a}b}{1+|a|^2|b|^2-\bar{a}b-a\bar{b}} \\
&< \frac{2-a\bar{b}-\bar{a}b}{2-\bar{a}b-a\bar{b}} \\
&= 1 \qquad (1.9)
\end{aligned}$$

From equations (1.8) and (1.9), we have
$$\begin{aligned}
\frac{|a-b|^2}{|1-\bar{a}b|^2} &< 1 \\
\frac{|a-b|}{|1-\bar{a}b|} &< 1
\end{aligned}$$

2. Prove Cauchy's inequality by induction.

Cauchy's inequality is
$$|a_1b_1+\cdots+a_nb_n|^2 \leqslant \left(|a_1|^2+\cdots+|a_n|^2\right)\left(|b_1|^2+\cdots+|b_n|^2\right)$$
which can be written more compactly as
$$\left|\sum_{i=1}^{n} a_ib_i\right|^2 \leqslant \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2.$$
For the base case, $i=1$, we have
$$|a_1b_1|^2 = (a_1b_1)(\bar{a}_1\bar{b}_1) = a_1\bar{a}_1b_1\bar{b}_1 = |a_1|^2|b_1|^2$$
so the base case is true. Now let the equality hold for all $k-1 \in \mathbb{Z}$ where $k-1 \leqslant n$. That is, we assume that
$$\left|\sum_{i=1}^{k-1} a_ib_i\right|^2 \leqslant \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2$$
to be true.
$$\begin{aligned}
\left|\sum_{i=1}^{k-1} a_ib_i\right|^2 + |a_kb_k|^2 &\leqslant \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2 + |a_kb_k|^2 \\
\left|\sum_{i=1}^{k} a_ib_i\right|^2 &\leqslant \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2 + (a_kb_k)(\bar{a}_k\bar{b}_k) \\
&= \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2 + |a_k|^2|b_k|^2 \\
&= \sum_{i=1}^{k}|a_i|^2\sum_{i=1}^{k}|b_i|^2
\end{aligned}$$
Therefore, by the principal of mathematical induction, Cauchy's inequality is true for all $n \geqslant 1$ for $n \in \mathbb{Z}^+$.

3. If $|a_i| < 1$, $\lambda_i \geqslant 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

Since $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geqslant 0$, $0 \leqslant \lambda_i < 1$. By the triangle inequality,

$$\begin{aligned}
|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| &\leqslant |\lambda_1||a_1| + \cdots + |a_n||\lambda_n| \\
&< \sum_{i=1}^{n} \lambda_i \\
&= 1
\end{aligned}$$

4. Show that there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leqslant |c|$. If this condition is fulfilled, what are the smallest and largest values $|z|$?

By the triangle inequality,

$$|z - a| + |z + a| \geqslant |(z - a) - (z + a)| = 2|a|$$

so

$$\begin{aligned}
2|c| &= |z - a| + |z + a| \\
&\geqslant |(z - a) - (z + a)| \\
&= 2|a|
\end{aligned}$$

Thus, $|c| \geqslant |a|$. For the second implication, if $a = 0$, the result follow. Suppose $a \neq 0$. Then let $z = |c|\frac{a}{|a|}$.

$$\begin{aligned}
2|c| &= |a|(|c|/|a| - 1) + |a|(|c|/|a| + 1) \\
&= |z - a| + |z + a|
\end{aligned}$$

The smallest and largest values of z can be found below.

$$\begin{aligned}
2|c| &= |z + a| + |z - a| \\
4|c|^2 &= \left(|z + a| + |z - a|\right)^2 \\
&= 2(|z|^2 + |a|^2) \\
&\leqslant 4(|z|^2 + |a|^2) \\
|c|^2 &\leqslant |z|^2 + |a|^2 \\
\sqrt{|c|^2 - |a|^2} &\leqslant |z|
\end{aligned}$$

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1. Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

2. Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1 a_2 + a_2 a_3 + a_1 a_3$.

3. Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

4. Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

1.2.2 The Binomial Equation

1. Express $\cos(3\varphi)$, $\cos(4\varphi)$, and $\sin(5\varphi)$ in terms of $\cos(\varphi)$ and $\sin(\varphi)$.

   For these problems, the sum addition identities will be employed; that is,

$$\begin{aligned}\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)\\ \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)\end{aligned}$$

   We can write $\cos(3\varphi)$ as $\cos(2\varphi + \varphi)$ so

$$\begin{aligned}\cos(3\varphi) &= \cos(2\varphi + \varphi)\\ &= \cos(2\varphi)\cos(\varphi) - \sin(2\varphi)\sin(\varphi)\\ &= \left[\cos^2(\varphi) - \sin^2(\varphi)\right]\cos(\varphi) - 2\sin(\varphi)\cos(\varphi)\sin(\varphi)\\ &= \cos^3(\varphi) - 3\sin^2(\varphi)\cos(\varphi)\end{aligned}$$

   For $\cos(4\varphi)$, we have

$$\begin{aligned}\cos(4\varphi) &= \cos(2\varphi)\cos(2\varphi) - \sin(2\varphi)\sin(2\varphi)\\ &= \left[\cos^2(\varphi) - \sin^2(\varphi)\right]^2 - 4\sin^2(\varphi)\cos^2(\varphi)\\ &= \cos^4(\varphi) + \sin^4(\varphi) - 6\sin^2(\varphi)\cos^2(\varphi)\end{aligned}$$

   For $\sin(5\varphi)$, we have

$$\begin{aligned}\sin(5\varphi) &= \sin(4\varphi)\cos(\varphi) + \sin(\varphi)\cos(4\varphi)\\ &= 2\sin(2\varphi)\cos(2\varphi)\left[\cos^2(\varphi) - \sin^2(\varphi)\right]\cos(\varphi) + \sin^5(\varphi) + \sin(\varphi)\cos^4(\varphi) - 6\sin^3(\varphi)\cos^2(\varphi)\\ &= 5\sin(\varphi)\cos^4(\varphi) - 10\sin^3(\varphi)\cos^2(\varphi) + \sin^5(\varphi)\end{aligned}$$

2. Simplify $1 + \cos(\varphi) + \cos(2\varphi) + \cdots + \cos(n\varphi)$ and $\sin(\varphi) + \cdots + \sin(n\varphi)$.

   Instead of considering the two separate series, we will consider the series

$$\begin{aligned}1 + \cos(\varphi) + i\sin(\varphi) + \cdots + \cos(n\varphi) + i\sin(n\varphi) &= 1 + e^{i\varphi} + e^{2i\varphi} + \cdots + e^{ni\varphi}\\ &= \sum_{k=0}^{n} e^{ki\varphi}\end{aligned}$$

   Recall that $\sum_{k=0}^{n-1} r^k = \frac{1-r^k}{1-r}$. So

$$\begin{aligned}&= \frac{1 - e^{i\varphi(n+1)}}{1 - e^{i\varphi}}\\ &= \frac{e^{i\varphi(n+1)} - 1}{e^{i\varphi} - 1}\end{aligned} \tag{1.10}$$

   Note that $\sin(\frac{\varphi}{2}) = \frac{e^{i\varphi/2} - e^{-i\varphi/2}}{2i}$ so $2ie^{i\varphi/2}\sin(\frac{\varphi}{2}) = e^{i\varphi} - 1$. We can now write equation (1.10) as

$$\begin{aligned}\sum_{k=0}^{n} e^{ki\varphi} &= \frac{e^{i\varphi(n+1)/2}\sin(\frac{\varphi(n+1)}{2})}{e^{i\varphi/2}\sin(\frac{\varphi}{2})}\\ &= \frac{\sin(\frac{\varphi(n+1)}{2})}{\sin(\frac{\varphi}{2})}e^{in\varphi/2}\end{aligned} \tag{1.11}$$

   By taking the real and imaginary parts of equation (1.11), we get the series for $\sum_{k=0}^{n}\cos(n\varphi)$ and $\sum_{k=0}^{n}\sin(n\varphi)$, respectively.

$$\begin{aligned}\sum_{k=0}^{n}\cos(n\varphi) &= \frac{\sin(\frac{\varphi(n+1)}{2})}{\sin(\frac{\varphi}{2})}\cos\left(\frac{n\varphi}{2}\right)\\ \sum_{k=0}^{n}\sin(n\varphi) &= \frac{\sin(\frac{\varphi(n+1)}{2})}{\sin(\frac{\varphi}{2})}\sin\left(\frac{n\varphi}{2}\right)\end{aligned}$$

3. Express the fifth and tenth roots of unity in algebraic form.

   To find the roots of unity, we are looking to solve $z^n = 1$. Let $z = e^{i\theta}$ and $1 = e^{2ik\pi}$. Then $\theta = \frac{2k\pi}{n}$. For the fifth roots of unity, $n = 5$ and $k = 0, 1, \ldots, 4$ so we have

$$\begin{aligned}
\omega_0 = e^0 &= \cos(0) + i\sin(0) \\
&= 1 \\
\omega_1 = e^{2\pi/5} &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\
\omega_2 = e^{4\pi/5} &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\
\omega_3 = e^{6\pi/5} &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\
\omega_4 = e^{8\pi/5} &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)
\end{aligned}$$

   Now we can plot the roots of unity on the unit circle.

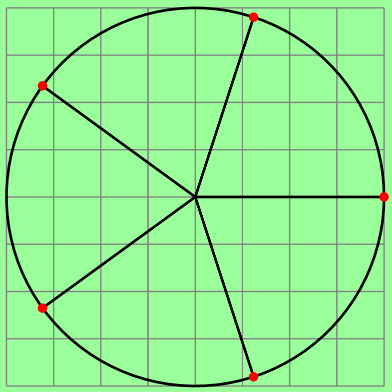

Figure 1.1: The fifth roots of unity.

   For the tenth roots of unity, $n = 10$ and $k = 0, 1, \ldots, 9$ so we have

$$\begin{aligned}
\omega_0 = e^0 &= \cos(0) + i\sin(0) \\
&= 1 \\
\omega_1 = e^{2\pi/10} &= \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) \\
\omega_2 = e^{4\pi/10} &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\
\omega_3 = e^{6\pi/10} &= \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right) \\
\omega_4 = e^{8\pi/10} &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\
\omega_5 = e^{10\pi/10} &= \cos(\pi) + i\sin(\pi) \\
&= -1 \\
\omega_6 = e^{12\pi/10} &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\
\omega_7 = e^{14\pi/10} &= \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right) \\
\omega_8 = e^{16\pi/10} &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) \\
\omega_9 = e^{18\pi/10} &= \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)
\end{aligned}$$

   Now we can plot the roots of unity on the unit circle.

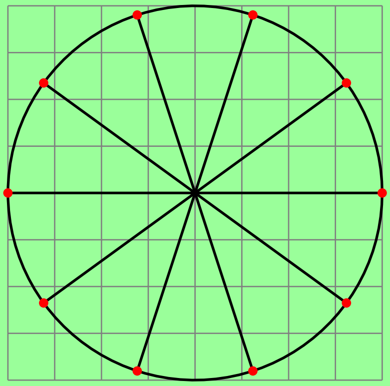

Figure 1.2: The tenth roots of unity.

4. If ω is given by $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$, prove that
$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0$$
for any integer h which is not a multiple of n.

Let $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ be written in exonential form as $\omega = e^{2\pi i/n}$. Then the series can be written as
$$\sum_{k=0}^{n-1} \left(e^{2\pi i h/n}\right)^k = \frac{e^{2ih\pi} - 1}{e^{2hi\pi/n} - 1}.$$
Since h is an integer, $e^{2ih\pi} = 1$; therefore, the series zero.

5. What is the value of
$$1 - \omega^h + \omega^{2h} - \cdots + (-1)^{n-1}\omega^{(n-1)h}?$$
We can represent this series similarly as
$$\sum_{k=0}^{n-1} \left(-e^{2\pi i h/n}\right)^k = \frac{(-1)^n e^{2ih\pi} - 1}{-e^{2hi\pi/n} - 1} = \frac{1 + (-1)^{n+1} e^{2ih\pi}}{1 + e^{2hi\pi/n}}.$$
Again, since h is an intger, we have that $e^{2ih\pi} = 1$ which leaves us with
$$\frac{1 + (-1)^{n+1}}{1 + e^{2hi\pi/n}} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{1+e^{2hi\pi/n}}, & \text{if } n \text{ is odd} \end{cases}$$

1.2.3 Analytic Geometry

1. When does $az + b\bar{z} + c = 0$ represent a line?

2. Write the equation of an ellipse, hyperbola, parabola in complex form.

For $x, y, h, k, a, b \in \mathbb{R}$ such that $a, b \neq 0$, we define a real ellipse as
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$
Let $z = \frac{x}{a} + i\frac{y}{b}$ and $z_0 = \frac{h}{a} + i\frac{k}{b}$. If we expand the equation for an ellipse, we have
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{h^2}{a^2} + \frac{k^2}{b^2} - \frac{2xh}{a^2} - \frac{2yk}{b^2} = 1.$$
Notice that $|z|^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ and $|z_0|^2 = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. Now, let's write the ellipse as
$$|z|^2 + |z_0|^2 - \frac{2xh}{a^2} - \frac{2yk}{b^2} + \frac{yh}{ab}i - \frac{yh}{ab}i + \frac{xk}{ab}i - \frac{xk}{ab}i = |z|^2 + |z_0|^2 - \bar{z}z_0 - z\bar{z}_0 = 1.$$
Thus, the equation of an ellipse in the complex plane is
$$(z - z_0)(\bar{z} - \bar{z}_0) = |z - z_0|^2 = 1 \Rightarrow |z - z_0| = 1$$

where z and z_0 are defined above. Additionally, the standard form of an ellipse in the complex plane is of the form

$$|z - a| + |z - b| = c$$

where $c > |a - b|$. Let a and b be the foci of a hyperbola. Then when the magnitude of the difference of z and the foci is a constant, we will have a hyperbola.

$$\big||z - a| - |z - b|\big| = c$$

3. Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

4. Prove analytically that the midpoints of parallel chords to a circle lie on a diameter perpendicular to the chords.

5. Show that all circles that pass through a and $1/\bar{a}$ intersect the circle $|z| = 1$ at right angles.

1.2.4 The Spherical Representation

1. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$.

2. A cube has its vertices on the sphere S and its edges parallel to the coordinate axes. Find the stereographic projections of the vertices.

3. Same problem for a regular tetrahedron in general position.

4. Let Z, Z' denote the stereographic projections of z, z', and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive

$$d(z, z') = \frac{2|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}.$$

5. Find the radius of the spherical image of the circle in the plane whose center is a and radius R.

   Let $z = a + R$ and $z' = a - R$. Then the distance $d(z, z') = 2R$.

$$\begin{aligned}
2R &= d(z, z') \\
R &= d(z, z')/2 \\
&= \frac{2|R|}{\sqrt{(1 + |a|^2 + |R|^2 + 2\Re\{a\bar{R}\})(1 + |a|^2 + |R|^2 - 2\Re\{a\bar{R}\})}} \\
&= \frac{2|R|}{\sqrt{(1 + |a|^2 + |R|^2)^2 - 4\Re^2\{a\bar{R}\}}}
\end{aligned}$$

2 Complex Functions

2.1 Introduction to the Concept of Analytical Function

2.1.1 Limits and Continuity

2.1.2 Analytic Functions

1. If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

   Let $g(w) = h(x,y) + it(x,y)$ and $f(z) = u(x,y) + iv(x,y)$ where $z = w = x + iy$ for $x, y \in \mathbb{R}$. Then

   $$(g \circ f)(z) = h(u(x,y), v(x,y)) + it(u(x,y), v(x,y)).$$

   Since f and g satisfy the Cauchy-Riemann equations,

   $$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} & \qquad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial h}{\partial x} &= \frac{\partial t}{\partial y} & \qquad \frac{\partial h}{\partial u} &= -\frac{\partial t}{\partial x} \end{aligned}$$

   The partial deivatives of $(g \circ f)(z)$ are

   $$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial x} & \qquad \frac{\partial t}{\partial y} &= \frac{\partial t}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial h}{\partial y} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial y} & \qquad \frac{\partial t}{\partial x} &= \frac{\partial t}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial x} \end{aligned}$$

   In order for $g(f(z))$ to be analytic, $\frac{\partial h}{\partial x} = \frac{\partial t}{\partial y}$ and $\frac{\partial h}{\partial y} = -\frac{\partial t}{\partial x}$. We can then write

   $$\begin{aligned} \frac{\partial h}{\partial x} - \frac{\partial t}{\partial y} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} - \frac{\partial t}{\partial u}\frac{\partial u}{\partial y} - \frac{\partial t}{\partial v}\frac{\partial v}{\partial y} \\ &= \underbrace{\frac{\partial h}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial t}{\partial u}\frac{\partial u}{\partial y}}_{\text{term 1}} + \underbrace{\frac{\partial h}{\partial v}\frac{\partial v}{\partial x} - \frac{\partial t}{\partial v}\frac{\partial v}{\partial y}}_{\text{term 2}} \end{aligned} \tag{2.1}$$

   In order for the right hand side of equation (2.1) to be zero, we need both terms to be zero.

   $$\begin{aligned} \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial t}{\partial u}\frac{\partial u}{\partial y} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial t}{\partial y}\frac{\partial u}{\partial u} \\ &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} \end{aligned} \tag{2.2}$$

   Equation (2.2) occurs since g is analytic and satisfies the Cauchy-Riemann equations.

   $$= 0$$

   For the second term in equation (2.1), we again use the analyticity of g.

   $$\begin{aligned} \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} - \frac{\partial t}{\partial v}\frac{\partial v}{\partial y} &= \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} - \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} \\ &= 0 \end{aligned}$$

   Therefore, from equation (2.1), we have

   $$\begin{aligned} \frac{\partial h}{\partial x} - \frac{\partial t}{\partial y} &= 0 \\ \frac{\partial h}{\partial x} &= \frac{\partial t}{\partial y} \end{aligned}$$

   By similar analysis, we are able to conclude that $\frac{\partial h}{\partial y} = -\frac{\partial t}{\partial x}$. Therefore, $g(f(z))$ satisfies the Cauchy-Riemann so it is analytic.

2. Verify Cauchy-Riemann's equations for the function z^2 and z^3.

   Let $z = x + iy$. Then $z^2 = x^2 - y^2 + 2xyi$ and $z^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$. For $f(z) = z^2$, the Cauchy-Riemann equations are

$$\begin{aligned} u_x &= 2x & v_y &= 2x \\ u_y &= -2y & -v_x &= -2y \end{aligned}$$

   Thus, the Cauchy-Riemann equation satisfied for $f(z) = z^2$. For $f(z) = z^3$, the Cauchy-Riemann equations are

$$\begin{aligned} u_x &= 3x^2 - 3y^2 & v_y &= 3x^2 - 3y^2 \\ u_y &= -6xy & -v_x &= -6xy \end{aligned}$$

   Thus, the Cauchy-Riemann equation satisfied for $f(z) = z^3$.

3. Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

   In order to be harmonic, $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ has to satisfy $\nabla^2 u = 0$ so

$$u_{xx} + u_{yy} = (3a + c)x + (3d + b)y = 0.$$

   Thus, $3a = -c$ and $3d = -b$ so

$$u(x, y) = ax^3 - 3axy^2 - 3dx^2y + dy^3.$$

   To find the harmonic conjugate $v(x, y$, we need to look at the Cauchy-Riemann equations. By the Cauchy-Riemann equations,

$$u_x = 3ax^2 - 3ay^2 - 6dxy = v_y.$$

   Then we can integrate with respect to y to find $v(x, y)$.

$$v(x, y) = \int (3ax^2 - 3ay^2 - 6dxy)dy = 3ax^2y - ay^3 - 3dxy^2 + g(x)$$

   Using the second Cauchy-Riemann, we have

$$v_x = 6axy - 3dy^2 + g'(x) = -u_y = 3dx^2 + 6axy - 3dy^2$$

   so $g'(x) = 3dx^2$. Then $g(x) = dx^3 + C$ and

$$v(x, y) = 3ax^2y - ay^3 - 3dxy^2 + dx^3 + C.$$

4. Show that an analytic function cannot have a constant absolute value without reducing to a constant.

   Let $f = u(x, y) + iv(x, y)$. Then the modulus of f is $|f| = \sqrt{u^2 + v^2}$. If the modulus of f is constant, then $u^2 + v^2 = c$ for some constant c. If $c = 0$, then $f = 0$ which is constant. Suppose $c \neq 0$. By taking the derivative with respect to x and y, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}(u^2 + v^2) \\ &= 2uu_x + 2vv_x \\ &= uu_x + vv_x \\ 0 &= \frac{\partial}{\partial y}(u^2 + v^2) \\ &= uu_y + vv_y \end{aligned}$$

   Since f is analytic, f satisfies the Cauchy-Riemann. That is, $u_x = v_y$ and $u_y = -v_x$.

$$uu_x - vu_y = 0 \tag{2.3a}$$

$$uu_y + vu_x = 0 \tag{2.3b}$$

Let's write equations (2.3a) and (2.3b) in matrix form. Then we have

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suppose the matrix is not invertible. Then $u^2 + v^2 = 0$. Since $u^2, v^2 \in \mathbb{R}$, $u^2, v^2 \geqslant 0$. Therefore, $u = v = 0$ so $f(z) = 0$. Now, suppose that the matrix is invertible. Then we have

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $f'(z) = 0$ and $f(z) = c$ for some constant c.

5. Prove rigorously that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

   Let $g(z) = \overline{f(\bar{z})}$ and suppose f is analytic. Then $g'(z)$ is

$$\begin{aligned} g'(z) &= \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{f(\bar{z} + \overline{\Delta z})} - \overline{f(\bar{z})}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \overline{\left[\frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}} \right]} \end{aligned}$$

   Since conjugation is continuous, we can move the limit inside the conjugation.

$$\begin{aligned} &= \overline{\lim_{\Delta z \to 0} \frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}}} \\ &= \overline{f'(\bar{z})} \end{aligned}$$

   Thus, g is differentiable with derivative $\overline{f'(\bar{z})}$. Suppose $\overline{f(\bar{z})}$ is analytic and let $\overline{g(\bar{z})} = f(z)$. Then by the same argument, f is differentiable with derivative $\overline{g'(\bar{z})}$. Therefore, $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

   We could also use the Cauchy-Riemann equations. Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ so $\bar{z} = x - iy$. Then $\overline{f(\bar{z})} = \alpha(x, y) - i\beta(x, y)$ where $\alpha(x, y) = u(x, -y)$ and $\beta(x, y) = v(x, -y)$. In order for both to be analytic, they both need to satisfy the Cauchy-Riemann equations. That is, $u_x = v_y$, $u_y = -v_x$, $\alpha_x = \beta_y$ and $\alpha_y = -\beta_x$.

$$\begin{aligned} u_x(x, y) &= v_y(x, y) \\ u_y(x, y) &= -v_x(x, y) \\ \alpha_x(x, y) &= u_x(x, -y) \\ \alpha_y(x, y) &= -u_y(x, -y) \\ -\beta_x(x, y) &= v_x(x, -y) \\ \beta_y(x, y) &= v_y(x, -y) \end{aligned}$$

   Suppose that $\overline{f(\bar{z})}$ satisfies the Cauchy-Riemann equations. Then $\alpha_x = u_x(x, -y) = v_y(x, -y) = \beta_y$ and $\alpha_y = -u_y(x, -y) = v_x(x, -y) = -\beta_x$. Therefore,

$$\begin{aligned} u_x(x, -y) &= v_y(x, -y) \\ u_y(x, -y) &= -v_x(x, -y) \end{aligned}$$

   which means $f(\bar{z})$ satisfies the Cauchy-Riemann equations. Now, recall that $|z| = |\bar{z}|$. Since $f(\bar{z})$ satisfies the Cauchy-Riemann equations, for an $\epsilon > 0$ there exists a $\delta > 0$ such that when $0 < |\Delta z| < \delta$, $|f(\bar{z}) - \bar{z}_0| = |f(z) - z_0| < \epsilon$. Thus, $\lim_{\Delta z \to 0} f(z) = z_0$ so $f(z)$ is analytic if $\overline{f(\bar{z})}$ is analytic.

6. Prove that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

   Since u is the real part of $f(z)$, $u(z) = u(x, y)$ where $z = x + iy$. Suppose $u(z)$ is harmonic. Then $u(z)$ satisfies Laplace equation.
$$\nabla^2 u(z) = u_{xx} + u_{yy} = 0$$
   Now, $u(\bar{z}) = u(x, -y)$ where $\frac{\partial^2}{\partial x^2}u(\bar{z}) = u_{xx}$ and $\frac{\partial^2}{\partial y^2}u(\bar{z}) = u_{yy}$ so
$$\nabla^2 u(\bar{z}) = u_{xx} + u_{yy} = 0.$$
   Since $u(z)$ is harmonic, $u_{xx} + u_{yy} = 0$ so it follows that $u(\bar{z})$ is harmonic as well.

7. Show that a harmonic function satisfies the formal differential equation
$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$
   Let u be a harmonic. Then $\nabla^2 u = 0$.
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \tag{2.4a}$$
$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \tag{2.4b}$$
   From equation (2.4a), we have
$$\frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)u = \frac{1}{2}(u_x + iu_y).$$
   Then we have
$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(u_x + iu_y) = \frac{1}{4}\big[u_{xx} + u_{yy} + i(u_{yx} - u_{xy})\big]$$
   Since u is a solution to the Laplace equation, u has continuous first and second derivatives. That is, $u \in C^2$ at a minimum. By Schwarz's theorem, $u_{xy} = u_{yx}$ so
$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$
   Schwarz's theorem states that if f is a function of two variables such that f_{xy} and f_{yx} both exist and are continuous at some point (x_0, y_0), then $f_{xy} = f_{yx}$.

2.1.3 Polynomials

2.1.4 Rational Functions

1. Use the method of the text to develop
$$\frac{z^4}{z^3 - 1} \qquad \text{and} \qquad \frac{1}{z(z+1)^2(z+2)^3}$$
   in partial fractions.

   Let $R(z) = \frac{z^4}{z^3-1} = z + \frac{z}{z^3-1}$. The poles of $R(z)$ occur when $z^3 = 1$. Then the distinct poles are $z = 1, e^{2i\pi/3}, e^{4i\pi/3}$. Let $H(z) = \frac{z}{z^3-1}$, $z \mapsto \beta_i + 1/w$, and $\beta_i \in \{1, e^{2i\pi/3}, e^{4i\pi/3}\}$.
$$\begin{aligned}
H(1 + 1/w) &= \frac{w}{3} - \frac{w}{3(3w^2 + 3w + 1)} \\
H(e^{2i\pi/3} + 1/w) &= \frac{w}{3e^{2i\pi/3}} - \frac{w}{3e^{2i\pi/3}(3e^{2i\pi/3}w^2 + 3e^{4i\pi/3}w + 1)} \\
H(e^{4i\pi/3} + 1/w) &= \frac{w}{3e^{4i\pi/3}} - \frac{w}{3e^{4i\pi/3}(3e^{4i\pi/3}w^2 + 3e^{2i\pi/3}w + 1)} \\
H(\beta_i + 1/w) &= \frac{w}{3\beta_i} - Q(w)
\end{aligned}$$

Then $G_i(w) = \frac{w}{3\beta_i}$ where $w \mapsto 1/(z-\beta_i)$. Now, $R(z) = G(z) + G_i[1/(z-\beta_i)]$ so we finally have that

$$R(z) = \frac{z^4}{z^3-1} = z + \frac{z}{z^3-1} = z + \sum_{i=1}^{3} \frac{1}{3\beta_i(z-\beta_i)}.$$

The second problem's numerator already is of a degree less than the denominator so we can proceed at once. Let $R(z) = \frac{1}{z(z+1)^2(z+2)^3}$. The poles of $R(z)$ are $\beta_i \in \{0, -1, -2\}$ and $z \mapsto \beta_i + 1/w$.

$$\begin{aligned}
R(1/w) &= \frac{w}{(1/w+1)^2(1/w+2)^3} \\
&= \frac{w^6}{(w+1)^2(2w+1)^3} \\
&= \frac{w}{8} + Q(w) \\
R(1/w-1) &= \frac{w^6}{(1-w)(w+1)^3} \\
&= 2w - w^2 + Q(w) \\
R(1/w-2) &= \frac{w^6}{(1-2w)(w-1)^2} \\
&= -\frac{17w}{8} - \frac{5w^2}{4} - \frac{w^3}{2} + Q(w)
\end{aligned}$$

Therefore, we can write

$$R(z) = \frac{1}{z(z+1)^2(z+2)^3} = \frac{1}{8z} + \frac{2}{z+1} - \frac{1}{(z+1)^2} - \frac{17}{8(z+2)} - \frac{5}{4(z+2)^2} - \frac{1}{2(z+2)^3}$$

2. Use the formula in the preceding exercise to prove that there exists a unique polynomial P of degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

3. What is the general form of a rational function which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles related to each other?

4. If a rational function is real on $|z| = 1$, how are the zeros and poles situated?

5. If $R(z)$ is a rational function of order n, how large and how small can the order of $R'(z)$ be?

   Let $R(z) = P(z)/Q(z)$ where $P(z)$ has degree n and $Q(z)$ has degree m. Let k be the degree of $R(z)$. Then $k = \max\{n, m\}$.

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2}$$

   Then we have four cases

   (a) Both $P(z)$ and $Q(z)$ are nonconstant.

   Then $P'(z)$ and $Q'(z)$ have degrees $n-1$ and $m-1$, respectively. Since we are looking only for the highest degree terms, we have

$$R'(z) = \frac{z^{n-1}z^m - z^n z^{m-1}}{z^{2m}}$$

   Therefore, the degree of $R'(z)$ is $k' = \max\{n+m-1, 2m\}$.

   (b) Suppose $P(z)$ is nonconstant and $Q(z)$ is a nonzero constant function.

$$R'(z) = \frac{z^{n-1} - z^n \cdot 0}{z^{0\cdot 2}}$$

   The degree of $R'(z)$ is $k' = n-1$.

   (c) Suppose $P(z)$ is a nonzero constant function and $Q(z)$ is a nonconstant.

$$R'(z) = \frac{0 \cdot z^m - z^{m-1}}{z^{2m}}$$

   The degree of $R'(z)$ is $k' = 2m$.

   (d) Suppose both $P(z)$ and $Q(z)$ are nonzero constant functions.

   In this case, $R'(z) = 0$ so $P(z) = a$ and $Q(z) = b$. Then the degree of $R'(z)$ is $k' = 0$.

2.2 Elementary Theory of Power Series

2.2.1 Sequences

2.2.2 Series

2.2.3 Uniform Convergence

1. Prove that a convergent sequence is bounded.

   Let $\{a_n\}$ be a convergent sequence and $\lim_{n\to\infty} a_n = a$. Let $\epsilon = 1$. Then there exists an $n > N$ such that $|a_n - a| < 1$.

   $$|a_n| = |a_n - a + a|$$

   By the triangle inequality, we have

   $$\begin{aligned} &\leqslant |a_n - a| + |a| \\ |a_n| - |a| &\leqslant |a_n - a| \end{aligned}$$

   Therefore, we have that

   $$\begin{aligned} |a_n| - |a| \leqslant |a_n - a| < 1 \\ |a_n| < 1 + |a| \end{aligned}$$

   For all $n > N$, $|a_n| < 1 + |a|$ so let $A = \max\{1 + |a|, |a_1|, \ldots, |a_N|\}$. Thus, $|a_n| < A$ for some finite A and hence $\{a_n\}$ is bounded by A.

2. If $\lim_{n\to\infty} z_n = A$, prove that

   $$\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + \cdots + z_n) = A.$$

   Given $\epsilon > 0$ there exists some $n > N$ such that

   $$|z_n - A| < \frac{N\epsilon}{2}.$$

   Now, since $\lim_{n\to\infty} z_n = A$ converges, it is Cauchy. Therefore, there exists $n, m > N$ such that

   $$|z_m - z_n| < \frac{N\epsilon}{2}.$$

   Repeating this we have that $|z_1 + \cdots + z_n - nA|$ or $|1/n(z_1 + \cdots + z_{n-1}) - A + (z_n - A)/n|$. For a fixed N, we can find n such that

   $$\sum_{i=1}^{n-1} |z_i - A| < \frac{N\epsilon}{2}$$

   We now have that

   $$\begin{aligned} |1/n(z_1 + \cdots + z_{n-1}) - A + (z_n - A)/n| &\leqslant \left|1/n \sum_{i=1}^{n-1} (z_i - A)\right| + 1/n|z_n - A| \qquad (2.5) \\ &\leqslant 1/n \sum_{i=1}^{n-1} |z_i - A| + 1/n|z_n - A| \\ &< 1/n\frac{N\epsilon}{2} + 1/n\frac{N\epsilon}{2} \\ &< \epsilon \end{aligned}$$

   Equation (2.5) can be written as $|1/n(z_1 + \cdots + z_n) - A| < \epsilon$ so

   $$\lim_{n\to\infty} 1/n(z_1 + \cdots + z_n) = A.$$

3. Show that the sum of an absolutely convergent series does not change if the terms are rearranged.

   Let $\sum a_n$ be an absolutely convergent series and $\sum b_n$ be its rearrangement. Since $\sum a_n$ converges absolutely, for $\epsilon > 0$, there exists a $n > N$ such that $|s_n - A| < \epsilon/2$ where s_n is the nth partial sum. Let t_n be the nth partial sum of $\sum b_n$. Then for some $n > N$

$$\begin{aligned} |t_n - A| &= |t_n - s_n + s_n - A| \\ &\leqslant |t_n - s_n| + |s_n - A| \\ &< |t_n - s_n| + \frac{\epsilon}{2} \end{aligned}$$

   Since $\sum a_n$ is absolutely convergent, $\sum_{k=n+1}^{\infty} |a_k|$ converges to zero. Let the remainder be r_n. Then for some $N > n, n_1$, $|r_n - 0| < \epsilon/2$. Let $M = \max\{k_1, k_2, \dots, k_N\}$. Then for some $n > M$, we have

$$|t_n - s_n| = \left|\sum_{n}^{N} a_n\right| \leqslant \sum |a_n| \leqslant \sum_{k=n+1}^{\infty} |a_n| = r_n < \frac{\epsilon}{2}$$

   Thus, $|t_n - s_n| < \epsilon$ and a rearrangement of an absolutely convergent series does not changes its sum.

4. Discuss completely the convergence and uniform convergence of the sequence $\{nz^n\}_{n=1}^{\infty}$.

   Consider when $|z| < 1$. Then $z^n = \frac{1}{w^n}$ where $|w| > 1$. By the ratio test, we have

$$\lim_{n\to\infty} \left|\frac{(n+1)w^n}{nw^{n+1}}\right| = \frac{1}{|w|} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{|w|}$$

   In order for convergence, the ratio test has to be less than one.

$$\frac{1}{|w|} = |z|$$

   which is less than one by our assumption so $\{nz^n\}$ converges absolutely in the disc less than one. Now, let's consider $|z| \geqslant 1$. By the ratio test, we get $\lim_{n\to\infty} |a_{n+1}/a_n| = |z| \geqslant 1$ by our assumption. When the limit is one, we can draw no conclusion about convergence, but when the limit is greater than one, the sequence diverges. For $|z| < 1$, $\epsilon > 0$, and $n > N$, $|nz^n - 0| < \epsilon$ for uniform convergence. Take $z = 9/10$, $n = 100$, and $\epsilon = 0.001$. Then

$$\begin{aligned} |nz^n| = n|z|^n &< \epsilon \\ |z|^n &< \frac{\epsilon}{n} \\ 0.000026 &\not< 0.00001 \end{aligned}$$

   Thus, the sequence is not uniformly convergent in the disc with radius less than one. Let's consider the closed disc $|z| \leqslant R$ where $R \in (0, 1)$. Now $|nz^n|$ is bounded above by a convergent geometric series, say $\sum r^n$ where $|r| < 1$. Then $|nz^n| < ar^n$ for $|z| \leqslant R$ and a a real constant. Let $M_n = ar^n$ where M_n is the M in the Weierstrass M-test. Thus, $\{nz^n\}$ is uniformly convergent in a closed disc less than one.

5. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1 + nx^2)}$$

   for real values of x.

   By the AM-GM inequality, $(x + y)/2 \geqslant \sqrt{xy}$, we have

$$1 + nx^2 \geqslant 2|x|\sqrt{n}$$

   or $\frac{1}{2|x|\sqrt{n}} \geqslant \frac{1}{1+nx^2}$. Let $f_n(x) = \frac{x}{n(1+nx^2)}$. Then

$$|f_n(x)| \leqslant \left|\frac{x}{2xn^{3/2}}\right| = \left|\frac{1}{2n^{3/2}}\right| = M_n$$

For a fixed x, $\sum|f_n(x)| \leqslant M_n < \infty$ so $\sum|f_n(x)|$ is absolutely convergent. Thus, $\sum f_n(x)$ is pointwise convergent to $f(x)$. Let $\epsilon > 0$ be given and $s_n = \sum_{k=1}^{n} f_k(x)$ be nth partial sum. Let $n > N$ such that

$$|f(x) - s_n| = \left|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \leqslant \sum_{k=n+1}^{\infty} |f_k(x)|$$

Since $\sum M_k$ converges to some limit, for n sufficiently large, $\sum_{k=n+1}^{\infty} M_k < \epsilon$. Select N such that this is true. Then

$$|f(x) - s_n| \leqslant \sum_{k=n+1}^{\infty} |f_k(x)| \leqslant \sum_{k=n+1}^{\infty} M_k < \epsilon$$

Therefore, $\sum f_n(x)$ where $f_n(x) = \frac{x}{n(1+nx^2)}$ is uniformly convergent by the Weierstrass M-test.

6. If $U = u_1 + u_2 + \cdots$, $V = v_1 + v_2 + \cdots$ are convergent series, prove that $UV = u_1v_1 + (u_1v_2 + u_2v_2) + (u_1v_3 + u_2v_2 + u_3v_1) + \cdots$ provided that at least one of the series is absolutely convergent. (It is easy if both series are absolutely convergent. Try to rearrange the proof so economically that the absolute convergence of the second series is not needed.)

2.2.4 Power Series

1. Expand $(1-z)^{-m}$, m a positive integer, in powers of z.

   The Binomial theorem states that $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$. In our case, we have

$$(1-z)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k} (-z)^k \tag{2.6}$$

   where $\binom{-m}{k} = (-1)^k \binom{m+k-1}{k}$. Then equation (2.6) can be written as

$$(1-z)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} z^k = 1 + mz + \frac{m(m+1)}{2!} z^2 + \cdots.$$

2. Expand $\frac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence?

   Let's just consider $\frac{1}{z+1}$ for the moment.

$$\frac{1}{z+1} = \frac{1}{z-1+2} = \frac{1/2}{1+\frac{z+1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z+1}{2}\right)^n$$

   From the full expressing, we obtain

$$\frac{2z+3}{z+1} = \frac{2z+3}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z+1}{2}\right)^n.$$

   The radius of convergence can be found by $1/R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Therefore, the radius of convergence is

$$R = 1/\limsup_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{1}{2^n}\right|} = |2| = 2$$

3. Find the radius of convergence of the following power series:

$$\sum n^p z^n, \quad \sum \frac{z^n}{n!}, \quad \sum n! z^n, \quad \sum q^{n^2} z^n, \quad \sum z^{n!}$$

   where $|q| < 1$.

   For $\sum n^p z^n$, we can use the inverse of argument of the ratio test to determine the radius of convergence; that is,

$$R = \lim_{n\to\infty} \left|\frac{n^p}{(n+1)^p}\right| = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1$$

For $\sum \frac{z^n}{n!}$, we can use the fact that the sum is e^z which is entire or the method used previously. Since e^z is entire, the radius of convergence is $R = \infty$.

$$R = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}\frac{n!(n+1)}{n!} = \infty$$

For $\sum n!z^n$, we use the modified ratio test again.

$$R = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = 0$$

For $\sum q^{n^2}z^n$, we will use the root test.

$$R = 1/\limsup_{n\to\infty}\sqrt[n]{|q^n|^n} = 1/\limsup_{n\to\infty}|q|^n$$

for $|q| < 1$, $R = \infty$, and for $|q| > 1$, $R = 0$. For $\sum z^{n!}$, we will use the root test.

$$R = 1/\limsup_{n\to\infty}\sqrt[n]{|z^{(n-1)!}|^n} = 1/\limsup_{n\to\infty}|z|^{(n-1)!}$$

When $|z| < 1$, $R = \infty$, and when $|z| > 1$, $R = 0$.

4. If $\sum a_n z^n$ has a radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$? of $\sum a_n^2 z^n$?

   Since $\sum a_n z^n$ has a radius of convergence R,

   $$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

   For $\sum a_n z^{2n} = z^2 \sum a_n z^n$, we have

   $$|z|^2 \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = |z|^2 R$$

   so the radius of convergence is $\sqrt{R}$. For $\sum a_n^2 z^n$, we have

   $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|^2 = R^2.$$

5. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

   Let's write out the first few terms of

   $$\sum n^3 a_n z^n = a_1 z + 8a_2 z^2 + 27a_3 z^3 + 64a_4 z^4 + \cdots$$

   Let's consider the first three derivatives of $f(z)$.

   $$\begin{aligned} f'(z) &= \sum n a_n z^{n-1} \\ zf'(z) &= \sum n a_n z^n \\ &= a_1 z + 2a_2 z^2 + 3a_3 z^3 + \cdots && (2.7) \\ f''(z) &= \sum n(n-1)a_n z^{n-2} \\ z^2 f''(z) &= \sum n(n-1)a_n z^n \\ &= 2a_2 z^2 + 6a_3 z^3 + 12a_4 z^4 + \cdots && (2.8) \\ f'''(z) &= \sum n(n-1)(n-2)a_n z^{n-3} \\ z^3 f'''(z) &= \sum n(n-1)(n-2)a_n z^n \\ &= 6a_3 z^3 + 24a_4 z^4 + 60a_5 z^5 + \cdots && (2.9) \end{aligned}$$

   If we add equations (2.7) to (2.9), we have

   $$zf'(z) + z^2 f''(z) + z^3 f'''(z) = a_1 z + 4a_2 z^2 + 15a_3 z^3 + \cdots \neq \sum n^3 a_n z^n$$

   However, consider $3z^2 f''(z) = 6a_2 z^2 + 18a_3 z^3 + 36a_4 z^4 + \cdots$. Then

   $$zf'(z) + 3z^2 f''(z) + z^3 f'''(z) = a_1 z + 8a_2 z^2 + 27a_3 z^3 + 64a_4 z^4 \cdots = \sum n^3 a_n z^n.$$

6. If $\sum a_n z^n$ and $\sum b_n z_n$ have radii of convergence R_1 and R_2, show that the radii of convergence of $\sum a_n b_n z^n$ is at least $R_1 R_2$.

   Let $\epsilon > 0$ be given. Then there exists $n > N$ such that

$$|a_n|^{1/n} < 1/R_1 + \epsilon, \qquad |b_n|^{1/n} < 1/R_2 + \epsilon$$

   since $\limsup_{n\to\infty} |a_n|^{1/n} = 1/R_1$ so $|a_n|^{1/n} < 1/R_1 + \epsilon$ and similarly for b_n. Multiplying we obtain

$$|a_n b_n|^{1/n} < \frac{1}{R_1 R_2} + \epsilon(1/R_1 + 1/R_2) + \epsilon^2$$

   Then

$$\frac{1}{R} \leqslant \frac{1}{R_1 R_2} \Rightarrow R_1 R_2 \leqslant R$$

7. If $\lim_{n\to\infty} |a_n|/|a_{n+1}| = R$, prove that $\sum a_n z^n$ has a radius of convergence of R.

   Let $\epsilon > 0$ be given. Suppose $|z| < R$. Pick ϵ such that $|z| < R - \epsilon$. Then for some $n > N$

$$\left|\left|\frac{a_n}{a_{n+1}}\right| - R\right| \leqslant R - \left|\frac{a_n}{a_{n+1}}\right| < \epsilon$$

$$R - \epsilon < \left|\frac{a_n}{a_{n+1}}\right| \tag{2.10}$$

   For $n > N$, we can write

$$\left|\frac{a_N}{a_n}\right| = \left|\frac{a_N a_{N+1} \cdots a_{n-1}}{a_{N+1} a_{N+2} \cdots a_n}\right| = \left|\frac{a_N}{a_{N+1}} \frac{a_{N+1}}{a_{N+2}} \cdots \frac{a_{n-1}}{a_n}\right| \tag{2.11}$$

   For $n > N$, we have that from equation (2.10), $R - \epsilon < \frac{a_N}{a_{N+1}}$. Thus, we can equation (2.11) as

$$(R - \epsilon)^{n-N} < \left|\frac{a_N}{a_n}\right|$$

$$|a_n| < \frac{|a_N|}{(R-\epsilon)^{n-N}}$$

$$|a_n z^n| < |a_N z^N| \left(\frac{|z|}{R-\epsilon}\right)^{n-N}$$

   Since ϵ was chosen such that $|z| < R - \epsilon$, we that $\frac{|z|}{R-\epsilon} < 1$ and

$$|a_n z^n| < |a_N z^N|$$

   where $|a_N z^N| < \infty$ since it is a convergent geometric series. Therefore, $\sum a_n z^n$ converges absolutely with a radius of convergence of R.

8. For what values of z is

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$$

   convergent?

   In order for series to converge $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|z/(z+1)|^n} = \left|\frac{z}{z+1}\right| < 1$$

   or $|z|^2 < (1+z)(1+\bar{z}) = 1 + 2\Re\{z\} + |z|^2$ so the series converges when

$$0 < 1 + 2\Re\{z\}.$$

9. Same question for

$$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}.$$

Consider the following two equations:

$$\begin{aligned} |1| &= |1+z^{2n}-z^{2n}| \\ &\leqslant |1+z^{2n}|+|z^{2n}| \\ 1-|z^{2n}| &\leqslant |1+z^{2n}| \qquad (2.12) \\ |z^{2n}| &= |1-1+z^{2n}| \\ &\leqslant |1+z^{2n}|+1 \\ |z^{2n}|-1 &\leqslant |1+z^{2n}| \qquad (2.13) \end{aligned}$$

From equations (2.12) and (2.13), the triangle inequality, we have that

$$\big|1-|z^{2n}|\big| \leqslant |1+z^{2n}|.$$

There exists an $m > 1$ such that

$$\frac{|z^{2n}|}{m} \leqslant \big|1-|z^{2n}|\big|.$$

By the root test,

$$\limsup_{n\to\infty} \sqrt[n]{\frac{m|z|^n}{|z^2|^n}} = \limsup_{n\to\infty} \frac{\sqrt[n]{m}}{|z|} = \frac{1}{|z|} < 1$$

When $|z| > 1$, the convergence of the ratio test $\frac{1}{|z|} < 1$ leads to $|z| > 1$. If $|z| < 1$, then $1/|z| > 1$ where we can write $1/|z| = |z_1|$. Since the choice dummy variables is arbitrary, $|z| < 1$. In other words, the series will converge when $|z| > 1$ or $|z| < 1$. Suppose $|z| = 1$. Then by the limit test,

$$\lim_{n\to\infty} \frac{1}{1^n + 1^{-n}} = \frac{1}{2} \neq 0;$$

therefore, the series diverges.

2.2.5 Abel's Limit Theorem

2.3 The Exponential and Trigonometric Functions

2.3.1 The Exponential

2.3.2 The Trigonometric Functions

1. Find the values of $\sin(i)$, $\cos(i)$, and $\tan(1+i)$.

For $\sin(i)$, we can use the identity $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. Then

$$\sin(i) = \frac{e^{-1}-e^{1}}{2i} = i\frac{e^{1}-e^{-1}}{2} = i\sinh(1).$$

Similarly, for $\cos(i)$, we have

$$\cos(i) = \frac{e^{-1}+e^{1}}{2} = \frac{e^{1}+e^{-1}}{2} = \cosh(1).$$

For $\tan(1+i)$, we can use the identity $\tan(z) = -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$. Then

$$\tan(1+i) = -i\frac{e^{i-1}-e^{1-i}}{e^{i-1}+e^{1-i}} = -i\tanh(i-1).$$

2. The hyperbolic cosine and sine are defined as $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and $\sinh(z) = \frac{e^z-e^{-z}}{2}$. Express them through $\cos(iz)$ and $\sin(iz)$. Derive the addition formulas, and formulas for $\cosh(2z)$ and $\sinh(2z)$.

   For the first part, we have

$$\begin{aligned}\cos(iz) &= \frac{e^{-z}+e^{z}}{2}\\ &= \cosh(z)\\ \sin(iz) &= \frac{e^{-z}-e^{z}}{2i}\\ &= i\frac{e^{z}-e^{-z}}{2}\\ &= i\sinh(z)\end{aligned}$$

   For cosh, we have that the addition formula is

$$\begin{aligned}\cosh(a+b) &= \cos[i(a+b)]\\ &= \frac{e^{a+b}+e^{-(a+b)}}{2}\\ &= \frac{2e^{a+b}+2e^{-(a+b)}}{4}\\ &= \left(e^{a+b}+e^{a-b}+e^{b-a}+e^{-(a+b)}+e^{a+b}-e^{a-b}-e^{b-a}+e^{-(a+b)}\right)/4\\ &= \frac{e^{a}+e^{-a}}{2}\frac{e^{b}+e^{-b}}{2}+\frac{e^{a}-e^{-a}}{2}\frac{e^{b}-e^{-b}}{2}\\ &= \cosh(a)\cosh(b)+\sinh(a)\sinh(b)\end{aligned}$$

   For sinh, we have that the addition formula is

$$\begin{aligned}\sinh(a+b) &= -i\sin[i(a+b)]\\ &= \frac{e^{-(a+b)}-e^{a+b}}{-2}\\ &= \frac{2e^{a+b}-2e^{-(a+b)}}{4}\\ &= \left(e^{a+b}+e^{a-b}-e^{b-a}-e^{-(a+b)}+e^{a+b}-e^{a-b}+e^{b-a}-e^{-(a+b)}\right)/4\\ &= \frac{e^{a}-e^{-a}}{2}\frac{e^{b}+e^{-b}}{2}+\frac{e^{a}+e^{-a}}{2}\frac{e^{b}-e^{-b}}{2}\\ &= \sinh(a)\cosh(b)+\cosh(a)\sinh(b)\end{aligned}$$

   For the double angle formulas, recall that $\cos(2z) = \cos^2(z) - \sin^2(z) = 2\cos^2(z) - 1 = 1 - 2\sin^2(z)$ and $\sin(2z) = 2\sin(z)\cos(z)$. Therefore, we have

$$\begin{aligned}\cosh(2z) &= \cos(2iz)\\ &= \cos^2(iz)-\sin^2(iz)\\ &= \left(\frac{e^{z}+e^{-z}}{2}\right)^2+\left(\frac{e^{z}-e^{-z}}{2}\right)^2\\ &= \cosh^2(z)+\sinh^2(z)\\ \cosh(2z) &= 2\cos^2(iz)-1\\ &= 2\cosh^2(z)-1\\ \cosh(2z) &= 1-2\sin^2(iz)\\ &= 1-2\sinh^2(z)\\ \sinh(2z) &= -i\sin(2iz)\\ &= -2i\sin(iz)\cos(iz)\\ &= 2\frac{e^{z}-e^{-z}}{2}\frac{e^{z}+e^{-z}}{2}\\ &= 2\sinh(z)\cosh(z)\end{aligned}$$

3. Use the addition formulas to separate $\cos(x+iy)$ and $\sin(x+iy)$ in real and imaginary parts.

$$\begin{aligned}
\cos(x+iy) &= \cos(x)\cos(iy) - \sin(x)\sin(iy) \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \\
\sin(x+iy) &= \sin(x)\cos(iy) + \sin(iy)\cos(x) \\
&= \sin(x)\cosh(y) + i\sinh(y)\cos(x)
\end{aligned}$$

4. Show that

$$|\cos(z)|^2 = \sinh^2(y) + \cos^2(x) = \cosh^2(y) - \sin^2(x) = (\cosh(2y) + \cos(2x))/2$$

and

$$|\sin(z)|^2 = \sinh^2(y) + \sin^2(x) = \cosh^2(y) - \cos^2(x) = (\cosh(2y) - \cos(2x))/2.$$

For the identities, recall that $\cosh^2(z) - \sinh^2(z) = 1$ and $\cos^2(z) + \sin^2(z) = 1$. Then for the first identity, we have

$$\begin{aligned}
|\cos(z)|^2 &= \cos(z)\cos(\bar{z}) \\
&= \left[\cos(x)\cosh(y) - i\sin(x)\sinh(y)\right]\left[\cos(x)\cosh(y) + i\sin(x)\sinh(y)\right] \\
&= \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) \\
&= \cos^2(x)(1+\sinh^2(y)) + \sin^2(x)\sinh^2(y) \\
&= \cos^2(x) + \sinh^2(y) \\
|\cos(z)|^2 &= \cos^2(x)\cosh^2(y) + \sin^2(x)(\cosh^2(y) - 1) \\
&= \cosh^2(y) - \sin^2(x) \\
|\cos(z)|^2 &= \\
&= (\cosh(2y) + \cos(2x))/2 \\
|\sin(z)|^2 &= \sin(z)\sin(\bar{z}) \\
&= \left[\sin(x)\cosh(y) + i\sinh(y)\cos(x)\right]\left[\sin(x)\cosh(y) - i\sinh(y)\cos(x)\right] \\
&= \sin^2(x)\cosh^2(y) + \sinh^2(y)\cos^2(x) \\
&= \sin^2(x)(1+\sinh^2(y)) + \sinh^2(y)\cos^2(x) \\
&= \sin^2(x) + \sinh^2(y) \\
|\sin(z)|^2 &= \sin^2(x)\cosh^2(y) + (\cosh^2(y) - 1)\cos^2(x) \\
&= \cosh^2(y) - \cos^2(x) \\
|\sin(z)|^2 &= \\
&= (\cosh(2y) - \cos(2x))/2
\end{aligned}$$

2.3.3 Periodicity

2.3.4 The Logarithm

1. For real y, show that every remainder in the series for $\cos(y)$ and $\sin(y)$ has the same sign as the leading term (this generalizes the inequalities used in the periodicity proof).

The series for both cosine and sine are

$$\begin{aligned}
\cos(y) &= \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k}}{(2k)!} \\
&= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots \\
\sin(y) &= \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k+1}}{(2k+1)!} \\
&= y - \frac{y^3}{3!} + \frac{y^6}{6!} - \cdots
\end{aligned}$$

We can write Taylor's formula as $f(y) = T_n(y) + R_n(y)$ where

$$f(y) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} y^k + \frac{1}{k!} \int_0^y (y-t)^k f^{(k+1)}(t)dt.$$

Now, we can write cosine and sine of y as

$$\cos(y) = \sum_{k=0}^{n} \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{n!} \int_0^y (y-t)^n \cos^{n+1}(t)dt$$
$$\sin(y) = \sum_{k=0}^{n-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{n!} \int_0^y (y-t)^n \sin^n(t)dt$$

For cosine and sine, let $n = 2m$ and $n = 2m-1$, respectively. Then

$$\cos(y) = \sum_{k=0}^{m} \frac{(-1)^k y^{2k}}{(2k)!} + \frac{1}{(2m)!} \int_0^y (y-t)^{2m} \cos^{2m+1}(t)dt$$
$$\sin(y) = \sum_{k=0}^{m-1} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + \frac{1}{(2m-1)!} \int_0^y (y-t)^{2m-1} \sin^{2m-1}(t)dt$$

2. Prove, for instance, that $3 < \pi < 2\sqrt{3}$.

3. Find the value of e^z for $z = -i\pi/2, 3i\pi/4, 2i\pi/3$.

$$\begin{aligned} e^{-i\pi/2} &= -i \\ e^{3i\pi/4} &= (-\sqrt{2} + i\sqrt{2})/2 \\ e^{2i\pi/3} &= (-1 + i\sqrt{3})/2 \end{aligned}$$

4. For that values of z is e^z equal to $2, -1, i, -i/2, -1-i, 1+2i$?

For all problems, $k \in \mathbb{Z}$.

$$\begin{aligned} e^z &= 2 \\ z &= \log(2) \\ e^z &= i \\ z &= \log|i| + i(\arg(i) + 2k\pi) \\ &= \frac{i\pi}{2}(1+4k) \\ e^z &= -1-i \\ z &= \log|-1-i| + i(\arg(-1-i) + 2k\pi) \\ &= \log(\sqrt{2}) - \frac{3i\pi}{4} + 2ki\pi \\ &= \frac{\log(2)}{2} - \frac{3i\pi}{4} + 2ki\pi \end{aligned}$$

$$\begin{aligned} e^z &= -1 \\ z &= \log(-1) \\ &= \log|i| + i(\arg(-1) + 2k\pi) \\ &= i\pi(1+2k\pi) \\ e^z &= \frac{-i}{2} \\ &= -\log(2) - \frac{i\pi}{2}(1+4k) \\ e^z &= 1+2i \\ z &= \log(\sqrt{5}) + i(\arctan(2) + 2k\pi) \\ &= \frac{\log(5)}{2} + i(\arctan(2) + 2k\pi) \end{aligned}$$

5. Find the real and imaginary parts of $\exp(e^z)$.

Let $z = x + iy$. Then

$$\begin{aligned} \exp(e^z) &= \exp\left[e^x(\cos(y) + i\sin(y)\right] \\ &= \exp(e^x\cos(y))\exp(ie^x\sin(y)) \\ &= \exp(e^x\cos(y))\left[\cos(e^x\sin(y)) + i\sin(e^x\sin(y))\right] \\ u(x,y) &= \exp(e^x\cos(y))\cos(e^x\sin(y)) \\ v(x,y) &= \exp(e^x\cos(y))\sin(e^x\sin(y)) \end{aligned}$$

where $u(x,y)$ is the real and $v(x,y)$ is the imaginary part of $\exp(e^z)$.

6. Determine all values of $2^i, i^i, (-1)^{2i}$.

   For all problems, $k \in \mathbb{Z}$.

$$\begin{aligned} z &= 2^i \\ &= \exp\left[i \log(2)\right] \\ &= \cos(\log(2)) + i \sin(\log(2)) \\ z &= i^i \\ &= \exp\left[i \log(i)\right] \\ &= \exp[-\pi(1+4k)/2] \\ z &= (-1)^{2i} \\ &= i^{4i} \\ &= (i^i)^4 \\ &= \exp[-2\pi(1+4k)] \end{aligned}$$

7. Determine the real and imaginary parts of z^z.

   Let $z = x + iy$ and $k \in \mathbb{Z}$.

$$\begin{aligned} z^z &= (x+iy)^{x+iy} \\ &= \exp\left[(x+iy)\log(x+iy)\right] \\ &= e^{x/2\log(x^2+y^2) - y(\arctan(y/x)+2k\pi)} \begin{bmatrix} \cos\big(x(\arctan(y/x)+2k\pi) + y/2\log(x^2+y^2)\big) + \\ i\sin\big(x(\arctan(y/x)+2k\pi) + y/2\log(x^2+y^2)\big) \end{bmatrix} \end{aligned}$$

   Thus, the real part is

$$u(x,y) = e^{x/2\log(x^2+y^2) - y(\arctan(y/x)+2k\pi)} \cos\big(x(\arctan(y/x)+2k\pi) + y/2\log(x^2+y^2)\big)$$

   and the imaginary part is

$$v(x,y) = e^{x/2\log(x^2+y^2) - y(\arctan(y/x)+2k\pi)} \sin\big(x(\arctan(y/x)+2k\pi) + y/2\log(x^2+y^2)\big)$$

8. Express $\arctan(w)$ in terms of the logarithm.

   Let $\arctan(w) = z$. Then $w = \tan(z)$. Recall that $\tan(z) = -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$. Now, let $e^{2iz} = x$. Then we have the following

$$w = -i\frac{x^2-1}{x^2+1}$$

   which leads to

$$e^{2iz} = \frac{i-w}{i+w}.$$

   By taking the log, we can recover z.

$$\begin{aligned} 2iz &= \log(i-w) - \log(i+w) \\ &= \log(i) + \log(1+iw) - \log(i) - \log(1-iw) \\ z &= \frac{i}{2}\left[\log(1-iw) - \log(1+iw)\right] \\ \arctan(w) &= z \\ &= \frac{i}{2}\left[\log(1-iw) - \log(1+iw)\right] \end{aligned}$$

9. Show how to define the "angles" in a triangle, bearing in mind that they should lie between 0 and π. With this definition, prove that the sum of the angles is π.

10. Show that the roots of the binomial equation $z^n = a$ are the vertices of a regular polygon (equal sides and angles).

Let $z = re^{i\theta}$. Then

$$r^n e^{i\theta n} = ae^{2i\pi k}.$$

Therefore, $r = a^{1/n}$ and $\theta = 2i\pi k/n$. Since r is just the radius, the roots will be located on a circle of radius r at $\exp(2i\pi k/n)$ for $k \in [0, n-1]$. Since each root are angle multiplies about the origin, they will be n equally spaced points. n equally spaced points will form the vertices of a regular n-gon.

3 Analytic Functions as Mappings

3.1 Elementary Point Set Topology

3.1.1 Sets and Elements

3.1.2 Metric Spaces

1. If S is a metric space with distance function $d(x, y)$, show that S with the distance function $\delta(x, y) = d(x, y)/[1 + d(x, y)]$ is also a metric space. The latter space is bounded in the sense that all distances lie under a fixed bound.

   Since $d(x, y)$ is a metric on S, $d(x, y)$ satisfies

   $$d(x, y) \geqslant 0, \text{ and zero only when } x = y \tag{3.1a}$$
   $$d(x, y) = d(y, x) \tag{3.1b}$$
   $$d(x, z) \leqslant d(x, y) + d(y, z) \tag{3.1c}$$

   By equation (3.1a), for $x = y$, $d(x, y) = 0$ so $\delta(x, y) = 0/1 = 0$, and when $x \neq y$, $d(x, y) > 0$ so $\delta(x, y) > 0$ since a positive number divided by a positive number is positive. We have that $\delta(x, y) \geqslant 0$ and equal zero if and only if $x = y$. By equation (3.1b), we have

   $$\begin{aligned}\delta(x, y) &= \frac{d(x, y)}{1 + d(x, y)}\\ &= \frac{d(y, x)}{1 + d(y, x)}\\ &= \delta(y, x)\end{aligned}$$

   For the triangle inequality, we have

   $$\frac{d(x, z)}{1 + d(x, z)} \leqslant \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}$$

   Let's multiple through by the product of all three denominators. After simplifying, we obtain

   $$d(x, z) \leqslant d(x, y) + d(y, z) + 2d(x, y)d(y, z) + d(x, y)d(y, z)d(x, z)$$

   We have already shown that $d(x, y) \geqslant 0$ and zero if and only if $x = y$. If $x = y = z$, the triangle inequality is vacuously true. When $x \neq y \neq z$, the triangle inequality follows since each distance is positive and equation (3.1c); that is,

   $$\delta(x, z) \leqslant \delta(x, y) + \delta(y, z).$$

2. Suppose that there are given two distance functions $d(x, y)$ and $d_1(x, y)$ on the same space S. They are said to be equivalent if they determine the same open sets. Show that d and d_1 are equivalent if to every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d_1(x, y) < \epsilon$, and vice versa. Verify that this condition is fulfilled in exercise 1.

   Let $\epsilon, \delta > 0$ be given. We can write $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)} = 1 - \frac{1}{1+d(x,y)}$. We need to find a δ such that whenever $d(x, y) < \delta$, $\delta(x, y) < \epsilon$.

   $$\begin{aligned}1 - \frac{1}{1 + d(x, y)} &< \epsilon\\ d(x, y) &< \frac{\epsilon}{1 - \epsilon}\end{aligned}$$

Let $\delta = \frac{\epsilon}{1-\epsilon}$. For $\epsilon < 1$, if $d(x,y) < \delta = \frac{\epsilon}{1-\epsilon}$, then

$$\delta(x,y) = 1 - \frac{1}{1+d(x,y)} < 1 - \frac{1}{1+\frac{\epsilon}{1-\epsilon}} = \epsilon.$$

If $\epsilon > 1$, $\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} < 1 < \epsilon$. For the reverse implication, we need to find a δ such that $d(x,y) < \epsilon$. Let $\delta = \frac{\epsilon}{1+\epsilon}$. For any $\epsilon > 0$, if $\delta(x,y) < \delta = \frac{\epsilon}{1+\epsilon}$, then

$$\begin{aligned}
\frac{d(x,y)}{1+d(x,y)} &< \frac{\epsilon}{1+\epsilon} \\
d(x,y)(1+\epsilon) &< \epsilon + \epsilon d(x,y) \\
d(x,y) &< \epsilon
\end{aligned}$$

as was needed to be shown. Therefore, $d(x,y)$ and $\delta(x,y)$ are equivalent metrics on S.

3. Show by strict application of the definition that the closure of $|z - z_0| < \delta$ is $|z - z_0| \leqslant \delta$.

4. If X is the set of complex numbers whose real and imaginary parts are rational, what is $\operatorname{Int} X, \bar{X}, \partial X$?

5. It is sometimes typographically simpler to write X' for $\sim X$. With this notation, how is $X'^{-'}$ related to X? Show that $X^{-'-'-'-'} = X^{-'-'}$.

6. A set is said to be discrete if all its points are isolated. Show that a discrete set in $\mathbb{R}$ or $\mathbb{C}$ is countable.

   Let S be a discrete set in $\mathbb{R}$ or $\mathbb{C}$. If $z \in S$, then for some $\epsilon_i > 0$, for $i \in \mathbb{Z}$, $N_{\epsilon_i}(z_i)$ is the i-th neighborhood of z_i. Since S is discrete, there exists an ϵ for each i such that the only point in $N_{\epsilon_i}(z_i)$ is z_i. Let ϵ_i be this ϵ. Consider the following function

$$f(i) = \begin{cases} 0, & i = 0 \\ 1, & i = 1 \\ 2i, & i > 0 \\ 2(-i)+1, & i < 0 \end{cases}$$

   We have put i in a one-to-one correspondence with $\mathbb{Z}^+$. Therefore, S is countable.

7. Show that the accumulation points of any set form a closed set.

   Let E be a set. Then E' is the set of accumulation (limit) points. If z_i is a limit point, $z_i \in E'$. Now, z_i are limit points of E as well. Then $\{z_i\} \to z$ where $z \in \bar{E}$. Therefore, $z \in E'$ so E' is closed.

3.1.3 Connectedness

1. If $X \subset S$, show that the relatively open (closed) subsets of X are precisely those sets that can be expressed as the intersection of X with an open (closed) subsets of S.

   Let $\{U_\alpha\}$ be the open sets of S such that $\bigcup_\alpha U_\alpha = S$. Then $X = X \cap \bigcup_\alpha U_\alpha = \bigcup_\alpha (X \cap U_\alpha)$. Let $\{A_n\} = \{X \cap U_\alpha\}$. Then A_n is relatively open in X since A_n belongs to the topology of X; that is, for each n, $A_n \subset X$ and $X = \bigcup_n A_n$.

2. Show that the union of two regions is a region if and only if they have a common point.

   For the first implication, $\Rightarrow$, suppose on the contrary that the union of two regions is a region and they have no point in common. Let A and B be these two nonempty regions. Since they share no point in common, $A \cap \bar{B} = B \cap \bar{A} = \varnothing$. Therefore, A and B are separated so they cannot be a region. We have reached a contradiction so if the union of two regions is a region, then they have a point in common. For the finally implication, suppose they have a point in common and the union of two regions is not a region. Since the union of two regions is not a region, the regions are separated. Let A and B be two nonempty separated regions. Since A and B are separated, $A \cap \bar{B} = B \cap \bar{A} = \varnothing$; therefore, A and B cannot have a point in common. We have reached contradiction so if they have a point in common, then the union of two regions is are a region.

3. Prove that the closure of a connected set is connected.

 Let T be a topological space such that $E, \bar{E} \subset T$. Let E be a connected set and suppose $\bar{E}$ is separated; that is, $\bar{E} = A \cup B$ where A, B are relatively open in $\bar{E}$, nonempty, and disjoint sets. Then there exists open sets U, V in T such that $A = U \cap \bar{E}$ and $B = V \cap \bar{E}$. Now, $A \subset U$, $B \subset V$, and $U, V \neq \varnothing$. Therefore, $U \cap E \neq \varnothing \neq V \cap E$ so $U \cap E$ and $V \cap E$ are nonempty, disjoint sets. Then $E = U \cup V$ so E is separated. We have, thus, reached a contradiction and the closure of connected set is also connected.

4. Let A be the set of points $(x, y) \in \mathbb{R}^2$ with $x = 0$, $|y| \leqslant 1$, and let B be the set with $x > 0$, $y = \sin(1/x)$. Is $A \cup B$ connected?

 This is known as the topologist's sine curve.

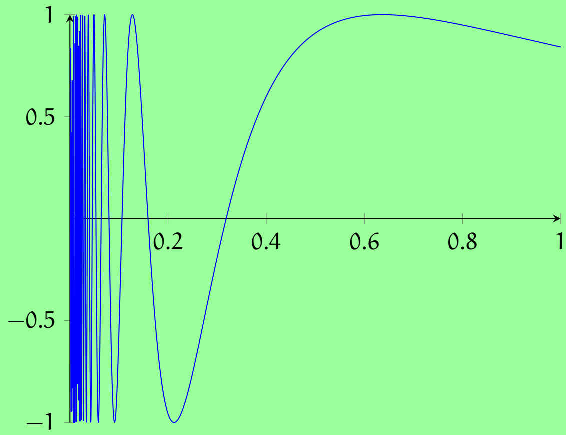

Figure 3.1: Topologist's sine curve plotted on the domain $x \in (0, 1]$.

 Let $S = A \cup B$. We claim that the closure of B in $\mathbb{R}^2$ is $\bar{B} = S$. Let $x \in S$. If $x \in B$, then take a constant sequence $\{x, x, \ldots\}$. If $x \in A$, then $x = (0, y)$ for $|y| \leqslant 1$ or said another way $y = \sin(\theta)$ for $\theta \in [-\pi, \pi]$. We can write $y = \sin(\theta)$ as $y = \sin(\theta + 2k\pi)$ for $k \in \mathbb{Z}^+$. Let $x_k = 1/(\theta + 2k\pi) > 0$. Then $y = \sin(1/x_k)$. Now $\{x_k\} \to 0$ when $k \to \infty$. Then $(x_k, \sin(1/x_k)) = (x_n, y) \to (0, y) \in \bar{B}$ since $|y| \leqslant 1$. Therefore, $S \subset \bar{B}$. Let $\{(x_n, y_n)\} \in S$ such that $\{(x_n, y_n)\} \to (x, y) \in \mathbb{R}^2$. Then $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. From the definition of the sets, $x \geqslant 0$ and $|y| \leqslant 1$ so $|y| = \lim_{n\to\infty}|y_n| \leqslant 1$. If $x = 0$, then $(0, y) \in S$ since $|y| \leqslant 1$. Suppose $x > 0$. Then there exist $m > N$ such that $x_m > 0$ for all $m > N$ so $(x_n, y_n) \in B$. Let $y_n = \sin(1/x_n)$ since $(x_n, y_n) \in B$. Notice that for $z \in (0, \infty)$, $\sin(1/z$ is continuous. Since $\{x_n\} \to x$ and $y_n = \sin(1/x_n)$, we have

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sin(1/x_n) = \sin(1/x).$$

 Thus, $(x, y) \in \bar{B} \subset S$ so $\bar{B} = S$. Since $A \cap \bar{B} = A \cap S = A \neq \varnothing$, S is connected. However, S is not path connected. That is, being connected doesn't imply path connectedness.

 Suppose S is path connected and there exists an $f\colon [0, 1] \to S$ such that $f(0) \in B$ and $f(1) \in A$. Since A is path connected, suppose $f(1) = (0, 1)$. Let $\epsilon = 1/2 > 0$. By continuity, for $\delta > 0$, $|f(t) - (0, 1)| < 1/2$ whenever $1 - \delta \leqslant t \leqslant 1$. Since f is continuous, the image of $f([1 - \delta, 1])$ is connected. Let $f(1 - \delta) = (x, y)$. Consider the composite of $f\colon [1 - \delta, 1] \to \mathbb{R}^2$ and its projection on the x-axis. Since both maps are continuous as well as their composite, the image of the composite map is a connected subset of $\mathbb{R}^1$ which contains zero and x. Now zero is the x-coordinate of $f(1)$ and x the x-coordinate of $f(1 - \delta)$. Since $\mathbb{R}^1$ is convex, connected sets are intervals. Then the set of x-coordinates for $f(1 - \delta)$ is $x_0 \in [0, x]$. For $x_0 \in (0, x]$, there exists $t \in [1 - \delta, 1]$ such that $f(t) = (x_0, \sin(1/x_0))$. If $x_0 = 1/(2k\pi - \pi/2)$ for $k \gg 1$, then $0 < x_0 < x$. Now $1/x_0 = \pi(4k - 1)/2$ which is a 2π mutliple of $-\pi/2$ for all k. Therefore, $\sin(1/x_0) = -1$ so $(x_0, \sin(1/x_0)) = (1/(2k\pi - \pi/2), -1)$ for some $t \in [1 - \delta, 1]$ which lies within a distance of $\epsilon = 1/2$ of $(0, 1)$. However, the distance between $(1/(2k\pi - \pi/2), -1)$ and $(0, 1)$ for large k is greater than 1 which is a contradiction. Thus, S cannot be path connected.

5. Let E be the set of points $(x, y) \in \mathbb{R}^2$ such that $0 \leqslant x \leqslant 1$ and either $y = 0$ or $y = 1/n$ for some positive integer n. What are the components of E? Are they all closed? Are they relatively open? Verify that E is not locally connected.

6. Prove that the components of a closed set are closed (use exercise 3).

7. A set is said to be *discrete* if all its points are isolated. Show that a discrete set in a separable metric space is countable.

3.1.4 Compactness

1. Given an alternate proof of the fact that every bounded sequence of complex numbers has a convergent subsequence (for instance by use of the limes inferior).

2. Show that the Heine-Borel property can also be expressed in the following manner: Every collection of closed sets with an empty intersection contains a finite subcollection with an empty intersection.

   The statement above is equivalent to: A collection $\mathcal{F}$ of closed subsets of a topological space $(X, \mathcal{T})$ has the finite intersection property if $\cap\mathcal{F}_\alpha \neq \varnothing$ for all finite subcollections $\mathcal{F}_\alpha \subset \mathcal{F}$. Show that $(X, \mathcal{T})$, a topological space, is compacy if and only if every family of closed sets $\mathcal{F} \subset \mathcal{P}(X)$ having the finite intersection property satisfies $\cap\mathcal{F} \neq \varnothing$.

   Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a collection of closed sets in X. Now

   $$\bigcap_\alpha F_\alpha = \varnothing \iff \bigcup_\alpha F_\alpha^c = X.$$

   Therefore, the set $\bigcup_\alpha F_\alpha^c$ is an open cover of X since F_α is closed. If the intersection of the set $\{F_{\alpha_n}\}$ is empty for a finite n, then $\bigcup_\alpha F_{\alpha_n}^c$ is a finite subcover of X. Then every open cover of X has a finite subcover if and only if every collection of closed sets with an empty intersection has a finite subcollection with an empty intersection. Thus, X is compact if and only if every collection of closed sets with the finite intersection property has a nonempty intersection.

3. Use compactness to prove that a closed bounded set of real numbers has a maximum.

   Since we are dealing with a set of real numbers, we are speaking of compact metric spaces. A subset E of a metric space X is compact if and only if every sequence in E has convergent subsequence in E (sequentially compact).

   First, we will show $\Rightarrow$ by contradiction. Let $\{x_n\}$ be a sequence in E. Suppose $\{x_n\}$ doesn't have a convergent subsequence in E. Then for $x \in E$, there exists $\epsilon > 0$ such that $x_n \in N_\epsilon(x)$ for only finitely many n. Then $N_\epsilon(x)$ would be an open cover of E which has no finite subcover. Therefore, E couldn't be compact contradicting the premise. Thus, if E is compact metric space, then E is sequentialy compact. In order to prove $\Leftarrow$, we need to prove that a sequentially compact set contains a countable dense subset.

   Lemma 3.1.4.1 : A sequentially compact set contains a countable dense subset (separable space).

   Let A be an infinite sequentially compact set. Since A is sequential compact, A is bounded; otherwise, we would have nonconvergent subsequences in A. Let $\{y_n\}$ be a dense sequence in A. Choose $y_1, y_2, \ldots, y_n$ of $\{y_n\}$. Let $\delta_n = \sup_{y \in A} \min_{k \leqslant n} d(y, y_n) > 0$. Let y_{n+1} be such that $d(y_{n+1}, y_k) \geqslant \delta/2$ for $k = 1, \ldots, n$. Since $\{y_n\}$ has a convergent subsequence, for all $\epsilon > 0$ there exist $m, n \in \mathbb{Z}^+$ such that $d(y_m, y_n) < \epsilon$. Then

   $$d(y, y_{n-1}) < \delta_{n-1}/2 < d(y_m, y_n) < \epsilon \iff \delta_{n-1} < 2\epsilon$$

   Thus, all $y \in A$ is in 2ϵ of y_k for $k < n$. Since $\epsilon > 0$ and arbitrary, $\{y_n\}$ is dense in A because every nonempty open set contains at least one element of the sequence.

   Now for $\Leftarrow$. Let F_α be an open cover E and let $\{y_n\}$ be a dense sequence. Let $r \in \mathbb{Q}$ and let G be the family of neighborhoods, $N_r(y_n)$, that are contained in F_α. Since $\mathbb{Q}$ is countable, G is countable. Let $x \in E$ and $x \in F_\alpha$. Then $N_\epsilon(x) \subset F_\alpha$ for $\epsilon > 0$. Since $\{y_n\}$ is dense in E, by lemma 3.1.4.1, $d(y, y_n) < \epsilon/2$ for some n. For all $r \in \mathbb{Q}$, $d(y_n, y) < r < \epsilon - d(y_n, y)$. Then $x \in N_r(y_n) \subset N_\epsilon(x) \subset G$. Since $x \in N_r(y_n) \in G$ and G is countable, we can find a finite subcover of G. Replace each G by F_α where $G \subset F_\alpha$ for some α. Then this set of F_α is a finite subcover. Thus, if E is a sequentially compact metric space, then E is compact.

4. If $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of nonempty compact sets, then the intersection $\bigcap_1^\infty E_n$ is not empty (Cantor's lemma). Show by example that this need not be true if the sets are merely closed.

   Consider the topological space $\mathbb{R}^1$. Let $E_n = \{n \in \mathbb{Z}^{\geqslant 0} \colon [n, \infty)\}$. Then $E_n \supset E_{n+1} \supset \cdots$. Since $(-\infty, n)$ is open in $\mathbb{R}^1$, $[n, \infty) = (-\infty, n)^c$ is closed. The infinite intersection of E_n is

   $$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} [n, \infty) = \varnothing.$$

   Thus, the statement isn't true if we consider only closed sets.

5. Let S be the set of all sequences $x = \{x_n\}$ of real numbers such that only a finite number of the x_n are $\neq 0$. Define $d(x, y) = \max|x_n - y_n|$. Is the space complete? Show that the δ-neighborhoods are not totally bounded.

3.1.5 Continuous Functions

3.1.6 Topological Spaces

3.2 Conformality