Solutions to Complex Analysis

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Contents

1 Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1. Find the values of

$$(1+2i)^3, \qquad \frac{5}{-3+4i}, \qquad \left(\frac{2+i}{3-2i}\right)^2, \qquad (1+i)^n + (1-i)^n$$

For the first problem, we have $(1+2i)^3 = (-3+4i)(1+2i) = -11-2i$. For the second problem, we should multiple by the conjugate $\bar{z} = -3-4i$.

$$\frac{5}{-3+4i}\frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \frac{-3}{5} - \frac{4}{5}i$$

For the third problem, we should first multiple by $\bar{z} = 3+2i$.

$$\frac{2+i}{3-2i}\frac{3+2i}{3+2i} = \frac{8+i}{13}$$

Now we need to just square the result.

$$\frac{1}{169}(8+i)^2 = \frac{63+16i}{169}$$

For the last problem, we will need to find the polar form of the complex numbers. Let $z_1 = 1+i$ and $z_2 = 1-i$. Then the modulus of $z_1 = \sqrt{2} = z_2$. Let ϕ_1 and ϕ_2 be the angles associated with z_1 and z_2, respectively. Then $\phi_1 = \arctan(1) = \frac{\pi}{4}$ and $\phi_2 = \arctan(-1) = \frac{-\pi}{4}$. Then $z_1 = \sqrt{2}e^{\pi i/4}$ and $z_2 = \sqrt{2}e^{-\pi i/4}$.

$$\begin{aligned}
z_1^n + z_2^n &= 2^{n/2}\left[e^{n\pi i/4} + e^{-n\pi i/4}\right] \\
&= 2^{n/2+1}\left[\frac{e^{n\pi i/4} + e^{-n\pi i/4}}{2}\right] \\
&= 2^{n/2+1}\cos\left(\frac{n\pi}{4}\right)
\end{aligned}$$

2. If $z = x+iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}$$

For z^4, we can use the binomial theorem since $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^n b^{n-k}$. Therefore,

$$(x+iy)^4 = \binom{4}{0}(iy)^4 + \binom{4}{1}x(iy)^3 + \binom{4}{2}x^2(iy)^2 + \binom{4}{3}x^3(iy) + \binom{4}{4}x^4 = y^4 - 4xy^3i - 6x^2y^2 + 4x^3yi + x^4$$

Then the real and imaginary parts are

$$\begin{aligned}
u(x,y) &= x^4 + y^4 - 6x^2y^2 \\
v(x,y) &= 4x^3y - 4xy^3
\end{aligned}$$

For second problem, we need to multiple by the conjugate $\bar{z}$.

$$\frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

so the real and imaginary parts are

$$u(x,y) = \frac{x}{x^2+y^2}$$
$$v(x,y) = \frac{-y}{x^2+y^2}$$

For the third problem, we have $\frac{x-1+iy}{x+1-iy}$. Then $\bar{z} = x+1+iy$.

$$\frac{x-1+iy}{x+1-iy}\frac{x+1+iy}{x+1+iy} = \frac{x^2-1+2xyi}{(x+1)^2+y^2}$$

Then real and imaginary parts are

$$u(x,y) = \frac{x^2-1}{(x+1)^2+y^2}$$
$$v(x,y) = \frac{2xy}{(x+1)^2+y^2}$$

For the last problem, we have

$$\frac{1}{z^2} = \frac{x^2-y^2-2xyi}{x^4+2x^2y^2+y^4}$$

so the real and imaginary parts are

$$u(x,y) = \frac{x^2-y^2}{x^4+2x^2y^2+y^4}$$
$$v(x,y) = \frac{-2xy}{x^4+2x^2y^2+y^4}$$

3. Show that $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3 = 1$ and $\left(\frac{\pm 1\pm i\sqrt{3}}{2}\right)^6 = 1$.

   Both problems will can be handled easily by converting to polar form. Let $z_1 = \frac{-1\pm i\sqrt{3}}{2}$. Then $|z_1| = 1$. Let ϕ_+ be the angle for the positive z_1 and ϕ_- for the negative. Then $\phi_+ = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$ and $\phi_- = \arctan(\sqrt{3}) = \frac{4\pi}{3}$. We can write $z_{1+} = e^{2i\pi/3}$ and $z_{1-} = e^{4i\pi/3}$.

$$\begin{aligned} z_{1+}^3 &= e^{2i\pi} \\ &= 1 \\ z_{1-}^3 &= e^{4i\pi} \\ &= 1 \end{aligned}$$

   Therefore, $z_1^3 = 1$. For the second problem, $\phi_{ij} = \pm\frac{\pi}{3}$ and $\pm\frac{2\pi}{3}$ for $i, j = +, -$ and the $|z_2| = 1$. When we raise z to the sixth poewr, the argument becomes $\pm 2\pi$ and $\pm 4\pi$.

$$e^{\pm 2i\pi} = e^{\pm 4i\pi} = z^6 = 1$$

1.1.2 Square Roots

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}$$

   For $\sqrt{i}$, we are looking for x and y such that

$$\begin{aligned} \sqrt{i} &= x+iy \\ i &= x^2-y^2+2xyi \\ x^2-y^2 &= 0 \qquad (1.1) \\ 2xy &= 1 \qquad (1.2) \end{aligned}$$

From equation (1.1), we see that $x^2 = y^2$ or $\pm x = \pm y$. Also, note that i is the upper half plane (UHP). That is, the angle is positive so $x = y$ and $2x^2 = 1$ from equation (1.2). Therefore, $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$. We also could have done this problem using the polar form of z. Let $z = i$. Then $z = e^{i\pi/2}$ so $\sqrt{z} = e^{i\pi/4}$ which is exactly what we obtained. For $\sqrt{-i}$, let $z = -i$. Then z in polar form is $z = e^{-i\pi/2}$ so $\sqrt{z} = e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$. For $\sqrt{1+i}$, let $z = 1+i$. Then $z = \sqrt{2}e^{i\pi/4}$ so $\sqrt{z} = 2^{1/4}e^{i\pi/8}$. Finally, for $\sqrt{\frac{1-i\sqrt{3}}{2}}$, let $z = \frac{1-i\sqrt{3}}{2}$. Then $z = e^{-i\pi/3}$ so $\sqrt{z} = e^{-i\pi/6} = \frac{1}{2}(\sqrt{3}-i)$.

2. Find the four values of $\sqrt[4]{-1}$.

   Let $z = \sqrt[4]{-1}$ so $z^4 = -1$. Let $z = re^{i\theta}$ so $r^4e^{4i\theta} = -1 = e^{i\pi(1+2k)}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{4}(1+2k) \end{aligned}$$

   where $k = 0, 1, 2, 3$. Since when $k = 4$, we have $k = 0$. Then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

$$z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$$

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

   Let $z = \sqrt[4]{i}$ and $z = re^{i\theta}$. Then $r^4e^{4i\theta} = i = e^{i\pi/2}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{8} \end{aligned}$$

   so $z = e^{i\pi/8}$. Now, let $z = \sqrt[4]{-i}$. Then $r^4e^{4i\theta} = e^{-i\pi/2}$ so $z = e^{-i\pi/8}$.

4. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

   The quadratic equation is $x = \frac{-b\pm\sqrt{b^2-ac}}{2}$. For the complex polynomial, we have

$$z = \frac{-\alpha - \beta i \pm \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}}{2}$$

   Let $a + bi = \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}$. Then

$$z = \frac{-\alpha - \beta \pm (a + bi)}{2}$$

1.1.3 Justification

1. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

   combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

1.1.4 Conjugation, Absolute Value

1. Verify by calculation the values of
$$\frac{z}{z^2+1}$$
for $z = x + iy$ and $\bar{z} = x - iy$ are conjugate.

   For z, we have that $z^2 = x^2 - y^2 + 2xyi$.
$$\begin{aligned}\frac{z}{z^2+1} &= \frac{x+iy}{x^2-y^2+1+2xyi} \\ &= \frac{x+iy}{x^2-y^2+1+2xyi}\frac{x^2-y^2+1-2xyi}{x^2-y^2+1-2xyi} \\ &= \frac{x(x^2-y^2+1)+2xy^2+iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2} \qquad (1.3)\end{aligned}$$
   For $\bar{z}$, we have that $\bar{z}^2 = x^2 - y^2 - 2xyi$.
$$\begin{aligned}\frac{\bar{z}}{\bar{z}^2+1} &= \frac{x-iy}{x^2-y^2+1-2xyi} \\ &= \frac{x-iy}{x^2-y^2+1-2xyi}\frac{x^2-y^2+1+2xyi}{x^2-y^2+1+2xyi} \\ &= \frac{x(x^2-y^2+1)+2xy^2-iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2} \qquad (1.4)\end{aligned}$$
   Therefore, we have that equations (1.3) and (1.4) are conjugates.

2. Find the absolute value (modulus) of
$$-2i(3+i)(2+4i)(1+i) \qquad \text{and} \qquad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}.$$
   When we expand the first problem, we have that
$$z_1 = -2i(3+i)(2+4i)(1+i) = 32+24i$$
   so
$$|z_1| = \sqrt{32^2+24^2} = 40.$$
   For the second problem, we have that
$$z_2 = \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)} = 2 - \frac{3}{2}i$$
   so
$$|z_2| = \sqrt{4+9/4} = \frac{5}{2}.$$

3. Prove that
$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$
   if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

   Recall that $|z|^2 = z\bar{z}$.
$$\begin{aligned}1^2 &= \left|\frac{a-b}{1-\bar{a}b}\right|^2 \\ 1 &= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\overline{\frac{a-b}{1-\bar{a}b}}\right) \\ &= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\frac{\bar{a}-\bar{b}}{1-a\bar{b}}\right)\end{aligned}$$

$$= \frac{a\bar{a} - a\bar{b} - \bar{a}b + b\bar{b}}{1 - \bar{a}b - a\bar{b} + a\bar{a}b\bar{b}} \tag{1.5}$$

If $|a| = 1$, then $|a|^2 = a\bar{a} = 1$ and similarly for $|b|^2 = 1$. Then equation (1.5) becomes

$$\frac{1 - a\bar{b} - \bar{a}b + b\bar{b}}{1 - \bar{a}b - a\bar{b} + b\bar{b}} \quad \text{and} \quad \frac{1 - a\bar{b} - \bar{a}b + a\bar{a}}{1 - \bar{a}b - a\bar{b} + a\bar{a}}$$

resepctively which is one. If $|a| = |b| = 1$, then $|a|^2 = |b|^2 = 1$ so equation (1.5) can be written as

$$\frac{2 - a\bar{b} - \bar{a}b}{2 - \bar{a}b - a\bar{b}}.$$

Therefore, we must have that $a\bar{b} + \bar{a}b \neq 2$.

4. Find the conditions under which the equation $az + b\bar{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

   Let $z = x + iy$. Then $az + b\bar{z} + c = a(x + iy) + b(x - iy) + c = 0$.

$$(a + b)x + c = 0 \tag{1.6a}$$
$$(a - b)y = 0 \tag{1.6b}$$

   Lets consider equation (1.6b). We either have that $a = b$ or $y = 0$. If $a = b$, then WLOG equation (1.6a) can be written as

$$x = \frac{-c}{2a}$$

   and $y \in \mathbb{R}$. For fixed a, b, c, we have infinitely many solutions when $a = b$ since $z = \frac{-c}{2a} + iy$ for $y \in \mathbb{R}$. If $y = 0$, then equation (1.6a) can be written as

$$x = \frac{-c}{a + b}.$$

   Therefore, $z = x$ and we have only one solution.

5. Prove that Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leqslant i \leqslant j \leqslant n} |a_i \bar{b}_j - a_j \bar{b}_i|^2.$$

1.1.5 Inequalities

1. Prove that

$$\left|\frac{a - b}{1 - \bar{a}b}\right| < 1$$

   if $|a| < 1$ and $|b| < 1$.

   From the properties of the modulus, we have that

$$\begin{aligned}
\left|\frac{a - b}{1 - \bar{a}b}\right| &= \frac{|a - b|}{|1 - \bar{a}b|} \\
&= \frac{|a - b|^2}{|1 - \bar{a}b|^2} \qquad (1.7) \\
&= \frac{(a - b)(\bar{a} - \bar{b})}{(1 - \bar{a}b)(1 - a\bar{b})} \\
&= \frac{|a|^2 + |b|^2 - a\bar{b} - \bar{a}b}{1 + |a|^2|b|^2 - \bar{a}b - a\bar{b}} \\
&< \frac{2 - a\bar{b} - \bar{a}b}{2 - \bar{a}b - a\bar{b}} \\
&= 1 \qquad (1.8)
\end{aligned}$$

From equations (1.7) and (1.8), we have

$$\frac{|a-b|^2}{|1-\bar{a}b|^2} < 1$$
$$\frac{|a-b|}{|1-\bar{a}b|} < 1$$

2. Prove Cauchy's inequality by induction.

3. If $|a_i| < 1$, $\lambda_i \geqslant 1$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

4. Show that there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leqslant |c|$. If this condition is fulfilled, what are the smallest and largest values $|z|$?

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1.2.2 The Binomial Equation

1.2.3 Analytic Geometry

1.2.4 The Spherical Representation