Solutions to Complex Analysis

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Contents

1 Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1. Find the values of

$$(1+2i)^3, \qquad \frac{5}{-3+4i}, \qquad \left(\frac{2+i}{3-2i}\right)^2, \qquad (1+i)^n + (1-i)^n$$

For the first problem, we have $(1+2i)^3 = (-3+4i)(1+2i) = -11-2i$. For the second problem, we should multiple by the conjugate $\bar{z} = -3-4i$.

$$\frac{5}{-3+4i}\frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \frac{-3}{5} - \frac{4}{5}i$$

For the third problem, we should first multiple by $\bar{z} = 3+2i$.

$$\frac{2+i}{3-2i}\frac{3+2i}{3+2i} = \frac{8+i}{13}$$

Now we need to just square the result.

$$\frac{1}{169}(8+i)^2 = \frac{63+16i}{169}$$

For the last problem, we will need to find the polar form of the complex numbers. Let $z_1 = 1+i$ and $z_2 = 1-i$. Then the modulus of $z_1 = \sqrt{2} = z_2$. Let ϕ_1 and ϕ_2 be the angles associated with z_1 and z_2, respectively. Then $\phi_1 = \arctan(1) = \frac{\pi}{4}$ and $\phi_2 = \arctan(-1) = \frac{-\pi}{4}$. Then $z_1 = \sqrt{2}e^{\pi i/4}$ and $z_2 = \sqrt{2}e^{-\pi i/4}$.

$$\begin{aligned} z_1^n + z_2^n &= 2^{n/2}\left[e^{n\pi i/4} + e^{-n\pi i/4}\right] \\ &= 2^{n/2+1}\left[\frac{e^{n\pi i/4} + e^{-n\pi i/4}}{2}\right] \\ &= 2^{n/2+1}\cos\left(\frac{n\pi}{4}\right) \end{aligned}$$

2. If $z = x+iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}$$

For z^4, we can use the binomial theorem since $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^n b^{n-k}$. Therefore,

$$(x+iy)^4 = \binom{4}{0}(iy)^4 + \binom{4}{1}x(iy)^3 + \binom{4}{2}x^2(iy)^2 + \binom{4}{3}x^3(iy) + \binom{4}{4}x^4 = y^4 - 4xy^3i - 6x^2y^2 + 4x^3yi + x^4$$

Then the real and imaginary parts are

$$\begin{aligned} u(x,y) &= x^4 + y^4 - 6x^2y^2 \\ v(x,y) &= 4x^3y - 4xy^3 \end{aligned}$$

For second problem, we need to multiple by the conjugate $\bar{z}$.

$$\frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

so the real and imaginary parts are

$$
\begin{aligned}
u(x,y) &= \frac{x}{x^2+y^2} \\
v(x,y) &= \frac{-y}{x^2+y^2}
\end{aligned}
$$

For the third problem, we have $\frac{x-1+iy}{x+1-iy}$. Then $\bar{z} = x+1+iy$.

$$
\frac{x-1+iy}{x+1-iy}\frac{x+1+iy}{x+1+iy} = \frac{x^2-1+2xyi}{(x+1)^2+y^2}
$$

Then real and imaginary parts are

$$
\begin{aligned}
u(x,y) &= \frac{x^2-1}{(x+1)^2+y^2} \\
v(x,y) &= \frac{2xy}{(x+1)^2+y^2}
\end{aligned}
$$

For the last problem, we have

$$
\frac{1}{z^2} = \frac{x^2-y^2-2xyi}{x^4+2x^2y^2+y^4}
$$

so the real and imaginary parts are

$$
\begin{aligned}
u(x,y) &= \frac{x^2-y^2}{x^4+2x^2y^2+y^4} \\
v(x,y) &= \frac{-2xy}{x^4+2x^2y^2+y^4}
\end{aligned}
$$

3. Show that $\left(\frac{-1\pm i\sqrt{3}}{2}\right)^3 = 1$ and $\left(\frac{\pm 1\pm i\sqrt{3}}{2}\right)^6 = 1$.

   Both problems will can be handled easily by converting to polar form. Let $z_1 = \frac{-1\pm i\sqrt{3}}{2}$. Then $|z_1| = 1$. Let ϕ_+ be the angle for the positive z_1 and ϕ_- for the negative. Then $\phi_+ = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$ and $\phi_- = \arctan(\sqrt{3}) = \frac{4\pi}{3}$. We can write $z_{1+} = e^{2i\pi/3}$ and $z_{1-} = e^{4i\pi/3}$.

$$
\begin{aligned}
z_{1+}^3 &= e^{2i\pi} \\
&= 1 \\
z_{1-}^3 &= e^{4i\pi} \\
&= 1
\end{aligned}
$$

   Therefore, $z_1^3 = 1$. For the second problem, $\phi_{ij} = \pm\frac{\pi}{3}$ and $\pm\frac{2\pi}{3}$ for $i,j = +,-$ and the $|z_2| = 1$. When we raise z to the sixth poewr, the argument becomes $\pm 2\pi$ and $\pm 4\pi$.

$$
e^{\pm 2i\pi} = e^{\pm 4i\pi} = z^6 = 1
$$

1.1.2 Square Roots

1. Compute

$$
\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}
$$

   For $\sqrt{i}$, we are looking for x and y such that

$$
\begin{aligned}
\sqrt{i} &= x+iy \\
i &= x^2-y^2+2xyi \\
x^2-y^2 &= 0 \qquad (1.1) \\
2xy &= 1 \qquad (1.2)
\end{aligned}
$$

From equation (1.1), we see that $x^2 = y^2$ or $\pm x = \pm y$. Also, note that i is the upper half plane (UHP). That is, the angle is positive so $x = y$ and $2x^2 = 1$ from equation (1.1). Therefore, $\sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$. We also could have done this problem using the polar form of z. Let $z = i$. Then $z = e^{i\pi/2}$ so $\sqrt{z} = e^{i\pi/4}$ which is exactly what we obtained. For $\sqrt{-i}$, let $z = -i$. Then z in polar form is $z = e^{-i\pi/2}$ so $\sqrt{z} = e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1 - i)$. For $\sqrt{1+i}$, let $z = 1 + i$. Then $z = \sqrt{2}e^{i\pi/4}$ so $\sqrt{z} = 2^{1/4}e^{i\pi/8}$. Finally, for $\sqrt{\frac{1-i\sqrt{3}}{2}}$, let $z = \frac{1-i\sqrt{3}}{2}$. Then $z = e^{-i\pi/3}$ so $\sqrt{z} = e^{-i\pi/6} = \frac{1}{2}(\sqrt{3} - i)$.

2. Find the four values of $\sqrt[4]{-1}$.

   Let $z = \sqrt[4]{-1}$ so $z^4 = -1$. Let $z = re^{i\theta}$ so $r^4e^{4i\theta} = -1 = e^{i\pi(1+2k)}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{4}(1 + 2k) \end{aligned}$$

   where $k = 0, 1, 2, 3$. Since when $k = 4$, we have $k = 0$. Then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

$$z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$$

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

   Let $z = \sqrt[4]{i}$ and $z = re^{i\theta}$. Then $r^4e^{4i\theta} = i = e^{i\pi/2}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{8} \end{aligned}$$

   so $z = e^{i\pi/8}$. Now, let $z = \sqrt[4]{-i}$. Then $r^4e^{4i\theta} = e^{-i\pi/2}$ so $z = e^{-i\pi/8}$.

4. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

   The quadratic equation is $x = \frac{-b\pm\sqrt{b^2-ac}}{2}$. For the complex polynomial, we have

$$z = \frac{-\alpha - \beta i \pm \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}}{2}$$

   Let $a + bi = \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}$. Then

$$z = \frac{-\alpha - \beta \pm (a + bi)}{2}$$

1.1.3 Justification

1. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

   combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

1.1.4 Conjugation, Absolute Value

1. Verify by calculation the values of
$$\frac{z}{z^2+1}$$
for $z = x + iy$ and $\bar{z} = x - iy$ are conjugate.

   For z, we have that $z^2 = x^2 - y^2 + 2xyi$.
$$\begin{aligned}\frac{z}{z^2+1} &= \frac{x+iy}{x^2-y^2+1+2xyi} \\ &= \frac{x+iy}{x^2-y^2+1+2xyi}\frac{x^2-y^2+1-2xyi}{x^2-y^2+1-2xyi} \\ &= \frac{x(x^2-y^2+1)+2xy^2+iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2} \end{aligned} \tag{1.3}$$

   For $\bar{z}$, we have that $\bar{z}^2 = x^2 - y^2 - 2xyi$.
$$\begin{aligned}\frac{\bar{z}}{\bar{z}^2+1} &= \frac{x-iy}{x^2-y^2+1-2xyi} \\ &= \frac{x-iy}{x^2-y^2+1-2xyi}\frac{x^2-y^2+1+2xyi}{x^2-y^2+1+2xyi} \\ &= \frac{x(x^2-y^2+1)+2xy^2-iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2} \end{aligned} \tag{1.4}$$

   Therefore, we have that equations (1.3) and (1.4) are conjugates.

2. Find the absolute value (modulus) of
$$-2i(3+i)(2+4i)(1+i) \qquad \text{and} \qquad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}.$$

   When we expand the first problem, we have that
$$z_1 = -2i(3+i)(2+4i)(1+i) = 32+24i$$
   so
$$|z_1| = \sqrt{32^2+24^2} = 40.$$

   For the second problem, we have that
$$z_2 = \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)} = 2 - \frac{3}{2}i$$
   so
$$|z_2| = \sqrt{4+9/4} = \frac{5}{2}.$$

3. Prove that
$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$
if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

   Recall that $|z|^2 = z\bar{z}$.
$$\begin{aligned} 1^2 &= \left|\frac{a-b}{1-\bar{a}b}\right|^2 \\ 1 &= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\overline{\frac{a-b}{1-\bar{a}b}}\right) \\ &= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\frac{\bar{a}-\bar{b}}{1-a\bar{b}}\right) \end{aligned}$$

$$= \frac{a\bar{a} - a\bar{b} - \bar{a}b + b\bar{b}}{1 - \bar{a}b - a\bar{b} + a\bar{a}b\bar{b}} \tag{1.5}$$

If $|a| = 1$, then $|a|^2 = a\bar{a} = 1$ and similarly for $|b|^2 = 1$. Then equation (1.5) becomes

$$\frac{1 - a\bar{b} - \bar{a}b + b\bar{b}}{1 - \bar{a}b - a\bar{b} + b\bar{b}} \quad \text{and} \quad \frac{1 - a\bar{b} - \bar{a}b + a\bar{a}}{1 - \bar{a}b - a\bar{b} + a\bar{a}}$$

resepctively which is one. If $|a| = |b| = 1$, then $|a|^2 = |b|^2 = 1$ so equation (1.5) can be written as

$$\frac{2 - a\bar{b} - \bar{a}b}{2 - \bar{a}b - a\bar{b}}.$$

Therefore, we must have that $a\bar{b} + \bar{a}b \neq 2$.

4. Find the conditions under which the equation $az + b\bar{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

   Let $z = x + iy$. Then $az + b\bar{z} + c = a(x + iy) + b(x - iy) + c = 0$.

$$(a + b)x + c = 0 \tag{1.6a}$$
$$(a - b)y = 0 \tag{1.6b}$$

   Lets consider equation (1.6b). We either have that $a = b$ or $y = 0$. If $a = b$, then WLOG equation (1.6a) can be written as
$$x = \frac{-c}{2a}$$
   and $y \in \mathbb{R}$. For fixed a, b, c, we have infinitely many solutions when $a = b$ since $z = \frac{-c}{2a} + iy$ for $y \in \mathbb{R}$. If $y = 0$, then equation (1.6a) can be written as

$$x = \frac{-c}{a + b}.$$

   Therefore, $z = x$ and we have only one solution.

5. Prove that Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leqslant i \leqslant j \leqslant n} |a_i \bar{b}_j - a_j \bar{b}_i|^2.$$

1.1.5 Inequalities

1. Prove that
$$\left|\frac{a - b}{1 - \bar{a}b}\right| < 1$$
   if $|a| < 1$ and $|b| < 1$.

   From the properties of the modulus, we have that

$$\begin{aligned}
\left|\frac{a - b}{1 - \bar{a}b}\right| &= \frac{|a - b|}{|1 - \bar{a}b|} \\
&= \frac{|a - b|^2}{|1 - \bar{a}b|^2} && (1.7) \\
&= \frac{(a - b)(\bar{a} - \bar{b})}{(1 - \bar{a}b)(1 - a\bar{b})} \\
&= \frac{|a|^2 + |b|^2 - a\bar{b} - \bar{a}b}{1 + |a|^2|b|^2 - \bar{a}b - a\bar{b}} \\
&< \frac{2 - a\bar{b} - \bar{a}b}{2 - \bar{a}b - a\bar{b}} \\
&= 1 && (1.8)
\end{aligned}$$

From equations (1.7) and (1.8), we have

$$\frac{|a-b|^2}{|1-\bar{a}b|^2} < 1$$
$$\frac{|a-b|}{|1-\bar{a}b|} < 1$$

2. Prove Cauchy's inequality by induction.

   Cauchy's inequality is

$$|a_1 b_1 + \cdots + a_n b_n|^2 \leqslant \left(|a_1|^2 + \cdots + |a_n|^2\right)\left(|b_1|^2 + \cdots + |b_n|^2\right)$$

   which can be written more compactly as

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leqslant \sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2.$$

   For the base case, $i = 1$, we have

$$|a_1 b_1|^2 = (a_1 b_1)(\bar{a}_1 \bar{b}_1) = a_1 \bar{a}_1 b_1 \bar{b}_1 = |a_1|^2 |b_1|^2$$

   so the base case is true. Now let the equality hold for all $k-1 \in \mathbb{Z}$ where $k-1 \leqslant n$. That is, we assume that

$$\left|\sum_{i=1}^{k-1} a_i b_i\right|^2 \leqslant \sum_{i=1}^{k-1}|a_i|^2 \sum_{i=1}^{k-1}|b_i|^2$$

   to be true.

$$\begin{aligned}
\left|\sum_{i=1}^{k-1} a_i b_i\right|^2 + |a_k b_k|^2 &\leqslant \sum_{i=1}^{k-1}|a_i|^2 \sum_{i=1}^{k-1}|b_i|^2 + |a_k b_k|^2 \\
\left|\sum_{i=1}^{k} a_i b_i\right|^2 &\leqslant \sum_{i=1}^{k-1}|a_i|^2 \sum_{i=1}^{k-1}|b_i|^2 + (a_k b_k)(\bar{a}_k \bar{b}_k) \\
&= \sum_{i=1}^{k-1}|a_i|^2 \sum_{i=1}^{k-1}|b_i|^2 + |a_k|^2|b_k|^2 \\
&= \sum_{i=1}^{k}|a_i|^2 \sum_{i=1}^{k}|b_i|^2
\end{aligned}$$

   Therefore, by the principal of mathematical induction, Cauchy's inequality is true for all $n \geqslant 1$ for $n \in \mathbb{Z}^+$.

3. If $|a_i| < 1$, $\lambda_i \geqslant 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

   Since $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geqslant 0$, $0 \leqslant \lambda_i < 1$. By the triangle inequality,

$$\begin{aligned}
|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| &\leqslant |\lambda_1||a_1| + \cdots + |a_n||\lambda_n| \\
&< \sum_{i=1}^{n} \lambda_i \\
&= 1
\end{aligned}$$

4. Show that there are complex numbers z satisfying

$$|z-a| + |z+a| = 2|c|$$

   if and only if $|a| \leqslant |c|$. If this condition is fulfilled, what are the smallest and largest values $|z|$?

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1. Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

2. Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_1a_3$.

3. Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

4. Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

1.2.2 The Binomial Equation

1. Express $\cos(3\varphi)$, $\cos(4\varphi)$, and $\sin(5\varphi)$ in terms of $\cos(\varphi)$ and $\sin(\varphi)$.

   For these problems, the sum addition identities will be employed; that is,

$$\begin{aligned}
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)
\end{aligned}$$

   We can write $\cos(3\varphi)$ as $\cos(2\varphi + \varphi)$ so

$$\begin{aligned}
\cos(3\varphi) &= \cos(2\varphi + \varphi) \\
&= \cos(2\varphi)\cos(\varphi) - \sin(2\varphi)\sin(\varphi) \\
&= \left[\cos^2(\varphi) - \sin^2(\varphi)\right]\cos(\varphi) - 2\sin(\varphi)\cos(\varphi)\sin(\varphi) \\
&= \cos^3(\varphi) - 3\sin^2(\varphi)\cos(\varphi)
\end{aligned}$$

   For $\cos(4\varphi)$, we have

$$\begin{aligned}
\cos(4\varphi) &= \cos(2\varphi)\cos(2\varphi) - \sin(2\varphi)\sin(2\varphi) \\
&= \left[\cos^2(\varphi) - \sin^2(\varphi)\right]^2 - 4\sin^2(\varphi)\cos^2(\varphi) \\
&= \cos^4(\varphi) + \sin^4(\varphi) - 6\sin^2(\varphi)\cos^2(\varphi)
\end{aligned}$$

   For $\sin(5\varphi)$, we have

$$\begin{aligned}
\sin(5\varphi) &= \sin(4\varphi)\cos(\varphi) + \sin(\varphi)\cos(4\varphi) \\
&= 2\sin(2\varphi)\cos(2\varphi)\left[\cos^2(\varphi) - \sin^2(\varphi)\right]\cos(\varphi) + \sin^5(\varphi) + \sin(\varphi)\cos^4(\varphi) - 6\sin^3(\varphi)\cos^2(\varphi) \\
&= 5\sin(\varphi)\cos^4(\varphi) - 10\sin^3(\varphi)\cos^2(\varphi) + \sin^5(\varphi)
\end{aligned}$$

2. Simplify $1 + \cos(\varphi) + \cos(2\varphi) + \cdots + \cos(n\varphi)$ and $\sin(\varphi) + \cdots + \sin(n\varphi)$.

   Instead of considering the two separate series, we will consider the series

$$\begin{aligned}
1 + \cos(\varphi) + i\sin(\varphi) + \cdots + \cos(n\varphi) + i\sin(n\varphi) &= 1 + e^{i\varphi} + e^{2i\varphi} + \cdots + e^{ni\varphi} \\
&= \sum_{k=0}^{n} e^{ki\varphi}
\end{aligned}$$

   Recall that $\sum_{k=0}^{n-1} r^k = \frac{1-r^k}{1-r}$. So

$$\begin{aligned}
&= \frac{1 - e^{i\varphi(n+1)}}{1 - e^{i\varphi}} \\
&= \frac{e^{i\varphi(n+1)} - 1}{e^{i\varphi} - 1} \qquad (1.9)
\end{aligned}$$

Note that $\sin(\frac{\theta}{2}) = \frac{e^{i\theta/2} - e^{-i\theta/2}}{2i}$ so $2ie^{i\theta/2}\sin(\frac{\theta}{2}) = e^{i\theta} - 1$. We can now write equation (1.9) as

$$\begin{aligned}\sum_{k=0}^{n} e^{ki\varphi} &= \frac{e^{i\varphi(n+1)/2}\sin\left(\frac{\varphi(n+1)}{2}\right)}{e^{i\varphi/2}\sin\left(\frac{\varphi}{2}\right)} \\ &= \frac{\sin\left(\frac{\varphi(n+1)}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)} e^{in\varphi/2} \end{aligned} \tag{1.10}$$

By taking the real and imaginary parts of equation (1.10), we get the series for $\sum_{k=0}^{n}\cos(n\varphi)$ and $\sum_{k=0}^{n}\sin(n\varphi)$, respectively.

$$\begin{aligned}\sum_{k=0}^{n}\cos(n\varphi) &= \frac{\sin\left(\frac{\varphi(n+1)}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}\cos\left(\frac{n\varphi}{2}\right) \\ \sum_{k=0}^{n}\sin(n\varphi) &= \frac{\sin\left(\frac{\varphi(n+1)}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}\sin\left(\frac{n\varphi}{2}\right)\end{aligned}$$

3. Express the fifth and tenth roots of unity in algebraic form.

To find the roots of unity, we are looking to solve $z^n = 1$. Let $z = e^{i\theta}$ and $1 = e^{2ik\pi}$. Then $\theta = \frac{2k\pi}{n}$. For the fifth roots of unity, $n = 5$ and $k = 0, 1, \ldots, 4$ so we have

$$\begin{aligned}\omega_0 = e^0 &= \cos(0) + i\sin(0) \\ &= 1 \\ \omega_1 = e^{2\pi/5} &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\ \omega_2 = e^{4\pi/5} &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\ \omega_3 = e^{6\pi/5} &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\ \omega_4 = e^{8\pi/5} &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)\end{aligned}$$

Now we can plot the roots of unity on the unit circle.

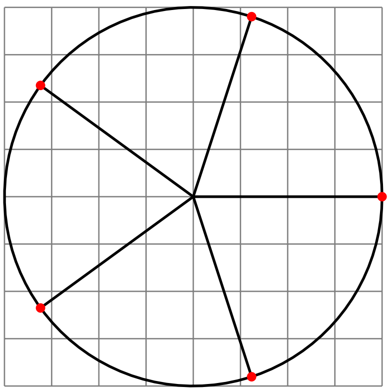

Figure 1.1: The fifth roots of unity.

For the tenth roots of unity, $n = 10$ and $k = 0, 1, \ldots, 9$ so we have

$$\begin{aligned}\omega_0 = e^0 &= \cos(0) + i\sin(0) \\ &= 1 \\ \omega_1 = e^{2\pi/10} &= \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) \\ \omega_2 = e^{4\pi/10} &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\end{aligned}$$

$$\begin{aligned}
\omega_3 = e^{6\pi/10} &= \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right) \\
\omega_4 = e^{8\pi/10} &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\
\omega_5 = e^{10\pi/10} &= \cos(\pi) + i\sin(\pi) \\
&= -1 \\
\omega_6 = e^{12\pi/10} &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\
\omega_7 = e^{14\pi/10} &= \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right) \\
\omega_8 = e^{16\pi/10} &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) \\
\omega_9 = e^{18\pi/10} &= \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)
\end{aligned}$$

Now we can plot the roots of unity on the unit circle.

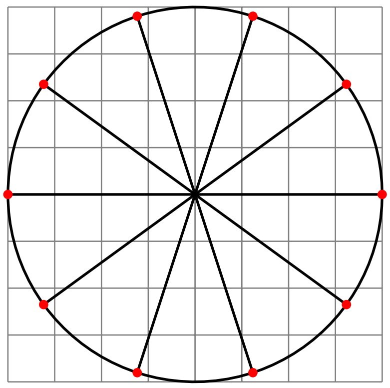

Figure 1.2: The tenth roots of unity.

4. If ω is given by $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, prove that

$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0$$

for any integer h which is not a multiple of n.

Let $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ be written in exonential form as $\omega = e^{2\pi i/n}$. Then the series can be written as

$$\sum_{k=0}^{n-1} \left(e^{2\pi i h/n}\right)^k = \frac{e^{2ih\pi} - 1}{e^{2hi\pi/n} - 1}.$$

Since h is an integer, $e^{2ih\pi} = 1$; therefore, the series zero.

5. What is the value of

$$1 - \omega^h + \omega^{2h} - \cdots + (-1)^{n-1}\omega^{(n-1)h}?$$

We can represent this series similarly as

$$\sum_{k=0}^{n-1} \left(-e^{2\pi i h/n}\right)^k = \frac{(-1)^n e^{2ih\pi} - 1}{-e^{2hi\pi/n} - 1} = \frac{1 + (-1)^{n+1} e^{2ih\pi}}{1 + e^{2hi\pi/n}}.$$

Again, since h is an intger, we have that $e^{2ih\pi} = 1$ which leaves us with

$$\frac{1 + (-1)^{n+1}}{1 + e^{2hi\pi/n}} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{1+e^{2hi\pi/n}}, & \text{if } n \text{ is odd} \end{cases}$$

1.2.3 Analytic Geometry

1.2.4 The Spherical Representation