Solutions to Ahlfors' Complex Analysis

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Contents

1 Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1. Find the values of

$$(1+2i)^3, \qquad \frac{5}{-3+4i}, \qquad \left(\frac{2+i}{3-2i}\right)^2, \qquad (1+i)^n+(1-i)^n$$

For the first problem, we have $(1+2i)^3 = (-3+4i)(1+2i) = -11-2i$. For the second problem, we should multiple by the conjugate $\bar{z} = -3-4i$.

$$\frac{5}{-3+4i}\frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \frac{-3}{5} - \frac{4}{5}i$$

For the third problem, we should first multiple by $\bar{z} = 3+2i$.

$$\frac{2+i}{3-2i}\frac{3+2i}{3+2i} = \frac{8+i}{13}$$

Now we need to just square the result.

$$\frac{1}{169}(8+i)^2 = \frac{63+16i}{169}$$

For the last problem, we will need to find the polar form of the complex numbers. Let $z_1 = 1+i$ and $z_2 = 1-i$. Then the modulus of $z_1 = \sqrt{2} = z_2$. Let ϕ_1 and ϕ_2 be the angles associated with z_1 and z_2, respectively. Then $\phi_1 = \arctan(1) = \frac{\pi}{4}$ and $\phi_2 = \arctan(-1) = \frac{-\pi}{4}$. Then $z_1 = \sqrt{2}e^{\pi i/4}$ and $z_2 = \sqrt{2}e^{-\pi i/4}$.

$$\begin{aligned} z_1^n + z_2^n &= 2^{n/2}\left[e^{n\pi i/4} + e^{-n\pi i/4}\right] \\ &= 2^{n/2+1}\left[\frac{e^{n\pi i/4} + e^{-n\pi i/4}}{2}\right] \\ &= 2^{n/2+1}\cos\left(\frac{n\pi}{4}\right) \end{aligned}$$

2. If $z = x+iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}$$

For z^4, we can use the binomial theorem since $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^n b^{n-k}$. Therefore,

$$(x+iy)^4 = \binom{4}{0}(iy)^4 + \binom{4}{1}x(iy)^3 + \binom{4}{2}x^2(iy)^2 + \binom{4}{3}x^3(iy) + \binom{4}{4}x^4 = y^4 - 4xy^3i - 6x^2y^2 + 4x^3yi + x^4$$

Then the real and imaginary parts are

$$\begin{aligned} u(x,y) &= x^4 + y^4 - 6x^2y^2 \\ v(x,y) &= 4x^3y - 4xy^3 \end{aligned}$$

For second problem, we need to multiple by the conjugate $\bar{z}$.

$$\frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

so the real and imaginary parts are

$$u(x, y) = \frac{x}{x^2 + y^2}$$
$$v(x, y) = \frac{-y}{x^2 + y^2}$$

For the third problem, we have $\frac{x-1+iy}{x+1-iy}$. Then $\bar{z} = x + 1 + iy$.

$$\frac{x - 1 + iy}{x + 1 - iy} \frac{x + 1 + iy}{x + 1 + iy} = \frac{x^2 - 1 + 2xyi}{(x + 1)^2 + y^2}$$

Then real and imaginary parts are

$$u(x, y) = \frac{x^2 - 1}{(x + 1)^2 + y^2}$$
$$v(x, y) = \frac{2xy}{(x + 1)^2 + y^2}$$

For the last problem, we have

$$\frac{1}{z^2} = \frac{x^2 - y^2 - 2xyi}{x^4 + 2x^2y^2 + y^4}$$

so the real and imaginary parts are

$$u(x, y) = \frac{x^2 - y^2}{x^4 + 2x^2y^2 + y^4}$$
$$v(x, y) = \frac{-2xy}{x^4 + 2x^2y^2 + y^4}$$

3. Show that $\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1$ and $\left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$.

   Both problems will can be handled easily by converting to polar form. Let $z_1 = \frac{-1 \pm i\sqrt{3}}{2}$. Then $|z_1| = 1$. Let ϕ_+ be the angle for the positive z_1 and ϕ_- for the negative. Then $\phi_+ = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$ and $\phi_- = \arctan(\sqrt{3}) = \frac{4\pi}{3}$. We can write $z_{1+} = e^{2i\pi/3}$ and $z_{1-} = e^{4i\pi/3}$.

$$\begin{aligned} z_{1+}^3 &= e^{2i\pi} \\ &= 1 \\ z_{1-}^3 &= e^{4i\pi} \\ &= 1 \end{aligned}$$

   Therefore, $z_1^3 = 1$. For the second problem, $\phi_{ij} = \pm\frac{\pi}{3}$ and $\pm\frac{2\pi}{3}$ for $i, j = +, -$ and the $|z_2| = 1$. When we raise z to the sixth poewr, the argument becomes $\pm 2\pi$ and $\pm 4\pi$.

$$e^{\pm 2i\pi} = e^{\pm 4i\pi} = z^6 = 1$$

1.1.2 Square Roots

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1 + i}, \qquad \sqrt{\frac{1 - i\sqrt{3}}{2}}$$

   For $\sqrt{i}$, we are looking for x and y such that

$$\sqrt{i} = x + iy$$
$$i = x^2 - y^2 + 2xyi$$
$$x^2 - y^2 = 0 \tag{1.1}$$
$$2xy = 1 \tag{1.2}$$

From equation (1.1), we see that $x^2 = y^2$ or $\pm x = \pm y$. Also, note that i is the upper half plane (UHP). That is, the angle is positive so $x = y$ and $2x^2 = 1$ from equation (1.1). Therefore, $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$. We also could have done this problem using the polar form of z. Let $z = i$. Then $z = e^{i\pi/2}$ so $\sqrt{z} = e^{i\pi/4}$ which is exactly what we obtained. For $\sqrt{-i}$, let $z = -i$. Then z in polar form is $z = e^{-i\pi/2}$ so $\sqrt{z} = e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$. For $\sqrt{1+i}$, let $z = 1+i$. Then $z = \sqrt{2}e^{i\pi/4}$ so $\sqrt{z} = 2^{1/4}e^{i\pi/8}$. Finally, for $\sqrt{\frac{1-i\sqrt{3}}{2}}$, let $z = \frac{1-i\sqrt{3}}{2}$. Then $z = e^{-i\pi/3}$ so $\sqrt{z} = e^{-i\pi/6} = \frac{1}{2}(\sqrt{3}-i)$.

2. Find the four values of $\sqrt[4]{-1}$.

   Let $z = \sqrt[4]{-1}$ so $z^4 = -1$. Let $z = re^{i\theta}$ so $r^4e^{4i\theta} = -1 = e^{i\pi(1+2k)}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{4}(1+2k) \end{aligned}$$

   where $k = 0, 1, 2, 3$. Since when $k = 4$, we have $k = 0$. Then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

$$z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$$

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

   Let $z = \sqrt[4]{i}$ and $z = re^{i\theta}$. Then $r^4e^{4i\theta} = i = e^{i\pi/2}$.

$$\begin{aligned} r^4 &= 1 \\ \theta &= \frac{\pi}{8} \end{aligned}$$

   so $z = e^{i\pi/8}$. Now, let $z = \sqrt[4]{-i}$. Then $r^4e^{4i\theta} = e^{-i\pi/2}$ so $z = e^{-i\pi/8}$.

4. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

   The quadratic equation is $x = \frac{-b\pm\sqrt{b^2-ac}}{2}$. For the complex polynomial, we have

$$z = \frac{-\alpha - \beta i \pm \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}}{2}$$

   Let $a + bi = \sqrt{\alpha^2 - \beta^2 - 4\gamma + i(2\alpha\beta - 4\delta)}$. Then

$$z = \frac{-\alpha - \beta \pm (a + bi)}{2}$$

1.1.3 Justification

1. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

   combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

1.1.4 Conjugation, Absolute Value

1. Verify by calculation the values of

$$\frac{z}{z^2+1}$$

   for $z = x + iy$ and $\bar{z} = x - iy$ are conjugate.

For z, we have that $z^2 = x^2 - y^2 + 2xyi$.

$$\begin{aligned}
\frac{z}{z^2+1} &= \frac{x+iy}{x^2-y^2+1+2xyi} \\
&= \frac{x+iy}{x^2-y^2+1+2xyi}\frac{x^2-y^2+1-2xyi}{x^2-y^2+1-2xyi} \\
&= \frac{x(x^2-y^2+1)+2xy^2+iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2}
\end{aligned} \tag{1.3}$$

For $\bar{z}$, we have that $\bar{z}^2 = x^2 - y^2 - 2xyi$.

$$\begin{aligned}
\frac{\bar{z}}{\bar{z}^2+1} &= \frac{x-iy}{x^2-y^2+1-2xyi} \\
&= \frac{x-iy}{x^2-y^2+1-2xyi}\frac{x^2-y^2+1+2xyi}{x^2-y^2+1+2xyi} \\
&= \frac{x(x^2-y^2+1)+2xy^2-iy(x^2-y^2+1-2x^2)}{(x^2-y^2+1)^2+4x^2y^2}
\end{aligned} \tag{1.4}$$

Therefore, we have that equations (1.3) and (1.4) are conjugates.

2. Find the absolute value (modulus) of

$$-2i(3+i)(2+4i)(1+i) \qquad \text{and} \qquad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}.$$

When we expand the first problem, we have that

$$z_1 = -2i(3+i)(2+4i)(1+i) = 32+24i$$

so

$$|z_1| = \sqrt{32^2+24^2} = 40.$$

For the second problem, we have that

$$z_2 = \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)} = 2 - \frac{3}{2}i$$

so

$$|z_2| = \sqrt{4+9/4} = \frac{5}{2}.$$

3. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Recall that $|z|^2 = z\bar{z}$.

$$\begin{aligned}
1^2 &= \left|\frac{a-b}{1-\bar{a}b}\right|^2 \\
1 &= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\overline{\frac{a-b}{1-\bar{a}b}}\right) \\
&= \left(\frac{a-b}{1-\bar{a}b}\right)\left(\frac{\bar{a}-\bar{b}}{1-a\bar{b}}\right) \\
&= \frac{a\bar{a}-a\bar{b}-\bar{a}b+b\bar{b}}{1-\bar{a}b-a\bar{b}+a\bar{a}b\bar{b}}
\end{aligned} \tag{1.5}$$

If $|a| = 1$, then $|a|^2 = a\bar{a} = 1$ and similarly for $|b|^2 = 1$. Then equation (1.5) becomes

$$\frac{1-a\bar{b}-\bar{a}b+b\bar{b}}{1-\bar{a}b-a\bar{b}+b\bar{b}} \qquad \text{and} \qquad \frac{1-a\bar{b}-\bar{a}b+a\bar{a}}{1-\bar{a}b-a\bar{b}+a\bar{a}}$$

resepctively which is one. If $|a| = |b| = 1$, then $|a|^2 = |b|^2 = 1$ so equation (1.5) can be written as

$$\frac{2-a\bar{b}-\bar{a}b}{2-\bar{a}b-a\bar{b}}.$$

Therefore, we must have that $a\bar{b}+\bar{a}b \neq 2$.

4. Find the conditions under which the equation $az + b\bar{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

   Let $z = x + iy$. Then $az + b\bar{z} + c = a(x+iy) + b(x-iy) + c = 0$.

$$(a+b)x + c = 0 \tag{1.6a}$$
$$(a-b)y = 0 \tag{1.6b}$$

   Lets consider equation (1.6b). We either have that $a = b$ or $y = 0$. If $a = b$, then WLOG equation (1.6a) can be written as

$$x = \frac{-c}{2a}$$

   and $y \in \mathbb{R}$. For fixed a, b, c, we have infinitely many solutions when $a = b$ since $z = \frac{-c}{2a} + iy$ for $y \in \mathbb{R}$. If $y = 0$, then equation (1.6a) can be written as

$$x = \frac{-c}{a+b}.$$

   Therefore, $z = x$ and we have only one solution.

5. Prove that Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2 - \sum_{1\leqslant i\leqslant j\leqslant n}|a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

   Let's consider

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 + \sum_{1\leqslant i\leqslant j\leqslant n}|a_i\bar{b}_j - a_j\bar{b}_i|^2 = \sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2.$$

   Then we can write the lefthand side as

$$\begin{aligned}
\left|\sum_{i=1}^{n} a_i b_i\right|^2 + \sum_{1\leqslant i\leqslant j\leqslant n}|a_i\bar{b}_j - a_j\bar{b}_i|^2 &= \sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} \bar{a}_j\bar{b}_j + \sum_{1\leqslant i\leqslant j\leqslant n}(a_i\bar{b}_j - a_j\bar{b}_i)(\bar{a}_i b_j - \bar{a}_j b_i) \\
&= \sum_{i,j=1}^{n} a_i b_i \bar{a}_j \bar{b}_j + \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right) - \sum_{i\leqslant j}\left(a_i\bar{a}_j b_i\bar{b}_j + \bar{a}_i a_j \bar{b}_i b_j\right) \\
&\qquad\qquad + \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right) \\
&= \sum_{i=j=1}^{n}|a_i|^2|b_i|^2 + \sum_{i\neq j}^{n} a_i b_i \bar{a}_j \bar{b}_j \\
&\qquad\qquad - \sum_{i\leqslant j}\left(a_i\bar{a}_j b_i\bar{b}_j + \bar{a}_i a_j \bar{b}_i b_j\right)
\end{aligned}$$

   For $i \neq j$, $\sum_{i\neq j}^{n} a_i b_i \bar{a}_j \bar{b}_j - \sum_{i<j}\left(a_i\bar{a}_j b_i\bar{b}_j + \bar{a}_i a_j \bar{b}_i b_j\right) = 0$. Thus, we now have

$$\sum_{i=1}^{n}|a_i|^2|b_i|^2 + \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right).$$

   When the indicies of both series on the right hand side coincide,

$$\sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2 = \sum_{i=1}^{n}|a_i|^2|b_i|^2. \tag{1.7}$$

   That is, both a_i and b_i index together on the left of side equation (1.7). When a_i and b_i dont index together on the left side of equation (1.7),

$$\sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2 = \sum_{i\leqslant j}\left(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2\right)$$

   as was needed to be shown.

1.1.5 Inequalities

1. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

From the properties of the modulus, we have that

$$\begin{aligned}
\left|\frac{a-b}{1-\bar{a}b}\right| &= \frac{|a-b|}{|1-\bar{a}b|} \\
&= \frac{|a-b|^2}{|1-\bar{a}b|^2} \qquad (1.8)\\
&= \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})} \\
&= \frac{|a|^2+|b|^2-a\bar{b}-\bar{a}b}{1+|a|^2|b|^2-\bar{a}b-a\bar{b}} \\
&< \frac{2-a\bar{b}-\bar{a}b}{2-\bar{a}b-a\bar{b}} \\
&= 1 \qquad (1.9)
\end{aligned}$$

From equations (1.8) and (1.9), we have

$$\begin{aligned}
\frac{|a-b|^2}{|1-\bar{a}b|^2} &< 1 \\
\frac{|a-b|}{|1-\bar{a}b|} &< 1
\end{aligned}$$

2. Prove Cauchy's inequality by induction.

Cauchy's inequality is

$$|a_1b_1+\cdots+a_nb_n|^2 \leqslant \left(|a_1|^2+\cdots+|a_n|^2\right)\left(|b_1|^2+\cdots+|b_n|^2\right)$$

which can be written more compactly as

$$\left|\sum_{i=1}^{n} a_ib_i\right|^2 \leqslant \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2.$$

For the base case, $i = 1$, we have

$$|a_1b_1|^2 = (a_1b_1)(\bar{a}_1\bar{b}_1) = a_1\bar{a}_1b_1\bar{b}_1 = |a_1|^2|b_1|^2$$

so the base case is true. Now let the equality hold for all $k-1 \in \mathbb{Z}$ where $k-1 \leqslant n$. That is, we assume that

$$\left|\sum_{i=1}^{k-1} a_ib_i\right|^2 \leqslant \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2$$

to be true.

$$\begin{aligned}
\left|\sum_{i=1}^{k-1} a_ib_i\right|^2 + |a_kb_k|^2 &\leqslant \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2 + |a_kb_k|^2 \\
\left|\sum_{i=1}^{k} a_ib_i\right|^2 &\leqslant \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2 + (a_kb_k)(\bar{a}_k\bar{b}_k) \\
&= \sum_{i=1}^{k-1}|a_i|^2\sum_{i=1}^{k-1}|b_i|^2 + |a_k|^2|b_k|^2 \\
&= \sum_{i=1}^{k}|a_i|^2\sum_{i=1}^{k}|b_i|^2
\end{aligned}$$

Therefore, by the principal of mathematical induction, Cauchy's inequality is true for all $n \geqslant 1$ for $n \in \mathbb{Z}^+$.

3. If $|a_i| < 1$, $\lambda_i \geqslant 0$ for $i = 1, \dots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

Since $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geqslant 0$, $0 \leqslant \lambda_i < 1$. By the triangle inequality,

$$\begin{aligned} |\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| &\leqslant |\lambda_1||a_1| + \cdots + |a_n||\lambda_n| \\ &< \sum_{i=1}^{n} \lambda_i \\ &= 1 \end{aligned}$$

4. Show that there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leqslant |c|$. If this condition is fulfilled, what are the smallest and largest values $|z|$?

By the triangle inequality,

$$|z - a| + |z + a| \geqslant |(z - a) - (z + a)| = 2|a|$$

so

$$\begin{aligned} 2|c| &= |z - a| + |z + a| \\ &\geqslant |(z - a) - (z + a)| \\ &= 2|a| \end{aligned}$$

Thus, $|c| \geqslant |a|$. If $|a| \leqslant |c|$, then let $z = |c|\frac{a}{|a|}$.

$$\begin{aligned} |z - a| + |z + a| &\geqslant |(z - a) + (z + a)| \\ &= 2||c|| \end{aligned}$$

since $\frac{a}{|a|}$ is a unit vector.

$$= 2|c|$$

Thus, $2|c| \geqslant 2|c|$ which is equality.

$$\begin{aligned} 2|c| &= |z + a| + |z - a| \\ 4|c|^2 &= \big(|z + a| + |z - a|\big)^2 \\ &= 2(|z|^2 + |a|^2) \\ &\leqslant 4(|z|^2 + |a|^2) \\ |c|^2 &\leqslant |z|^2 + |a|^2 \\ \sqrt{|c|^2 - |a|^2} &\leqslant |z| \end{aligned}$$

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1. Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

2. Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1 a_2 + a_2 a_3 + a_1 a_3$.

3. Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

4. Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

1.2.2 The Binomial Equation

1. Express $\cos(3\varphi)$, $\cos(4\varphi)$, and $\sin(5\varphi)$ in terms of $\cos(\varphi)$ and $\sin(\varphi)$.

   For these problems, the sum addition identities will be employed; that is,

$$\begin{aligned}\cos(\alpha\pm\beta) &= \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)\\ \sin(\alpha\pm\beta) &= \sin(\alpha)\cos(\beta)\pm\sin(\beta)\cos(\alpha)\end{aligned}$$

   We can write $\cos(3\varphi)$ as $\cos(2\varphi+\varphi)$ so

$$\begin{aligned}\cos(3\varphi) &= \cos(2\varphi+\varphi)\\ &= \cos(2\varphi)\cos(\varphi)-\sin(2\varphi)\sin(\varphi)\\ &= \left[\cos^2(\varphi)-\sin^2(\varphi)\right]\cos(\varphi)-2\sin(\varphi)\cos(\varphi)\sin(\varphi)\\ &= \cos^3(\varphi)-3\sin^2(\varphi)\cos(\varphi)\end{aligned}$$

   For $\cos(4\varphi)$, we have

$$\begin{aligned}\cos(4\varphi) &= \cos(2\varphi)\cos(2\varphi)-\sin(2\varphi)\sin(2\varphi)\\ &= \left[\cos^2(\varphi)-\sin^2(\varphi)\right]^2-4\sin^2(\varphi)\cos^2(\varphi)\\ &= \cos^4(\varphi)+\sin^4(\varphi)-6\sin^2(\varphi)\cos^2(\varphi)\end{aligned}$$

   For $\sin(5\varphi)$, we have

$$\begin{aligned}\sin(5\varphi) &= \sin(4\varphi)\cos(\varphi)+\sin(\varphi)\cos(4\varphi)\\ &= 2\sin(2\varphi)\cos(2\varphi)\left[\cos^2(\varphi)-\sin^2(\varphi)\right]\cos(\varphi)+\sin^5(\varphi)+\sin(\varphi)\cos^4(\varphi)-6\sin^3(\varphi)\cos^2(\varphi)\\ &= 5\sin(\varphi)\cos^4(\varphi)-10\sin^3(\varphi)\cos^2(\varphi)+\sin^5(\varphi)\end{aligned}$$

2. Simplify $1+\cos(\varphi)+\cos(2\varphi)+\cdots+\cos(n\varphi)$ and $\sin(\varphi)+\cdots+\sin(n\varphi)$.

   Instead of considering the two separate series, we will consider the series

$$\begin{aligned}1+\cos(\varphi)+i\sin(\varphi)+\cdots+\cos(n\varphi)+i\sin(n\varphi) &= 1+e^{i\varphi}+e^{2i\varphi}+\cdots+e^{ni\varphi}\\ &= \sum_{k=0}^{n}e^{ki\varphi}\end{aligned}$$

   Recall that $\sum_{k=0}^{n-1}r^k=\frac{1-r^k}{1-r}$. So

$$\begin{aligned}&= \frac{1-e^{i\varphi(n+1)}}{1-e^{i\varphi}}\\ &= \frac{e^{i\varphi(n+1)}-1}{e^{i\varphi}-1}\end{aligned}\tag{1.10}$$

   Note that $\sin(\frac{\varphi}{2})=\frac{e^{i\varphi/2}-e^{-i\varphi/2}}{2i}$ so $2ie^{i\varphi/2}\sin(\frac{\varphi}{2})=e^{i\varphi}-1$. We can now write equation (1.10) as

$$\begin{aligned}\sum_{k=0}^{n}e^{ki\varphi} &= \frac{e^{i\varphi(n+1)/2}\sin\left(\frac{\varphi(n+1)}{2}\right)}{e^{i\varphi/2}\sin\left(\frac{\varphi}{2}\right)}\\ &= \frac{\sin\left(\frac{\varphi(n+1)}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}e^{in\varphi/2}\end{aligned}\tag{1.11}$$

   By taking the real and imaginary parts of equation (1.11), we get the series for $\sum_{k=0}^{n}\cos(n\varphi)$ and $\sum_{k=0}^{n}\sin(n\varphi)$, respectively.

$$\begin{aligned}\sum_{k=0}^{n}\cos(n\varphi) &= \frac{\sin\left(\frac{\varphi(n+1)}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}\cos\left(\frac{n\varphi}{2}\right)\\ \sum_{k=0}^{n}\sin(n\varphi) &= \frac{\sin\left(\frac{\varphi(n+1)}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}\sin\left(\frac{n\varphi}{2}\right)\end{aligned}$$

3. Express the fifth and tenth roots of unity in algebraic form.

To find the roots of unity, we are looking to solve $z^n = 1$. Let $z = e^{i\theta}$ and $1 = e^{2ik\pi}$. Then $\theta = \frac{2k\pi}{n}$. For the fifth roots of unity, $n = 5$ and $k = 0, 1, \dots, 4$ so we have

$$\begin{aligned}
\omega_0 = e^0 &= \cos(0) + i\sin(0) \\
&= 1 \\
\omega_1 = e^{2\pi/5} &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\
\omega_2 = e^{4\pi/5} &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\
\omega_3 = e^{6\pi/5} &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\
\omega_4 = e^{8\pi/5} &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)
\end{aligned}$$

Now we can plot the roots of unity on the unit circle.

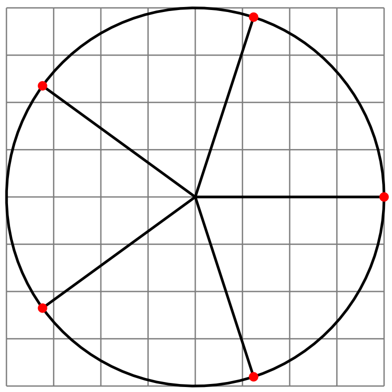

Figure 1.1: The fifth roots of unity.

For the tenth roots of unity, $n = 10$ and $k = 0, 1, \dots, 9$ so we have

$$\begin{aligned}
\omega_0 = e^0 &= \cos(0) + i\sin(0) \\
&= 1 \\
\omega_1 = e^{2\pi/10} &= \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) \\
\omega_2 = e^{4\pi/10} &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\
\omega_3 = e^{6\pi/10} &= \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right) \\
\omega_4 = e^{8\pi/10} &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\
\omega_5 = e^{10\pi/10} &= \cos(\pi) + i\sin(\pi) \\
&= -1 \\
\omega_6 = e^{12\pi/10} &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\
\omega_7 = e^{14\pi/10} &= \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right) \\
\omega_8 = e^{16\pi/10} &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) \\
\omega_9 = e^{18\pi/10} &= \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)
\end{aligned}$$

Now we can plot the roots of unity on the unit circle.

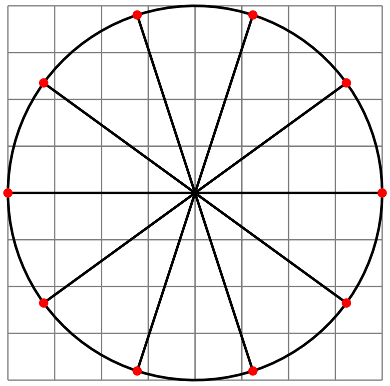

Figure 1.2: The tenth roots of unity.

4. If ω is given by $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, prove that

$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0$$

for any integer h which is not a multiple of n.

Let $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ be written in exonential form as $\omega = e^{2\pi i/n}$. Then the series can be written as

$$\sum_{k=0}^{n-1} \left(e^{2\pi ih/n}\right)^k = \frac{e^{2ih\pi} - 1}{e^{2hi\pi/n} - 1}.$$

Since h is an integer, $e^{2ih\pi} = 1$; therefore, the series zero.

5. What is the value of

$$1 - \omega^h + \omega^{2h} - \cdots + (-1)^{n-1}\omega^{(n-1)h}?$$

We can represent this series similarly as

$$\sum_{k=0}^{n-1} \left(-e^{2\pi ih/n}\right)^k = \frac{(-1)^n e^{2ih\pi} - 1}{-e^{2hi\pi/n} - 1} = \frac{1 + (-1)^{n+1}e^{2ih\pi}}{1 + e^{2hi\pi/n}}.$$

Again, since h is an intger, we have that $e^{2ih\pi} = 1$ which leaves us with

$$\frac{1 + (-1)^{n+1}}{1 + e^{2hi\pi/n}} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{1 + e^{2hi\pi/n}}, & \text{if } n \text{ is odd} \end{cases}$$

1.2.3 Analytic Geometry

1. When does $az + b\bar{z} + c = 0$ represent a line?

2. Write the equation of an ellipse, hyperbola, parabola in complex form.

For $x, y, h, k, a, b \in \mathbb{R}$ such that $a, b \neq 0$, we define a real ellipse as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Let $z = \frac{x}{a} + i\frac{y}{b}$ and $z_0 = \frac{h}{a} + i\frac{k}{b}$. If we expand the equation for an ellipse, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{h^2}{a^2} + \frac{k^2}{b^2} - \frac{2xh}{a^2} - \frac{2yk}{b^2} = 1.$$

Notice that $|z|^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ and $|z_0|^2 = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. Now, let's write the ellipse as

$$|z|^2 + |z_0|^2 - \frac{2xh}{a^2} - \frac{2yk}{b^2} + \frac{yh}{ab}i - \frac{yh}{ab}i + \frac{xk}{ab}i - \frac{xk}{ab}i = |z|^2 + |z_0|^2 - \bar{z}z_0 - z\bar{z}_0 = 1.$$

Thus, the equation of an ellipse in the complex plane is

$$(z - z_0)(\bar{z} - \bar{z}_0) = |z - z_0|^2 = 1 \Rightarrow |z - z_0| = 1$$

where z and z_0 are defined above. Additionally, the standard form of an ellipse in the complex plane is of the form

$$|z - a| + |z - b| = c$$

where $c > |a - b|$. Let a and b be the foci of a hyperbola. Then when the magnitude of the difference of z and the foci is a constant, we will have a hyperbola.

$$\big||z - a| - |z - b|\big| = c$$

3. Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

4. Prove analytically that the midpoints of parallel chords to a circle lie on a diameter perpendicular to the chords.

5. Show that all circles that pass through a and $1/\bar{a}$ intersect the circle $|z| = 1$ at right angles.

1.2.4 The Spherical Representation

1.

2 Complex Functions

2.1 Introduction to the Concept of Analytical Function

2.1.1 Limits and Continuity

No problem set in Ahlfors.

2.1.2 Analytic Functions

1. If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

2. Verify Cauchy-Riemann's equations for the function z^2 and z^3.

   Let $z = x + iy$. Then $z^2 = x^2 - y^2 + 2xyi$ and $z^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$. For $f(z) = z^2$, the Cauchy-Riemann equations are

$$\begin{aligned} u_x &= 2x \qquad & v_y &= 2x \\ u_y &= -2y \qquad & -v_x &= -2y \end{aligned}$$

   Thus, the Cauchy-Riemann equation satisfied for $f(z) = z^2$. For $f(z) = z^3$, the Cauchy-Riemann equations are

$$\begin{aligned} u_x &= 3x^2 - 3y^2 \qquad & v_y &= 3x^2 - 3y^2 \\ u_y &= -6xy \qquad & -v_x &= -6xy \end{aligned}$$

   Thus, the Cauchy-Riemann equation satisfied for $f(z) = z^3$.

3. Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

   In order to be harmonic, $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ has to satisfy $\nabla^2 u = 0$ so

$$u_{xx} + u_{yy} = (3a + c)x + (3d + b)y = 0.$$

   Thus, $3a = -c$ and $3d = -b$ so

$$u(x,y) = ax^3 - 3axy^2 - 3dx^2y + dy^3.$$

   To find the harmonic conjugate $v(x,y$, we need to look at the Cauchy-Riemann equations. By the Cauchy-Riemann equations,

$$u_x = 3ax^2 - 3ay^2 - 6dxy = v_y.$$

   Then we can integrate with respect to y to find $v(x,y)$.

$$v(x,y) = \int (3ax^2 - 3ay^2 - 6dxy)dy = 3ax^2y - ay^3 - 3dxy^2 + g(x)$$

   Using the second Cauchy-Riemann, we have

$$v_x = 6axy - 3dy^2 + g'(x) = -u_y = 3dx^2 + 6axy - 3dy^2$$

   so $g'(x) = 3dx^2$. Then $g(x) = dx^3 + C$ and

$$v(x,y) = 3ax^2y - ay^3 - 3dxy^2 + dx^3 + C.$$

4. Show that an analytic function cannot have a constant absolute value without reducing to a constant.

   Let $f = u(x, y) + iv(x, y)$. Then the modulus of f is $|f| = \sqrt{u^2 + v^2}$. If the modulus of f is constant, then $u^2 + v^2 = c$ for some constant c. If $c = 0$, then $f = 0$ which is constant. Suppose $c \neq 0$. By taking the derivative with respect to x and y, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}(u^2 + v^2) \\ &= 2uu_x + 2vv_x \\ &= uu_x + vv_x \\ 0 &= \frac{\partial}{\partial y}(u^2 + v^2) \\ &= uu_y + vv_y \end{aligned}$$

   Since f is analytic, f satisfies the Cauchy-Riemann. That is, $u_x = v_y$ and $u_y = -v_x$.

$$uv_y + vv_x = 0 \tag{2.1a}$$
$$-uv_x + vv_y = 0 \tag{2.1b}$$

   Setting equation (2.1a) equal to equation (2.1b), we have

$$v_x(u + v) + v_y(u - v) = 0.$$

   Now, either v_x and v_y are zero, v_x and $u - v$ are zero, v_y and $u + v$ are zero, or $u + v$ and $u - v$ are zero. If $v_x = v_y = 0$, then f is constant. If $v_x = 0$ and $u - v = 0$, then $u_y = 0$ and $u = v$. Since $u = v$ and $v_x = 0$, then so does $u_x = 0$ and it also follows that $v_y = 0$; thus, f is a constant. By the same argument, f is a constant when $v_y = 0$ and $u + v = 0$. If $u + v = 0$ and $u - v = 0$, then $u = \pm v$ so $u = v = 0$ and f is a constant.

5. Prove rigorously that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

   Let $g(z) = \overline{f(\bar{z})}$ and suppose f is analytic. Then $g'(z)$ is

$$\begin{aligned} g'(z) &= \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{f(\bar{z} + \overline{\Delta z})} - \overline{f(\bar{z})}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left[\overline{\frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}}}\right] \end{aligned}$$

   Since conjugation is continuous, we can move the limit inside the conjugation.

$$\begin{aligned} &= \overline{\lim_{\Delta z \to 0} \frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}}} \\ &= \overline{f'(\bar{z})} \end{aligned}$$

   Thus, g is differentiable with derivative $\overline{f'(\bar{z})}$. Suppose $\overline{f(\bar{z})}$ is analytic and let $\overline{g(\bar{z})} = f(z)$. Then by the same argument, f is differentiable with derivative $\overline{g'(\bar{z})}$. Therefore, $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

   We could also use the Cauchy-Riemann equations. Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ so $\bar{z} = x - iy$. Then $\overline{f(\bar{z})} = \alpha(x, y) - i\beta(x, y)$ where $\alpha(x, y) = u(x, -y)$ and $\beta(x, y) = v(x, -y)$. In order for both to be analytic, they both need to satisfy the Cauchy-Riemann equations. That is, $u_x = v_y$, $u_y = -v_x$, $\alpha_x = \beta_y$ and $\alpha_y = -\beta_x$.

$$\begin{aligned} u_x(x, y) &= v_y(x, y) \\ u_y(x, y) &= -v_x(x, y) \\ \alpha_x(x, y) &= u_x(x, -y) \\ \alpha_y(x, y) &= -u_y(x, -y) \end{aligned}$$

$$-\beta_x(x,y) = \nu_x(x,-y)$$
$$\beta_y(x,y) = \nu_y(x,-y)$$

Suppose that $\overline{f(\bar{z})}$ satisfies the Cauchy-Riemann equations. Then $\alpha_x = u_x(x,-y) = \nu_y(x,-y) = \beta_y$ and $\alpha_y = -u_y(x,-y) = \nu_x(x,-y) = -\beta_x$. Therefore,

$$u_x(x,-y) = \nu_y(x,-y)$$
$$u_y(x,-y) = -\nu_x(x,-y)$$

which means $f(\bar{z})$ satisfies the Cauchy-Riemann equations. Now, recall that $|z| = |\bar{z}|$. Since $f(\bar{z})$ satisfies the Cauchy-Riemann equations, for an $\epsilon > 0$ there exists a $\delta > 0$ such that when $0 < |\Delta z| < \delta$, $|f(\bar{z}) - \bar{z}_0| = |f(z) - z_0| < \epsilon$. Thus, $\lim_{\Delta z \to 0} f(z) = z_0$ so $f(z)$ is analytic if $\overline{f(\bar{z})}$ is analytic.

6. Prove that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

   Since u is the real part of $f(z)$, $u(z) = u(x,y)$ where $z = x + iy$. Suppose $u(z)$ is harmonic. Then $u(z)$ satisfies Laplace equation.

$$\nabla^2 u(z) = u_{xx} + u_{yy} = 0$$

   Now, $u(\bar{z}) = u(x,-y)$ where $\frac{\partial^2}{\partial x^2}u(\bar{z}) = u_{xx}$ and $\frac{\partial^2}{\partial y^2}u(\bar{z}) = u_{yy}$ so

$$\nabla^2 u(\bar{z}) = u_{xx} + u_{yy} = 0.$$

   Since $u(z)$ is harmonic, $u_{xx} + u_{yy} = 0$ so it follows that $u(\bar{z})$ is harmonic as well.

7. Show that a harmonic function satisfies the formal differential equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

   Let u be a harmonic. Then $\nabla^2 u = 0$.

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \tag{2.2a}$$
$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \tag{2.2b}$$

   From equation (2.2a), we have

$$\frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)u = \frac{1}{2}(u_x + iu_y).$$

   Then we have

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(u_x + iu_y) = \frac{1}{4}\big[u_{xx} + u_{yy} + i(u_{yx} - u_{xy})\big]$$

   Since u is a solution to the Laplace equation, u has continuous first and second derivatives. That is, $u \in C^2$ at a minimum. By Schwarz's theorem, $u_{xy} = u_{yx}$ so

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

   Schwarz's theorem states that if f is a function of two variables such that f_{xy} and f_{yx} both exist and are continuous at some point (x_0, y_0), then $f_{xy} = f_{yx}$.

2.1.3 Polynomials

No problem set in Ahlfors.

2.1.4 Rational Functions

1. Use the method of the text to develop

$$\frac{z^4}{z^3-1} \qquad \text{and} \qquad \frac{1}{z(z+1)^2(z+2)^3}$$

in partial fractions.

Let $R(z) = \frac{z^4}{z^3-1}$. Then we need to find $G(z)$ and $H(z)$ such that $G(z) + H(z) = R(z)$. $G(z)$ is the singular part of $R(z)$ and the degree of $G(z)$ is order of the pole at infinity. Dividing z^4 by $z^3 - 1$, we get

$$R(z) = z + \frac{z}{z^3-1}$$

so $G(z) = z$ and $H(z) = \frac{z}{z^3-1}$. The order of the pole at infinity is one.

2. If Q is a polynomial with distinct roots $\alpha_1, \dots, \alpha_n$, and if P is a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

Let's multiple by $Q(z)$. We then have

$$P(z) = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} Q(z)$$

which are both polynomials of degree less than n and agreeing at $z = \alpha_k$.

3. Use the formula in the preceeding exercise to prove that there exists a unique polynomial P or degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

4. What is the general form of a rational function which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles related to each other?

5. If a rational function is real on $|z| = 1$, how are the zeros and poles situated?

6. If $R(z)$ is a rational function of order n, how large and how small can the order of $R'(z)$ be?

2.2 Elementary Theory of Power Series

2.2.1 Sequences

No problem sets in Ahlfors.

2.2.2 Series

No problem sets in Ahlfors.

2.2.3 Uniform Convergence

1. Prove that a convergent sequence is bounded.

Let $\{a_n\}$ be a convergent sequence and $\lim_{n\to\infty} a_n = a$. Let $\epsilon = 1$. Then there exists an $n > N$ such that $|a_n - a| < 1$.

$$|a_n| = |a_n - a + a|$$

By the triangle inequality, we have

$$\leqslant |a_n - a| + |a|$$
$$|a_n| - |a| \leqslant |a_n - a|$$

Therfore, we have that

$$\begin{gathered}|a_n| - |a| \leqslant |a_n - a| < 1 \\ |a_n| < 1 + |a|\end{gathered}$$

For all $n > N$, $|a_n| < 1 + |a|$ so let $A = \max\{1 + |a|, |a_1|, \ldots, |a_N|\}$. Thus, $|a_n| < A$ for some finite A and hence $\{a_n\}$ is bounded by A.

2. If $\lim_{n\to\infty} z_n = A$, prove that

$$\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + \cdots + z_n) = A.$$

Given $\epsilon > 0$ there exists some $n > N$ such that

$$|z_n - A| < \frac{N\epsilon}{2}.$$

Now, since $\lim_{n\to\infty} z_n = A$ converges, it is Cauchy. Therefore, there exists $n, m > N$ such that

$$|z_m - z_n| < \frac{N\epsilon}{2}.$$

Repeating this we have that $|z_1 + \cdots + z_n - nA|$ or $|1/n(z_1 + \cdots + z_{n-1}) - A + (z_n - A)/n|$. For a fixed N, we can find n such that

$$\sum_{i=1}^{n-1} |z_i - A| < \frac{N\epsilon}{2}$$

We now have that

$$\begin{aligned}|1/n(z_1 + \cdots + z_{n-1}) - A + (z_n - A)/n| &\leqslant \left|1/n \sum_{i=1}^{n-1} (z_i - A)\right| + 1/n|z_n - A| \\ &\leqslant 1/n \sum_{i=1}^{n-1} |z_i - A| + 1/n|z_n - A| \\ &< 1/n\frac{N\epsilon}{2} + 1/n\frac{N\epsilon}{2} \\ &< \epsilon\end{aligned} \tag{2.3}$$

Equation (2.3) can be written as $|1/n(z_1 + \cdots + z_n) - A| < \epsilon$ so

$$\lim_{n\to\infty} 1/n(z_1 + \cdots + z_n) = A.$$

3. Show that the sum of an absolutely convergent series does not change if the terms are rearranged.

Let $\sum a_n$ be an absolutely convergent series and $\sum b_n$ be its rearrangement. Since $\sum a_n$ converges absolutely, for $\epsilon > 0$, there exists a $n > N$ such that $|s_n - A| < \epsilon/2$ where s_n is the nth partial sum. Let t_n be the nth partial sum of $\sum b_n$. Then for some $n > N$

$$\begin{aligned}|t_n - A| &= |t_n - s_n + s_n - A| \\ &\leqslant |t_n - s_n| + |s_n - A| \\ &< |t_n - s_n| + \frac{\epsilon}{2}\end{aligned}$$

Since $\sum a_n$ is absolutely convergent, $\sum_{k=n+1}^{\infty} |a_k|$ converges to zero. Let the remainder be r_n. Then for some $N > n, n_1$, $|r_n - 0| < \epsilon/2$. Let $M = \max\{k_1, k_2, \ldots, k_N\}$. Then for some $n > M$, we have

$$|t_n - s_n| = \left|\sum_{n}^{N} a_n\right| \leqslant \sum |a_n| \leqslant \sum_{k=n+1}^{\infty} |a_n| = r_n < \frac{\epsilon}{2}$$

Thus, $|t_n - s_n| < \epsilon$ and a rearrangement of an absolutely convergent series does not changes its sum.

4. Discuss completely the convergence and uniform convergence of the sequence $\{nz^n\}_{n=1}^{\infty}$.

Consider when $|z| < 1$. Then $z^n = \frac{1}{w^n}$ where $|w| > 1$. By the ratio test, we ahve

$$\lim_{n\to\infty}\left|\frac{(n+1)w^n}{nw^{n+1}}\right| = \frac{1}{|w|}\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{|w|}$$

In order for convergence, the limit of ratio test has to be less than one.

$$\frac{1}{|w|} = |z|$$

which is less than one by our assumption so $\{nz^n\}$ converges absolutely in the disc less than one. Now, let's consider $|z| \geqslant 1$. By the ratio test, we get $\lim_{n\to\infty}|a_{n+1}/a_n| = |z| \geqslant 1$ by our assumption. When the limit is one, we can draw no conclusion about convergence, but when the limit is greater than one, the sequence diverges. For $|z| < 1$, $\epsilon > 0$, and $n > N$, $|nz^n - 0| < \epsilon$ for uniform convergence. Take $z = 9/10$, $n = 100$, and $\epsilon = 0.001$. Then

$$\begin{gathered}|nz^n| = n|z|^n < \epsilon \\ |z|^n < \frac{\epsilon}{n} \\ 0.000026 \not< 0.00001\end{gathered}$$

Thus, the sequence is not uniformly convergent in the disc with radius less than one. Let's consider the closed disc $|z| \leqslant R$ where $R \in (0,1)$. Now $|nz^n|$ is bounded above by a convergent geometric series, say $\sum r^n$ where $|r| < 1$. Then $|nz^n| < ar^n$ for $|z| \leqslant R$ and a a real constant. Let $M_n = ar^n$ where M_n is the M in the Weierstrass M-test. Thus, $\{nz^n\}$ is uniformly convergent in a closed disc less than one.

5. Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty}\frac{x}{n(1+nx^2)}$$

for real values of x.

By the AM-GM inequality, $(x+y)/2 \geqslant \sqrt{xy}$, we have

$$1 + nx^2 \geqslant 2|x|\sqrt{n}$$

or $\frac{1}{2|x|\sqrt{n}} \geqslant \frac{1}{1+nx^2}$. Let $f_n(x) = \frac{x}{n(1+nx^2)}$. Then

$$|f_n(x)| \leqslant \left|\frac{x}{2xn^{3/2}}\right| = \left|\frac{1}{2n^{3/2}}\right| = M_n$$

For a fixed x, $\sum|f_n(x)| \leqslant M_n < \infty$ so $\sum|f_n(x)|$ is absolutely convergent. Thus, $\sum f_n(x)$ is pointwise convergent to $f(x)$. Let $\epsilon > 0$ be given and $s_n = \sum_{k=1}^{n} f_k(x)$ be nth partial sum. Let $n > N$ such that

$$|f(x) - s_n| = \left|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \leqslant \sum_{k=n+1}^{\infty}|f_k(x)|$$

Since $\sum M_k$ converges to some limit, for n sufficiently large, $\sum_{k=n+1}^{\infty} M_k < \epsilon$. Select N such that this is true. Then

$$|f(x) - s_n| \leqslant \sum_{k=n+1}^{\infty}|f_k(x)| \leqslant \sum_{k=n+1}^{\infty} M_k < \epsilon$$

Therefore, $\sum f_n(x)$ where $f_n(x) = \frac{x}{n(1+nx^2)}$ is uniformly convergent by the Weierstrass M-test.

6. If $U = u_1 + u_2 + \cdots$, $V = v_1 + v_2 + \cdots$ are convergent series, prove that $UV = u_1v_1 + (u_1v_2 + u_2v_2) + (u_1v_3 + u_2v_2 + u_3v_1) + \cdots$ provided that at least one of the series is absolutely convergent. (It is easy if both series are absolutely convergent. Try to rearrange the proof so economically that the absolute convergence of the second series is not needed.)