Principles of Mathematical Analysis
Chapter Theorems and Examples

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Contents

1 The Real and Complex Number Systems

1.1. Prove that there is no rational p such that $p^2 = 2$.

Suppose p is rational. Then $p = m/n$ where $m, n \in \mathbb{Z}$ such that both m and n are not even.

$$\begin{aligned} p^2 &= \frac{m^2}{n^2} = 2 \\ &= m^2 \ = 2n^2 \end{aligned} \tag{1.1}$$

Therefore, since $m^2 = 2n^2$, m^2 is even and m is even. Since m is even, we can write $m = 2a$ so $m^2 = 4a^2$ and m^2 is divisible by 4. Now that the left hand side of equation (1.1) is divisible by 4 so is the right hand side. Thus, we have reached a contradiction since this would imply both m and n are even.

Another common way to prove this is to use the Fundamental Theorem of Arithmetic.

Theorem 1: Fundamental Theorem of Arithmetic

Also known as the Unique Prime Factorization Theorem states that for every $z \in \mathbb{Z}^+$ either is prime or can be expressed as the product of prime numbers and that this representation is unique up to the order of the factors.

Again, suppose $p = m/n$ where $m, n \in \mathbb{Z}$ are co-prime. From equation (1.1), we have

$$2n^2 = m^2.$$

By the FTA, both n and m can be expressed as unique product of prime factors. Let p_i be the ith prime for n and q_i be the ith prime for b where α_i is ith power of n and β_i is the ith power m. Then

$$\begin{aligned} n^2 &= p_1^{2\alpha_1} \cdot p_2^{2\alpha_2} \cdots p_k^{2\alpha_k} \\ m^2 &= q_1^{2\beta_1} \cdot q_2^{2\beta_2} \cdots q_l^{2\beta_l} \end{aligned}$$

Let p_j and q_j be the representation of prime factorization of 2. Then we have

$$\begin{aligned} 2 \cdot 2^{2\alpha_j} &= 2^{2\beta_j} \\ 2^{2\alpha_j+1} &= 2^{2\beta_j} \end{aligned} \tag{1.2}$$

From equation (1.2), we have $2\alpha_j + 1$ must equal $2\beta_j$ but this cannot be the case since $2\alpha_j + 1$ is clearly odd and $2\beta_j$ is even. Thus, we have reached a contradiction and p is irrational.

1.9 Let A be the set of all positive of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$.

a. Consider the sets A and B as subsets of the ordered set $\mathbb{Q}$. The set A is bounded above. In fact, the upper bounds of A are exactly the members of B. Since B contains no smallest member, A has *no least upper bound in* $\mathbb{Q}$.

   Similarly, B is bounded below: The set of all lower bounds of B consists of A and of all $r \in \mathbb{Q}$ with $r \leqslant 0$. Since A has no largest member, B *has no greatest lower bound in* $\mathbb{Q}$.

b. If $\alpha = \sup E$ exists, then α may or may not be a member of E. For instance, let E_1 be the set of all $r \in \mathbb{Q}$ with $r < 0$. Let E_2 be the set of all $r \in \mathbb{Q}$ with $r \leqslant 0$. Then

$$\sup E_1 = \sup E_2 = 0,$$

   and $0 \notin E_1$, $0 \in E_2$.

c. Let E consist of all numbers $1/n$, where $n = 1, 2, 3, \ldots$. Then $\sup E = 1$, which is in E, and $\inf E = 0$, which is not in E.