

# SOLUTIONS TO MUNKRES' TOPOLOGY

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# 1 Set Theory and Logic

Skipped for now

## 1.1 Fundamental Concepts



## 2 Topological Spaces and Continuous Functions

### 2.1 Topological Spaces

### 2.2 Basis for a Topology

1. Let  $X$  be a topological space; let  $A$  be a subset of  $X$ . Suppose that for each  $x \in A$  there is an open set  $U$  containing  $x$  such that  $U \subset A$ . Show  $A$  is open in  $X$ .

For each  $x$  in  $A$ , there exists an open set  $U_x$  such that  $x \in U_x \subset A$ . Then  $\bigcup_{x \in A} U_x \subseteq A$ . Since  $U_x$  is an open set containing all  $x \in A$ ,  $A = \bigcup_{x \in A} U_x$ ; that is,  $A$  is the arbitrary union of a collection of open set so  $A$  is open in  $X$ .

2. Consider the nine topologies on the set  $X = \{a, b, c\}$  indicated in examaple 1 of chapter 2 section 2.1. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

Let  $X = \{a, b, c\}$ . Next, let's index the nine topologies.

$$\begin{array}{lll} \mathcal{T}_1 = \{\emptyset, X\} & \mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\} & \mathcal{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} \\ \mathcal{T}_4 = \{\emptyset, \{b\}, X\} & \mathcal{T}_5 = \{\emptyset, \{a\}, \{b, c\}, X\} & \mathcal{T}_6 = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} \\ \mathcal{T}_7 = \{\emptyset, \{a, b\}, X\} & \mathcal{T}_8 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} & \mathcal{T}_9 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\} \end{array}$$

A topology  $\mathcal{T}_i$  is comparable with  $\mathcal{T}_j$  for  $i \neq j$  if either  $\mathcal{T}_i \supset \mathcal{T}_j$  or  $\mathcal{T}_i \subset \mathcal{T}_j$ . Now  $\mathcal{T}_1 \subset \mathcal{T}_j$  for  $i \in [2, 9]$ . Therefore,  $\mathcal{T}_1$  is coarser than  $\mathcal{T}_i$  so  $\mathcal{T}_i$  is finer than  $\mathcal{T}_1$ . We will next look at  $\mathcal{T}_9$ . For  $i \in [1, 8]$ ,  $\mathcal{T}_i \subset \mathcal{T}_9$ . Therefore,  $\mathcal{T}_9$  is finer than  $\mathcal{T}_i$  for  $i \in [1, 8]$ .  $\mathcal{T}_2$  is comparable with  $\mathcal{T}_7$  and  $\mathcal{T}_8$  For  $\mathcal{T}_8$ ,  $\mathcal{T}_2 \subset \mathcal{T}_8$  so  $\mathcal{T}_8$  is finer than  $\mathcal{T}_2$ , but  $\mathcal{T}_7 \subset \mathcal{T}_2$  so  $\mathcal{T}_2$  is finer than  $\mathcal{T}_7$ .  $\mathcal{T}_3$  is comparable with  $\mathcal{T}_4$ ,  $\mathcal{T}_6$ , and  $\mathcal{T}_7$ . For  $i = 4, 7$ ,  $\mathcal{T}_i \subset \mathcal{T}_3$  so  $\mathcal{T}_3$  is finer than  $\mathcal{T}_i$  for  $i = 4, 7$ , but  $\mathcal{T}_3 \subset \mathcal{T}_6$  so  $\mathcal{T}_6$  is finer  $\mathcal{T}_3$ .  $\mathcal{T}_4$  is comparable to  $\mathcal{T}_6$  and  $\mathcal{T}_8$  and  $\mathcal{T}_4 \subset \mathcal{T}_i$  for  $i = 6, 8$  so  $\mathcal{T}_i$  is finer than  $\mathcal{T}_4$ .  $\mathcal{T}_5$  and  $\mathcal{T}_6$  are only comparable to  $\mathcal{T}_9$  and we have already determined the comparability of  $\mathcal{T}_9$ .  $\mathcal{T}_7$  is comparable to  $\mathcal{T}_8$  and  $\mathcal{T}_7 \subset \mathcal{T}_8$  so  $\mathcal{T}_8$  is finer than  $\mathcal{T}_7$ .  $\mathcal{T}_8$  has been compared with all possible topologies by now.

3. Show that the collection  $\mathcal{T}_c$  given in example 4 of chapter 2 section 2.1 is a topology on the set  $X$ . Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on  $X$ ?

Example 4 states: Let  $X$  be a set; let  $\mathcal{T}_c$  be the collection of all subsets  $U$  of  $X$  such that  $X - U$  is either countable or is all of  $X$ . Then  $\mathcal{T}_c$  is a topology on  $X$ .

First, we need to determine if  $X, \emptyset \in \mathcal{T}_c$ . We have that  $X - X = \emptyset$ , and since  $\emptyset$  is finite,  $\emptyset$  is countable so  $X - X = \emptyset \in \mathcal{T}_c$ . Now,  $X - \emptyset = X$  which is all of  $X$  so  $X - \emptyset = X \in \mathcal{T}_c$ . Let  $\{U_\alpha\}$  be an indexed family of nonempty elements in  $\mathcal{T}_c$ . Then

$$X - \bigcup_{\alpha} U_{\alpha} = \bigcap_{\alpha} (X - U_{\alpha})$$

By definition of the problem,  $U_{\alpha}$  is such that  $X - U_{\alpha}$  is either countable or all of  $X$ . Since  $X - U_{\alpha}$  is countable,  $\bigcap_{\alpha} (X - U_{\alpha})$ . Therefore  $\bigcup_{\alpha} U_{\alpha} \in \mathcal{T}_c$ . Now, let's take a finite intersections of elements of  $\mathcal{T}_c$ . Then

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$$

Agian, we have that  $X - U_i$  is countable and the finite union of countable sets is countable so  $\bigcap_{i=1}^n U_i \in \mathcal{T}_c$  and  $\mathcal{T}_c$  is a topology on  $X$ . For the second part, let  $X = \mathbb{R}$ . Let  $U_1 = \mathbb{R} \setminus \{x\}$  where  $x$  can be any real number you choose. Then  $X - U_1 = \{x\}$  which is a single element and is hence finite. Therefore,  $\mathcal{T}_\infty$  is not a topology on  $X$ .

4. (a) If  $\mathcal{T}_\alpha$  is a family of topologies on  $X$ , show that  $\bigcap \mathcal{T}_\alpha$  is a topology on  $X$ . Is  $\bigcup \mathcal{T}_\alpha$  a topology on  $X$ ?

Since, for each  $\alpha$ ,  $\mathcal{T}_\alpha$  is a topology,  $X, \emptyset \in \mathcal{T}_\alpha$  so  $X, \emptyset \in \bigcap \mathcal{T}_\alpha$ . Let  $U_n \in \bigcap \mathcal{T}_\alpha$  be basis elements of  $\bigcap \mathcal{T}_\alpha$ . Then  $U_n$ , for each  $\alpha$ , exists in  $\mathcal{T}_\alpha$ . Therefore,  $U_n$  is a basis element for each  $\mathcal{T}_\alpha$ .  $\bigcap_1^k U_i$  is the finite intersection of basis elements of  $\mathcal{T}_\alpha$ ; thus,  $\bigcap_1^k U_i \in \mathcal{T}_\alpha$ . Let's consider the arbitrary intersection of  $U_n$ . For each  $\alpha$ ,  $U_n$  exist in  $\mathcal{T}_\alpha$ . Then  $\bigcup U_n \in \mathcal{T}_\alpha$ . Hence  $\bigcap \mathcal{T}_\alpha$  is a topology on  $X$ . For the union, let  $X = \{a, b, c\}$  and  $\mathcal{T}_\alpha = \{\mathcal{T}_1, \mathcal{T}_2\}$  where  $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$  and  $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$ . Then  $\bigcup \mathcal{T}_\alpha = \{\emptyset, \{a\}, \{b\}, X\}$ . Now the union of elements in  $\mathcal{T}_\alpha$  must be in  $\mathcal{T}_\alpha$ . However,  $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_\alpha$ . Hence, the union is not a topology on  $X$ .

- (b) Let  $\{\mathcal{T}_\alpha\}$  be a family of topologies on  $X$ . Show that there is a unique smallest topology on  $X$  containing all the collections  $\mathcal{T}_\alpha$ , and a unique largest topology contained in all  $\mathcal{T}_\alpha$ .

- (c) If  $X = \{a, b, c\}$ , let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , and the largest topology contained in  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

The smallest topology containing  $\mathcal{T}_1$  and  $\mathcal{T}_2$  is  $\mathcal{T}_s = \{\emptyset, \{a, b\}, \{b, c\}, X\}$  and the largest contained in is  $\mathcal{T}_l = \{\emptyset, \{a\}, X\}$ .

5. Show that if  $\mathcal{A}$  is a basis for a topology on  $X$ , then the topology generated by  $\mathcal{A}$  equals the intersection of all topologies on  $X$  that contain  $\mathcal{A}$ . Prove the same if  $\mathcal{A}$  is a subbasis.

Let  $\mathcal{T}$  be the topology generated by  $\mathcal{A}$  and let  $\{\mathcal{T}_\alpha\}$  be the family of topologies that contain  $\mathcal{A}$ . Since  $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha$  and  $\mathcal{A} \subset \mathcal{T}$ , we have that  $\bigcap \mathcal{T}_\alpha \subset \mathcal{T}$ . Let  $U$  be an open set in  $\mathcal{A}$ . Then  $U \in \mathcal{T}_\alpha$  for all  $\alpha$  but  $\mathcal{T}_\alpha$  contains  $\mathcal{A}$  so  $\mathcal{T} \subset \bigcap \mathcal{T}_\alpha$ . Therefore,  $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ .

6. Show that the topologies on  $\mathbb{R}_\ell$  and  $\mathbb{R}_K$  are not comparable.

The lower limit topology,  $\mathbb{R}_\ell$ , is  $[a, b) = \{x \mid a \leq x < b\}$  and the  $K$ -topology,  $\mathbb{R}_K$ , is the collection of all open intervals  $(a, b)$  along with all sets of the form  $(a, b) - K$  where  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ . Consider the basis element  $[2, 3)$  in the lower limit topology.  $[2, 3)$  is open in  $\mathbb{R}_\ell$ , but I claim it isn't open in  $\mathbb{R}_K$ . Let  $\mathcal{B}$  be a basis element for  $\mathbb{R}_K$ . Suppose  $2 \in \mathcal{B} \subset [2, 3)$ . Basis elements in  $\mathbb{R}_K$  are of the form  $(a, b)$  or  $(a, b) - K$ . If  $2 \in \mathcal{B}$ , then  $a < 2$ . Therefore, there exists an  $x$  such that  $\max(a, 1) < x < 2$  so  $x \notin K$ . Then  $x$  is in the basis element  $\mathcal{B}$ , but  $x \notin [2, 3)$ . Thus,  $[2, 3)$  is not open in  $\mathbb{R}_K$ . Consider the basis element  $(-1, 1) - K$ . Now,  $(-1, 1) - K$  is open in  $\mathbb{R}_K$ , but I claim it isn't open in  $\mathbb{R}_\ell$ . Let  $\mathcal{B}$  be a basis element of  $\mathbb{R}_\ell$  such that  $0 \in \mathcal{B} \subset ((-1, 1) - K)$ . Let  $\mathcal{B} = [a, b)$  be a basis element containing zero. Then  $a \leq 0$  and  $b > 0$ . Let  $n$  be an integer such that  $nb > 1 \iff 1/n < b$ . Then  $1/n \in K$  and  $1/n \in \mathcal{B}$ . Therefore,  $1/n \notin (-1, 1)$  so  $\mathcal{B} \not\subset ((-1, 1) - K)$ .

7. Consider the following topologies on  $\mathbb{R}$ :

$\mathcal{T}_1$  = the standard topology, having all sets  $(a, b)$  as a basis,

$\mathcal{T}_2$  = the topology of  $\mathbb{R}_K$ ,

$\mathcal{T}_3$  = the finite complement topology, having all sets  $\mathcal{T} = \{X \subset \mathbb{R} : X = \emptyset \text{ or } \mathbb{R} \setminus X \text{ is finite}\}$

$\mathcal{T}_4$  = the upper limit topology, having all sets  $(a, b]$  as a basis,

$\mathcal{T}_5$  = the topology having all sets  $(-\infty, a) = \{x \mid x < a\}$  as a basis.

Determine, for each of these topologies, which of the others it contains.

Lemma 13.4: The topologies of  $\mathbb{R}_\ell$  and  $\mathbb{R}_K$  are strictly finer than the standard topology on  $\mathbb{R}$ , but are not comparable with one another where  $\mathbb{R}_\ell$  is the lower limit topology. However, from lemma 13.4, we can show that upper limit topology is strictly finer than the standard topology. Therefore, we know that  $\mathcal{T}_1 \subset \mathcal{T}_2, \mathcal{T}_4$ .

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis that generates the standard topology on  $\mathbb{R}$ .

- (b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis that generates a topology different from the lower limit topology on  $\mathbb{R}$ .



## 2.3 The Order Topology

## 2.4 The Product Topology on $X \times Y$

## 2.5 The Subspace Topology

1. Show that if  $Y$  is a subspace of  $X$ , and  $A$  is a subset of  $Y$ , then the topology  $A$  inherits as a subspace of  $Y$  is the same as the topology it inherits as a subspace of  $X$ .

Let  $\mathcal{T}_X$  be the topology on  $X$  and  $\mathcal{U}$  be a basis of  $\mathcal{T}_X$ . Then  $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}_X\}$ . Let  $V = Y \cap U$ . Then  $\mathcal{T}_A = \{A \cap V : V \in \mathcal{T}_Y\}$ . We now have that  $A \cap V = A \cap Y \cap U$ , and since  $A \subset Y \subset X$ ,  $A \cap Y = A$ . Then  $\mathcal{T}_A^X = \{A \cap U : U \in \mathcal{T}_X\}$  and  $\mathcal{T}_A \subset \mathcal{T}_A^X$ . Let  $U \cap A \in \mathcal{T}_A^X$ . Since  $U$  is open in  $X$ ,  $U \cap Y$  is open relative to  $Y$  so  $U \cap Y \in \mathcal{T}_Y$ . Then  $U \cap Y \cap A \in \mathcal{T}_A$  since  $U \cap Y \in \mathcal{T}_Y$  and  $U \cap Y \cap A = U \cap A \in \mathcal{T}_A^X$ . Therefore,  $\mathcal{T}_A = \mathcal{T}_A^X$  and the topology  $A$  inherits as a subspace of  $Y$  is the same as the topology it inherits as a subspace of  $X$ .

2. If  $\mathcal{T}$  and  $\mathcal{T}'$  are topologies on  $X$  and  $\mathcal{T}'$  is strictly finer than  $\mathcal{T}$ ,  $\mathcal{T}' \supset \mathcal{T}$ , what can you say about the corresponding subspace topologies on the subset  $Y$  of  $X$ ?

The inherited topologies can be the same.

3. Consider the set  $Y = [-1, 1]$  as a subspace of  $\mathbb{R}$ . Which of the following sets are open in  $Y$ ? Which are open in  $\mathbb{R}$ ?

$$A = \{x : 1/2 < |x| < 1\}$$

$$B = \{x : 1/2 < |x| \leq 1\}$$

$$C = \{x : 1/2 \leq |x| < 1\}$$

$$D = \{x : 1/2 \leq |x| \leq 1\}$$

$$E = \{x : 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}^+\}$$

4. A map  $f : X \rightarrow Y$  is said to be an *open map* if for every open set  $U$  of  $X$ , the set  $f(U)$  is open in  $Y$ . Show that  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are open maps.
5. Let  $X$  and  $X'$  denote a single set in the topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively; let  $Y$  and  $Y'$  denote a single set in the topologies  $\mathcal{U}$  and  $\mathcal{U}'$ , respectively. Assume these sets are nonempty.
  - (a) Show that if  $\mathcal{T}' \supset \mathcal{T}$  and  $\mathcal{U}' \supset \mathcal{U}$ , then the product topology on  $X' \times Y'$  is finer than the product topology on  $X \times Y$ .
  - (b) Does the converse of exercise 5 (a) hold? Justify your answer.
6. Show that the countable collection

$$\{(a, b) \times (c, d) : a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ rational}\}$$

is a basis for  $\mathbb{R}^2$ .

7. Let  $X$  be an ordered set. If  $Y$  is a proper subset of  $X$  that is convex in  $X$ , does it follow that  $Y$  is an interval or a ray in  $X$ ?
8. If  $L$  is a straight line in the plane, describe the topology  $L$  inherits as a subspace of  $\mathbb{R}_\ell \times \mathbb{R}$  and as a subspace of  $\mathbb{R}_\ell \times \mathbb{R}_\ell$ . In each case it is a familiar topology.
9. Show that the dictionary order topology on the set  $\mathbb{R} \times \mathbb{R}$  is the same as the product topology  $\mathbb{R}_d \times \mathbb{R}$ , where  $\mathbb{R}_d$  denotes  $\mathbb{R}$  in the discrete topology. Compare this topology with the standard topology on  $\mathbb{R}^2$ .
10. Let  $I = [0, 1]$ . Compare the product topology on  $I \times I$ , the dictionary order topology on  $I \times I$ , and the topology  $I \times I$  inherits as a subspace of  $\mathbb{R} \times \mathbb{R}$  in the dictionary order topology.