Solutions to Munkres' Topology

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Contents

1 Set Theory and Logic

Skipped for now

1.1 Fundamental Concepts

2 Topological Spaces and Continuous Functions

2.1 Topological Spaces

2.2 Basis for a Topology

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U contianing x such that $U \subset A$. Show A is open in X.

   For each x in A, there exists an open set U_x such that $x \in U_x \subset A$. Then $\bigcup_{x \in A} U_x \subseteq A$. Since U_x is an open set containing all $x \in A$, $A = \bigcup_{x \in A} U_x$; that is, A is the arbitrary union of a collection of open set so A is open in X.

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in examaple 1 of chapter 2 section 2.1. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

   Let $X = \{a, b, c\}$. Next, let's index the nine topologies.

$\mathcal{T}_1 = \{\varnothing, X\}$	$\mathcal{T}_2 = \{\varnothing, \{a\}, \{a, b\}, X\}$	$\mathcal{T}_3 = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$
$\mathcal{T}_4 = \{\varnothing, \{b\}, X\}$	$\mathcal{T}_5 = \{\varnothing, \{a\}, \{b, c\}, X\}$	$\mathcal{T}_6 = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$
$\mathcal{T}_7 = \{\varnothing, \{a, b\}, X\}$	$\mathcal{T}_8 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$	$\mathcal{T}_9 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$

   A topology $\mathcal{T}_i$ is comparable with $\mathcal{T}_j$ for $i \neq j$ if either $\mathcal{T}_i \supset \mathcal{T}_j$ or $\mathcal{T}_i \subset \mathcal{T}_j$. Now $\mathcal{T}_1 \subset \mathcal{T}_j$ for $i \in [2, 9]$. Therefore, $\mathcal{T}_1$ is coarser than $\mathcal{T}_i$ so $\mathcal{T}_i$ is finer than $\mathcal{T}_1$. We will next look at $\mathcal{T}_9$. For $i \in [1, 8]$, $\mathcal{T}_i \subset \mathcal{T}_9$. Therefore, $\mathcal{T}_9$ is finer than $\mathcal{T}_i$ for $i \in [1, 8]$. $\mathcal{T}_2$ is comparable with $\mathcal{T}_7$ and $\mathcal{T}_8$ For $\mathcal{T}_8$, $\mathcal{T}_2 \subset \mathcal{T}_8$ so $\mathcal{T}_8$ is finer than $\mathcal{T}_2$, but $\mathcal{T}_7 \subset \mathcal{T}_2$ so $\mathcal{T}_2$ is finer than $\mathcal{T}_7$. $\mathcal{T}_3$ is comparable with $\mathcal{T}_4$, $\mathcal{T}_6$, and $\mathcal{T}_7$. For $i = 4, 7$, $\mathcal{T}_i \subset \mathcal{T}_3$ so $\mathcal{T}_3$ is finer than $\mathcal{T}_i$ for $i = 4, 7$, but $\mathcal{T}_3 \subset \mathcal{T}_6$ so $\mathcal{T}_6$ is finer $\mathcal{T}_3$. $\mathcal{T}_4$ is comparable to $\mathcal{T}_6$ and $\mathcal{T}_8$ and $\mathcal{T}_4 \subset \mathcal{T}_i$ for $i = 6, 8$ so $\mathcal{T}_i$ is finer than $\mathcal{T}_4$. $\mathcal{T}_5$ and $\mathcal{T}_6$ are only comparable to $\mathcal{T}_9$ and we have already determined the comparability of $\mathcal{T}_9$. $\mathcal{T}_7$ is comparable to $\mathcal{T}_8$ and $\mathcal{T}_7 \subset \mathcal{T}_8$ so $\mathcal{T}_8$ is finer than $\mathcal{T}_7$. $\mathcal{T}_8$ has been compared with all possible topologies by now.

3. Show that the collection $\mathcal{T}_c$ given in example 4 of chapter 2 section 2.1 is a topology on the set X. Is the collection

   $$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

   a topology on X?

   Example 4 states: Let X be a set; let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Then $\mathcal{T}_c$ is a topology on X.

   First, we need to determine if $X, \varnothing \in \mathcal{T}_c$. We have that $X - X = \varnothing$, and since $\varnothing$ is finite, $\varnothing$ is countable so $X - X = \varnothing \in \mathcal{T}_c$. Now, $X - \varnothing = X$ which is all of X so $X - \varnothing = X \in \mathcal{T}_c$. Let $\{U_\alpha\}$ be an indexed family of nonempty elements in $\mathcal{T}_c$. Then

   $$X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha)$$

   By definition of the problem, U_α is such that $X - U_\alpha$ is either countable or all of X. Since $X - U_\alpha$ is countable, $\bigcap_\alpha (X - U_\alpha)$. Therefore $\bigcup_\alpha U_\alpha \in \mathcal{T}_c$. Now, let's take a finite intersections of elements of $\mathcal{T}_c$. Then

   $$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

   Agian, we have that $X - U_i$ is countable and the finite union of countable sets is countable so $\bigcap_1^n U_i \in \mathcal{T}_c$ and $\mathcal{T}_c$ is a topology on X. For the second part, let $X = \mathbb{R}$. Let $U_1 = \mathbb{R} \setminus \{x\}$ where x can be any real number you choose. Then $X - U_1 = \{x\}$ which is a single element and is hence finite. Therefore, $\mathcal{T}_\infty$ is not a topology on X.

4. (a) If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

   Since, for each α, $\mathcal{T}_\alpha$ is a topology, $X, \varnothing \in \mathcal{T}_\alpha$ so $X, \varnothing \in \bigcap \mathcal{T}_\alpha$. Let $U_n \in \bigcap \mathcal{T}_\alpha$ be basis elements of $\bigcap \mathcal{T}_\alpha$. Then U_n, for each α, exists in $\mathcal{T}_\alpha$. Therefore, U_n is a basis element for each $\mathcal{T}_\alpha$. $\bigcap_1^k U_i$ is the finite intersection of basis elements of $\mathcal{T}_\alpha$; thus, $\bigcap_1^k U_i \in \mathcal{T}_\alpha$. Let's consider the arbitrary intersection of U_n. For each α, U_n exist in $\mathcal{T}_\alpha$. Then $\bigcup U_n \in \mathcal{T}_\alpha$. Hence $\bigcap \mathcal{T}_\alpha$ is a topology on X. For the union, let $X = \{a, b, c\}$ and $\mathcal{T}_\alpha = \{\mathcal{T}_1, \mathcal{T}_2\}$ where $\mathcal{T}_1 = \{\varnothing, \{a\}, X\}$ and $\mathcal{T}_2 = \{\varnothing, \{b\}, X\}$. Then $\bigcup \mathcal{T}_\alpha = \{\varnothing, \{a\}, \{b\}, X\}$. Now the union of elements in $\mathcal{T}_\alpha$ must be in $\mathcal{T}_\alpha$. However, $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_\alpha$. Hence, the union is not a topology on X.

   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

   (c) If $X = \{a, b, c\}$, let

   $$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \qquad \text{and} \qquad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$.

   The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\mathcal{T}_s = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and the largest is $\mathcal{T}_l = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$.

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

6. Show that the topologies on $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

7. Consider the following topologies on $\mathbb{R}$:

   $\mathcal{T}_1 =$ the standard topology,
   $\mathcal{T}_2 =$ the topology of $\mathbb{R}_K$,
   $\mathcal{T}_3 =$ the finite complement topology,
   $\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as a basis,
   $\mathcal{T}_5 =$ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as a basis.

   Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

   $$\mathcal{B} = \{(a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$$

   is a basis that generates the standard topology on $\mathbb{R}$.

   (b) Show that the collection

   $$\mathcal{C} = \{[a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$$

   is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.