Solutions to Munkres' Topology

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2.1 Topological Spaces

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1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U contianing x such that $U \subset A$. Show A is open in X.

   For each x in A, there exists an open set U_x such that $x \in U_x \subset A$. Then $\bigcup_{x \in A} U_x \subseteq A$. Since U_x is an open set containing all $x \in A$, $A = \bigcup_{x \in A} U_x$; that is, A is the arbitrary union of a collection of open set so A is open in X.

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in examaple 1 of chapter 2 section 2.1. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

   Let $X = \{a, b, c\}$. Next, let's index the nine topologies.

   $\mathcal{T}_1 = \{\varnothing, X\}$ $\quad$ $\mathcal{T}_2 = \{\varnothing, \{a\}, \{a, b\}, X\}$ $\quad$ $\mathcal{T}_3 = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$

   $\mathcal{T}_4 = \{\varnothing, \{b\}, X\}$ $\quad$ $\mathcal{T}_5 = \{\varnothing, \{a\}, \{b, c\}, X\}$ $\quad$ $\mathcal{T}_6 = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$

   $\mathcal{T}_7 = \{\varnothing, \{a, b\}, X\}$ $\quad$ $\mathcal{T}_8 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ $\quad$ $\mathcal{T}_9 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$

   A topology $\mathcal{T}_i$ is comparable with $\mathcal{T}_j$ for $i \neq j$ if either $\mathcal{T}_i \supset \mathcal{T}_j$ or $\mathcal{T}_i \subset \mathcal{T}_j$. Now $\mathcal{T}_1 \subset \mathcal{T}_j$ for $i \in [2, 9]$. Therefore, $\mathcal{T}_1$ is coarser than $\mathcal{T}_i$ so $\mathcal{T}_i$ is finer than $\mathcal{T}_1$. We will next look at $\mathcal{T}_9$. For $i \in [1, 8]$, $\mathcal{T}_i \subset \mathcal{T}_9$. Therefore, $\mathcal{T}_9$ is finer than $\mathcal{T}_i$ for $i \in [1, 8]$. $\mathcal{T}_2$ is comparable with $\mathcal{T}_7$ and $\mathcal{T}_8$ For $\mathcal{T}_8$, $\mathcal{T}_2 \subset \mathcal{T}_8$ so $\mathcal{T}_8$ is finer than $\mathcal{T}_2$, but $\mathcal{T}_7 \subset \mathcal{T}_2$ so $\mathcal{T}_2$ is finer than $\mathcal{T}_7$. $\mathcal{T}_3$ is comparable with $\mathcal{T}_4$, $\mathcal{T}_6$, and $\mathcal{T}_7$. For $i = 4, 7$, $\mathcal{T}_i \subset \mathcal{T}_3$ so $\mathcal{T}_3$ is finer than $\mathcal{T}_i$ for $i = 4, 7$, but $\mathcal{T}_3 \subset \mathcal{T}_6$ so $\mathcal{T}_6$ is finer $\mathcal{T}_3$. $\mathcal{T}_4$ is comparable to $\mathcal{T}_6$ and $\mathcal{T}_8$ and $\mathcal{T}_4 \subset \mathcal{T}_i$ for $i = 6, 8$ so $\mathcal{T}_i$ is finer than $\mathcal{T}_4$. $\mathcal{T}_5$ and $\mathcal{T}_6$ are only comparable to $\mathcal{T}_9$ and we have already determined the comparability of $\mathcal{T}_9$. $\mathcal{T}_7$ is comparable to $\mathcal{T}_8$ and $\mathcal{T}_7 \subset \mathcal{T}_8$ so $\mathcal{T}_8$ is finer than $\mathcal{T}_7$. $\mathcal{T}_8$ has been compared with all possible topologies by now.

3. Show that the collection $\mathcal{T}_c$ given in example 4 of chapter 2 section 2.1 is a topology on the set X. Is the collection

   $$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

   a topology on X?

   Example 4 states: Let X be a set; let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Then $\mathcal{T}_c$ is a topology on X.

   First, we need to determine if $X, \varnothing \in \mathcal{T}_c$. We have that $X - X = \varnothing$, and since $\varnothing$ is finite, $\varnothing$ is countable so $X - X = \varnothing \in \mathcal{T}_c$. Now, $X - \varnothing = X$ which is all of X so $X - \varnothing = X \in \mathcal{T}_c$. Let $\{U_\alpha\}$ be an indexed family of nonempty elements in $\mathcal{T}_c$. Then

   $$X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha)$$

   By definition of the problem, U_α is such that $X - U_\alpha$ is either countable or all of X. Since $X - U_\alpha$ is countable, $\bigcap_\alpha (X - U_\alpha)$. Therefore $\bigcup_\alpha U_\alpha \in \mathcal{T}_c$. Now, let's take a finite intersections of elements of $\mathcal{T}_c$. Then

   $$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

   Agian, we have that $X - U_i$ is countable and the finite union of countable sets is countable so $\bigcap_1^n U_i \in \mathcal{T}_c$ and $\mathcal{T}_c$ is a topology on X. For the second part, let $X = \mathbb{R}$. Let $U_1 = \mathbb{R} \setminus \{x\}$ where x can be any real number you choose. Then $X - U_1 = \{x\}$ which is a single element and is hence finite. Therefore, $\mathcal{T}_\infty$ is not a topology on X.

4. (a) If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

   Since, for each α, $\mathcal{T}_\alpha$ is a topology, $X, \varnothing \in \mathcal{T}_\alpha$ so $X, \varnothing \in \bigcap \mathcal{T}_\alpha$. Let $U_n \in \bigcap \mathcal{T}_\alpha$ be basis elements of $\bigcap \mathcal{T}_\alpha$. Then U_n, for each α, exists in $\mathcal{T}_\alpha$. Therefore, U_n is a basis element for each $\mathcal{T}_\alpha$. $\bigcap_1^k U_i$ is the finite intersection of basis elements of $\mathcal{T}_\alpha$; thus, $\bigcap_1^k U_i \in \mathcal{T}_\alpha$. Let's consider the arbitrary intersection of U_n. For each α, U_n exist in $\mathcal{T}_\alpha$. Then $\bigcup U_n \in \mathcal{T}_\alpha$. Hence $\bigcap \mathcal{T}_\alpha$ is a topology on X. For the union, let $X = \{a, b, c\}$ and $\mathcal{T}_\alpha = \{\mathcal{T}_1, \mathcal{T}_2\}$ where $\mathcal{T}_1 = \{\varnothing, \{a\}, X\}$ and $\mathcal{T}_2 = \{\varnothing, \{b\}, X\}$. Then $\bigcup \mathcal{T}_\alpha = \{\varnothing, \{a\}, \{b\}, X\}$. Now the union of elements in $\mathcal{T}_\alpha$ must be in $\mathcal{T}_\alpha$. However, $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_\alpha$. Hence, the union is not a topology on X.

   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

   (c) If $X = \{a, b, c\}$, let

   $$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \qquad \text{and} \qquad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$.

   The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is $\mathcal{T}_s = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and the largest is $\mathcal{T}_l = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$.

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

   Let $\mathcal{T}$ be the be the topology generated by $\mathcal{A}$ and let $\{\mathcal{T}_\alpha\}$ be the family of topologies that contain $\mathcal{A}$. Since $\mathcal{A} \subset \cap \mathcal{T}_\alpha$ and $\mathcal{A} \subset \mathcal{T}$, we have that $\cap \mathcal{T}_\alpha \subset \mathcal{T}$. Let U be an open set in $\mathcal{A}$. Then $\cup U \in \mathcal{T}_\alpha$ for all α but $\mathcal{T}_\alpha$ contains $\mathcal{A}$ so $\mathcal{T} \subset \cap \mathcal{T}_\alpha$. Therefore, $\mathcal{T} = \cap \mathcal{T}_\alpha$.

6. Show that the topologies on $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

   The lower limit topology, $\mathbb{R}_\ell$, is $[a, b) = \{x \mid a \leqslant x < b\}$ and the K-topology, $\mathbb{R}_K$, is the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$ where $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

7. Consider the following topologies on $\mathbb{R}$:

   $\mathcal{T}_1 =$ the standard topology, having all sets (a, b) as a basis,
   $\mathcal{T}_2 =$ the topology of $\mathbb{R}_K$,
   $\mathcal{T}_3 =$ the finite complement topology, having all sets $\mathcal{T} = \{X \subset \mathbb{R} \colon X = \varnothing \text{ or } \mathbb{R} \setminus X \text{ is finite}\}$
   $\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as a basis,
   $\mathcal{T}_5 =$ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as a basis.

   Determine, for each of these topologies, which of the others it contains.

   Lemma 13.4: The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparbale with one another where $\mathbb{R}_\ell$ is the lower limit topology. However, from lemma 13.4, we can show that upper limit topology is strictly finer than the standard topology. Therefore, we know that $\mathcal{T}_1 \subset \mathcal{T}_2, \mathcal{T}_4$.

8. (a) Apply Lemma 13.2 to show that the countable collection

   $$\mathcal{B} = \{(a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$$

   is a basis that generates the standard topology on $\mathbb{R}$.

   (b) Show that the collection

   $$\mathcal{C} = \{[a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$$

   is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

2.3 The Order Topology

2.4 The Product Topology on $X \times Y$

2.5 The Subspace Topology

1. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

   Let $\mathcal{T}_X$ be the topology on X and U be a basis of $\mathcal{T}_X$. Then $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}_X\}$. Let $V = Y \cap U$. Then $\mathcal{T}_A = \{A \cap V : V \in \mathcal{T}_Y\}$. We now have that $A \cap V = A \cap Y \cap U$, and since $A \subset Y \subset X$, $A \cap Y = A$. Then $\mathcal{T}_A^X = \{A \cap U : U \in \mathcal{T}_X\}$ and $\mathcal{T}_A \subset \mathcal{T}_A^X$. Let $U \cap A \in \mathcal{T}_A^X$. Since U is open in X, $U \cap Y$ is open relative to Y so $U \cap Y \in \mathcal{T}_Y$. Then $U \cap Y \cap A \in \mathcal{T}_A$ since $U \cap Y \in \mathcal{T}_Y$ and $U \cap Y \cap A = U \cap A \in \mathcal{T}_A$. Therefore, $\mathcal{T}_A = \mathcal{T}_A^X$ and the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

2. It $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, $\mathcal{T}' \supset \mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned}
A &= \{x : 1/2 < |x| < 1\} \\
B &= \{x : 1/2 < |x| \leqslant 1\} \\
C &= \{x : 1/2 \leqslant |x| < 1\} \\
D &= \{x : 1/2 \leqslant |x| \leqslant 1\} \\
E &= \{x : 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}^+\}
\end{aligned}$$

4. A map $f : X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

5. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.
   (a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
   (b) Does the converse of exercise 5 (a) hold? Justify your answer.

6. Show that the countable collection

$$\{(a, b) \times (c, d) : a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ rational}\}$$

   is a basis for $\mathbb{R}^2$.

7. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

9. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.