

SOLUTIONS TO MUNKRES' TOPOLOGY

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1 Set Theory and Logic

Skipped for now

1.1 Fundamental Concepts

2 Topological Spaces and Continuous Functions

2.1 Topological Spaces

2.2 Basis for a Topology

1. Let X be a topological space; let A be a subset of X . Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show A is open in X .

For each x in A , there exists an open set U_x such that $x \in U_x \subset A$. Then $\bigcup_{x \in A} U_x \subseteq A$. Since U_x is an open set containing all $x \in A$, $A = \bigcup_{x \in A} U_x$; that is, A is the arbitrary union of a collection of open set so A is open in X .

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in examaple 1 of chapter 2 section 2.1. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.

Let $X = \{a, b, c\}$. Next, let's index the nine topologies.

$$\begin{array}{lll} \mathcal{T}_1 = \{\emptyset, X\} & \mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\} & \mathcal{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} \\ \mathcal{T}_4 = \{\emptyset, \{b\}, X\} & \mathcal{T}_5 = \{\emptyset, \{a\}, \{b, c\}, X\} & \mathcal{T}_6 = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} \\ \mathcal{T}_7 = \{\emptyset, \{a, b\}, X\} & \mathcal{T}_8 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} & \mathcal{T}_9 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\} \end{array}$$

A topology \mathcal{T}_i is comparable with \mathcal{T}_j for $i \neq j$ if either $\mathcal{T}_i \supset \mathcal{T}_j$ or $\mathcal{T}_i \subset \mathcal{T}_j$. Now $\mathcal{T}_1 \subset \mathcal{T}_j$ for $i \in [2, 9]$. Therefore, \mathcal{T}_1 is coarser than \mathcal{T}_i so \mathcal{T}_i is finer than \mathcal{T}_1 . We will next look at \mathcal{T}_9 . For $i \in [1, 8]$, $\mathcal{T}_i \subset \mathcal{T}_9$. Therefore, \mathcal{T}_9 is finer than \mathcal{T}_i for $i \in [1, 8]$. \mathcal{T}_2 is comparable with \mathcal{T}_7 and \mathcal{T}_8 For \mathcal{T}_8 , $\mathcal{T}_2 \subset \mathcal{T}_8$ so \mathcal{T}_8 is finer than \mathcal{T}_2 , but $\mathcal{T}_7 \subset \mathcal{T}_2$ so \mathcal{T}_2 is finer than \mathcal{T}_7 . \mathcal{T}_3 is comparable with \mathcal{T}_4 , \mathcal{T}_6 , and \mathcal{T}_7 . For $i = 4, 7$, $\mathcal{T}_i \subset \mathcal{T}_3$ so \mathcal{T}_3 is finer than \mathcal{T}_i for $i = 4, 7$, but $\mathcal{T}_3 \subset \mathcal{T}_6$ so \mathcal{T}_6 is finer \mathcal{T}_3 . \mathcal{T}_4 is comparable to \mathcal{T}_6 and \mathcal{T}_8 and $\mathcal{T}_4 \subset \mathcal{T}_i$ for $i = 6, 8$ so \mathcal{T}_i is finer than \mathcal{T}_4 . \mathcal{T}_5 and \mathcal{T}_6 are only comparable to \mathcal{T}_9 and we have already determined the comparability of \mathcal{T}_9 . \mathcal{T}_7 is comparable to \mathcal{T}_8 and $\mathcal{T}_7 \subset \mathcal{T}_8$ so \mathcal{T}_8 is finer than \mathcal{T}_7 . \mathcal{T}_8 has been compared with all possible topologies by now.

3. Show that the collection \mathcal{T}_c given in example 4 of chapter 2 section 2.1 is a topology on the set X . Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X ?

Example 4 states: Let X be a set; let \mathcal{T}_c be the collection of all subsets U of X such that $X - U$ is either countable or is all of X . Then \mathcal{T}_c is a topology on X .

First, we need to determine if $X, \emptyset \in \mathcal{T}_c$. We have that $X - X = \emptyset$, and since \emptyset is finite, \emptyset is countable so $X - X = \emptyset \in \mathcal{T}_c$. Now, $X - \emptyset = X$ which is all of X so $X - \emptyset = X \in \mathcal{T}_c$. Let $\{U_\alpha\}$ be an indexed family of nonempty elements in \mathcal{T}_c . Then

$$X - \bigcup_{\alpha} U_{\alpha} = \bigcap_{\alpha} (X - U_{\alpha})$$

By definition of the problem, U_{α} is such that $X - U_{\alpha}$ is either countable or all of X . Since $X - U_{\alpha}$ is countable, $\bigcap_{\alpha} (X - U_{\alpha})$. Therefore $\bigcup_{\alpha} U_{\alpha} \in \mathcal{T}_c$. Now, let's take a finite intersections of elements of \mathcal{T}_c . Then

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$$

Agian, we have that $X - U_i$ is countable and the finite union of countable sets is countable so $\bigcap_{i=1}^n U_i \in \mathcal{T}_c$ and \mathcal{T}_c is a topology on X . For the second part, let $X = \mathbb{R}$. Let $U_1 = \mathbb{R} \setminus \{x\}$ where x can be any real number you choose. Then $X - U_1 = \{x\}$ which is a single element and is hence finite. Therefore, \mathcal{T}_∞ is not a topology on X .

4. (a) If \mathcal{T}_α is a family of topologies on X , show that $\bigcap \mathcal{T}_\alpha$ is a topology on X . Is $\bigcup \mathcal{T}_\alpha$ a topology on X ?

Since, for each α , \mathcal{T}_α is a topology, $X, \emptyset \in \mathcal{T}_\alpha$ so $X, \emptyset \in \bigcap \mathcal{T}_\alpha$. Let $U_n \in \bigcap \mathcal{T}_\alpha$ be basis elements of $\bigcap \mathcal{T}_\alpha$. Then U_n , for each α , exists in \mathcal{T}_α . Therefore, U_n is a basis element for each \mathcal{T}_α . $\bigcap_1^k U_i$ is the finite intersection of basis elements of \mathcal{T}_α ; thus, $\bigcap_1^k U_i \in \mathcal{T}_\alpha$. Let's consider the arbitrary intersection of U_n . For each α , U_n exist in \mathcal{T}_α . Then $\bigcup U_n \in \mathcal{T}_\alpha$. Hence $\bigcap \mathcal{T}_\alpha$ is a topology on X . For the union, let $X = \{a, b, c\}$ and $\mathcal{T}_\alpha = \{\mathcal{T}_1, \mathcal{T}_2\}$ where $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$. Then $\bigcup \mathcal{T}_\alpha = \{\emptyset, \{a\}, \{b\}, X\}$. Now the union of elements in \mathcal{T}_α must be in \mathcal{T}_α . However, $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_\alpha$. Hence, the union is not a topology on X .

- (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X . Show that there is a unique smallest topology on X containing all the collections \mathcal{T}_α , and a unique largest topology contained in all \mathcal{T}_α .

- (c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing \mathcal{T}_1 and \mathcal{T}_2 , and the largest topology containing \mathcal{T}_1 and \mathcal{T}_2 .

The smallest topology containing \mathcal{T}_1 and \mathcal{T}_2 is $\mathcal{T}_s = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and the largest is $\mathcal{T}_l = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$.

5. Show that if \mathcal{A} is a basis for a topology on X , then the topology generated by \mathcal{A} equals the intersection of all topologies on X that contain \mathcal{A} . Prove the same if \mathcal{A} is a subbasis.

Let \mathcal{T} be the topology generated by \mathcal{A} and let $\{\mathcal{T}_\alpha\}$ be the family of topologies that contain \mathcal{A} . Since $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha$ and $\mathcal{A} \subset \mathcal{T}$, we have that $\bigcap \mathcal{T}_\alpha \subset \mathcal{T}$. Let U be an open set in \mathcal{A} . Then $U \in \mathcal{T}_\alpha$ for all α but \mathcal{T}_α contains \mathcal{A} so $\mathcal{T} \subset \bigcap \mathcal{T}_\alpha$. Therefore, $\mathcal{T} = \bigcap \mathcal{T}_\alpha$.

6. Show that the topologies on \mathbb{R}_ℓ and \mathbb{R}_K are not comparable.

The lower limit topology, \mathbb{R}_ℓ , is $[a, b) = \{x \mid a \leq x < b\}$ and K -topology, \mathbb{R}_K , is the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$ where $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

7. Consider the following topologies on \mathbb{R} :

\mathcal{T}_1 = the standard topology,

\mathcal{T}_2 = the topology of \mathbb{R}_K ,

\mathcal{T}_3 = the finite complement topology,

\mathcal{T}_4 = the upper limit topology, having all sets $(a, b]$ as a basis,

\mathcal{T}_5 = the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as a basis.

Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis that generates the standard topology on \mathbb{R} .

- (b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis that generates a topology different from the lower limit topology on \mathbb{R} .