NUPOC Study Guide Solutions

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Contents

1 Mathematics, Calculus, and Differential Equations

1. What is a solution to the equation $(1-y)^2 + 2xy = 0$.

   It is apparent that $(1,1)$ and $(1,0)$ can't be solutions. Therefore, the only two solutions to consider are $(1,i)$ and $(i,1)$. If we take $(i,1)$, then our equation becomes

   $$2i = 0$$

   which isn't true. Thus, the solution must be $(1,i)$. Let's check to see.

   $$\begin{aligned}(1-i)^2 + 2i &= 0\\ 1 - 2i - 1 + 2i &= 0\end{aligned}$$

   As we can see, everything checks out.

2. The set of all points $P(x,y)$ in a plane, such that the difference of their distance from two fixed points is a positive constant is called?

   Let's start be looking at a diagram of the situation. Also, note that we can always make a change of coordinates that puts our two fixed points on the x axis line.

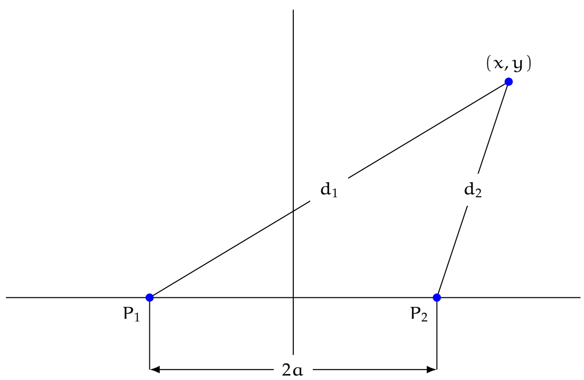


Figure 1: The geometry of two fixed points from the point (x,y).

   Now, let $P_1 = (-a, 0)$ and $P_2 = (a, 0)$. We can find both d_1 and d_2 by use of the distance formula.

   $$\begin{aligned}d_1 &= \sqrt{(x+a)^2 + y^2}\\ d_1 &= \sqrt{(x-a)^2 + y^2}\end{aligned}$$

   Using the premise of the question that the difference of their distance is a positive constant, we have

   $$R = \sqrt{(x+a)^2 + y^2} - \sqrt{(x-a)^2 + y^2}$$

   Using algebra, we can begin to re-write our equation above.

   $$\begin{aligned}\left(R + \sqrt{(x-a)^2 + y^2}\right)^2 &= (x+a)^2 + y^2\\ R^2 + 2R\sqrt{(x-a)^2 + y^2} + (x-a)^2 + y^2 &= (x+a)^2 + y^2\\ \frac{R}{2}\sqrt{(x-a)^2 + y^2} &= ax - \left(\frac{R}{2}\right)^2\\ \left(\frac{R}{2}\right)^2\left((x-a)^2 + y^2\right) &= a^2x^2 - 2ax\left(\frac{R}{2}\right)^2 + \left(\frac{R}{2}\right)^4\\ x^2\left(\left(\frac{R}{2}\right)^2 - a^2\right) + \left(\frac{R}{2}\right)^2 y^2 &= \left(\frac{R}{2}\right)^2\left(\left(\frac{R}{2}\right)^2 - a^2\right)\\ \frac{x^2}{\left(\frac{R}{2}\right)^2} + \frac{y^2}{\left(\frac{R}{2}\right)^2 - a^2} &= 1 \qquad (1)\end{aligned}$$

At this point, we have an equation for an ellipse. Unfortunately, we have one question that needs to be cleared up. Is $\left(\frac{R}{2}\right)^2 - a^2 > 0$? Given an arbitrary triangle with sides a, b, and c, we know by the triangle inequality that the addition of two sides of the triangle is always greater than or equal to the remaining side. That is,

$$a + b \geqslant c$$

or any permutation of the sides. We can know apply this result to our triangle.

$$d_1 + d_2 \geqslant 2a$$
$$d_1 + 2a \geqslant d_2 \tag{2}$$
$$d_2 + 2a \geqslant d_1 \tag{3}$$

Let's re-write equations (2) and (3).

$$2a \geqslant d_2 - d_1$$
$$2a \geqslant d_1 - d_2$$

From these two equations, we have that $2a \geqslant |d_1 - d_2|$. Since the distance between difference of the two fixed points is a positive constant, $|d_1 - d_2| = R$. Thus,

$$2a \geqslant R \Rightarrow a \geqslant \frac{R}{2}.$$

Let's go back to equation (1). Since we know now that $a \geqslant \frac{R}{2}$, we must re-write equation (1).

$$\frac{x^2}{\left(\frac{R}{2}\right)^2} - \frac{y^2}{\left(\frac{R}{2}\right)^2 - a^2} = 1$$

which is the equation of a hyperbola.

3. A propeller plane and a jet travel 3000 miles. The velocity of the plane is 1/3 the velocity of the jet. It takes the prop plane 10 hours longer to complete the trip. What is the velocity of the jet?

   We can model the time it takes the prop plane to complete the journey as $t_p = t_j + 10$, and the distance is simply the velocity times time. Since both the prop plane and the jet travels the same distance,

$$v_j t_j = \frac{v_j}{3}(t_j + 10)$$
$$t_j = 5$$

   So we have found the time it take the jet to complete the trip is 5 hours. Since $3000 = v_j t_j$ and we now know the time of the jet, we can find that the velocity is $v_j = 600\ \mathrm{mph}$.

4. What is the center of $x^2 + y^2 - 2x - 4y - 17 = 0$?

   For this problem, we will use the technique of completing the square.

$$x^2 - 2x + y^2 - 4y = 17$$

   For completing the square, we must add and subtract $\left(\frac{b}{2}\right)^2$ to the quadratic polynomials.

$$x^2 - 2x + 1 - 1 + y^2 - 4y + 4 - 4 = 17$$
$$(x-1)^2 + (y-2)^2 = 22$$

   We now have an equation of a circle centered at $(1, 2)$ with radius $\sqrt{22}$.

5. Rewrite with nothing higher than order 2:

$$\frac{a^4 + b^4}{a^2 + b^2}$$

   Without using Complex variables, neither the numerator nor the denominator readily factors. We could use long division, but there is another way. For instance, take the denominator $a^2 + b^2$. What happens when we square it? Well, we get $a^4 + b^4 + 2a^2b^2$. Therefore, $(a^2 + b^2)^2 - 2a^2b^2 = a^4 + b^4$. By making this substitution, we obtain

$$a^2 + b^2 - \frac{2a^2b^2}{a^2 + b^2}$$

   which has no order higher than 2.

6. Simplify $\frac{3+2i}{3-2i}$

   To simplify a complex fraction, we need to first multiple by the conjugate where the conjugate of $a+bi$ is defined as $a-bi$.

$$\frac{(3+2i)^2}{(3-2i)(3+2i)} = \frac{5+12i}{11}$$

7. Solve the following linear system

$$\begin{aligned} 5x-4y+2z &= 0 \\ -3x+4y &= 6 \\ x+4z &= 6 \end{aligned}$$

   For this problem, we can use matrix row operations to determine the tuple (x, y, z).

$$\begin{bmatrix} 5 & -4 & 2 & 0 \\ -3 & 4 & 0 & 6 \\ 1 & 0 & 4 & 6 \end{bmatrix}$$

   Our goal here is to obtain zeros under every pivot position where the pivot positions are a_{ii} where $i = 1$, 2, and 3. Using row operations, let $r_2 = r_2 + 3r_3$ where each row is written is lowest form.

$$\begin{bmatrix} 5 & -4 & 2 & 0 \\ 0 & 1 & 3 & 6 \\ 1 & 0 & 4 & 6 \end{bmatrix}$$

   Then take $r_1 = r_1 - 5r_3$.

$$\begin{bmatrix} 0 & 2 & 9 & 15 \\ 0 & 1 & 3 & 6 \\ 1 & 0 & 4 & 6 \end{bmatrix}$$

   We can interchange row one and row three but let's wait on that. Finally, take $r_1 = r_1 - 2r_2$.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 6 \\ 1 & 0 & 4 & 6 \end{bmatrix}$$

   The top row tells use that $z = 1$. From row two, we have that $y = 3$, and from row three, $x = 2$. That is, our tuple is $(2, 3, 1)$.

8. What is a logarithm? How is e, the natural logarithm base, defined?

   Let $\log_a(b) = x$. Then the logarithm identifies the power a needs to be raised to equal b. In other words, $a^x = b$. The natural logarithm is log base, $\log_e(a) = x \Rightarrow e^x = a$. In calculus, the natural log is defined as

$$\ln(x) = \int_0^x \frac{1}{t}\,dt.$$

9. The number of square feet in a circle is equal to the number of in feet of the circle's circumference. What is the circle's radius?

   By the premise, we have $2\pi r = \pi r^2$. Solving for r, we obtain $r(r-2) = 0$. So $r = 0, 2$. Zero is a trivial solution so $r = 2$.

10. Derive the equation of a circle around any point.

    Let (h, k) be the center of the circle and (x, y) be a point on the circle. Let's define the radius to be r. Then the distance between any point (x, y) and the center is

$$r = \sqrt{(x-h)^2 + (y-k)^2} \Rightarrow r^2 = (x-h)^2 + (y-k)^2$$

    which is the equation of a circle around the center (h, k) with radius r.

11. Given a closed box, where the length is twice the height, the width is 10m less than the length, and the surface area is 10 times the width times the height, what are the dimensions?

The surface of a rectangular box is defined as

$$SA = 2h\ell + 2w\ell + 2hw.$$

For our box, $\ell = 2h$, $w = 2h - 10$, and $SA = 10(2h - 10)h$.

$$\begin{aligned} SA &= 2h\ell + 2w\ell + 2hw \\ &= 2\left(2h^2 + 4h^2 - 20h + 2h^2 - 10h\right) \\ &= 16h^2 - 60h \\ 20h^2 - 100h &= 16h^2 - 60h \\ 4h^2 - 40h &= 0 \\ h(h - 10) &= 0 \end{aligned}$$

Therefore, $h = 10$ so $\ell = 20$ and $w = 10$.

12. Derive the quadratic equation.

The general form of a parabolic equation is $ax^2 + bx + c = 0$. First, we will divide by a and then complete the square.

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\ x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

13. What geometric surface encloses the maximum volume with the minimum surface area? How do you prove it?

This will be added in last.

14. What type of smooth curve would go through these points: $(0, 4)$ and $(\pm 2, 0)$? What would the equation be?

Since we are given the x-intercepts, we know the function is of the form $f(x) = (x - 2)(x + 2) = x^2 - 4$ but this function has a y-intercept of -4 not 4. If we take the negative of $f(x)$, we obtain $f(x) = -x^2 + 4$ which is a parabola that opens down with the appropriate x and y intercepts.

15. Find the area of items 15(a) and 15(b) using calculus and also derive the formulas for the volume of items 15(c) and 15(d).

    (a) Triangle

    For any triangle on the xy-plane, we can always perform a change in coordinates so that one of the vertices is at the origin and one edge is along the x axis. Therefore, we will only consider a triangle that his this geometry.

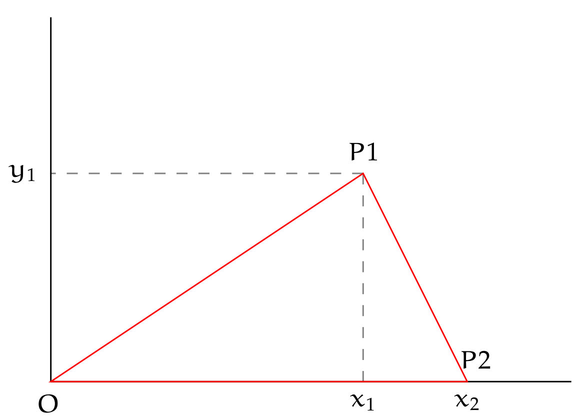


Figure 2: An arbitrary triangle with an edge on the x axis starting at the origin.

Before we begin, let's perform our sanity check. The area of the triangle is $A = \frac{x_2 y_1}{2}$; therefore, the formula that we are going to derive from calculus better equal this. The integral defines the area under the curve. That is, we can come up with an integral over our domain that should be the area of a triangle. First, let's determine the lines OP1 and P1P2.

$$\begin{aligned}
\text{OP1} &\Rightarrow f_1(x) = \frac{y_1}{x_1} x \\
\text{P1P2} &\Rightarrow f_2(x) = \frac{y_1}{x_1 - x_2} x - \frac{y_1 x_2}{x_1 - x_2}
\end{aligned}$$

We can break up our integral over the domain of the triangle as

$$\begin{aligned}
\int_0^{x_1} f_1(x) dx + \int_{x_1}^{x_2} f_2(x)\ dx &= \int_0^{x_1} \frac{y_1}{x_1} x\ dx + \int_{x_1}^{x_2} \left(\frac{y_1}{x_1 - x_2} x - \frac{y_1 x_2}{x_1 - x_2} \right) dx \\
&= \frac{y_1}{2x_1} x^2 \Big|_0^{x_1} + \frac{y_1}{x_1 - x_2} \left[\frac{x^2}{2} - x_2 x \right|_{x_1}^{x_2} \\
&= \frac{y_1 x_1}{2} + \frac{y_1}{x_1 - x_2} \left(\frac{x_2^2 - x_1^2}{2} - x_2(x_2 - x_1) \right) \\
&= \frac{y_1 x_1}{2} + y_1 x_2 - \frac{y_1(x_2 + x_1)}{2} \\
&= \frac{y_1 x_2}{2}
\end{aligned}$$

(b) Circle

From item 10, we previously determined the equation for a circle with center (h, k) and radius a is

$$a^2 = (x - h)^2 + (y - k)^2.$$

Let's solve this equation for y. Then $y(x) = \pm\sqrt{a^2 - (x - h)^2} + k$. By performing a change of coordinates, we can write our equations as $y(x) = \pm\sqrt{a^2 - x^2}$. That is, a circle with origin $(0, 0)$. Let's convert to polar coordinates. Let $x = a \sin(\theta)$ where the bounds of integration go from $x \in [-a, a]$ to $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Moreover, we can use the symmetry property of the circle to take 4 times the integral with bound $\theta \in \left[0, \frac{\pi}{2}\right]$. Then $dx = a \cos(\theta)\ d\theta$.

$$\begin{aligned}
4 \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2(\theta)} (a \cos(theta)\ d\theta) &= 4a^2 \int_0^{\pi/2} \sqrt{1 - \sin^2(\theta)} \cos(\theta)\ d\theta \\
&= 4a^2 \int_0^{\pi/2} \cos^2(\theta)\ d\theta
\end{aligned}$$

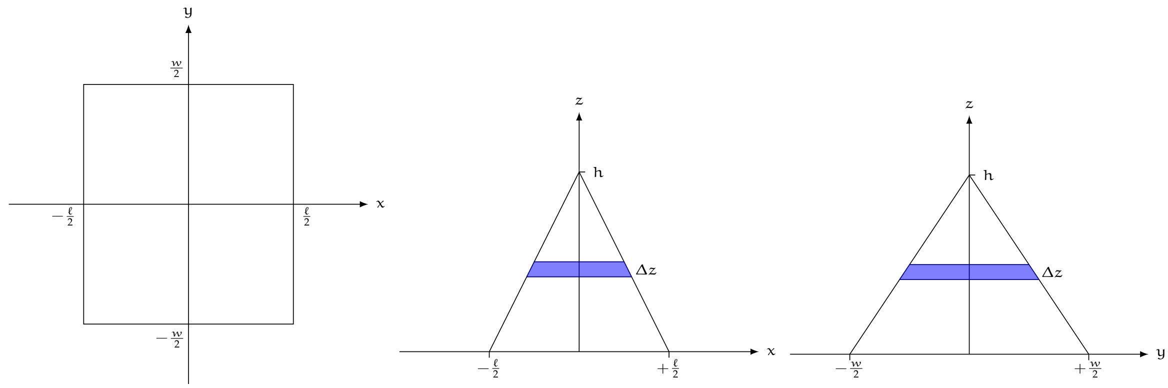
Recall that $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$. Then

$$\begin{aligned}
&= 2a^2 \int_0^{\pi/2} (1 + \cos(2\theta))\ d\theta \\
&= \pi a^2 + \left[a^2 \sin(2\theta)\right|_0^{\pi/2} \\
&= \pi a^2
\end{aligned}$$

Again, we can do a sanity check. The area of a circle is $A = \pi r^2$. Since are radius is a, we have obtained the correct formula for the area of a circle.

(c) Pyramid

Volume is the integral of the area. Let's take a look in the xz, yz, and xy plane of pyramid with length ℓ, width w, and height h.



(a) xy plane (b) xz plane (c) yz plane

Figure 3: 2D views of a 3D pyramid

The differential area Δz is simply $A = x \cdot y$ where x and y depend on the height of pyramid. Let's find the line that goes through h and $\frac{\ell}{2}$ and $\frac{w}{2}$, respectively.

$$z(x) = -\frac{2h}{\ell}x + h \tag{4}$$

$$z(y) = -\frac{2h}{w}y + h \tag{5}$$

Since x and y depend on z, we need to write x and y in terms of z from equations (4) and (5). Before we do, notice that equations (4) and (5) only govern half of the pyramid in their respective plane. Therefore, $A = (2x)(2y)$ to make up for this discrepancy.

$$2x = \ell\left(1 - \frac{z}{h}\right)$$
$$2y = w\left(1 - \frac{z}{h}\right)$$

So $A = w\ell\left(1 - \frac{z}{h}\right)^2$.

$$\begin{aligned} V &= \int_D A \, dA \\ &= w\ell \int_0^h \left(1 - \frac{z}{h}\right)^2 dz \\ &= w\ell \int_0^h \left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right) dz \\ &= w\ell\left(h - h + \frac{h}{3}\right) \\ &= \frac{w\ell h}{3} \end{aligned}$$

(d) Cone

In common usage, cones are assumed to be right circular. That is, the base is a circle of some radius r and right implies height of the cone runs through the center at right angles straight up. Then for a cone, the view from the xz and yx planes are the same.

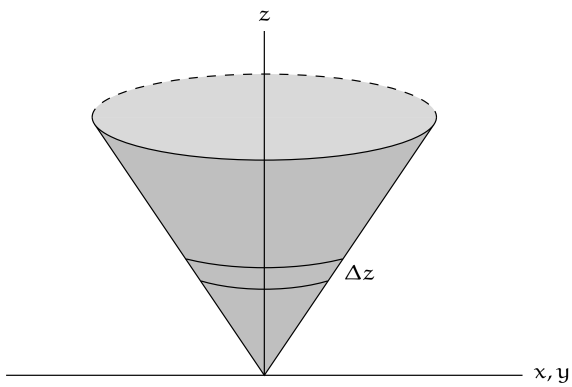


Figure 4: A right circular cone with height h.

In this case, our differential area is circle that depends on its height; that is, the radius of the circle changes. Since the area of a circle is $A = \pi r^2$ and our radius depends on z, we have $A(z) = \pi(r(z))^2$. Let's assume our circle has a radius of ℓ at the base which is the top of the cone since it is inverted. Then we have a line from the origin to $x, y = \pm\frac{\ell}{2}$.

$$z(x) = \frac{2h}{\ell}x \tag{6}$$
$$z(y) = \frac{2h}{\ell}y \tag{7}$$

Solving for z in equation (6), we will obtain $x = \frac{\ell}{2h}z$. As in the previous problem, this is only considering half of the cone. Therefore, $r = 2x = \frac{\ell}{h}z$.

$$\begin{aligned} V &= \int_D hA \; dA \\ &= \frac{\pi\ell^2}{h^2}\int_0^h z^2 \; dz \\ &= \frac{\pi\ell^2 h}{3} \end{aligned}$$

16. Draw the following curves and find the area between them:

(a) $y = x^3$ and $y = x^2$

In order to find the area between the two curves, we need to determine the domain of interest. By setting the functions equal to each other, we will be able to find their points of intersection. In this problem, this is a trivial task and one can easily see that the domain is $x \in [0, 1]$.

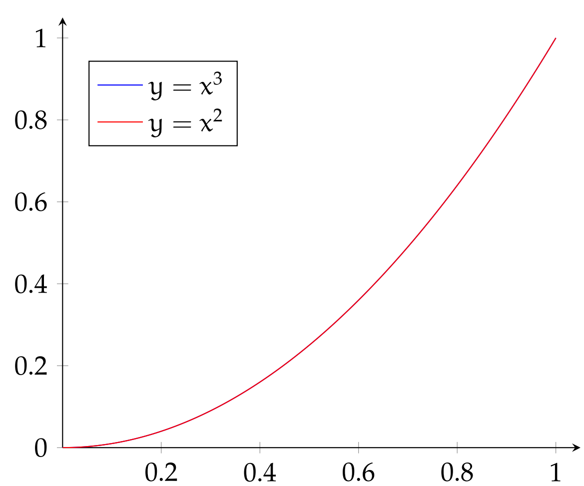


Figure 5

The area between to curves is

$$\int_D (f(x) - g(x))\,dx$$

where $f(x)$ is the top curve and $g(x)$ is the bottom curve.

$$\int_0^1 (x^2 - x^3)\,dx = \frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_0^1 = \frac{1}{12}$$

(b) $y = x^2$ and $y = x$

As with the previous part, we have that $x \in [0, 1]$.

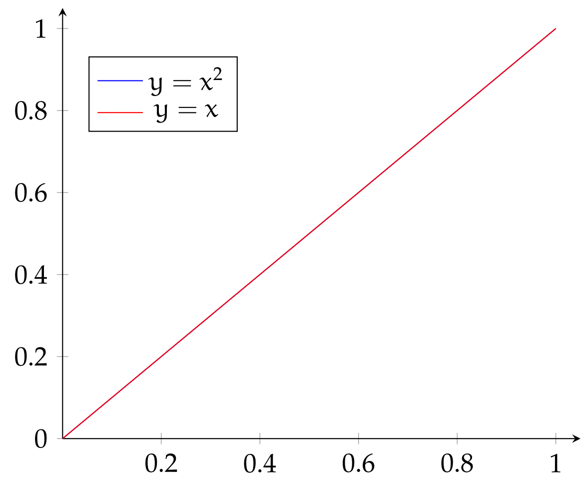


Figure 6

In this case, we have

$$\int_0^1 (x - x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{6}.$$

17. Plot $f(x) = x^2 + x - 6$. Find the area between the x axis on the top, the line $y = -4$ on the bottom, and the graph on each side.

When dealing with a parabola, we can try to factor, complete the square, and/or use the quadratic equation. In this problem, $f(x)$ is factorable.

$$f(x) = (x-3)(x+2)$$

We now know that the x-intercepts are $x = 3, -2$ and the y-intercept is $y = -6$. To find the vertex of the parabola, we need to complete the square which leads to the following form of $f(x)$.

$$f(x) = \left(x + \frac{1}{2}\right)^2 - \frac{25}{4}$$

Thus, the vertex is $\left(-\frac{1}{2}, -\frac{25}{4}\right)$.

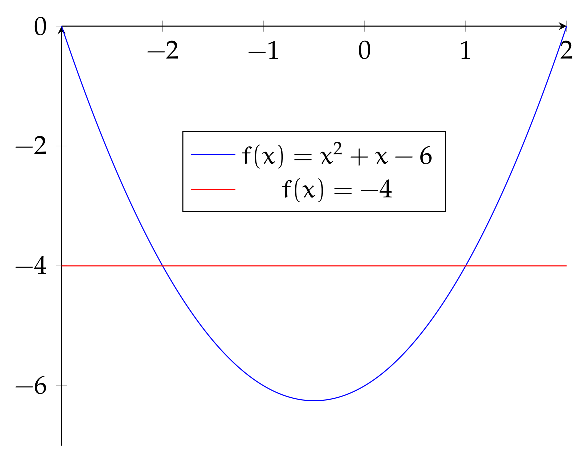


Figure 7

Since we want to find the area of the both pieces of the divided parabola, let's start by first finding the total area of the parabola below the x axis.

$$\int_{-3}^{2} (-x^2 - x + 6)\ dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\Big|_{-3}^{2} = \frac{125}{6}$$

Now, once we find one of the other areas, we can subtract it from the total area to get the other side. We need to find where the function $f(x) = -4$ intersects the parabola.

$$x^2 + x - 6 = -4 \Rightarrow x^2 + x - 2 = (x+2)(x-1)$$

So intersection occurs at $x = -2$ and $x = 1$.

$$A_{\text{below}} = \int_{-2}^{1} (-4 - x^2 - x + 6)\ dx = \int_{-2}^{1} (-x^2 - x + 2)\ dx = \frac{9}{2}$$

Therefore, the area bounded by the x axis and $f(x) = -4$ is $A_{\text{above}} = \frac{49}{3}$.

18. Rotate $y = \frac{1}{x}$ about the x axis and find the volume from 1 to infinity.

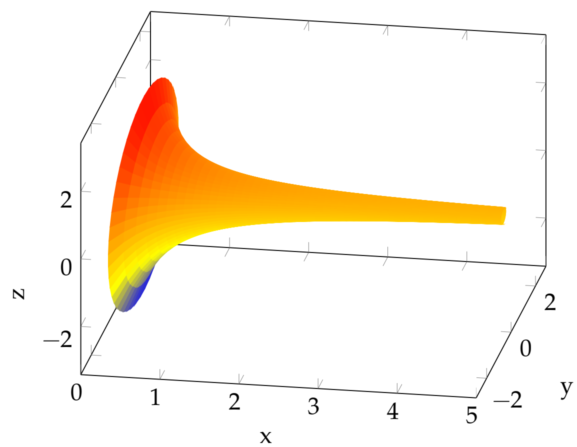


Figure 8: $\frac{1}{x}$ rotated about the x axis on the domain $x \in [0.5, 5]$.

In order to find the volume, we need to determine the differential we will use. Therefore, let's take a look at our solid in the xy plane.

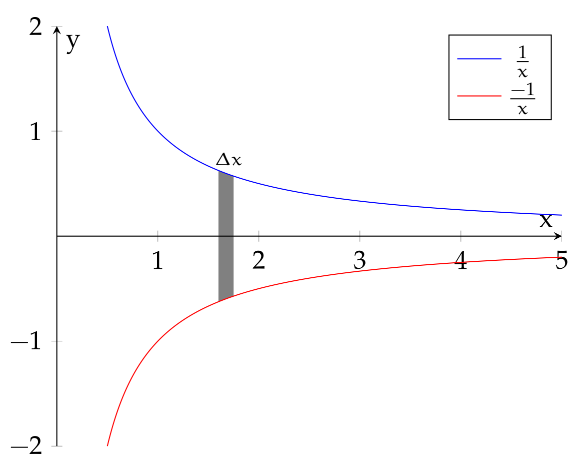


Figure 9: A two dimensional view of the solid.

The volume of the solid is $V = \int_D A\ dA$ where the area is πr^2. As Δx moves from $[1, \infty)$, the differential circle changes accordingly. That is, $r(x) = \frac{1}{x}$.

$$V = \pi \int_{1}^{\infty} \frac{1}{x^2}\ dx = \frac{-\pi}{x}\Big|_{1}^{\infty} = \pi$$

This problem is traditionally known as Gabriel's Horn which has infinite surface area and finite volume.

19. Determine the area between two concentric circles of radii 1 and 2, respectively, using calculus.

Before we use calculus to find the area, we can determine the area from geometry. With this solution, we will be able to check are result from our calculus approach. The area between the two circles is simply the difference or $A = \pi(r_1^2 - r_2^2) = 3\pi$.

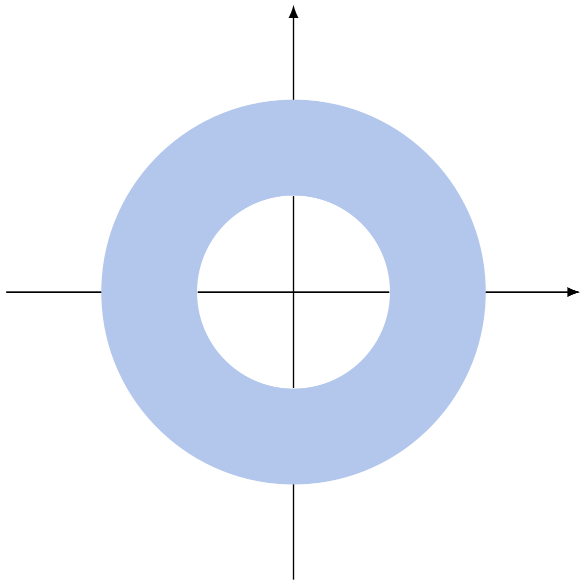


Figure 10: The area between two concentric circles of radii 1 and 2.

The equations for two circles of radii 1 and 2, respectively, centered at the origin are

$$x^2 + y^2 = 4$$
$$x_1^2 + y_1^2 = 1$$

which we derived in item 10. Solving for y, we have

$$y = \pm\sqrt{4 - x^2}$$
$$y_1 = \pm\sqrt{1 - x_1^2}$$

Due to the nature of the problem, polar coordinates are preferred. Therefore, let $x = 2\sin(\theta)$ and $x_1 = \sin(\theta)$. Then $dx = 2\cos(\theta)d\theta$ and $dx_1 = \cos(\theta)d\theta$. Additionally, our bounds of integration are now $x \in [-2, 2] \to \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ which is only the right half of the circle. Since a circle is symmetric, we only need to integrate from $\theta \in \left[0, \frac{\pi}{2}\right]$. This is only a quarter of the area so need to multiple this integral by 4.

$$A = 4\int_0^{\pi/2} (4\cos^2(\theta) - \cos^2(\theta))d\theta = 4\left[\pi - \frac{\pi}{4}\right] = 3\pi$$

20. Integrate the following:

(a) $\int(x\sin(x))\,dx$

To integrate the above integral, we need to use integration by parts. Recall that integration by parts is

$$uv - \int v\,du.$$

Let $x = u$ and $dv = \sin(x)$. Then $dx = du$ and $v = -\cos(x)$.

$$-x\cos(x) + \int \cos(x)\,dx = -x\cos(x) + \sin(x) + C$$

(b) $\int x\sqrt{x^2 - 4}\,dx$

This is integral is done by u-substitution. Let $u = x^2 - 4$. Then $du = 2xdx \Rightarrow dx = \frac{du}{2}$.

$$\frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2 - 4)^{3/2} + C$$

(c) $\int \frac{e^4 - \frac{3}{x}}{x^2}\ dx$

Here we have a straight forward integral. All we need to do is write fraction as a difference of two fractions.

$$\int \left(\frac{e^4}{x^2} - \frac{3}{x^3} \right) dx = \frac{-e^4}{x} + \frac{3}{2x^2} + C$$

(d) $\int (e^{-x} + 3x^2)\ dx$

Again, we have another trivial integral.

$$\int (e^{-x} + 3x^2)\ dx = -e^{-x} + x^3 + C$$

(e) $\int (x \sin^2(x) + x^3)\ dx$

For this integral, recall $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$. The

$$\frac{1}{2} \int (x - x\cos(2x))\ dx + \int x^3\ dx = \frac{x^2}{4} - \frac{1}{2} \int x \cos(2x)\ dx + \frac{x^4}{4} + C$$

As in the item 20(a), we need to use integration by parts again. Let $u = x$ and $dv = \cos(2x)$. Then $du = dx$ and $v = \frac{1}{2}\sin(2x)$.

$$\frac{x^2 + x^4 - x\sin(2x) + C}{4} + \frac{1}{4} \int \sin(2x)\ dx = \frac{x^2 + x^4 - x\sin(2x) + C}{4} - \frac{\cos(2x)}{8}$$

(f) $\int \sec(u) \tan(u)\ du$

This is a standard substitution problem since $\frac{d}{du}(\sec(u)) = \sec(u)\tan(u)$.

$$\int \sec(u)\tan(u)\ du = \sec(u) + C$$

(g) $\int x e^x\ dx$

This is another integration by parts. Let $u = x$ and $dv = e^x$. Then $du = dx$ and $v = e^x$.

$$xe^x - \int e^x\ dx = e^x(x - 1) + C$$

(h) $\int (y+3)(y+1)\ dy$

For this problem, it will serve us better to multiple the factored polynomial out. Thus, $y^2 + 4y + 3 = (y+3)(y+1)$.

$$\int (y^2 + 4y + 3)\ dy = \frac{y^3}{3} + 2y^2 + 3y + C$$

(i) $\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} r \sin(\theta)\ d\phi d\theta dr$

Iterative integrals work exactly like single integrals. In these situations, we treat the additional parameter to what we are integrating over as a constant.

$$\begin{aligned}
\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} r \sin(\theta)\ d\phi d\theta dr &= \frac{\pi}{2} \int_0^R \int_0^{\pi/2} r \sin(\theta)\ d\theta dr \\
&= -\frac{\pi}{2} \int_0^R \cos(\theta)\Big|_0^{\pi/2}\ dr \\
&= \frac{\pi}{2} \int_0^R r\ dr \\
&= \frac{\pi R^2}{4} + C
\end{aligned}$$

(j) $\int (2x + 1)\ dx$

This last integral is just a basic integration.

$$\int (2x + 1)\ dx = x^2 + x + C$$

21. Take the derivative with respect to x of the following:

(a) $\cos^4(x)\sin(x)$

A derivative of a product of functions requires the product rule,

$$\frac{d}{dx}(f(x)g(x)) = f'g + fg'.$$

Furthermore, since we have power of cosine, we will need to also use the chain rule,

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x))g'(x).$$

Therefore, the derivative is

$$\frac{d}{dx}\big(\cos^4(x)\sin(x)\big) = -4\cos^3(x)\sin^2(x) + \cos^5(x).$$

(b) $\frac{ae^{-bx}}{cx^2}$

For this problem, we can use the quotient or product rule. To use the product rule, we need to write the expression as $\frac{a}{c}e^{-bx}x^{-2}$ and proceed as before, but I will use the quotient rule here,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'g - fg'}{g^2}.$$

Thus, the derivative is

$$\frac{d}{dx}\left[\frac{ae^{-bx}}{cx^2}\right] = \frac{-a(be^{-bx}x^2 + 2e^{-bx}x)}{cx^4} = \frac{-a(be^{-bx}x + 2e^{-bx})}{cx^3}.$$

(c) $5x^4$

Here the derivative is simply $\frac{d}{dx}(5x^4) = 20x^3$.

(d) $x\sqrt{x^2-4}$

Again, we will use the product and chain rule.

$$\frac{d}{dx}\big(x\sqrt{x^2-4}\big) = \sqrt{x^2-4} + \frac{x^2}{\sqrt{x^2-4}}$$

(e) $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$

$$\begin{aligned}
\frac{d}{dx}(\sin(x)) &= \cos(x) \\
\frac{d}{dx}(\cos(x)) &= -\sin(x) \\
\frac{d}{dx}(\tan(x)) &= \sec^2(x) \\
\frac{d}{dx}(\cot(x)) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \\
&= -\csc^2(x) \\
\frac{d}{dx}(\sec(x)) &= \sec(x)\tan(x) \\
\frac{d}{dx}(\csc(x)) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= -\cot(x)\csc(x)
\end{aligned}$$

(f) $\ln(x)$ and 10^x

For the natural log, the derivative rule is

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'}{f}.$$

Therefore, $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$. For the second expression, we can derive the rule.

$$\begin{aligned} y &= a^{f(x)} \\ \ln(y) &= f(x)\ln(a) \\ \frac{y'}{y} &= f'(x)\ln(a) \\ y' &= yf'(x)\ln(a) \\ y' &= a^{f(x)}f'(x)\ln(a) \end{aligned}$$

Thus, our second derivative is $\frac{d}{dx}(10^x) = 10^x \ln(10)$.

(g) $x + x^3 + \sin(x)\cos(x) + \sin(x)$

This is should be a trivial exercise after doing the previous derivatives.

$$\frac{d}{dx}f(x) = 1 + 3x^2 + \cos^2(x) - \sin^2(x) + \cos(x)$$

(h) $x^5 + \cos(x)e^x + \sin\left(\frac{x^2}{3}\right)$

Here the derivative is

$$\frac{d}{dx}f(x) = 5x^4 + e^x\cos(x) - e^x\sin(x) + \frac{2x}{3}\cos\left(\frac{x^2}{3}\right).$$

(i) $x^{1/2} + x^2\sin^2(x)$

The final derivative is

$$\frac{d}{dx}f(x) = \frac{1}{2\sqrt{x}} + 2x\sin^2(x) + 2x^2\sin(x)\cos(x).$$

22. What is an integral? How is it used? What is the difference between a definite and an indefinite integral?

An integral is an anti-derivative. For simplicity, one can think of it as the inverse of a derivative. Integrals are primarily used to find areas, volumes, and surfaces. A definite integral has bounds of integration which finds the constant of integration whereas indefinite integrals have no bounds. Since there are no bounds for the indefinite integrals, we must always remember to put $+C$ for the constant of integration.

23. What is a derivative? How is it used? What is a differential? What is the significance of the first and second derivatives?

By definition, a derivative is

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

In other words, the derivative is slope of the tangent line or we can say the instantaneous rate of change of the function. Derivatives can be used to find velocity if we differentiate position and acceleration when we differentiate velocity. A differential is an infinitesimal width of distance. The first derivative of a function identifies critical points that identify maximum and minimums. The second derivative of a function identifies critical points that identify inflection points which is where concavity changes.

24. Prove that the derivative of x^2 is $2x$.

To prove $2x$ is the derivative of x^2, we need to use the definition.

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} &= \lim_{\Delta x \to 0} \frac{2(\Delta x)x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x \end{aligned}$$

25. What is the $\lim_{x\to 0} \frac{\sin(x)}{x}$?

When we take the limit as shown, we obtain $\frac{0}{0}$ which is an indeterminant form. A technique we can use for indeterminate forms is L'Hôpital's rule.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x} \\ &= \lim_{x\to 0} \cos(x) \\ &= 1\end{aligned}$$

26. Be able to integrate or differentiate by using parts, chain rule, or quotient rule.

All techniques we used in previous problems.

27. Draw the following curves. Plot any max, min, and points of inflection.

(a) $f(x) = e^{-x^2}$

To find a maximum or minimum, we need to set $f'(x) = 0$.

$$f'(x) = -2xe^{-x^2} = 0$$

Since the exponential is never zero, we have $-2x = 0$ or $x = 0$. By testing points to the left and right of zero, we can determine what is occurring at $x = 0$.

$$\begin{aligned}f'(1) &< 0 \\ f'(-1) &> 0\end{aligned}$$

Since $f'(-1) > 0$, we know that the slope of the tangent line is positive to the right of zero, and since $f'(1) < 0$, we know the slope is negative to the left of zero. That is, we have a slope that raises and then falls. Thus, we have a maximum at $x = 0$. For inflection points, we need to set $f''(x) = 0$.

$$f''(x) = 2e^{-x^2}(2x^2 - 1)$$

We then have $2x^2 - 1 = 0$ so $x = \pm\frac{1}{\sqrt{2}}$ as our inflection points.

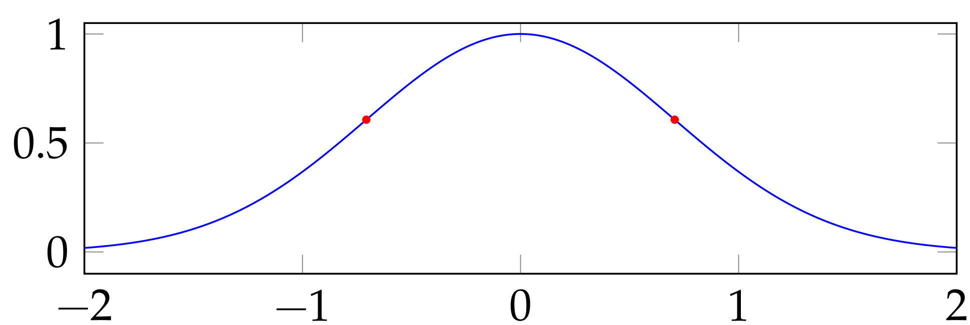


Figure 11: The plot of $f(x) = e^{-x^2}$ from $x \in [-3, 3]$.

(b) $f(x) = a\sin(x)$ for $x \in [0, 2\pi]$

Again, we need to find $f'(x) = 0$ and $f''(x) = 0$.

$$f'(x) = a\cos(x) = 0$$

So $x = \frac{\pi}{2} + \pi k$ where $k \in \mathbb{Z}$. Since we are only dealing $x \in [0, 2\pi]$, $k = 0$ and 1. therefore, $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. The question remains now are these x points maxima or minima? Let's test a point to the left and right of both critical values to find out.

$$f'\left(\frac{\pi}{4}\right) = +$$

$$f'(\frac{3\pi}{4}) = -$$
$$f'(\frac{5\pi}{4}) = +$$

Therefore, at $x = \frac{\pi}{2}$, we have a maximum, and at $x = \frac{\pi}{3}$, we have a minimum. Next, we need to determine if we have any inflection points and if so where.

$$f''(x) = -a\sin(x) = 0$$

We have zeros when $x = 0$, π, and 2π. Since $x = 0$ and 2π are the endpoints, they can't be points of inflection. Thus, our only point of inflection occurs at $x = \pi$.

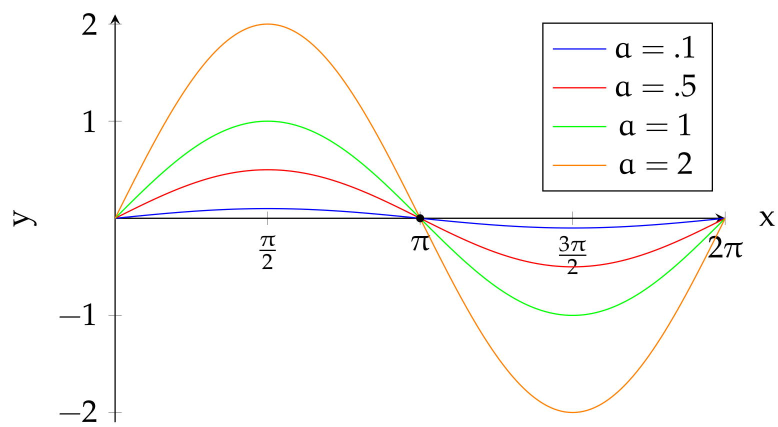


Figure 12: The plot of $f(x) = a\sin(x)$ for different values of a.

(c) $f(x) = e^{\pi/2}$

Since the first and second derivatives are a constant, zero, we have no maxima, minima, or inflection points.

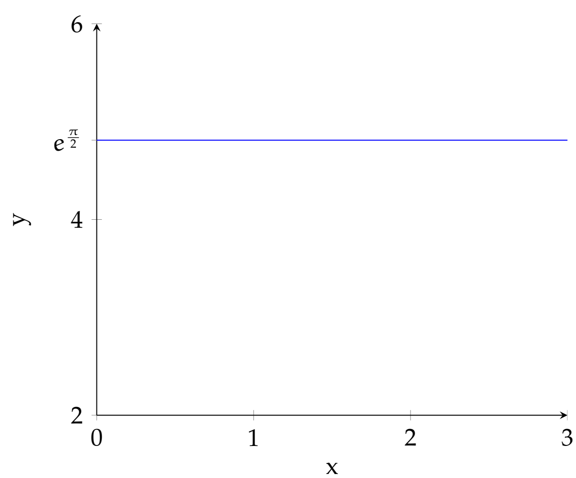


Figure 13: The plot of $f(x) = e^{\frac{\pi}{2}}$ which is a horizontal line.

(d) $f(x) = 3x^2 - 17x - 10$

For the first derivative, we have

$$f'(x) = 6x - 17 = 0.$$

Our only critical point is then $x = \frac{17}{6}$.

$$f'(0) = -$$
$$f'(3) = +$$

Therefore, $x = \frac{17}{6}$ is a minimum. Since the second derivative is a constant, we have no inflection points.

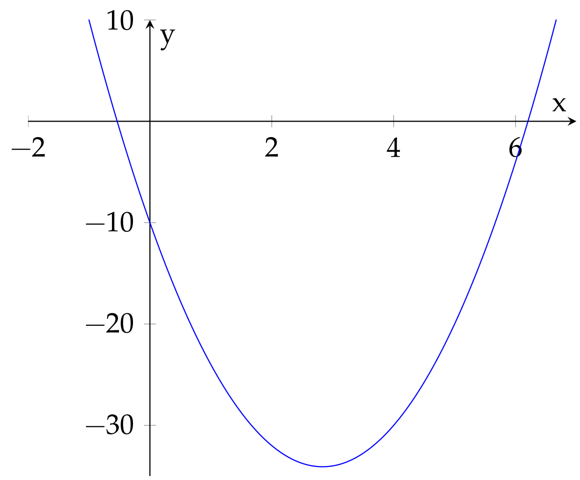


Figure 14: The plot of $f(x) = 3x^2 - 17x - 10$.

(e) $f(x) = x^3 - x^2$

In this case, we have

$$f'(x) = 3x^2 - 2x$$
$$f''(x) = 6x - 2$$

The critical points for the first derivative are $x = 0$ and $\frac{2}{3}$.

$$f'(-1) = +$$
$$f'(\frac{1}{2}) = -$$
$$f'(1) = +$$

Therefore, $x = 0$ is a maximum and $x = \frac{2}{3}$ is a minimum. A change in concavity occurs at $x = \frac{1}{3}$.

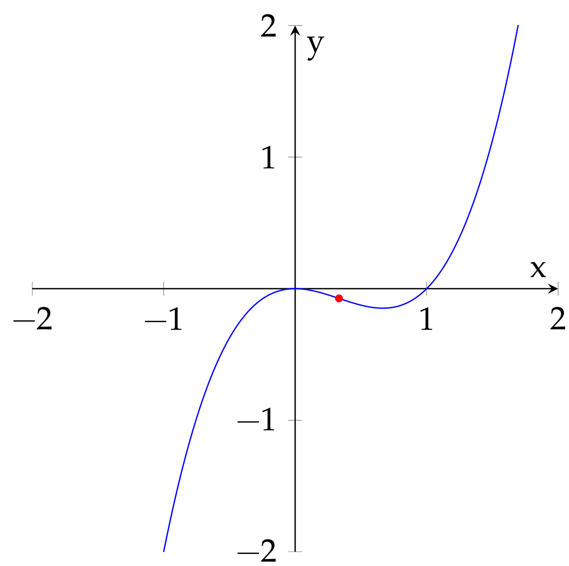


Figure 15: The plot of $f(x) = x^3 - x^2$.

(f) $f(x) = x^2e^{-x^2}$

For this equation, our first and second derivatives are

$$f'(x) = 2xe^{-x^2}(1 - x^2)$$

$$f''(x) = 2e^{-x^2}(2x^4 - 5x^2 + 1)$$

Then the critical points are $x = 0$ and ± 1.

$$f'(-2) = +$$
$$f'(\frac{-1}{2}) = -$$
$$f'(\frac{1}{2}) = +$$
$$f'(2) = -$$

We have maxima at $x = \pm 1$ and a minimum at $x = 0$. For our inflection points, we need to solve the quartic polynomial

$$2x^4 - 5x^2 + 1 = 0.$$

Let $y = x^2$. Then $2y^2 - 5y + 1 = 0$. By the quadratic equation,

$$y = \frac{5 \pm \sqrt{17}}{4}.$$

Since $y = x^2$, we have that our inflection points are at

$$x = \pm\frac{\sqrt{5 \pm \sqrt{17}}}{2}.$$

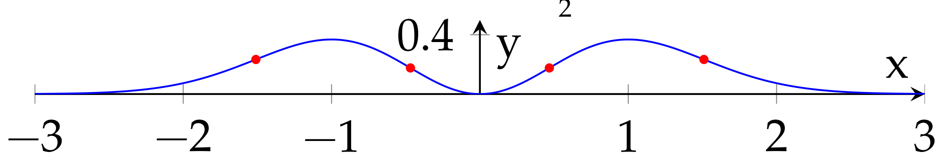


Figure 16: The plot of $f(x) = x^2e^{-x^2}$.

28. Analyze the curve $y = 1 + e^{-x}$ by finding the first two derivatives, maxima, minima, and inflection points.

    The first derivatives for y are

$$y' = -e^{-x}$$
$$y'' = e^{-x}$$

Since $e^{\pm x} > 0$, we have no maximums (minimums) and inflection points. Additionally, as $x \to \infty$, $y \to 1$. Therefore, we have a horizontal asymptote at $y = 1$.

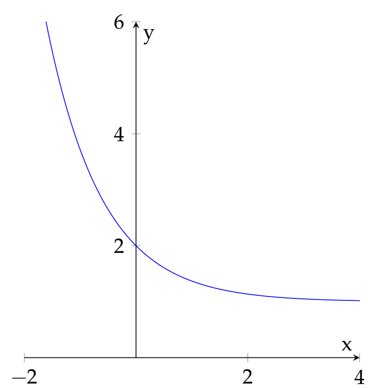


Figure 17: The plot of $y = 1 + e^{-x}$.

29. Find the max or min of a parabola and determine if it is a max or min.

    The most general parabola is $f(x) = ax^2 + bx + c$. The first derivative is then

$$f'(x) = 2ax + b.$$

A maximum (minimum) will occur at $x = \frac{-b}{2a}$. We have a few cases to consider in order to determine if we have a maximum (minimum).

Case 1: Both $a, b > 0$.

In this case, x_c (x critical) is negative.

$$f'(\frac{-b}{a}) = -$$
$$f'(0) = +$$

Therefore, $x = \frac{-b}{2a}$ is a minimum.

Case 2: $a > 0$ and $b < 0$.

In this case x_c, is positive.

$$f'(\frac{-b}{a}) = +$$
$$f'(0) = -$$

Therefore, $x = \frac{-b}{2a}$ is a minimum.

Case 3: $a < 0$ and $b > 0$.

In this case x_c, is positive.

$$f'(\frac{-b}{a}) = -$$
$$f'(0) = +$$

Therefore, $x = \frac{-b}{2a}$ is a maximum.

Case 4: Both $a, b < 0$.

In this case x_c, is negative.

$$f'(\frac{b}{a}) = +$$
$$f'(0) = -$$

Therefore, $x = \frac{-b}{2a}$ is a maximum.

30. Using calculus, derive the formula for the exposed surface area of a ball floating in water.

First, let's consider the geometric formula of surface area for a sphere (ball). The surface area of a sphere is circumference times arc length,

$$SA = 2\pi r\ell.$$

For an arbitrary sphere, we need to determine what the radius and arc length are. Let' start with the arc length, ℓ.

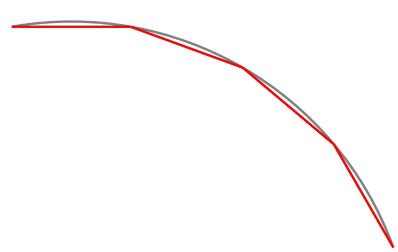


Figure 18: An arc on interval $x \in [a, b]$ with $n = 4$.

Let the end points of the estimated arc be labelled x_i where $i = 1, 2, 3, 4$, and 5. Then the distance of any line segment is $d_k = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$; therefore, the arc length, ℓ, is

$$\sum_{k=1}^{N} d_k. \tag{8}$$

By the Mean Value Theorem, we have that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

on where f is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $a < b$. Let $a = x_i$ and $b = x_{i+1}$. Then

$$f'(x_k) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \Rightarrow f'(x_k)(x_{i+1} - x_i) = f(x_{i+1}) - f(x_i)$$

Let $\Delta x = x_{i+1} - x_i$. Then we can write equation (8) as

$$\sum_{k=1}^{N} \sqrt{(\Delta x)^2\big(1 + (f'(x_k))^2\big)} = \sum_{k=1}^{N} \Delta x \sqrt{\big(1 + (f'(x_k))^2\big)}.$$

As $N \to \infty$, $\Delta x \ll 1$ and we can write the summation as the integral

$$\lim_{N\to\infty} \sum_{k=1}^{N} \Delta x \sqrt{\big(1 + (f'(x_k))^2\big)} = \int_a^b \Delta x \sqrt{\big(1 + (f'(x))^2\big)}\, dx.$$

Now, we have that the radius is

$$r = \frac{f(x_i) + f(x_{i+1})}{2},$$

but since $\Delta x \ll 1$, $f(x_i) \approx f(x_{i+1})$. Thus, $r = f(x)$. Finally, we have that the total surface area of a sphere is

$$SA = 2\pi \lim_{N\to\infty} f(x) \sum_{k=1}^{N} \Delta x \sqrt{1 + \big(f'(x_k)\big)^2} = 2\pi \int_a^b f(x) \sqrt{1 + \big(f'(x)\big)^2}\, dx.$$

Let's find the total sphere area of an arbitrary sphere. Suppose the sphere as radius, $r = R$. Then in the xy plane the equation for the circle is $x^2 + y^2 = R^2$ or $y = f(x) = \pm\sqrt{R^2 - x^2}$.

$$\begin{aligned}
f'(x) &= \frac{-x}{\sqrt{R^2 - x^2}} \\
\big(f'(x)\big)^2 &= \frac{x^2}{R^2 - x^2} \\
\sqrt{1 + \big(f'(x)\big)^2} &= \frac{R}{\sqrt{R^2 - x^2}} \\
f(x)\sqrt{1 + \big(f'(x)\big)^2} &= R
\end{aligned}$$

After grinding through the above algebra, we have a much easier integral to solve.

$$SA = 2\pi R \int_{-R}^{R} dx = 4\pi R^2$$

Now the only question remains is how to find the total exposed surface area of a ball floating in water. For this, we will assume the ball is in placid water. That is, the amount of surface area exposed doesn't change. Let h be the height of the water line, and let $y = 0$ correspond to the mid point on the ball. We have to two cases to consider, because when $h = 0$, the surface area exposed is $2\pi R^2$.

Case 1: When $h > 0$.

Since h corresponds to y, we need to determine the x associated with this y for the bounds of integration.

$$h = \pm\sqrt{R^2 - x^2} \Rightarrow x = \pm\sqrt{R^2 - h^2}$$

Then the exposed surface area is

$$\begin{aligned}
SA &= 2\pi R \int_{-\sqrt{R^2-h^2}}^{\sqrt{R^2-h^2}} dx \\
&= 4\pi R \sqrt{R^2 - h^2} \qquad (9)
\end{aligned}$$

Case 2: When $h < 0$.

For this case, it will be $2\pi R^2$ plus equation (9).

31. Solve the following differential equations:

(a) $y'' + 6y' + 9 = 5$

To solve this differential equation, let's first combine the constants.

$$y'' + 6y' + 4 = 0$$

We can then treat this differential equation as quadratic polynomial. That is, $m^2 + 6m + 4 = 0$. Then $y(x) = Ae^{m_1 x} + Be^{m_s x}$ so all we need to do now is find the zeros of the polynomial.

$$m = -3 \pm \sqrt{5}$$
$$y(x) = Ae^{x(\sqrt{5}-3)} + Be^{-x(3+\sqrt{5})}$$

(b) $\frac{dN}{dt} = -2N$

For this ODE, we can use separation of variables.

$$\int \frac{dN}{N} = -2\int dt$$
$$\ln(N) = -2t + C$$
$$N(t) = Ce^{-2t}$$

(c) $y' = xy^3$ where $y(0) = 1$

Again we can use separation of variables.

$$\int \frac{dy}{y^3} = \int x\, dx$$
$$\frac{-1}{2y^2} = \frac{x^2}{2} + C$$
$$y(x) = \frac{1}{\sqrt{C - x^2}}$$

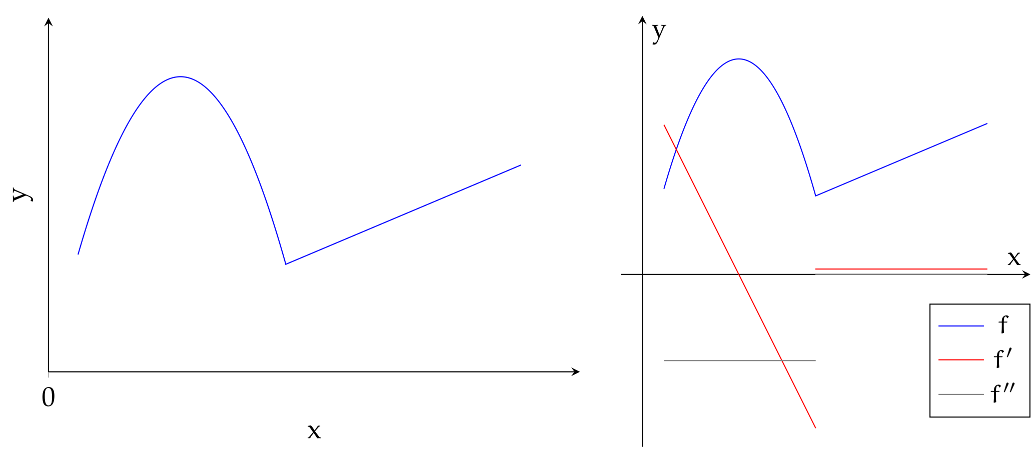
Let's now use the initial condition to find the constant of integration.

$$y(0) = \frac{1}{C} = 1$$

Therefore, $C = 1$ and the exact solution is

$$y(x) = \frac{1}{\sqrt{1 - x^2}}.$$

32. For the following curve, plot the first and second derivatives.



(a) Problem statement plot

(b) Solution plot

Figure 19: Plots for the problem and solution.

33. Given 80 feet of fencing, what is the maximum area that you can enclose along a wall?

Here we are given an optimization problem. Let's construct a diagram of what we know.

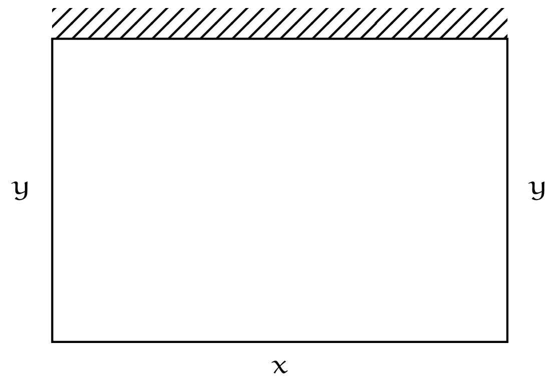


Figure 20: A rectangular area enclosed along a wall.

We can use calculus or algebra to solve to solve this problem.

Case 1: Algebra only.

The perimeter of our fenced in area is $P = 2y + x = 80$ and the area is $A = xy$. Therefore, we can write $x = 80 - 2y$ and $A = 80y - 2y^2$.

$$2y(40 - y) = 0 \Rightarrow y = 0,\ 40$$

We have that y intercepts of the parabola are $y = 0$ and 40. Since this parabola opens down, we have a maximum, and this maximum is when $y \in (0, 40)$. Again, since we are dealing with a parabola, we know it is symmetric about line that bisects it at the vertex (maximum in our case). Therefore, $y = 20$ and $x = 40$ so the maximum area is $A = 20 \cdot 40 = 800$ square feet.

Case 2: Calculus.

With calculus, we will have the same equation for area, $A = 80y - 2y^2$.

$$\frac{dA}{dy} = 80 - 4y$$

So the critical value is $y = 20$ which is a maximum since the parabola opens down. Thus, $x = 40$ and the maximum area is $A = 800$ square feet.

34. Given the figure below, determine the value of x so when the corners are removed and flaps folded up, the five-sided box formed will have the maximum volume.

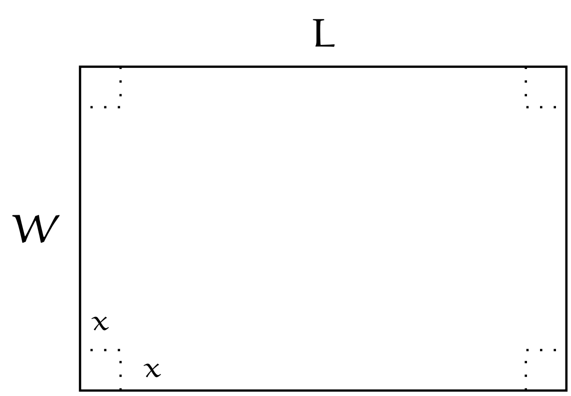


Figure 21: A rectanlge with length, L and width, W.

What we have here is an optimization problem. Let $\ell = L - 2x$ and $w = W - 2x$. Then $V = \ell w h$.

$$V(x) = (L - 2x)(W - 2x)x$$

$$= 4x^3 - 2x^2(L + W) + LWx \tag{10}$$
$$V' = 12x^2 - 4x(L + W) + LW$$

We need to find the x critical value the will maximize the volume.

$$x = \frac{L + W \pm \sqrt{L^2 - LW + W^2}}{6}$$

Let $x_+ = \frac{L+W+\sqrt{L^2-LW+W^2}}{6}$ and $x_- = \frac{L+W-\sqrt{L^2-LW+W^2}}{6}$. Since equation (10) is a cubic with the leading power positive, $4x^3$, we know from a sketch of cubics of this form that the maximum occurs when $x \in (0, x_-)$ and a minimum is when $x \in (x_-, x_+)$. Thus, the the volume of the box is maximized when $x = x_-$.

35. Two runners start at a distance of 10 miles from each other. They run towards each other at a constant velocity of 5 mph. A fly takes off from runner one's nose at $t = 0$. The fly has a constant velocity of 20 mph and flies between the runners. Find the total distance that the fly has traveled when the runners collide.

    We will assume the runner's velocity behaves much like the Dirac delta function. That is, at the moment the time starts to roll from $t = 0$ to $t > 0$, the runners are moving at $v_r = 5$ mph. Therefore, this problem has to potential solutions depending on your view. We will consider both cases.

Case 1: At $t = 0$, the runner is at 5 mph.

Since the runner is at 5 mph when the fly takes off, the fly is has a velocity of $v_f = 25$ mph. The runners will collide in 1 hour since each runner will cover 5 miles in that time. Thus, the fly will have travelled 25 miles.

Case 2: At $t = 0$, the runner is at 0 mph.

In this scenario, the fly will only have a velocity of $v_f = 20$ mph, so in 1 hour, the fly will have travelled 20 miles.

36. What is a Laplace transform, a Fourier transform, or a Taylor series? How are each used?

    The Laplace transform is defined as
    $$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt$$
    where $s = \sigma + i\omega$. The Fourier transform is defined as
    $$\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x)e^{-2\pi i x\xi}\, d\xi.$$
    A Taylor series is when you represent a function as an infinite series around some point say, a. For example, let f be an arbitrary continuous function with continuous derivatives; in other words, let f belong to C^∞. Then the Taylor series of f is
    $$\sum_{n=1}^\infty \frac{f^{(n)}(a)}{n!}(x - a)^n$$
    where $f^{(n)}$ is the nth derivative of f. Laplace transforms are used to solve ordinary differential equations and to find transfer functions in control systems. The transfer function is defined as the Laplace transform of the output over the Laplace transform of the input with zero initial conditions. The Fourier transfer is used to solve partial differential equations on infinite domains. Taylor series are used in perturbations. We can expand the Taylor series of the function in question and take only the linear terms, quadratic, or etc. depending on the desired level of accuracy.

37. When do you use L'Hôpital's rule?

    L'Hôpital's rule is used in solving limits that produces indeterminate forms. The indeterminate forms we will usually see are $0/0$, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0.

38. What is the probability of throwing one 7 with two dice?

    The probability of rolling a 7 with two dice is simply the number of ways a 7 can be rolled divided by the number of ways two dice can be rolled. The number of ways to roll a seven is 6 and the total possible rolls is 36. Thus, the probability of rolling a 7 with two dice is $\frac{1}{6}$.

39. If the population doubles in two years, how long does it take to triple?

We will assume that the change in population is proportional to a constant multiple of the population.

$$\frac{dP}{dt} \propto kP \Rightarrow \int \frac{dP}{P} = k \int dt$$

Then $P(t) = Ce^{kt}$. We know that $P(0) = P = C$ and $P(2) = 2P$.

$$P(t) = Pe^{kt}$$

We now need to find k so we can determine the time that produces $3P$.

$$\begin{aligned} 2P &= Pe^{2k} \\ 2 &= e^{2k} \\ k &= \frac{1}{2}\ln(2) \end{aligned}$$

Let's find the appropriate time now.

$$\begin{aligned} 3P &= P\exp\left(\frac{1}{2}\ln(2)t\right) \\ \ln(3) &= \frac{1}{2}\ln(2)t \\ t &= 2\frac{\ln(3)}{\ln(2)} \\ &\approx 3.16993 \end{aligned}$$

It will take approximately 3.16993 years for the population to triple.

40. Find $f(x)$ which best describes the following graph where A represents area.

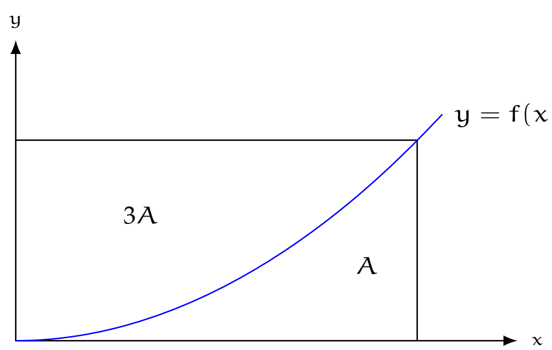


Figure 22: The plot of $f(x)$.

Let (x_1, y_1) be the point in the northeast corner of the rectangle. Since integral finds the area under the curve, we know that

$$A = \int_0^{x_1} f(x)\,dx.$$

Additionally, the area in the upper half is

$$3A = \int_0^{x_1} (y_1 - f(x))\,dx \Rightarrow 3A = x_1y_1 - \int_0^{x_1} f(x)\,dx.$$

Therefore, $A = \frac{1}{4}x_1y_1$ and

$$\int_0^{x_1} f(x)\,dx = \frac{x_1y_1}{4}.$$

41. Use a first order differential equation to find the function to represent current with respect to time and to find the time constant to the circuit.

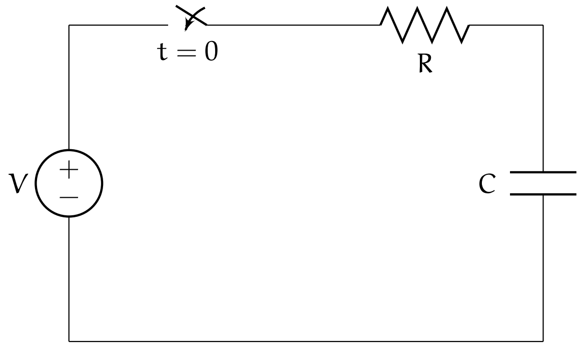


Figure 23: Electrical diagram

For $t > 0$,

$$V = iR + \frac{1}{C}\int i(t)\ dt \tag{11}$$

where $i(t) = \frac{dq}{dt}$. We can differeniate equation (11) to obatin

$$0 = \dot{i}R + \frac{i}{C}.$$

We can solve for current as a function of time now since $Rm + \frac{1}{C} = 0$ so $i(t) = Ae^{\frac{-t}{RC}}$. When $t = 0$, $i(0) = A = i_0$. Thus, $i(t) = i_0 e^{\frac{-t}{RC}}$.

42. Show how to solve a differential equation with matrices.

Let's consider the matrix differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a square matrix. Let's assume that $\mathbf{A}$ is diagonalizable.

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

Then the solution to the matrix differential equation is

$$\mathbf{x} = \sum_{i=1}^{N} c_i e^{\lambda_i t}\mathbf{v_i}$$

c_i are constants, λ_i is the ith eigenvalue of $\mathbf{A}$, and $\mathbf{v}_i$ is the ith eigenvector.

43. Find the sum of $\sum\limits_{n=1}^{100} n$.

Thanks to Gauss, we know that sums of this form can be found using the formula

$$\sum_{n=1}^{N} n = \frac{n}{2}(n+1).$$

Thus, $\sum\limits_{n=1}^{100} n = 5050$.

44. Given the figure below with uniform mass, what is the y coordinate of the center of gravity?

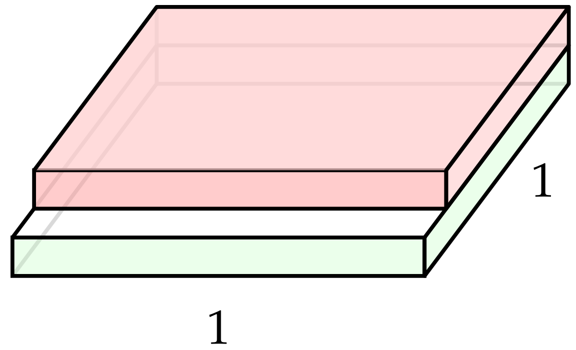


Figure 24: The height of both blocks are a and the indentation of the smaller block is a.

Suppose the density is ρ. Then $m = V\rho$ where V is the volume. The volume of the bottom block is a, and the volume of the top block is $a(1-a)$. Then $m_b = a\rho$, $m_t = a\rho(1-a)$, and $M = m_b + m_t = a\rho(2-a)$. Now let's assume the x axis runs along the front and y along the side. Then the center for both the top and bottom blocks are $x = \frac{1}{2}$.

$$\bar{y} = \frac{M_x}{M} = \frac{1}{a\rho(2-a)}$$

For practice, let's find the x coordinate for the center of gravity. The center for the bottom block is $y_b = \frac{1}{2}$ and the top is $y_t = \frac{1-a}{2}$.

$$\bar{x} = \frac{M_y}{M} = \frac{1+1-a}{2a\rho(2-a)} = \frac{1}{2a\rho}$$

45. Describe how to classify differential equations.

    Differential equations can be classified into three main categories–ordinary differential equations (ODE), partial differential equations (PDE), and differential algebraic equations (DAE). Additionally, ODE and PDE can be classified as linear, nonlinear, homogeneous, and nonhomogeneous. PDE can then be classified as elliptic, parabolic, and hyperbolic.

46. Solve $\ddot{x} + 5\dot{x} + 6x = e^{-t}$.

    First, let's determine the complementary solution; that is, the solution to the homogeneous differential equation.

    $$m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0$$

    Therefore, $x_c = Ae^{-3x} + Be^{-2x}$. For the particular solution, let $y_p = Ce^{-t}$. Then $\dot{x}_p' = -Ce^{-t}$ and $\ddot{x}_p' = Ce^{-t}$.

    $$\begin{aligned} C(e^{-t} - 5e^{-t} + 6e^{-t}) &= e^{-t} \\ 2Ce^{-t} &= e^{-t} \end{aligned}$$

    So $C = \frac{1}{2}$ and $x_p = \frac{1}{2}e^{-t}$.

    $$x(t) = Ae^{-3t} + Be^{-2t} + \frac{1}{2}e^{-t}$$

47. What is the Laplace transform of $f(t) = t$?

    Let's apply the definition.

    $$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\, dt$$

    We will have to use integration by parts where $u = t$, $du = dt$, $dv = e^{-st}$, and $v = \frac{-1}{s}e^{-st}$.

    $$\begin{aligned} &= \frac{-t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt \\ &= \frac{-1}{s^2}e^{-st}\Big|_0^\infty \\ &= \frac{1}{s^2} \end{aligned}$$

48. Solve $y'' + 4y' + 3y = \sin(x)$.

Again, we will start by finding y_c.

$$m^2 + 4m + 3 = 0 \Rightarrow (m+3)(m+1) = 0$$

So $y_c = Ae^{-3x} + Be^{-x}$. Let $y_p = C_1 \sin(x) + C_2 \cos(x)$. Then $y_p' = C_1 \cos(x) - C_2 \sin(x)$ and $y_p'' = -C_1 \sin(x) - C_2 \cos(x)$.

$$\begin{aligned} \sin(x)(3C_1 - 4C_2 - C_1) + \cos(x)(3C_2 + 4C_1 - C_2) &= \sin(x) \\ \sin(x)(2C_1 - 4C_2) + \cos(x)(2C_2 + 4C_1) &= \sin(x) \end{aligned}$$

We can use linear algebra to find the coefficients C_1 and C_2.

$$\begin{bmatrix} 2 & -4 & 1 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{5} \end{bmatrix}$$

Therefore, $C_2 = \frac{-1}{5}$ and $C_1 = \frac{1}{10}$ so $y_p = \frac{1}{10}\sin(x) - \frac{1}{5}\cos(x)$.

$$y(t) = Ae^{-3x} + Be^{-x} + \frac{1}{10}\sin(x) - \frac{1}{5}\cos(x)$$

49. Solve $\dot{x} = \frac{x}{k}$.

We can use separation of variables here.

$$\int \frac{dx}{x} = \frac{1}{k}\int dt \Rightarrow \ln(x) = \frac{t}{k} + C$$

Therefore, the solution is $x(t) = Ce^{\frac{t}{k}}$.

50. Solve the general and specific homogeneous equation with derivatives:

$$\frac{dy}{dx} + Ky = 10$$

The homogeneous equation is $\frac{dy}{dx} = -Ky$. We can use separation of variables here as well.

$$\int \frac{dy}{y} = -K\int dt \Rightarrow y(t) = Ce^{-Kt}$$

Let $y_p = A$. Then $y_p' = 0$. We have that $KA = 10 \Rightarrow A = \frac{10}{K}$.

$$y(x) = \frac{10}{K} + Ce^{-Kt}$$

51. Explain how to solve the following differential equation $A'' + A' + A = 0$.

The roots of the polynomial form the constants, α, to the general solution,

$$A(x) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}.$$

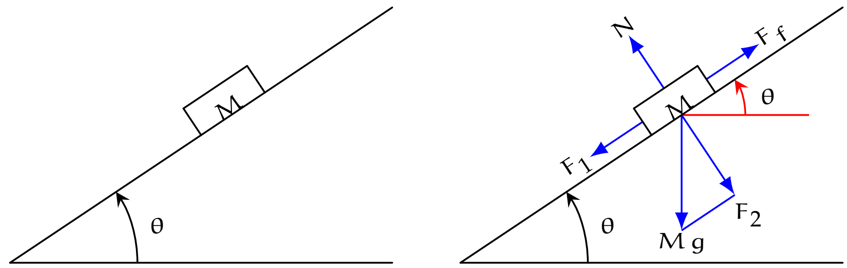
52. Solve $y - 3y' = 0$ with $y(0) = 3$.

We have $-3m + 1 = 0$ so $y(x) = Ae^{\frac{1}{3}x}$. Using the initial condition, $y(0) = A = 3$.

$$y(x) = 3e^{\frac{1}{3}x}$$

2 Physics

1. What must the angle θ be in order for the block of mass M to start sliding when $\mu = 0.8$?



(a) What is giving (b) Forces drawn

Figure 25: The picture of the problem statement and forces drawn to help solve the problem.

Let $\mathbf{N}$ be the normal force, $\mathbf{F}_f = \mathbf{N}\mu$ be the frictional force, and $M\mathbf{g}$ be the weight of the block. We will define our coordinate system such that positive x is in the direction the box will move once it over comes friction, and positive y in the direction of $\mathbf{N}$. What we need to know now is $\mathbf{F}_1$ and $\mathbf{F}_2$. From the figure 25b, we see that $\mathbf{F}_1$ is parallel to line between $M\mathbf{g}$ and $\mathbf{F}_2$. From geometry, we can determine that the angle between $M\mathbf{g}$ and $\mathbf{F}_2$ is also θ. First, let's consider the 90° angle formed by $\mathbf{F}_2$ and the ramp. From the red line to the ramp, we have a θ; therefore, the angle between the red line and $\mathbf{F}_2$ is $90 - \theta$. Additionally, we have that the angle between the red line and $M\mathbf{g}$ is also 90. Let ψ be the angle between $M\mathbf{g}$ and $\mathbf{F}_2$. Then $\psi + 90 - \theta = 90$. Thus, $\psi = \theta$ as was needed to be shown. By determining the force from $\mathbf{F}_2$ to $M\mathbf{g}$, we have the force $\mathbf{F}_1$ since $\psi = \theta$. From the properties of a right triangle, we have that

$$\begin{aligned}\mathbf{F}_2 &= M\mathbf{g}\cos(\theta)\\ \mathbf{F}_1 &= M\mathbf{g}\sin(\theta)\end{aligned}$$

By Newton's second law, $\sum \mathbf{F} = m\mathbf{a}$, we have that

$$Mg\sin(\theta) - N\mu = M\ddot{x} \tag{12a}$$
$$N - Mg\cos(\theta) = M\ddot{y} \tag{12b}$$

Let's assume there is no acceleration. Then our equation (12) can be written as

$$Mg\sin(\theta) = N\mu \tag{13a}$$
$$N = Mg\cos(\theta) \tag{13b}$$

When we combine equations (13a) and (13b), we obtain the following expression.

$$\begin{aligned}\tan(\theta) &= \mu\\ \tan(\theta) &= 0.8\end{aligned}$$

Therefore, $\theta > \arctan(0.8) \approx 38.6598^\circ$.

2. Find the final velocity of M for both elastic collision and inelastic collision.

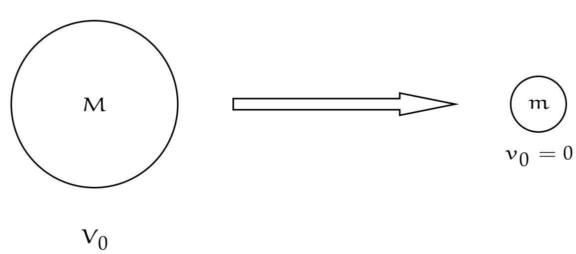


Figure 26: A ball of mass M with speed V_0 collides with a ball of mass m and speed of $v_0 = 0$.

For elastic collision, we have that the Conservation of Kinetic Energy and the Conservation of Momentum hold.

$$\begin{aligned}\text{CoKE}: \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 &= \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2\\ \text{CoM}: m_1v_1 + m_2v_2 \qquad &= m_1v_{1f} + m_2v_{2f}\end{aligned}$$

Let's apply these definitions to our problem now.

$$\begin{aligned} MV_0^2 &= MV_{0f}^2 + mv_{0f}^2 \\ M(V_0^2 - V_{0f}^2) &= mv_{0f}^2 \end{aligned} \tag{14}$$

Recall that $V_0^2 - V_{0f}^2 = (V_0 - V_{0f})(V_0 + V_{0f})$.

$$\begin{aligned} MV_0 &= MV_{0f} + mv_{0f} \\ M(V_0 - V_{0f}) &= mv_{0f} \end{aligned} \tag{15}$$

Let's divide equation (14) by equation (15).

$$\frac{V_0^2 - V_{0f}^2}{V_0 - V_{0f}} = v_{0f} \Rightarrow V_0 + V_{0f} = v_{0f}$$

Therefore, the velocity of M after collision is $V_{0f} = v_{0f} - V_0$. For inelastic collision, the Conservation of Kinetic Energy doesn't hold. In an inelastic collision, the objects stick together after impact. Therefore, when we use the CoM equation, the mass of the final state will be $m + M$.

$$MV_0 = (M + m)V_{0f} \Rightarrow V_{0f} = \frac{MV_0}{M + m}$$

3. Describe the motion of the block-spring assembly when the block is displaced 4 inches from the equilibrium position.

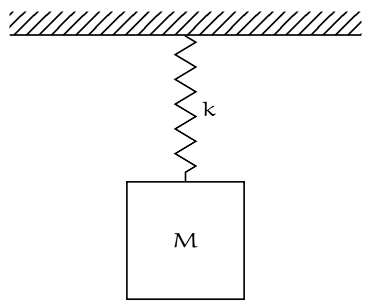


Figure 27: A block of mass M connected be a spring with spring constant k.

In the x direction, we will have no movement; thus, the only direction of movement is in the y direction. In the y direction, according to Newton's second law, we have a simple harmonic oscillator.

$$-ky = m\ddot{y} \Rightarrow y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$. Prior to the block being released, the velocity is 0; that is, $\dot{y}(0) = 0$. Since the block is displaced 4 inches, our initial condition for displacement is $y(0) = -4$. Using the initial conditions, we have the following

$$\begin{aligned} \dot{y}(0) &= B\omega = 0 \\ y(0) &= A \quad = -4 \end{aligned}$$

Therefore, the equation is $y(t) = -4\cos(\omega t)$. When the block is displaced, it will oscillate with period

$$\omega t = 2\pi \Rightarrow t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

4. How does the gravitational force vary between two masses if distance is doubled? How does electrostatic force vary between two charged particles if the distance is doubled? Explain using both equations and physical applications.

The gravitational force between two objects is

$$F_g = \frac{Gm_1m_2}{r^2}.$$

Let's determine what happens when we double the distance. Let $r = 2r$. Then the denominator is $4r^2$. By doubling the distance between the masses, the force decreases by 4. Since the electrostatic force between two charged particles is

$$F_e = \frac{k|q_1q_2|}{d^2},$$

doubling the distance will still decrease the force by 4.

5. Given the figure 28, calculate the distance traveled by the ball being thrown off the monument.

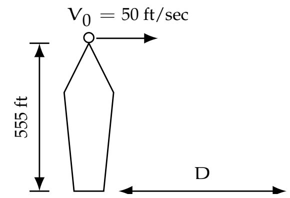


Figure 28: A ball being thrown from a monument.

For this problem, we will assume that there is no acceleration in the x direction, no air resistance, and $v_{y0} = 0$. Let $(x_0, y_0) = (0, 0)$ and $g = -9.8\ m/s^2$.

$$\begin{aligned} a_y(t) &= g \\ \int a_y(t)\ dt &= \int g\ dt \\ v_y(t) &= gt + v_{y0} \\ \int v_y(t)\ dt &= \int (gt + v_{y0})\ dt \\ y(t) &= \frac{g}{2}t^2 + v_{y0}t + y_0 \end{aligned}$$

By starting with acceleration in the y direction, we were able to integrate to obtain the position function for y. When $y(t) = -555$ ft, we will have the time it takes for the ball to strike the ground. We need to convert feet to meters. The conversion for is $\frac{1600\ m}{5280\ ft}$, so when $y(t) = \frac{-1850}{111}$.

$$y(t) = -4.9t_i^2 = \frac{-1850}{11} \Rightarrow t_i = 5.85857\ \text{s}$$

where t_i is the time of impact. Since we have no acceleration in the x direction, we have

$$\begin{aligned} a_x(t) &= 0 \\ v_x(t) &= v_{x0} \\ &= 50 \\ x(t) &= 50t + x_0 \end{aligned}$$

Let's convert 50 ft/sec to meters per sec. $\frac{50*1600\ m}{5280} = \frac{500}{33}\ m/s$.

$$x(t) = \frac{500}{33}t$$

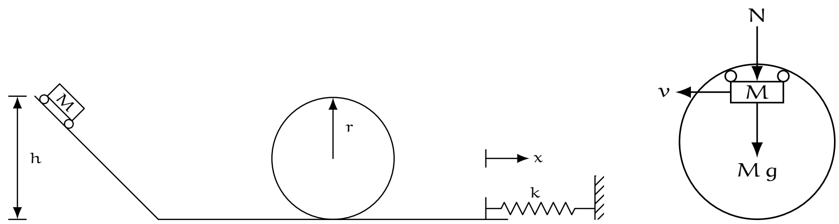
Finally, we can determine the distance the ball traveled from $x(t_i)$.

$$x(5.85857) = 88.7662\ m$$

6. A spaceship is accelerating at 1000 m/s^2. How much force is required from the backthrusters to completely stop the spaceship?

   Let M be the mass of the spaceship. By Newton's second law, we have that $F_s = 1000M\ Kg \cdot m/s^2$. Therefore, the force required to stop the spaceship is $1000M\ Kg \cdot m/s^2$. If spaceship was moving a some velocity, v_0, it would take a force larger than $1000M\ Kg \cdot m/s^2$ to stop the spacecraft.

7. Find h such that the car will make it around the loop without falling. Find the x that occurs when the car impacts the spring.



(a) Prior to being released

(b) At the apex of the loop.

Figure 29: Cart moving through a loop from an inclined starting position.

When the cart is moving through the loop, we have centripetal acceleration; that is, $a = \frac{v^2}{r} = \ddot{y}$. Now let's used Newton's second law.

$$\begin{aligned} M\ddot{y} &= N + Mg \\ \frac{Mv^2}{r} &= N + Mg \end{aligned}$$

What is the minimum speed that the cart must travel to not fall off? The minimum speed occurs when $N = 0$ so $v^2 = gr$ which is also the minimum kinetic energy since

$$\frac{1}{2}Mv^2 = \frac{1}{2}Mgr.$$

We can now use the Conservation of Energy. The initial kinetic energy of the system is zero and the initial potential energy is Mgh.

$$\begin{aligned} Mgh_i &= Mgh_f + \frac{1}{2}Mv_f^2 \\ Mgh &= 2Mgr + \frac{1}{2}Mgr \end{aligned}$$

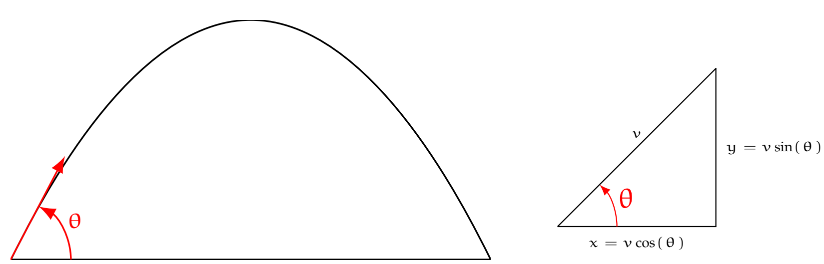
$$h = \frac{5}{2}r$$

The cart will make it around the loop when $h \geqslant \frac{5}{2}r$. For the second part of the question, recall that the energy of a compressed spring is $\frac{1}{2}kx^2$. Suppose the spring is located at $h = 0$. Then by the Conservation of Energy, we have

$$Mgh = \frac{1}{2}kx^2.$$

Since $h \geqslant \frac{5}{2}r$, we have that $x \geqslant \sqrt{\frac{5Mgr}{k}}$.

8. What angle will give the maximum range for a projectile neglecting air resistance? What would happen if air resistance occurred?



(a) The flight path of a projectile.

(b) Components of the velocity vector.

Figure 30: Projectile motion

We will assume the only acceleration is in the y direction and it is gravity, $\mathbf{a} = -9.8\ m/s^2$.

$$a_x = 0$$

$$\begin{aligned}
v_x &= \int 0 \, dt \\
&= v_{0x} \\
x(t) &= \int v_{0x} \, dt \\
&= v_{0x} t + x_0 \qquad (16) \\
a_y &= -9.8 \\
v_y &= -9.8 \int dt \\
&= -9.8t + v_{0y} \\
y(t) &= -4.9t^2 + v_{0y} t + y_0 \qquad (17)
\end{aligned}$$

We now have our equations of motion, equations (16) and (17). Let $(x_0, y_0) = (0, 0)$. Then equations (16) and (17) can be written as

$$\begin{aligned}
x(t) &= v_{0x} t \\
&= v \cos(\theta) t \\
y(t) &= v \sin(\theta) t - 4.9t^2
\end{aligned}$$

We need to determine the time it takes for $y = 0$ or when the projectile impacts the ground. Distance is simply the rate times the time, $d = r \cdot t$, where $r = v \cos(\theta)$ and

$$0 = v \sin(\theta) - 4.9t^2 \Rightarrow t = \frac{v \sin(\theta)}{4.9}.$$

Thus, distance is now a function of the angle, $d(\theta) = rt$. To maximize the range, we need to find the maximum θ. Let's take the derivative of d and set it equal to zero.

$$\begin{aligned}
d' = \frac{v^2}{4.9}(\cos^2(\theta) - \sin^2(\theta)) = 0 \\
\tan(\theta) = 1
\end{aligned}$$

A projectile being fired only occurs when $\theta \in \left[0, \frac{\pi}{2}\right]$ since we can always change our coordinates; therefore, $\theta = \frac{\pi}{4}$. If we consider air resistance, an angle less than $\frac{\pi}{4}$ will produce a maximum range. This is due to the fact that a lower trajectory reduces the time and distance over which the drag force due to air is acting. This is the basic understanding, but this isn't entirely true. An article, from 1997, by Dr.s Richard Price and Joseph Romano, has determined that this isn't always the case. The article can be found at `http://www.physics.rutgers.edu/~zapolsky/381/aim.pdf`. It goes on to say that drag is proportional the nth power of velocity, drag $\propto v^n$. The critical value for n is $n_{crit} \approx 3.5$. For $n > n_{crit}$, angles greater than $\frac{\pi}{4}$ can achieve a max range.

9. If a piece of paper is put on a full glass of water and inverted, what happens? Why?

   By paper, I assume we mean something along the lines of an index card. The water will stay in the cup. This occurs because we have a higher air pressure pushing against the index card on the bottom of the cup. In the cup, there may be a small pocket of air, but it is of a lower air pressure. The force from atmospheric pressure holds the index card up, and the lower pressure in the glass prevents the waters weight from pushing the card down.

10. Given a hollow and a solid cylinder of equal masses that are placed on an inclined plane with both cylinders having equal radii. Which cylinder will reach the bottom of the plane first?

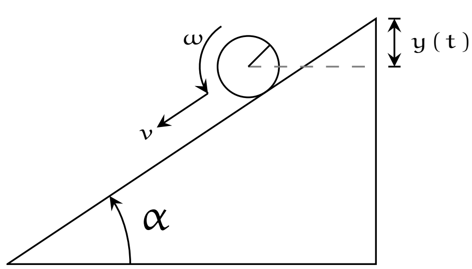


Figure 31: A cylinder rolling down and inclined plane.

The kinetic energy of the cylinder is the energy of the translational motion plus the rotational energy.

$$\mathsf{KE} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where I is the moment of inertia and ω is the angular velocity. Recall that the relation between angular and translational velocity is $v = \omega \cdot R$. Since a rotation through an angle θ causes the cylinder to travel, a distance is traveled which we refer to has arc length, $s = \theta \cdot R$. Then $\frac{ds}{dt} = v$ and $\frac{d\theta}{dt} = \omega$. Let's use the Conservation of Energy. Initial we will have no kinetic energy; therefore, we have

$$mgy(t) = \frac{1}{2}mv^2 + \frac{1}{2}\omega^2 = \frac{v^2}{2}\Big(m + I\frac{1}{R^2}\Big).$$

By solving for velocity, we have

$$v^2 = \frac{2gy(t)}{1 + I\frac{1}{mR^2}}.$$

The vertical velocity is $\frac{dy}{dt}$ where $y = s\sin(\alpha)$, and the translational velocity is $\frac{dy}{dt} = v\sin(\alpha)$. Then

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I\frac{1}{mR^2}}}\sqrt{y(t)}\sin(\alpha)$$

where $y(0) = 0$. We know have a differential equation to solve.

$$y(t) = \left[\sqrt{\frac{2g}{1 + \frac{I}{mR^2}}}\frac{\sin(\alpha)t}{2} + C\right]^2$$

At $y(0)$, $C = 0$.

$$y(t) = \frac{g\sin^2(\alpha)t^2}{2(1 + \frac{I}{mR^2})}$$

Suppose the height of the cylinder is h. Then the time required to roll down the incline plane is

$$t = \frac{\sqrt{2(1 + \frac{I}{mr^2})}}{\sqrt{g}\sin(\alpha)}.$$

Then the cylinder with the smallest $\frac{I}{mR^2}$ reach the bottom first. Let's assume the the cylinder has constant density, the cylinder as length or height ℓ if it is set on, and the cylinder has radius R. Since the cylinder has constant density, $\rho = 1$. We can calculate the moment inertia by

$$I = \iiint_D r^2 \, dm.$$

Since $\rho = \frac{m}{V}$, $dm = \rho \, dV$.

$$\begin{aligned} I_s &= \int_0^h \int_0^{2\pi} \int_0^R r^2 r \, dr d\theta dx \\ &= \frac{R^4 \pi h}{2} \end{aligned}$$

We now need to know the mass of the solid cylinder.

$$\begin{aligned} m_s &= \int_0^h \int_0^{2\pi} \int_0^R r \, dr d\theta dx \\ &= R^2 \pi h \end{aligned}$$

Then

$$\frac{I_s}{m_s r^2} = \frac{R^4 \pi h}{2R^2 \pi h R^2} = \frac{1}{2}.$$

For the hollow cylinder, let the inner radius be a where $0 < a < R$.

$$I_h = \int_0^h \int_0^{2\pi} \int_a^R r^2 r \, dr d\theta dx$$

$$= \frac{\pi h}{2}(R^4 - a^4)$$

$$m_h = \int_0^h \int_0^{2\pi} \int_0^R r \, dr d\theta dx$$

$$= \pi h(R^2 - a^2)$$

Then

$$\frac{I_h}{m_h r^2} = \frac{(R^4 - a^4)\pi h}{2R^2(R^2 - a^2)\pi h} = \frac{1}{2}\left(1 + \left(\frac{a}{r^2}\right)^2\right).$$

Since $a \neq 0$, $\frac{I_h}{m_h r^2} > \frac{I_s}{m_s r^2}$. Therefore, the solid cylinder will reach the bottom first.

11. In figure 32, find the position of the electron when it hits the screen. Will it hit the screen? What two variables can you change to determine where the electron will hit? (Assume that d and L are fixed)

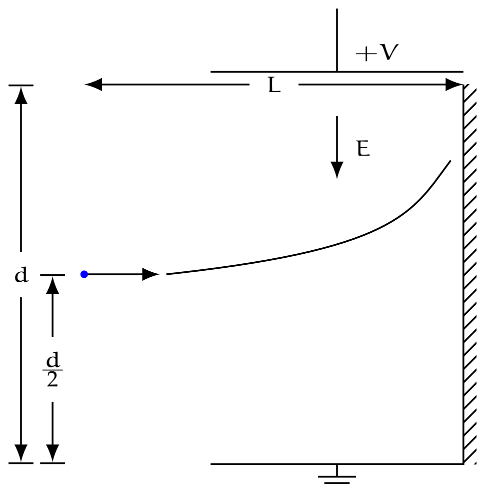


Figure 32: An electron from a cathode ray tube TV moving towards the screen.

Let's assume the electron has mass m. Since the field E is downward and the electron is negative, there will be an upwards force on the electron. Since $d = v_{0x}t$, we have that $t = \frac{d}{v_{0x}}$ but $t = \frac{L}{v_{0x}}$. The upward force on the electron is $F_e = eE$. By Newton's second law,

$$F_e = ma \Rightarrow a = \frac{eE}{m}.$$

When the electron enters the field, $v_{0y} = 0$ and $y_0 = 0$. Then $y(t) = a\frac{t^2}{2} = \frac{eEl^2}{2mv_{0x}^2}$. This is the distance above $\frac{d}{2}$ where the electron strikes. If $\frac{eEl^2}{2mv_{0x}^2} > \frac{d}{2}$, then $\frac{eEl^2}{mv_{0x}^2} > d$ which is above the screen so the electron misses. Suppose E is uniform. Then $E = \frac{+V}{d}$ and

$$y(t) = \frac{e(+V)L^2}{2mdv_{0x}^2}. \tag{18}$$

In equation (18), L, m, and d are fixed. By changing the top plate potential, $+V$, and the velocity of the electron, we can alter the the location of the electron striking the screen.

12. Given the following data from a projectile, find the height of the parabola when $t = 4$.

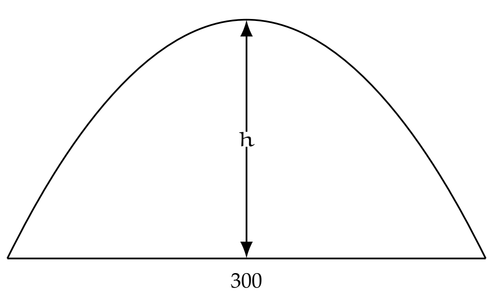


Figure 33: The flight of the projectile in 4 seconds.

We will be assuming there is no air resistance. Let the initial position be $(0,0)$ and the final position be $(300,0)$. Acceleration due to gravity is $g = -9.8\ m/s^2$.

$$
\begin{aligned}
y(t) &= y_0 + v_{0y}t + g\frac{t^2}{2} \\
y(4) &= 4v_{0y} - 4.9 \cdot 16 \quad &= 0 \\
v_{0y} &= 19.6 \\
y'(t) &= 19.6 - 9.8t &= 0 \\
t &= 2
\end{aligned}
$$

When $t = 2$, the projectile will reach its maximum height.

$$y_{max} = 19.6\ m$$

Thus, the height of the parabola is $19.6\ m$.

13. List and discuss Newton's Laws of Motion.

    Newton's first law: When viewed in an inertial reference frame, an object either remains at rest or moves at a constant velocity, unless acted upon by an external force.

    Newton's second law: $\sum \mathbf{F} = m\mathbf{a}$

    Newton's third law: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

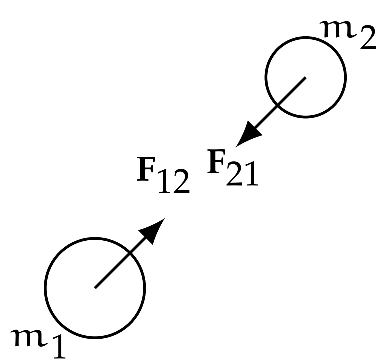


Figure 34: Diagram of Newton's third law.

From Newton's third law, we have $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

14. A bullet with a mass of 10 g and a velocity of 1000 m/s embeds in a wooden block with a mass of 1000 g suspended by a rope. How high will the block swing in the vertical direction.

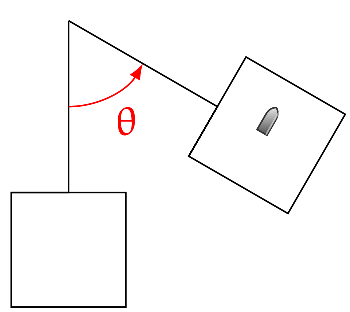


Figure 35: Ballistic pendulum

Since the bullet embeds in the block, we have inelastic collision. At t_0, the bullet is moving at 1000 m/s with a mass of 10 g, the block has $v_{0p} = 0$, and vertical height $h = 0$. Let the final height and velocity be h_f and v_{fp}, respectively.

We will assume there is no air resistance acting on the bullet. By Conservation of Momentum, we have $p_{1i} + p_{2i} = p_f$. Let $p_1 = m_b v_{0b}$ and $p_2 = m_p v_{0p}$. Since the bullet embeds into the block, $p_f = (m_b + m_p)v_f$. We can now solve for the final velocity.

$$v_f = \frac{1000}{101} m/s$$

For the Conservation of Energy, we will examine the system immediately after impact and when the pendulum reaches its maximum height. Immediately after impact, the pendulum has no potential energy. At maximum height, the pendulum has no kinetic energy. By the Conservation of Energy, we have $KE_{ib} + PE_{ib} = KE_{fp} + PE_{fp}$.

$$\frac{1}{2}(m_b + m_p)v_f^2 = (m_b + m_p)gh$$

where g is gravity and h is the maximum height. The only unknown is h; therefore, we can solve for h.

$$h = \frac{1}{2g}v_f^2 = \frac{1}{2 \cdot 9.8}\left(\frac{1000}{101}\right)^2 m$$

15. Given the following rocket sled with initial velocity equal to v_0, find the total distance the sled travels.

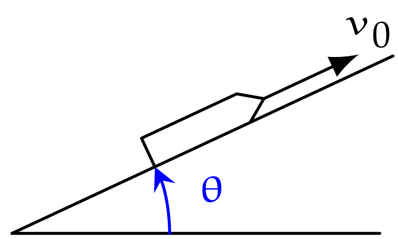


Figure 36: Rocket sled

We will assume the sled has mass m and that there is no air resistance. At $t = 0$, the sled is at rest at $0, 0$. There will only be acceleration in the y direction, gravity. The velocity in the x and y direction are $v_x = v_0 \cos(\theta)$ and $v_y = v_0 \sin(\theta)$, respectively. We have integrated from acceleration and previously shown that

$$x(t) = x_0 + v_x t + a_x \frac{t^2}{2}$$
$$y(t) = y_0 + v_y t + a_y \frac{t^2}{2}$$

Using what we know, we have

$$x(t) = v_0 \cos(\theta)t$$
$$y(t) = v_0 \sin(\theta)t - 4.9t^2$$

When $y = 0$, the sled will impact the ground.

$$t_i = \frac{v_0 \sin(\theta)}{4.9} s$$

Therefore, the total distance the sled travels is

$$x(t_i) = \frac{v_0^2 \cos(\theta)\sin(\theta)}{4.9} m.$$

16. Find the time it takes to hit the ground.

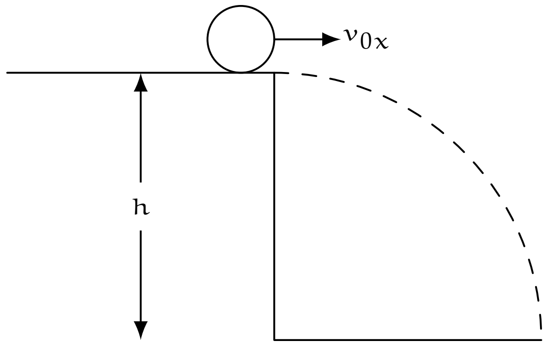


Figure 37: A ball with initial velocity v_{0x}.

Again, we will assume we only have acceleration due to gravity in the y direction and there is no air resistance. Let the initial position of the ball be $(0, 0)$. The initial velocity in the y direction is $v_{0y} = 0$. Thus, we have

$$\begin{aligned} x(t) &= v_{0x}t \\ y(t) &= a_y\frac{t^2}{2} \\ &= -4.9t^2 \end{aligned}$$

The ball has to drop h m for it to strike the ground.

$$t = \sqrt{\frac{h}{4.9}}\,s$$

17. A man has a velocity $v_0 = 3$ m/s and starts z m behind a bus with initial velocity $v_{0b} = 0$ at $t = 0$. The bus accelerates with acceleration $a = 1$ m/s^2. Does the man catch the bus?

    We will assume that the man is not accelerating and air resistance is negligible. Our equations of motion for the man and the bus are then

$$\begin{aligned} m(t) &= m_0 + v_0t + a_m\frac{t^2}{2} \\ &= 3t \\ b(t) &= b_0 + v_{0b} + a_b\frac{t^2}{2} \\ &= z + \frac{t^2}{2} \end{aligned}$$

    If the man is to catch the bus, the distance he travels must equal the distance the bus travels; that is, $m(t) = b(t)$.

$$\frac{t^2}{2} - 3t + z = 0$$

    By the quadratic equation, we have

$$t = 3 \pm \sqrt{9 - 2z}.$$

    So $\frac{9}{2} \geqslant z$. In order for the man to catch the bus and barring the bus starting behind the man, $0 < z \leqslant \frac{9}{2}$.

18. A block of mass M_1 is attached by string to a support. The block is raised to a height of H and released. It then strikes a block of mass M_2 on a frictionless surface. Find the velocity of the block M_2, assuming a totally elastic collision.

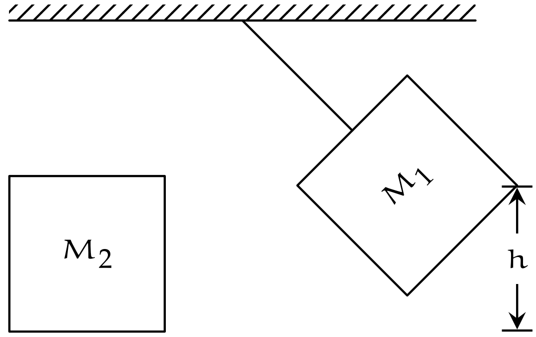


Figure 38: M_1 positioned at a height of h.

With elastic collision, we have the Conservation of Momentum and Kinetic Energy. The block of mass M_2 has initial velocity of 0. Therefore, by the Conservation of Momentum and KE, we have

$$\begin{aligned} M_1 v_{1i} + M_2 v_{2i} &= M_1 v_{1f} + M_2 v_{2f} \\ M_1(v_{1i} - v_{1f}) &= M_2 v_{2f} \qquad (19) \\ M_1 v_{1i}^2 + M_2 v_{2i}^2 &= M_1 v_{1f}^2 + M_2 v_{2f}^2 \\ M_1(v_{1i}^2 - v_{1f}^2) &= M_2 v_{2f}^2 \qquad (20) \end{aligned}$$

The difference of squares can be factor to $v_{1i}^2 - v_{1f}^2 = (v_{1i} - v_{1f})(v_{1i} + v_{1f})$. Let's divide equation (20) by equation (19).

$$\frac{v_{1i}^2 - v_{1f}^2}{v_{1i} - v_{1f}} = v_{2f}$$

So the final velocity of the second mass after collision is $v_{2f} = v_{1i} + v_{1f}$.

19. Given the following setup, why will only one ball swing out?

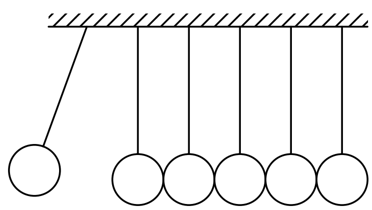


Figure 39: Newton's Cradle

First, assume each ball has mass m. At rest, $v_0 = 0$. Also, suppose n balls fly out instead of one. Then by the Conservation of Momentum $mv = nmu$, and by the Conservation of Energy, $\frac{1}{2}mv^2 = \frac{n}{2}mu^2$. We have two equations with two unknowns.

$$\begin{aligned} v &= nu \\ v^2 &= nu^2 \end{aligned}$$

It is easy to that the only solution is when $u = v$ and $n = 1$. Therefore, we have reached a contradiction and only one ball will fly out.

20. A 10 g bullet with velocity of 1000 m/s strikes a 100 g block at rest. What is their combined velocity? Can you work the problem using the principle of Conservation of Momentum? Energy?

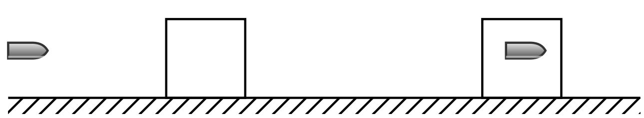


Figure 40: A bullet embedding into a block.

We will assume the block is on a frictionless surface. This is a case of inelastic collision. Let m be the mass of the bullet and M the mass of the block. By Conservation of Momentum, we have

$$mv + MV = (m + M)v_f.$$

Since the block is initially at rest, our equation is

$$10 \cdot 1000 = 10 \cdot 11v_f \Rightarrow v_f = \frac{1000}{11}\,\mathrm{m/s}.$$

Since we have inelastic, we can't use Conservation of Energy.

21. What is momentum and how is it related to Newton's second law?

    Linear momentum is the product of mass times velocity of an object, $\mathbf{p} = m\mathbf{v}$. Newton's second law is $\sum \mathbf{F} = m\mathbf{a}$ but acceleration is the derivative of velocity.

    $$m\mathbf{a} = \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}\mathbf{p} = \sum \mathbf{F}$$

22. What is the maximum altitude reached when $W = 100$ lb and $v_0 = 100$ ft/s.

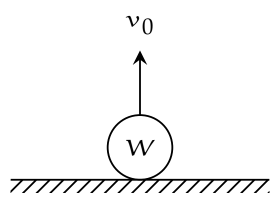


Figure 41: An object moving up with velocity of v_0.

    Here we can use the Conservation of Energy. Initially the potential energy will be zero, and when the object reaches its maximum height, the kinetic energy will be zero. Thus, we have

    $$\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g} = \frac{100^2}{64}\,\mathrm{ft}.$$

23. A mass is dropped from a height of H. What is the velocity of the mass just before it hits the ground?

    For this problem, we can use the Conservation of Energy. Initially the kinetic energy is zero and the final potential energy is zero.

    $$mgh = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gh}$$

24. Consider the following pendulum system:

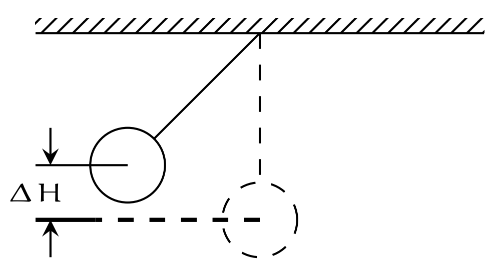


Figure 42: Simple pendulum

    (a) If the bob is release from rest, what is maximum velocity attained?

        We will need to assume there is no air resistance and the bob has mass m. Since the bob is released from rest, the initial potential energy is zero. At the bob's maximum height, the kinetic energy is also zero. By the Conservation of Energy, we have

        $$\frac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh}.$$

(b) What assumptions are made in the answer?

The assumptions we made were that air resistance is negligible and the bob has mass m.

(c) What difference does it make if the system is in vacuum?

By assuming we have no air resistance, we are assuming it is in a vacuum. Therefore, there would no difference.

(d) Suppose a second mass m was suspended at the lowest point, what would be the velocities of both masses after the collision?

Recall that for elastic collisions we can use the Conservation of Energy and Momentum. At $t = 0$, the suspended bob has no potential energy, and the bob hanging at equilibrium has kinetic energy. Let the suspended bob have mass m_1 and the bob hanging at equilibruim have mass m_2. Then

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$m_1(v_{1i} - v_{2i}) = m_2 v_{2f} \tag{21}$$
$$m_1 v_{1i}^2 + m_2 v_{2i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2$$
$$m_1(v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2 \tag{22}$$

We can now divide equation (22) by equation (21) and solve for the final velocities.

$$v_{1i} + v_{1f} = v_{2f}$$

From item 24(a), we have that the bobs initial velocity is $v_{1i} = \sqrt{2gh}$. Thus, the final velocities are

$$v_{1f} = v_{2f} - \sqrt{2gh}$$
$$v_{2f} = v_{1f} + \sqrt{2gh}$$

(e) What if the collision was non-elastic?

For inelastic collision, our Conservation equations become

$$m_1 v_{1i} = (m_1 + m_2) v_f$$
$$m_1 v_{1i}^2 = (m_1 + m_2) v_f^2$$

Then we have

$$v_{1i} = v_f = \sqrt{2gh}.$$

25. Given a spring with displacement force $F = e^x$, determine the energy required to move the block three units.

The force of the spring opposes F so the sum of forces is $e^x - kx$. Then the work is

$$W = \int_0^3 (e^x - kx)dx = e^x - \frac{kx^2}{2}\Big|_0^3 = e^3 - \frac{9k}{2} - 1.$$

26. Define the following:

(a) **Work:** a force is to do work when it acts on a body, and there is displacement of the point of application in the direction of the force.

(b) **Energy:** energy is a fundamental property of any physical object or system of objects which is used to describe and predict its interaction with other objects.

(c) **Power:** power is defined as the amount of energy consumed per unit time.

2.1 Wave Properties and Oscillations

1. What is the oscillation period of a simple harmonic oscillator?

A simple harmonic oscillator can be defined as a mass on a spring. Then by Hooke's law and Newton's second law, we have

$$m\ddot{x} = -kx.$$

The solution to the following ODE is $x(t) = A\cos(\omega t) + B\sin(\omega t)$ where $\omega = \sqrt{\frac{k}{m}}$. Let t be the period of the oscillations. Then

$$\omega t = 2\pi \Rightarrow t = \frac{2\pi}{\omega}$$

so $t = 2\pi\sqrt{\frac{m}{k}}$.

2. Derive the period of a simple pendulum?

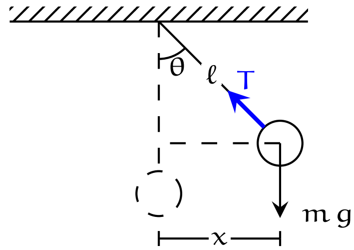


Figure 43: A simple pendulum with a small angle approximation.

For this problem, we will assume there is no air resistance and a small angle approximation which will become apparent later. From the figure 43, we see that $x = \ell\sin(\theta)$ and $y = \ell(1-\cos(\theta))$. We can use Newton's second law for both the x and y motion.

$$\begin{aligned} m\ddot{x} &= F_x \\ &= -T\sin(\theta) \\ m\ddot{y} &= F_y \\ &= T\cos(\theta) - mg \end{aligned}$$

Now we need to find $\ddot{x}$ and $\ddot{y}$.

$$\begin{aligned} \dot{x} &= \ell\dot{\theta}\cos(\theta) \\ \ddot{x} &= \ell\ddot{\theta}\cos(\theta) - \ell\dot{\theta}^2\sin(\theta) \\ \dot{y} &= \ell\dot{\theta}\sin(\theta) \\ \ddot{y} &= \ell\ddot{\theta}\sin(\theta) + \ell\dot{\theta}^2\cos(\theta) \end{aligned}$$

We can now construct our equations of motion.

$$m\ell(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)) = -T\sin(\theta) \tag{23}$$

$$m\ell(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)) = T\cos(\theta) - mg \tag{24}$$

We can now multiple equation (23) by cosin, equation (24) by sine, and then add equations (23) and (24) together.

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0 \tag{25}$$

Equation (25) is a nonlinear equation. Since we are assuming small angle approximations, $\theta \ll 1$, let's look at the power series for sine.

$$\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Therefore, $\sin(\theta) \approx \theta$ and equation (25) is

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0.$$

The solution to ordinary differential equation is $\Theta(t) = A\cos(\omega t) + B\sin(\omega t)$ where $\omega = \sqrt{\frac{g}{\ell}}$. Then the period is $\omega t = 2\pi$ so $t = 2\pi\sqrt{\frac{\ell}{g}}$.

3. Explain the difference between light and radio waves.

   The main difference is waves length. Radio waves start around 1 cm whereas visible light is between 380 nm and 740 nm. Additionally, radio waves are created by the acceleration of electrons in radio antenna, and light waves are created by the oscillation of the electrons within the atoms.

4. What is the relationship between frequency and wavelenght?

   Frequency is $f = \frac{1}{t}$ where t is the period, and the wave length λ is defined as $\lambda = \frac{c}{f}$ and c is the speed of light.

5. What is the frequency of a 5 Å wavelength emission?

   Since 5 Å is a wavelength, we know from item 4 that $5 = \frac{c}{f}$.

$$f = \frac{c}{5\text{Å}} = \frac{3 \times 10^8 \text{ m/s}}{5 \times 10^{-10} \text{ m}} = \frac{3}{500\text{s}}$$

   Therefore, the frequency is $\frac{3}{500}$ Hz.

6. Contrast light and sound waves. How do they propagate energy? Do they travel at different speeds in different media? Why?

   Sound requires a meduim to propage. Progation occurs from vibration which produces a mechanical wave of pressure and displacement through the medium. Light doesn't require a medium. Propation occurs from the vibration of an electric charge. Additionally, light travels exremely fast, $c = 3 \times 10^8$ m/s. Light waves are electromagnetic waves consisting of varying electric and magnetic fields. Yes, when waves (sound or light) enter a medium, they may slow down or speed up.

7. Define Doppler Shift.

   The Doppler Shift is a change in frequency due to the Doppler Effect. Therefore, we need to understand what the Doppler Effect is. The Doppler Effect is an increase (or decrease) in the frequency of sound, light, or other waves as the source and the observer move toward (or away from) each other. The effect causes the sudden change in pitch noticeable in a passing siren, as well as the redshift seen by astromers.

8. Arrange the following electro-magnetic radiation in order of increasing frequency: X-rays, gamma rays, infrared radiation, and visible light.

   Correct order is: gamma rays, X-rays, visible light, and infrared radiation

9. State Snell's Law.

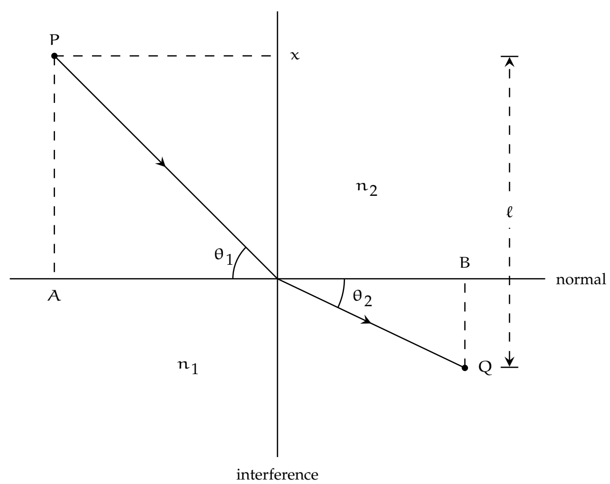


Figure 44: A ray refracted through a medium.

Let n_1 be the first medium and n_2 be the second medium. Let $v_1 = \frac{c}{n_1}$ and $v_2 = \frac{c}{n_2}$ where v_i is the speed of light in the ith medium. Then $n_1, n_2 \geqslant 1$. The time taken to move from P to Q is

$$T = \frac{d_1}{v_1} + \frac{d_2}{v_2}.$$

By the distance formula, $d_1 = \sqrt{A^2 + x^2}$ and $d_2 = \sqrt{B^2 + (\ell - x)^2}$. Thus,

$$T = \frac{\sqrt{A^2 + x^2}}{v_1} + \frac{\sqrt{B^2 + (\ell - x)^2}}{v_2}.$$

Next, let's minimize the transient time; that is, we need to set $\frac{dT}{dx} = 0$.

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{A^2 + x^2}} - \frac{\ell - x}{v_2\sqrt{B^2 + (\ell - x)^2}} = 0$$

Then $\sin(\theta_1) = \frac{x}{\sqrt{A^2+x^2}}$ and $\sin(\theta_2) = \frac{\ell - x}{\sqrt{B^2+(\ell-x)^2}}$.

$$\begin{aligned}\frac{dT}{dx} &= \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2} = 0\\ \Rightarrow \frac{\sin(\theta_1)}{v_1} &= \frac{\sin(\theta_2)}{v_2}\\ \Rightarrow \frac{\sin(\theta_1)}{\sin(\theta_2)} &= \frac{n_2}{n_1}\end{aligned} \tag{26}$$

where equation (26) is Snell's law.

10. Draw a picture of a fish in water and show where you would throw a spear to hit it. Where does the fish appear? Why? How do n and C relate to refraction?

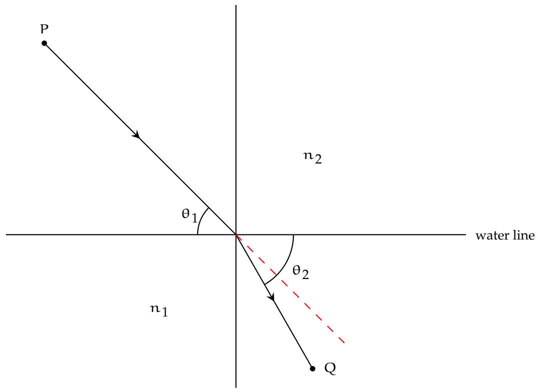
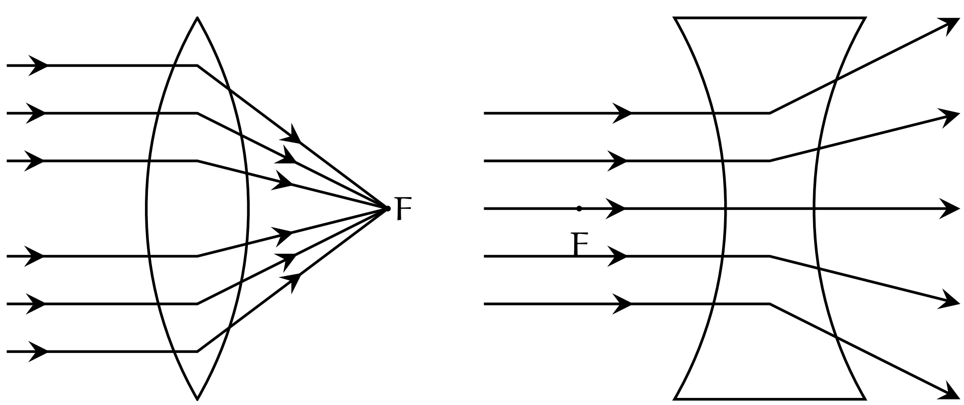


Figure 45: A fish under water.

Let line PO where O is the origin be the line of site. Then Q is where the fish actually is and the red dashed line is where the fish would appear to be. Therefore, we will want to throw the spear along the solid black line. This occurs because we are going from the medium of air into water. The speed of light through the medium is $v = \frac{c}{n}$ where n is the index of refraction. Thus, $n = \frac{c}{v}$ where c is the speed of light.

11. Draw a concave and convex lens. What effect would each have on paraxial rays? Why?



(a) Convex lens

(b) Concave lens

Figure 46: Paraxial rays moving through convex and concave lenses.

With a convex lens, the paraxial rays would be refracted towards the focus on the opposing side, see figure 46a. For the concave lencs, the paraxial rays would refracted along the angle of incoming sides focus, see figure 46b. This occurs because of Fermat's principle or the principle of least time. Fermat's principle says that the path taken between two points by a ray of light is the path that can be traversed in the least time. Snell's law is the solution to Fermat's principle. Therefore, concave and convex lenses do what they do based on the geometry of their lens grind, in conjuction with Snell's law.

12. What does a diffraction grating do, and what is it used for? Are there circumstances under which light must be considered a particle? When?

    Diffraction grating separates light of different wave lengths. Think of a prism. Diffraction gratings are used in monochromators, spectrometers, and lasers to name a few. Ligth should be considered a particle in the photoelectric effect. More can be read on this topic here `http://en.wikipedia.org/wiki/Photoelectric_effect`