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1 Gravity

Newton's Law of Gravitation is $F = -\frac{Gm_1m_2}{r^2}$ and gravitational potential is $\mathcal{V} = -\frac{Gm_1m_2}{r}$. From figure 1.0.1, we can write the differential potential energy due to a differential volume, $d\forall$. Let $dv = -\frac{GMm}{r}$ and $dM = \rho\, d\forall$ where ρ is constant. Then

$$\begin{aligned}
V &= -\int_\forall \frac{GdMm}{r} \\
&= -Gm\int_\forall \frac{\rho\, d\forall}{r} \\
&= -Gm\iiint \frac{\rho}{r}(r')^2\sin(\theta)\, d\theta\, d\varphi\, dr' \\
&= -\rho Gm\iiint \frac{(r')^2\sin(\theta)}{r}\, d\theta\, d\varphi\, dr'
\end{aligned}$$

The Law of Cosine for figure can be written as $r^2 = R^2 + r'^2 - 2Rr'\cos(\theta)$ so $r = \sqrt{R^2 + r'^2 - 2Rr'\cos(\theta)}$. Therefore, we can substitute r into our integral equation.

$$V = -\rho Gm\iiint \frac{(r')^2\sin(\theta)}{\sqrt{R^2 + r'^2 - 2Rr'\cos(\theta)}}\, d\theta\, d\varphi\, dr' \tag{1.0.1}$$

where the region of integration is $0 < r' < a$, $0 < \theta < \pi$, and $0 < \varphi < 2\pi$.

$$\begin{aligned}
V &= -\rho Gm\iiint \frac{(r')^2\sin(\theta)}{R\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}}\, d\theta\, d\varphi\, dr' \\
&= -2\pi\rho Gm\iint \frac{(r')^2\sin(\theta)}{R\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}}\, d\theta\, dr'
\end{aligned} \tag{1.0.2}$$

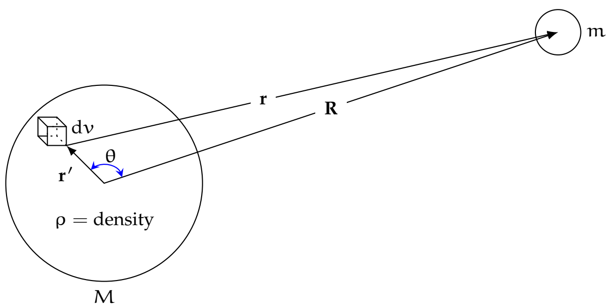


Figure 1.0.1: The gravitational potential of two masses M and m.

From equation (1.0.2), let $u = \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}$ so $du = \frac{r'\sin(\theta)}{R\sqrt{1+\left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}}\,d\theta$.

$$d\theta = \frac{R\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}}{r'\sin(\theta)}\,dr$$

Making the substitution with u and du, we now have

$$\begin{aligned}
V &= -2\pi\rho G m \iint r'\,du\,dr' \\
&= -2\pi\rho G m \int r'\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}\Big|_0^{\pi}\,dr' \\
&= -2\pi\rho G m \int r'\left[\sqrt{1 + \left(\frac{r'}{R}\right)^2 + 2\frac{r'}{R}} - \sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}}\right] dr' \\
&= -2\pi\rho G m \int r'\left[\sqrt{\frac{(r'+R)^2}{R^2}} - \sqrt{\frac{(r'-R)^2}{R^2}}\right] dr' \\
&= -\frac{2\pi\rho G m}{R}\int r'\left[r' + R - \sqrt{(r'-R)^2}\right] dr' \qquad (1.0.3)
\end{aligned}$$

Since $0 < r' < a < R$, we can write $\sqrt{(r'-R)^2} = \sqrt{(R-r')^2} = R - r'$.

$$\begin{aligned}
V &= -\frac{2\pi\rho G m}{R}\int r'\big[r' + R - (R - r')\big]\,dr' \\
&= -\frac{4\pi\rho G m}{R}\int r'^2\,dr' \\
&= -\frac{4\pi a^3}{3}\frac{G m \rho}{R} \qquad (1.0.4)
\end{aligned}$$

The density, ρ, is defined as $\rho = \frac{M}{V}$ where the volume of a sphere is $V = \frac{4\pi r^3}{3}$. When we make this final substitution, we have the desired result, $V = -\frac{GmM}{R}$.

Now let's take the general case when $\mathbf{R} = \mathbf{x}$. That is, let's look at the gravitational potential for an arbitrary spheroid. Then

$$V(\mathbf{x}) = \frac{GMm}{x}\left[1 + \sum_{n=1}^{\infty} J_n\left(\frac{a_0}{x}\right)^n P_n(\cos\theta)\right] \qquad (1.0.5)$$

where a_0 is the mean radius of the body M, θ is the angular location of m, J_n is a constant (zonal harmonic), and P_n is the Legendre Polynomials of order "n". Recall that for the Law of Cosine, we had $r = \sqrt{R^2 + r'^2 - 2Rr'\cos(\theta)} = R\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}$. Then we have the generating function:

$$\frac{1}{\sqrt{1 + \left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos(\theta)}} = \sum_{n=0}^{\infty} J_n\left(\frac{a_0}{x}\right)^n P_n(\cos(\theta)). \qquad (1.0.6)$$

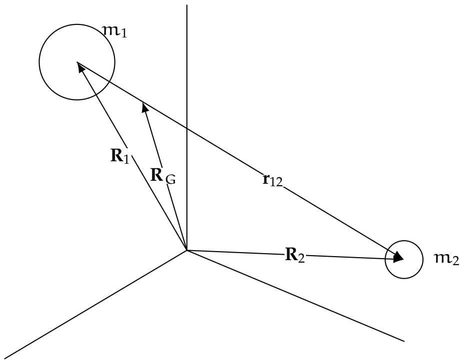


Figure 1.0.2: Vector location of masses m_1 and m_2 with relation to the center of gravity vector

Let M be the total of the above system. Then $\mathbf{R}_G$ is the sum of the moments divided by the total mass. That is,

$$\mathbf{R}_G = \frac{\sum m_i}{M} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2}.$$

The velocity of the center of gravity is simply the first derivative of $\mathbf{R}_G$.

$$\begin{aligned}\mathbf{V}_G &= \dot{\mathbf{R}}_G \\ &= \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2}\end{aligned} \tag{1.0.7}$$

and then $\mathbf{A}_G = \dot{\mathbf{V}}_G = \ddot{\mathbf{R}}_G$. By Newton's 2^{nd} of Motion, we have that

$$\begin{aligned}\mathbf{F}_1 &= m_1\ddot{\mathbf{R}}_1 \\ &= \frac{Gm_1m_2}{r_{12}^3}\mathbf{r}_{12} \\ &= \frac{Gm_1m_2}{r_{12}^2}\left(\frac{\mathbf{r}_{12}}{r_{12}}\right) \qquad (1.0.8)\\ \mathbf{F}_2 &= \frac{Gm_1m_2}{r_{12}^2}\left(-\frac{\mathbf{r}_{12}}{r_{12}}\right) \qquad (1.0.9)\end{aligned}$$

From figure 1.0.3, we know that $\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1$ so $\ddot{\mathbf{r}} = \ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1$. Note that $m_1\ddot{\mathbf{R}}_1 = \frac{Gm_2}{r^2}m_1\left(\frac{\mathbf{r}}{r}\right) \Rightarrow \ddot{\mathbf{R}}_1 = \frac{Gm_2}{r^3}\mathbf{r}$. Using Newton's Law of Universal Gravitation, we can write

$$\begin{aligned}\ddot{\mathbf{r}} &= -\frac{Gm_1}{r^3}\mathbf{r} - \frac{Gm_2}{r^3}\mathbf{r} \\ &= -\frac{G(m_1 + m_2)}{r^3}\mathbf{r}\end{aligned} \tag{1.0.10}$$

From equation (1.0.10), let $\mu = G(m_1 + m_2)$ be the gravitational parameter. In the case of a planet m_1 and a spacecraft m_2, m_2's mass is negligible so $\mu \approx Gm_1$. Now we can

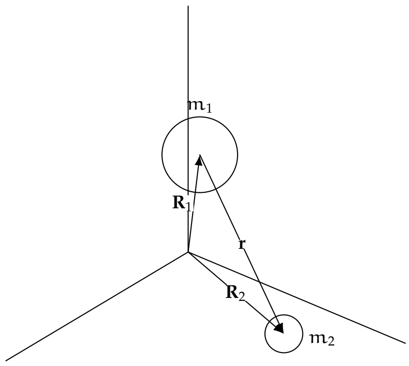


Figure 1.0.3: Two bodies in 3 space.

write the governing equation of motion:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

which is a nonlinear second order differential equation. Now let's look at the Conservation of Mechanical Energy of chapter 1.

$$\begin{aligned}\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} &= 0\\ \dot{\mathbf{r}}\cdot\left(\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r}\right) &= 0\\ \dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\mu}{r^3}\dot{\mathbf{r}}\cdot\mathbf{r} &= 0\end{aligned}$$

We claim that $\dot{\mathbf{r}}\cdot\mathbf{r} = \dot{r}r$.

$$\begin{aligned}\frac{d}{dt}(r^2) = 2r\dot{r} \text{ and } \frac{d}{dt}(\mathbf{r}\cdot\mathbf{r})\\ = 2\mathbf{r}\cdot\dot{\mathbf{r}}\end{aligned}$$

Since $r^2 = \mathbf{r}\cdot\mathbf{r}$, $\dot{\mathbf{r}}\cdot\mathbf{r} = \dot{r}r$ as was needed to be shown. Similarly, we can show that $\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\dot{r}^2)$ since $\frac{d}{dt}(\dot{r}^2) = 2\dot{r}\ddot{r} = \frac{d}{dt}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) = 2\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) + \frac{\mu}{r^3}\dot{r}r &= 0\\ \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) + \frac{\mu}{r^2}\dot{r} &= 0\\ \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2\right) - \mu\frac{d}{dt}\left(\frac{1}{r}\right) &= 0\\ \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 - \mu\frac{1}{r}\right) &= 0 \qquad (1.0.11)\end{aligned}$$

If we integrate both sides of equation (1.0.11), we end up with

$$\frac{\dot{r}^2}{2} - \frac{\mu}{r} = \frac{v^2}{2} - \frac{\mu}{r}$$

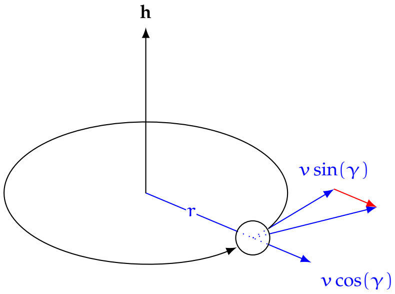


Figure 1.0.4: Angular momentum vector of a rotating body.

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} \tag{1.0.12}$$

where $\mathcal{E}$ is the energy which is constant.

$$\mathcal{E} = \begin{cases} \text{a closed orbit (ellipse),} & \text{if } \mathcal{E} < 0 \\ \text{an open orbit (hyperbola),} & \text{if } \mathcal{E} > 0 \\ \text{an escape trajectory (parabola),} & \text{if } \mathcal{E} = 0 \end{cases} \tag{1.0.13}$$

Now let's take the cross product of $\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = 0$ with $\mathbf{r}$.

$$\begin{aligned} \ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} &= 0 \\ \mathbf{r} \times \left(\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r}\right) &= 0 \\ \mathbf{r} \times \ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} \times \mathbf{r} &= 0 \\ \mathbf{r} \times \ddot{\mathbf{r}} &= 0 \end{aligned}$$

Here we claim that $\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \times \ddot{\mathbf{r}}$ which can be easily verified. Therefore,

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) &= 0 \\ \mathbf{r} \times \mathbf{v} &= \mathbf{h} \end{aligned} \tag{1.0.14}$$

where h is the angular momentum which is conserved. Additionally, by the definition of the cross product, we can write $\mathbf{h} = \mathbf{r} \times \mathbf{v} = rv\sin(\theta)\hat{\mathbf{n}}$ or $h = rv\sin(\theta)$.

Kepler's 2$^{\text{nd}}$ Law

A line joining a planet and the Sun sweeps out equal areas during equal time intervals (see figure 1.0.5).

For small time,

$$dA \approx \frac{1}{2}bh \qquad \text{(area of a triangle)}$$

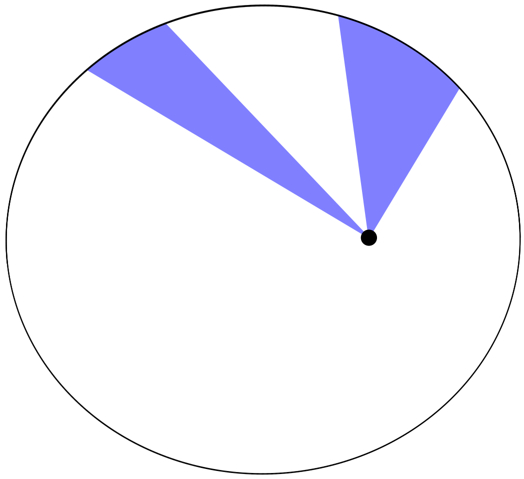


Figure 1.0.5: Equal areas being swept out in equal times.

$$
\begin{aligned}
&= \frac{1}{2}\mathbf{v}_{\perp}\mathbf{r}\,\mathrm{d}t \\
\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}\mathbf{v}_{\perp}\mathbf{r} \\
&= \frac{1}{2}rv\sin\theta \\
&= \frac{1}{2}|\mathbf{r}\times\mathbf{v}| \\
&= \frac{1}{2}\mathbf{h}
\end{aligned}
$$

Consider $\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$ crossed with $\mathbf{h}$.

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r^3}\mathbf{h}\times\mathbf{r} \tag{1.0.15}$$

First, let's write the LHS of equation (1.0.15) as $\ddot{\mathbf{r}}\times\mathbf{h} = \ddot{\mathbf{r}}\times\mathbf{h} + \dot{\mathbf{r}}\times\dot{\mathbf{h}} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}\times\mathbf{h})$. Next, let's write the RHS of equation (1.0.15) as $\mathbf{h}\times\mathbf{r} = \mathbf{r}\times\mathbf{v}\times\mathbf{r} = \mathbf{v}r^2 - \mathbf{r}(rv)$.

$$
\begin{aligned}
\frac{\mu}{r^3}\mathbf{h}\times\mathbf{r} &= \frac{\mu}{r}\mathbf{v} - \frac{v\mu}{r^2}\mathbf{r} \\
&= \frac{\mu}{r}\dot{\mathbf{r}} - \frac{\mu\dot{r}}{r^2}\mathbf{r} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\mathbf{r}}{r}\right)
\end{aligned} \tag{1.0.16}
$$

We can now write equation (1.0.15) as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}\times\mathbf{h}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\mathbf{r}}{r}\right) \\
\dot{\mathbf{r}}\times\mathbf{h} &= \mu\frac{\mathbf{r}}{r} + \mathbf{b}
\end{aligned} \tag{1.0.17}
$$

where $\mathbf{b}$ is a constant vector. Let $\mathbf{b} = \mu\mathbf{e}$ where $\mathbf{e}$ is the eccentricity vector which points in the direction of periapsis. Now let's dot $\mathbf{r}$ with the LHS of equation (1.0.17). For the LHS, we have

$$\begin{aligned} \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) &= \mathbf{h} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) \\ &= \mathbf{h} \cdot (\mathbf{r} \times \mathbf{v}) \\ &= h^2 \end{aligned} \tag{1.0.18}$$

Let ν be the angle between $\mathbf{r}$ and $\mathbf{e}$. Then our equation becomes

$$\begin{aligned} h^2 &= \mu\frac{\mathbf{r} \cdot \mathbf{r}}{r} + e\mathbf{r} \cdot \mathbf{e} \\ &= \mu r + \mu e r \cos(\nu) \\ r &= \frac{h^2}{\mu(1 + e\cos(\nu))} \end{aligned} \tag{1.0.19}$$

which is the trajectory equation.

2 Elliptical Orbit

Elliptical orbits are orbits in which the functional form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or can be written as a vector valued function of the form $\mathbf{r}(t) = a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j}$.

1. Conservation of Mechanic Energy: $\mathcal{E} = \dfrac{v^2}{2} - \dfrac{\mu}{r} < 0$
2. Conservation of Angular Momentum: $\mathbf{h} = \mathbf{r} \times \mathbf{v}$
3. The Trajectory Equation: $r = \dfrac{h^2}{\mu(1 + e\cos(\nu))}$

From the figure 2.0.1, we that

4. p is the semi-latus rectum,
5. the distance from the focus to a is r_p (radius at periapsis),
6. the distance from the origin to c is ae where e is the eccentricity, and
7. $0 < e = 1 - \frac{c}{a} < 1$.

We can then write $r = \frac{p}{1+e\cos(\nu)}$ and $p = a(1 - e^2)$. Moreover, $p = \frac{h^2}{\mu}$. At periapsis, $\nu = 0$, and at apoapsis, $\nu = \pi$ so

$$r_p = \frac{p}{1+e} = a(1-e) \tag{2.0.1}$$
$$r_a = \frac{p}{1-e} = a(1+e). \tag{2.0.2}$$

Now we can take the ratio of r_p and r_a.

$$\frac{r_p}{r_a} = \frac{1-e}{1+e}$$
$$e = \frac{r_a - r_p}{r_a + r_p} \tag{2.0.3}$$

Let's examine the specific energy at periapsis. Then $v = v_p$ and $r = r_p$. Recall that $h = rv\sin\theta$. At periapsis, the angle between r and v is $\theta = \frac{\pi}{2}$ so $h = rv_p \Rightarrow v_p = \frac{h}{r_p}$. Now we can write the specific energy as

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$$

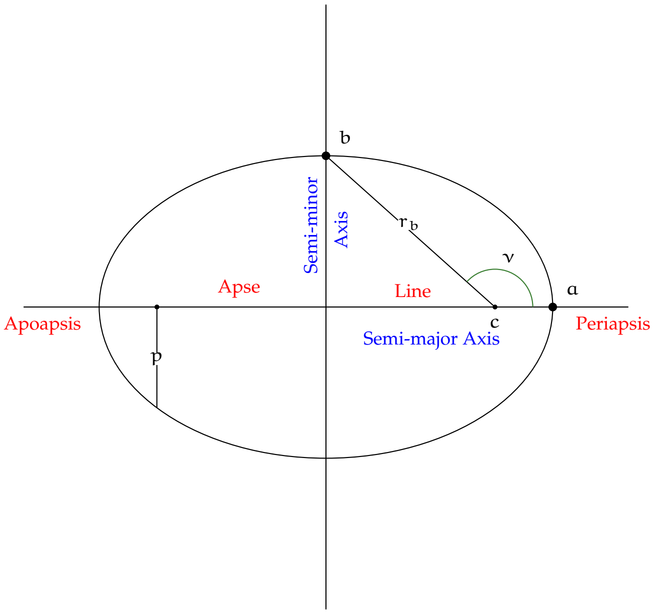


Figure 2.0.1: The geometry of an ellipse.

$$= \frac{v_p^2}{2} - \frac{\mu}{r_p}$$
$$= \frac{h^2}{2r_p^2} - \frac{\mu}{r_p} \tag{2.0.4}$$

Using equation (2.0.4) and the relation that $p = \frac{h^2}{\mu}$ at periapsis, we have $h^2 = p\mu$ but $p = a(1 - e^2)$ so $h^2 = a(1 - e^2)u$.

$$\begin{aligned}
\mathcal{E} &= \frac{h^2}{2r_p^2} - \frac{\mu}{r_p} \\
&= \frac{a(1-e^2)\mu}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} \\
&= \frac{(1-e^2)\mu - 2\mu(1-e)}{2a(1-e)^2} \\
&= \frac{\mu(1+e-2)}{2a(1-e)} \\
&= \frac{\mu(e-1)}{2a(1-e)} \\
&= -\frac{\mu}{2a}
\end{aligned} \tag{2.0.5}$$

Recall equation (1.0.17). Then $\mu\mathbf{e} = (\dot{\mathbf{r}} \times \mathbf{h}) - \mu\frac{\mathbf{r}}{r}$.

$$\mu\mathbf{e} = \dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}$$

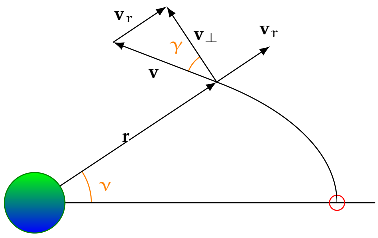


Figure 2.0.2: The components of the velocity vector $\mathbf{v}$ consist of the perpendicular and the parallel velocities.

$$
\begin{aligned}
&= \dot{\mathbf{r}} \times \mathbf{r} \times \dot{\mathbf{r}} - \mu \frac{\mathbf{r}}{r} \\
&= v^2 \mathbf{r} - (rv)\mathbf{v} - \mu \frac{\mathbf{r}}{r} \\
&= \left(v^2 - \frac{\mu}{r} \right) \mathbf{r} - (rv)\mathbf{v}
\end{aligned} \tag{2.0.6}
$$

At this point in the notes, given $\mathbf{r}$ and $\mathbf{v}$, we are able to find

I. $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$

II. $\mathbf{e} = \frac{1}{\mu} \left[\left(v^2 - \frac{\mu}{r} \right) \mathbf{r} - (rv)\mathbf{v} \right]$ and $e = \|\mathbf{e}\|$.

III. $p = a(1 - e^2) \Rightarrow r = \frac{p}{1 + e\cos\nu}$ so we can also find the true anomaly ν.

IV. $\mathbf{h} = \mathbf{r} \times \mathbf{v}$

In the diagram above, $\|\mathbf{v}_r\| = \|\mathbf{v}_\|\|$.

$$
\begin{aligned}
v_\perp &= \frac{h}{r} & \qquad v_r &= \dot{r} \\
&= \frac{\mu}{h}(1 + e\cos(\nu)) & &= \frac{d}{dt}\left[\frac{h^2}{\mu(1 + e\cos(\nu))} \right] \\
& & &= -\frac{h^2 e \dot{\nu} \sin(\nu)}{\mu(1 + e\cos(\nu))^2} \\
& & &= -\frac{h}{r^2} \frac{h^2 e \sin(\nu)}{\mu(1 + e\cos(\nu))^2} \\
& & &= -\frac{\mu^2(1 + e\cos(\nu))^2}{h^4} \frac{h^3 e \sin(\nu)}{\mu(1 + e\cos(\nu))^2} \\
& & &= \frac{\mu e \sin(\nu)}{h}
\end{aligned}
$$

We were able to make the substitution $\dot{\nu} = \frac{h}{r^2}$. Let ω be the angular velocity. Then $\omega = r\frac{d\nu}{dt} = r\dot{\nu}$. ω is related to the tangential velocity, $v_\perp$, by $v_\perp = r\dot{\nu}$.

$$
\tan(\gamma) = \frac{v_r}{v_\perp} = \frac{e\sin(\nu)}{1 + e\cos(\nu)} \tag{2.0.7}
$$

Recall that $\frac{dA}{dt} = \frac{1}{2}h$ and let T be the period of the ellipse where A_0 be the total area of the ellipse.

$$\int_0^T dt = \frac{2}{h}\int_0^{A_0} dA$$
$$T = \frac{2A_0}{h} \tag{2.0.8}$$

The area of an ellipse is $A_0 = \pi ab$ and $b = a\sqrt{1-e^2}$. Recall $\frac{h^2}{\mu} = a(1-e^2) \Rightarrow h = \sqrt{\mu a(1-e^2)}$. So the period of an ellipse is

$$T = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}}$$
$$= \frac{2\pi}{\sqrt{\mu}}a^{3/2}. \tag{2.0.9}$$

2.1 GEO Synchronous Earth Orbits

First, we need to note the difference in a sidereal day and synodic day. A sidereal day is the time it takes for the Earth to rotate 360° on its axis, $T = 23.93$ hours. A synodic day is the time it takes for the Sun to appear in the same place overhead, $T = 24$ hours. Now, for a circular orbit, $T = \frac{2\pi}{\mu}r^{3/2}$. To put a satellite in GEO around the Earth, $T = 23.93$ hours where $\mu = G(m_\oplus + m_{\text{satellite}}) \approx \mu_\oplus = 398600\frac{\text{km}^3}{\text{s}^2}$. Now all we need to do is solve for r.

$$r = \left(\frac{23.93 \cdot 3600\sqrt{398600}}{2\pi}\right)^{2/3}$$
$$= 42158.9 \text{ km}$$

The radius of the Earth is 6378 km so the altitude of a satellite in GEO is 35780.9 km.

Example Problem 1 Elliptical Orbit:

For an elliptical orbit show that the orbital speed as a function of the true anomaly is

$$v = \frac{\mu}{h}\sqrt{e^2 + 2e\cos(\nu) + 1}.$$

Plot the normalized speed $v^* = \frac{vh}{\mu}$ as a function of the true anomaly for different eccentricities between zero and unity.

First, we need to note that $v_r = \frac{\mu}{h}e\sin(\nu)$ and $v_\perp = \frac{\mu(1+e\cos(\nu))}{h}$.

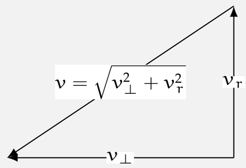


Figure 2.1.1: Components of velocity vector example.

By substitution, we have

$$
\begin{aligned}
v &= \sqrt{v_r^2 + v_\perp^2} \\
&= \sqrt{\left(\frac{\mu}{h} e \sin(\nu)\right)^2 + \left(\frac{\mu(1 + e\cos(\nu))}{h}\right)^2} \\
&= \frac{\mu}{h}\sqrt{(e\sin(\nu))^2 + (1 + e\cos(\nu))^2} \\
&= \frac{\mu}{h}\sqrt{e^2(\sin^2(\nu) + \cos^2(\nu)) + 1 + 2e\cos(\nu)} \\
&= \frac{\mu}{h}\sqrt{e^2 + 2e\cos(\nu) + 1}
\end{aligned}
$$

For the plot of $v^* = \frac{vh}{\mu} = \sqrt{e^2 + 2e\cos(\nu) + 1}$, see figure 2.1.2.

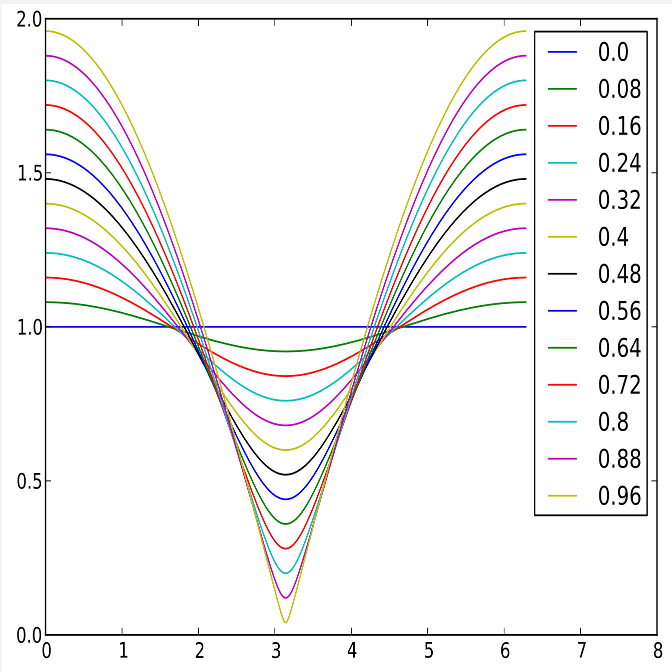


Figure 2.1.2: Velocity profiles for $\nu \in [0, 2\pi]$ and $e \in [0, 1]$.

3 Parabolic Trajectories

A parabolic orbit is when the eccentricity, $e = 1$, and $\mathcal{E} = 0$. Since we are dealing with a parabola, we can relate the directrix with the semi-latus rectum, p, and r_p. That is, $r_p = \frac{p}{2} \Rightarrow p = 2r_p$. Since $\mathcal{E} = 0 = \frac{v^2}{2} - \frac{\mu}{r}$, $v = \sqrt{\frac{2\mu}{r}}$ which is the speed of a parabolic path. For a parabolic trajectory, we can write the flight angle as:

$$\tan\gamma = \frac{\sin\nu}{1+\cos\nu}. \tag{3.0.1}$$

Using the following identities, we can simplify the right hand side of equation (3.0.1).

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{3.0.2}$$
$$\cos(2\theta) = 2\cos^2(\theta) - 1 \tag{3.0.3}$$

That is, $\tan\gamma = \tan\frac{\nu}{2}$ so $\gamma = \frac{\nu}{2}$.

Example Problem 2 Parabolic Trajectory:

What minimum velocity, relative to the Earth, is required to escape the solar system along a parabolic path from Earth orbit?

Figure 3.0.2: A parabolic trajectory from Earth.

The velocity of the Earth is $v_\oplus = \sqrt{\frac{\mu_\odot}{2a_\oplus}} = 29.784$ km/s where $\mu_\odot = 132712440018$ and $a_\oplus = 149.6 \times 10^6$. Now $v_{esc} = \sqrt{2}v_\oplus = 42.1212$ km/s.

$$\begin{aligned} v_{esc} &= v_{rel} + v_\oplus \\ v_{rel} &= v_{esc} - v_\oplus \\ &= 42.1212 - 29.784 \\ &= 12.337\text{km/s} \end{aligned}$$

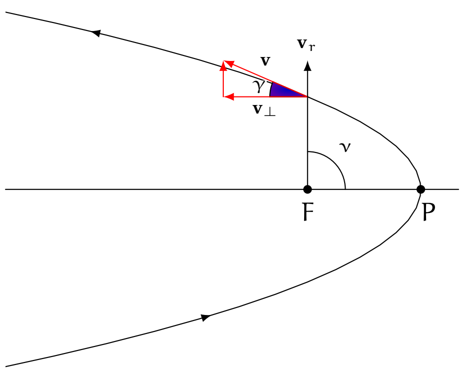


Figure 3.0.1: The geometry of a parabolic trajectory.

4 Hyperbolic Trajectories

$$\begin{aligned} \nu_\infty &= \text{The true anomaly} & \Delta &= a\sqrt{e^2-1} \text{ The aiming radius} \\ \delta &= 2\sin^{-1}\left(\frac{1}{e}\right) \text{ The turn angle} & c &= \text{Distance from } F \text{ to } F_2 \\ e &> 1 & a &= \frac{h^2}{\mu(e^2-1)} \\ \mathcal{E} &> 0 & v_\infty &= \frac{\mu}{h}\sqrt{e^2-1} \text{ The hyperbolic excess speed} \end{aligned}$$

See figure 4.0.1.

Example Problem 3 Hyperbolic Trajectories:

A meteoroid on a hyperbolic orbit is first observed approaching the Earth when it is 402,000 km from the center of the Earth with a true anomaly of 150°. If the speed of the meteoroid at that time is 2.23 km/s:

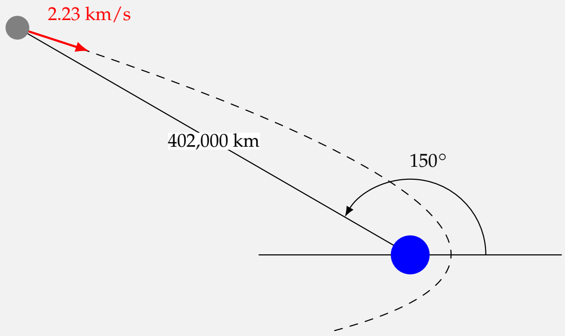


Figure 4.0.2: An incoming hyperbolic trajectory approaching Earth.

(a) Calculate the eccentricity of the trajectory

The energy for the orbit is

$$\begin{aligned} \mathcal{E} &= \frac{v^2}{2} - \frac{\mu}{r} \\ &= \frac{2.23^2}{2} - \frac{398600}{402000} \\ &= 2.48645 - 0.991542 \\ &= 1.4949 \text{km}^2/\text{s}^2 \end{aligned}$$

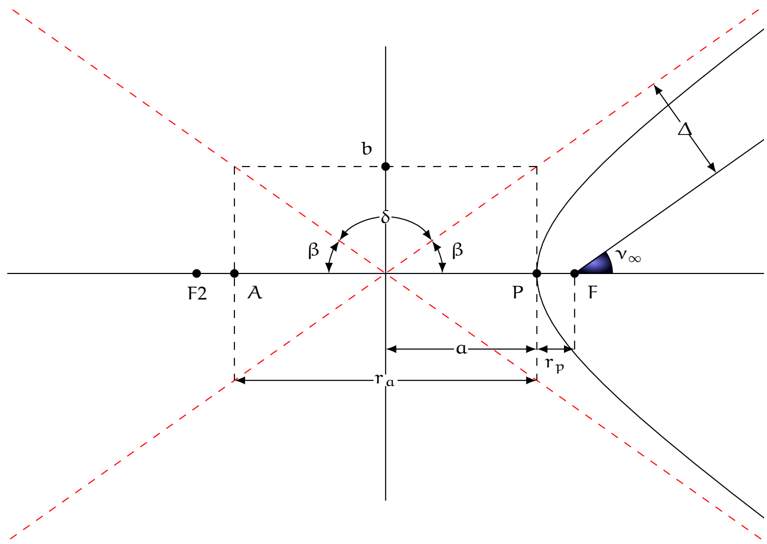


Figure 4.0.1: The geometry of a hyperbolic trajectory.

We can write $a = \frac{h^2}{\mu(1-e^2)} \Rightarrow h^2 = a\mu(1-e^2)$ and $\mathcal{E} = -\frac{\mu}{2a}$.

$$\begin{aligned} h^2 &= a\mu(1-e^2) \\ &= \frac{\mu^2(e^2-1)}{2\mathcal{E}} \\ &= \frac{398600^2(e^2-1)}{2 \times 1.4949} \\ &= 5.31413 \times 10^{10}(e^2-1) \end{aligned}$$

The orbital equation is $r = \frac{h^2}{\mu(1+e\cos\nu)} \Rightarrow h^2 = r\mu(1+e\cos\nu)$.

$$\begin{aligned} h^2 &= r\mu(1+e\cos\nu) \\ &= 402000 \cdot 3986000\left[1 + e\cos\left(\frac{5\pi}{6}\right)\right] \\ &= 1.60237 \times 10^{11}\left(1 - \frac{e\sqrt{3}}{2}\right) \end{aligned}$$

Now, we can equate the equations for h^2 and solve for e.

$$5.31413 \times 10^{10}(e^2-1) = 1.60237 \times 10^{11}\left(1 - \frac{e\sqrt{3}}{2}\right)$$

$$
\begin{aligned}
e^2 - 1 &= 3.0153 - \frac{3.01533e\sqrt{3}}{2} \\
e^2 + 2.61133e - 4.0153 &= 0 \\
e &= \frac{-2.61133 \pm \sqrt{2.61133^2 + 4 \cdot 4.0153}}{2} \\
e &= -3.69731, 1.08601
\end{aligned}
$$

Since $e \geqslant 0$, the eccentricity of the orbit is $e = 1.08601$.

(b) Calculate the altitude at closest approach

We can use e from (a) to solve for h^2.

$$
\begin{aligned}
h^2 &= r\mu(1 + e\cos\nu) \\
&= 402000 \cdot 398600 \left(1 - \frac{1.08601\sqrt{3}}{2}\right) \\
&= 97639.155 \text{ km}^2/\text{s}^2
\end{aligned}
$$

Now

$$
\begin{aligned}
r_p &= \frac{h^2}{\mu(1+e)} \\
&= \frac{97639.155}{398600 \cdot 2.08601} \\
&= 11465.6\text{km}
\end{aligned}
$$

So the altitude from the Earth is $r_p - r_e$.

$$11465.6 - 6378 = 5087.59 \text{ km}$$

(c) Calculate the speed at closest approach

Velocity at closet approach can be found from

$$
\begin{aligned}
v &= \frac{h}{r_p} \\
&= \frac{\sqrt{97639.155}}{11464} \\
&= 8.5158 \text{ km/s}
\end{aligned}
$$

(d) Plot the trajectory to scale

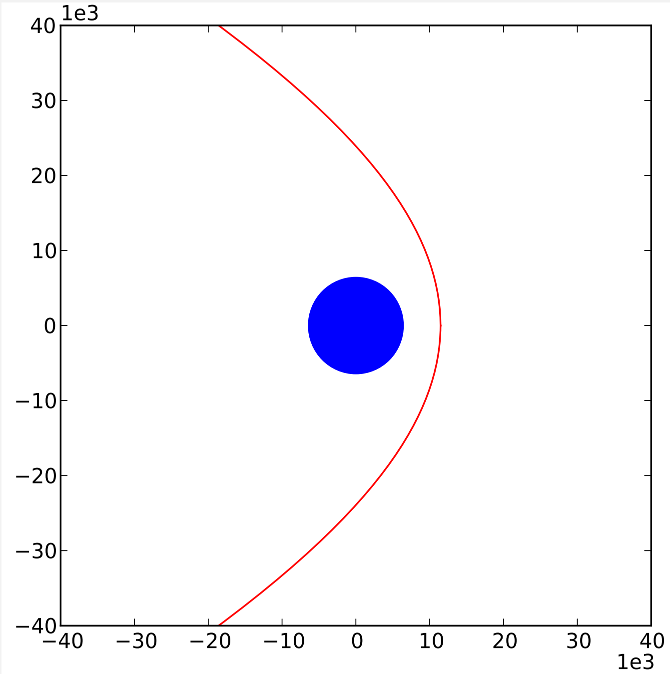


Figure 4.0.3: Meteorite on a hyperbolic orbit near Earth.

(e) Calculate the hyperbolic excess speed

The equation for the hyperbolic excess speed is

$$\begin{aligned} v_\infty &= \frac{\mu}{h}\sqrt{e^2 - 1} \\ &= \frac{398600}{\sqrt{97639.155}}\sqrt{1.08601^2 - 1} \\ &= 1.72932 \text{ km/s} \end{aligned}$$

Using a scientific software package, to develop a computer program to compute the trajectory of meteorite from the equation of motion

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = 0$$

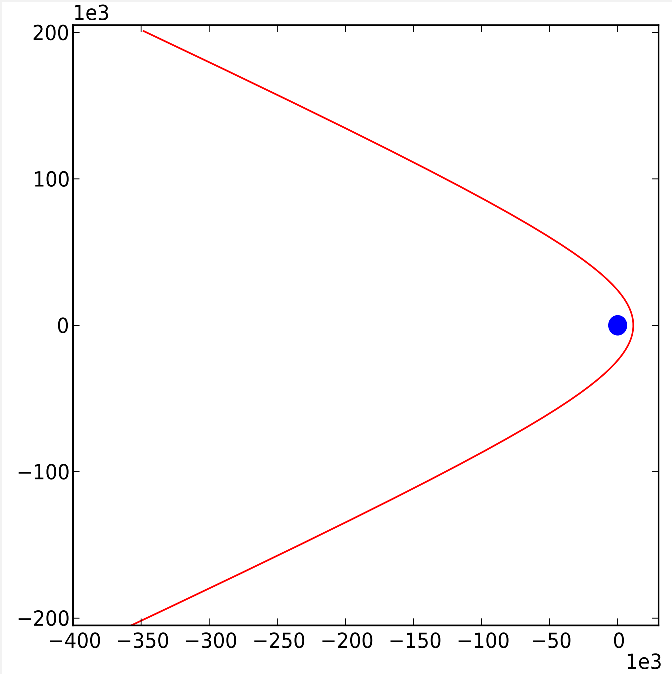


Figure 4.0.4: The trajectory using the 3 body numerical simulation.

Refer to Python file example3notes2.py: where $r(0) = rx$, $r(1) = ry$, and $r2(0) = 0$ are the initial position vector, $v(0) = vx$, $v(1) = vy$, and$v(2) = 0$ are the initial velocity vector, and **r** and **v** were found from

$$\mathbf{r} = r\cos(\nu)\hat{\mathbf{p}} + r\sin(\nu)\hat{\mathbf{q}}$$
$$\mathbf{v} = \frac{\mu}{h}\Big[\sin(\nu)\hat{\mathbf{p}} - (e + \cos(\nu))\hat{\mathbf{q}}\Big]$$

Example Problem 4 Hyperbolic Venus Fly By:

An interplanetary probe intended for the outer solar system is launched from Earth. Initially it is sent towards Venus on a flyby trajectory in order to perform a gravity assist maneuver see figure 4.0.5. Referring to the diagram below, suppose the spacecraft enters the sphere of influence of Venus ($r_{SOI} = 616000$ km) with a heliocentric speed of 37.7 km/s and a heliocentric flight path angle $\sigma_2 = 20°$. Suppose that the hyperbolic trajectory asymptote is designed with an aiming radius of $\Delta = 1.5$ planetary radii ($r_{♀} = 6,051.8$ km). For this problem, the heliocentric velocity of Venus is 35.022km/s and the gravitational parameter for Venus is $\mu_{♀} = 324859\text{km}^3/\text{s}^2$.

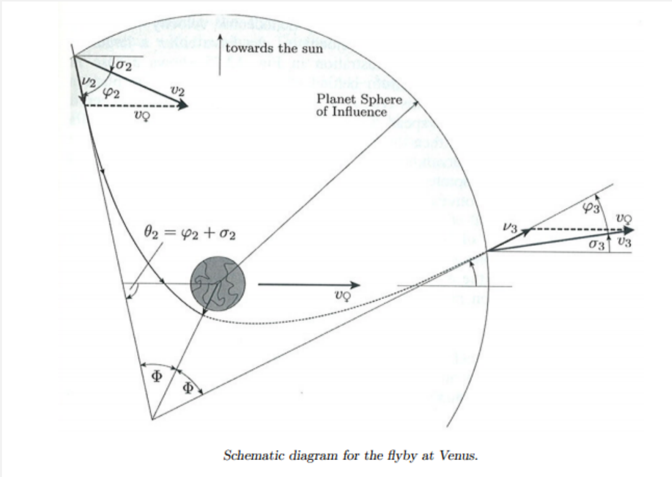


Figure 4.0.5: Depiction of the flyby of Venus.

(a) What is the arrival/departure speed v_2 and v_3 relative to Venus?

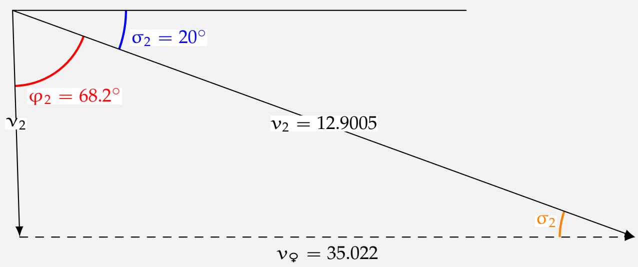


Figure 4.0.6: Close up of the geometry of the incoming spacecraft.

The speed in the x direction is $\mathbf{v}_x = 37.7 \cos \frac{\pi}{9}$ km/s but this isn't the speed relative to Venus see figure 4.0.6.

$$\begin{aligned}\mathbf{v}_{x\text{rel}} &= 37.7 \cos \frac{\pi}{9} - 35.022 \\ &= 0.404412 \text{ km/ s}\end{aligned}$$

The speed in the y direction relative to Venus is $\mathbf{v}_y = -37.7 \sin \frac{\pi}{9}$ so the relative arrival velocity vector to Venus is

$$\begin{aligned}\mathbf{v}_2 &= v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \\ &= 0.404412 \hat{\mathbf{i}} - 12.8942 \hat{\mathbf{j}} + 0 \hat{\mathbf{k}}\end{aligned}$$

So the relative arrival speed to Venus is $\|\mathbf{v}_2\| = \sqrt{0.404412^2 + 12.8942^2} = 12.9005$ km/s which is also the departure speed.

(b) What is the eccentricity of the fly-by trajectory?

From the energy equation, we have

$$\begin{aligned}\mathcal{E} &= \frac{v_2^2}{2} - \frac{\mu_♀}{♀\mathrm{SOI}} \\ &= \frac{12.9005^2}{2} - \frac{324859}{616000} \\ &= 82.6841\end{aligned}$$

Now we can use $\mathcal{E} = \frac{\mu_♀}{2a}$, to solve for the semi-major axis, a.

$$a = \frac{\mu_♀}{2\mathcal{E}} = 1964.46 \text{ km}.$$

Finally, we can use the equation for the aiming radius to the eccentricity, e.

$$\begin{aligned}\Delta &= 1.5 \cdot 6051.8 \\ &= 9077.7 \\ 9077.7 &= a\sqrt{e^2 - 1} \\ e &= \sqrt{\left(\frac{9077.7}{1964.46}\right)^2 + 1} \\ &= 4.72793\end{aligned}$$

(c) What is the magnitude of the heliocentric velocity as it leaves the sphere of influence? What percentage increase in heliocentric velocity is obtained?

The magnitude of the heliocentric velocity is

$$v_{3\odot} = \sqrt{35.022^2 + v_2^2 + 2v_2 \cdot 35.022\cos(\pi - \delta - \theta_2)} \text{ km/s}$$

where $\theta_2 = \sigma_2 + \varphi_2$ and $\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$.

$$\begin{aligned}\delta &= 2\sin^{-1}\left(\frac{1}{e}\right) \\ &= 0.426237 \text{ rad} \quad \text{and} \\ \varphi_2 &= \cos^{-1}\left(\frac{\mathbf{v}_2 \cdot \mathbf{v}_♀}{v_2 v_♀}\right) - \frac{\pi}{9} \\ &= 1.19038 \text{ rad}\end{aligned}$$

Therefore, $v_{3\odot} = 41.7203$ km/s. The percentage increase is

$$\frac{41.7203 - 37.7}{37.7} \times 100\% = 10.6638\%.$$

5 Restricted 3-Body Problem

$$r_{12} = \sqrt{(x_1 - x_2)^2} \qquad x_1 = \text{Is the } x \text{ location of } m_1$$
$$x_2 = x_1 + r_{12} \qquad \text{relative to the center of gravity.}$$
$$x_1 = \frac{-m_2}{m_1 + m_2} r_{12} \qquad \pi_1 = \frac{m_1}{m_1 + m_2}$$
$$x_2 = \frac{m_1}{m_1 + m_2} r_{12} \qquad \pi_2 = \frac{m_2}{m_1 + m_2}$$
$$0 = m_1 x_1 + m_2 x_2$$

We can describe the position of m as $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ in relation to the center of gravity, i.e., the origin.

$$\begin{aligned}\mathbf{r}_1 &= (x - x_1)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\ &= (x + \pi_2 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \end{aligned} \tag{5.0.1}$$

$$\begin{aligned}\mathbf{r}_2 &= (x - x_2)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\ &= (x - \pi_1 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \end{aligned} \tag{5.0.2}$$

Let's define the absolute acceleration where ω is the initial angular velocity which is constant. Then $\omega = \frac{2\pi}{T}$.

$$\ddot{\mathbf{r}}_{\text{abs}} = \mathbf{a}_{\text{rel}} + \mathbf{a}_{\text{CG}} + \Omega \times (\Omega \times \mathbf{r}) + \dot{\Omega} \times \mathbf{r} + 2\Omega \times \mathbf{v}_{\text{rel}} \tag{5.0.3}$$

where

$$\mathbf{a}_{\text{rel}} = \text{Rectilinear acceleration relative to the frame} \qquad \Omega \times (\Omega \times \mathbf{r}) = \text{Centripetal acceleration}$$
$$2\Omega \times \mathbf{v}_{\text{rel}} = \text{Coriolis acceleration}$$

Since the velocity of the center of gravity is constant, $\mathbf{a}_{\text{CG}} = 0$, and $\dot{\Omega} = 0$ since the angular velocity of a circular orbit is constant. Therefore, equation (5.0.3) becomes:

$$\ddot{\mathbf{r}} = \mathbf{a}_{\text{rel}} + \Omega \times (\Omega \times \mathbf{r}) + 2\Omega \times \mathbf{v}_{\text{rel}} \tag{5.0.4}$$

where

$$\Omega = \Omega\hat{\mathbf{k}} \tag{5.0.5}$$
$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{5.0.6}$$

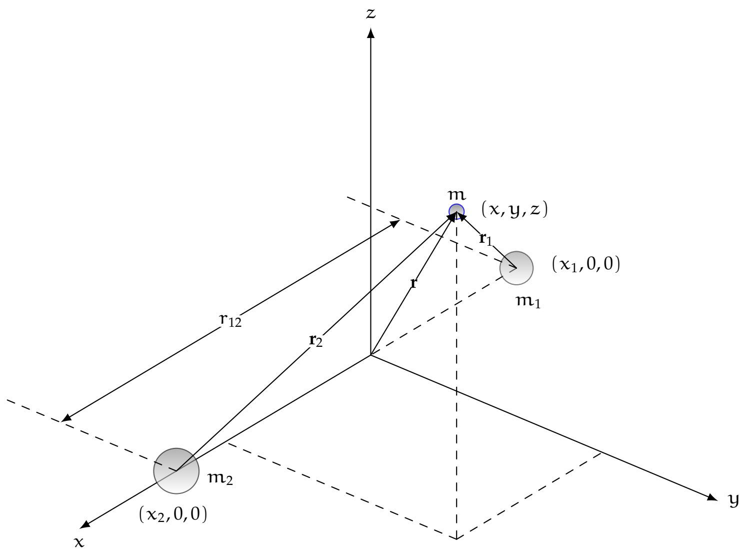


Figure 5.0.1: Diagram of the restricted 3 body problem.

$$\dot{\mathbf{r}} = \mathbf{v}_{CG} + \Omega \times \mathbf{r} + \mathbf{v}_{rel} \tag{5.0.7}$$

$$\mathbf{v}_{rel} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{5.0.8}$$

$$\mathbf{a}_{rel} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{5.0.9}$$

After substituting equation (5.0.5), equation (5.0.6), equation (5.0.8), and equation (5.0.9) into equation (5.0.4), we obtain

$$\ddot{\mathbf{r}} = \left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}. \tag{5.0.10}$$

Newton's 2nd Law of Motion is $m\mathbf{a} = \mathbf{F}_1 + \mathbf{F}_2$ where $\mathbf{F}_1 = -\frac{Gm_1 m}{r_1^3}\mathbf{r}_1$ and $\mathbf{F}_2 = -\frac{Gm_2 m}{r_2^3}\mathbf{r}_2$. Let $\mu_1 = Gm_1$ and $\mu_2 = Gm_2$.

$$\begin{aligned}
m\mathbf{a} &= \mathbf{F}_1 + \mathbf{F}_2 \\
m\mathbf{a} &= -\frac{m\mu_1}{r_1^3}\mathbf{r}_1 - \frac{m\mu_2}{r_2^3}\mathbf{r}_2 \\
\mathbf{a} &= -\frac{\mu_1}{r_1^3}\mathbf{r}_1 - \frac{\mu_2}{r_2^3}\mathbf{r}_2 \\
\left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} &= -\frac{\mu_1}{r_1^3}\mathbf{r}_1 - \frac{\mu_2}{r_2^3}\mathbf{r}_2 \\
\left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} &= \begin{array}{l} -\frac{\mu_1}{r_1^3}\left[(x + \pi_2 r_{12})\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right] \\ -\frac{\mu_2}{r_2^3}\left[(x - \pi_1 r_{12})\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right] \end{array}
\end{aligned} \tag{5.0.11}$$

Now all we have to do is equate the coefficients.

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{5.0.12}$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \tag{5.0.13}$$

$$\ddot{z} = -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z \tag{5.0.14}$$

We now have system of nonlinear ODEs. The standard approach is to start by identifying the fixed points (Lagrange points). The Lagrange points occur when $\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0$ which brings us to the following system of ODEs.

$$\begin{aligned}
-\Omega^2 x &= -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \\
-\Omega^2 y &= -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \\
0 &= -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z
\end{aligned}$$

We can clearly see from the last equation that $z = 0$. We can simplify the last two ODEs further by using the relations $\Omega = \sqrt{\frac{\mu}{r_{12}^3}}$, $\pi = 1 - \pi_2$, $\pi_1 = \frac{\mu_1}{\mu}$, and $\pi_2 = \frac{\mu_2}{\mu}$.

$$\begin{aligned}
-\frac{\mu}{r_{12}^3}x &= -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \\
\frac{x}{r_{12}^3} &= \frac{\mu_1}{\mu}\frac{1}{r_1^3}(x + \pi_2 r_{12}) + \frac{\mu_2}{\mu}\frac{1}{r_2^3}[x - (1 - \pi_2)r_{12}] \\
\frac{x}{r_{12}^3} &= \pi_1\frac{1}{r_1^3}(x + \pi_2 r_{12}) + \pi_2\frac{1}{r_2^3}[x - r_{12} + \pi_2 r_{12}] \\
\frac{x}{r_{12}^3} &= \frac{(1 - \pi_2)}{r_1^3}(x + \pi_2 r_{12}) + \frac{\pi_2}{r_2^3}[x - r_{12} + \pi_2 r_{12}] \qquad (5.0.15) \\
-\frac{\mu}{r_{12}^3}y &= -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \\
\frac{1}{r_{12}^3} &= \frac{\mu_1}{\mu}\frac{1}{r_1^3} + \frac{\mu_2}{\mu}\frac{1}{r_2^3} \\
\frac{1}{r_{12}^3} &= \frac{\pi_1}{r_1^3} + \frac{\pi_2}{r_2^3} \\
\frac{1}{r_{12}^3} &= \frac{1 - \pi_2}{r_1^3} + \frac{\pi_2}{r_2^3} \qquad (5.0.16)
\end{aligned}$$

We have ended up with the following two equations (5.0.15) and (5.0.16). We can solve the equations simultaneously.

$$\begin{bmatrix} (1 - \pi_2)(x + \pi_2 r_{12}) & \pi_2(x - r_{12} + \pi_2 r_{12}) & -x \\ 1 - \pi_2 & \pi_2 & -1 \end{bmatrix} \overbrace{\Rightarrow}^{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \tag{5.0.17}$$

Finally, we have that $\frac{1}{r_1^3} = \frac{1}{r_{12}^3}$ and $\frac{1}{r_2^3} = \frac{1}{r_{12}^3}$ so

$$\frac{1}{r_1^3} = \frac{1}{r_2^3} = \frac{1}{r_{12}^3}$$
$$r_1 = r_2 = r_{12}.$$

Recall equations (5.0.1) and (5.0.2) which we can now rewrite with by using the previously obtained information: $z = 0$, $r_1 = r_2 = r_{12}$, and $\pi_1 = 1 - \pi_2$.

$$\begin{aligned}
\mathbf{r}_1 &= (x + \pi_2 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} \\
r_1 &= \sqrt{(x + \pi_2 r_{12})^2 + y^2} \\
r_1^2 &= (x + \pi_2 r_{12})^2 + y^2 \\
r_{12}^2 &= (x + \pi_2 r_{12})^2 + y^2 && (5.0.18) \\
\mathbf{r}_2 &= (x - \pi_1 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} \\
r_{12}^2 &= (x + \pi_2 r_{12} - r_{12})^2 + y^2 && (5.0.19)
\end{aligned}$$

Next we need to set the two equations (5.0.18) and (5.0.19) equal to each other. However, remember that the solution to a square root involves both plus and minus. If we take the positive solution, we will end up $r_{12} = 0$ which is certainly not the case.

$$\begin{aligned}
x + \pi_2 r_{12} - r_{12} &= -x - \pi_2 r_{12} \\
x &= \frac{r_{12}}{2} - \pi_2 r_{12} && (5.0.20)
\end{aligned}$$

Lastly, $r_{12}^2 = (x + \pi_2 r_{12} - r_{12})^2 + y^2 \Rightarrow y = \pm\frac{r_{12}\sqrt{3}}{2}$. We now have 2 Lagrange points, namely L_4 and L_5.

$$L_4 : \left(\frac{r_{12}}{2} - \pi_2 r_{12}, \frac{r_{12}\sqrt{3}}{2}, 0\right) \quad \text{and} \quad L_5 : \left(\frac{r_{12}}{2} - \pi_2 r_{12}, -\frac{r_{12}\sqrt{3}}{2}, 0\right)$$

Since $r_1 = r_2 = r_{12}$, these points form 2 equilateral triangles with the m_1 and m_2. To find the remaining Lagrange points, we let $y = z = 0$. Then $\mathbf{r}_1 = (x + \pi_2 r_{12})\hat{\mathbf{i}}$ and $\mathbf{r}_2 = (x - \pi_1 r_{12})\hat{\mathbf{i}}$. Let $\xi = \frac{x}{r_{12}}$.

$$\begin{aligned}
-\Omega^2 x &= -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x + \pi_2 r_{12} - r_{12}) \\
-\frac{\mu}{r_{12}^3} x &= -\frac{\mu_1}{|x + \pi_2 r_{12}|^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{|x + \pi_2 r_{12} - r_{12}|^3}(x + \pi_2 r_{12} - r_{12}) \\
0 &= \frac{\pi_1}{|x + \pi_2 r_{12}|^3}(x + \pi_2 r_{12}) + \frac{\pi_2}{|x + \pi_2 r_{12} - r_{12}|^3}(x + \pi_2 r_{12} - r_{12}) - \frac{x}{r_{12}^3} \\
0 &= \frac{(1 - \pi_2)}{|x + \pi_2 r_{12}|^3}(x + \pi_2 r_{12}) + \frac{\pi_2}{|x + \pi_2 r_{12} - r_{12}|^3}(x + \pi_2 r_{12} - r_{12}) - \frac{x}{r_{12}^3}
\end{aligned}$$

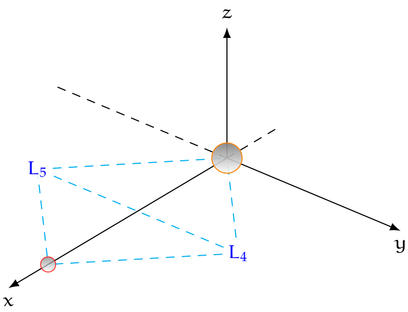


Figure 5.0.2: L_4 and L_5 Lagrange points of a 2 body system.

$$\begin{aligned}
0 &= \frac{(1-\pi_2)}{|r_{12}\xi + \pi_2 r_{12}|^3}(r_{12}\xi + \pi_2 r_{12}) + \frac{\pi_2}{|r_{12}\xi + \pi_2 r_{12} - r_{12}|^3}(r_{12}\xi + \pi_2 r_{12} - r_{12}) - \frac{r_{12}\xi}{r_{12}^3} \\
0 &= \frac{(1-\pi_2)}{r_{12}^2|\xi + \pi_2|^3}(\xi + \pi_2) + \frac{\pi_2}{r_{12}^2|\xi + \pi_2 - 1|^3}(\xi + \pi_2 - 1) - \frac{\xi}{r_{12}^2} \\
0 &= \frac{(1-\pi_2)}{|\xi + \pi_2|^3}(\xi + \pi_2) + \frac{\pi_2}{|\xi + \pi_2 - 1|^3}(\xi + \pi_2 - 1) - \xi \\
f(\xi) &= \frac{(1-\pi_2)}{|\xi + \pi_2|^3}(\xi + \pi_2) + \frac{\pi_2}{|\xi + \pi_2 - 1|^3}(\xi + \pi_2 - 1) - \xi \qquad (5.0.21)
\end{aligned}$$

The roots of equation (5.0.21) are the Lagrange points L_1, L_2 and L_3.

5.1 Lagrange Points for the Earth-Moon System

Example Problem 5 Lagrange Points L_1, L_2, and L_3:

Refer to the Python file example5notes.py for the calculation of L_1, L_2, and L_3. See figure 5.1.1 for a plot of the $f(\xi)$.

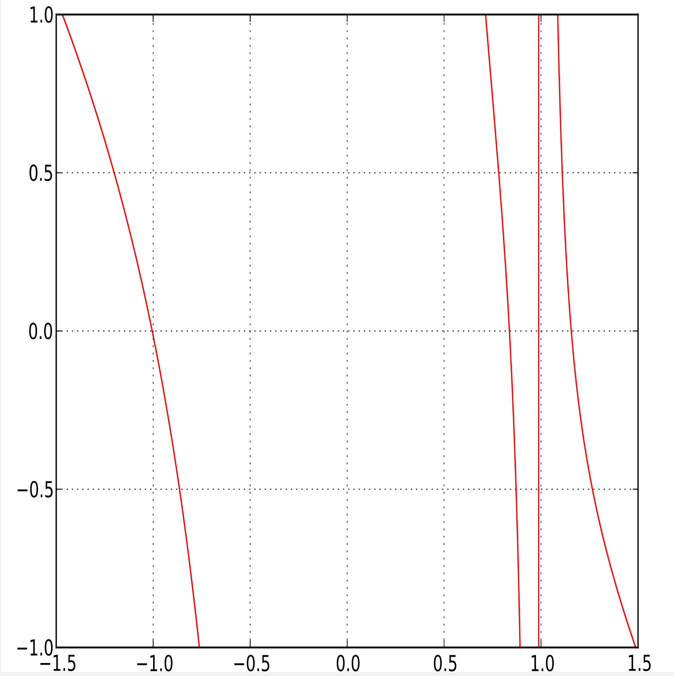


Figure 5.1.1: The ξ intercepts of $f(\xi)$.

To find L_1, L_2, and L_3 of the Earth-Moon system, we simply multiple ξ times r_{12}.

$$\begin{aligned} L_1 &= 0.8369 \cdot r_{12} \\ &= 321709.713544 \\ L_2 &= 1.15568 \cdot r_{12} \\ &= 444244.584579 \\ L_3 &= -1.005062 \cdot r_{12} \\ &= -386346.120068 \end{aligned}$$

5.2 Jacobi Constant

In order to find the Jacobi constant, we need to turn back to our our original system of ODEs:

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{5.2.1}$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \tag{5.2.2}$$

$$\ddot{z} = -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z \tag{5.2.3}$$

Let's multiple equation (5.2.1) by $\dot{x}$, equation (5.2.2) by $\dot{y}$, and equation (5.2.3) by $\dot{z}$.

$$\dot{x}\ddot{x} - 2\Omega\dot{x}\dot{y} - \Omega^2\dot{x}x = -\dot{x}\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \dot{x}\frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{5.2.4}$$

$$\dot{y}\ddot{y} + 2\Omega\dot{y}\dot{x} - \Omega^2\dot{y}y = -\frac{\mu_1}{r_1^3}\dot{y}y - \frac{\mu_2}{r_2^3}\dot{y}y \tag{5.2.5}$$

$$\dot{z}\ddot{z} = -\frac{\mu_1}{r_1^3}\dot{z}z - \frac{\mu_2}{r_2^3}\dot{z}z \tag{5.2.6}$$

Now we can add the equations (5.2.4) to (5.2.6) together.

$$\begin{aligned}
\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} - \Omega^2(\dot{x}x + \dot{y}y) &= -\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)(\dot{x}x + \dot{y}y + \dot{z}z) + \left(\frac{\pi_1\mu_2}{r_2^3} - \frac{\pi_2\mu_1}{r_1^3}\right) r_{12}\dot{x} \\
\frac{1}{2}\frac{d}{dt}(v^2) - \frac{1}{2}\Omega^2\frac{d}{dt}(r^2) &= -\frac{\mu_1}{r_1^3}[(x + \pi_2 r_{12})\dot{x} + \dot{y}y + \dot{z}z] - \frac{\mu_2}{r_2^3}[(x - \pi_1 r_{12})\dot{x} + \dot{y}y + \dot{z}z] \\
\frac{d}{dt}\left[\frac{v^2}{2} - \frac{1}{2}\Omega^2 r^2 - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}\right] &= 0 \\
\frac{v^2}{2} - \frac{1}{2}\Omega^2 r^2 - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} &= C \\
v^2 &= \Omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C
\end{aligned}$$

where

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
\dot{r}r &= \dot{x}x + \dot{y}y \\
r_1^2 &= (x + \pi_2 r_{12})^2 + y^2 + z^2 \\
\dot{r}_1 &= \frac{1}{r_1}\left[\dot{x}(x + \pi_2 r_{12}) + \dot{y}y + \dot{z}z\right] \\
r_2^2 &= (x - \pi_1 r_{12})^2 + y^2 + z^2 \\
\dot{r}_2 &= \frac{1}{r_2}\left[\dot{x}(x - \pi_1 r_{12}) + \dot{y}y + \dot{z}z\right]
\end{aligned}$$

Since $v^2 \geqslant 0$, $\Omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \geqslant 0$ too. When $\Omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C = 0$, we have zero velocity surfaces.

Example Problem 6 Lagrange Point L_4:

Using a scientific software package, develop a computer program to compute the trajectory of a spacecraft using the restricted three-body equations of motion. Use this program to design a trajectory from Earth to the Earth-Moon Lagrange point L_4 starting at a 200km altitude burnout point. The mission design requirement is that the trajectory should take the coasting spacecraft to within 500km of L_4 with a relative speed of no more than 1 km/s.

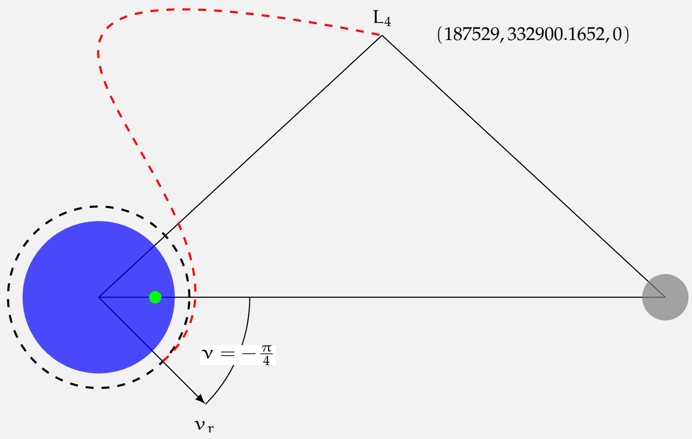


Figure 5.2.1: Mission to Earth-Moon L_4 point.

First, we need to determine the location of L_4. Since L_4 is lying in the the xy-plane, the z-coordinate is 0. The center of gravity is located 1707km inside the Earth. Therefore, the center of Earth is at $(-4671, 0, 0)$ km. To find the y-coordinate, we have $\frac{\sqrt{3}}{2}384400 = 332900.1652$ km. Using Pythagoras's Theorem, $x = \sqrt{384400^2 - 332900^2} \approx 187529$ km. That is, L_4 is at $(187529, 332900, 0)$. With this information, we can find Jacobi's constant

$$\begin{aligned}\Omega^2(x_{L_4}^2 + y_{L_4}^2) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C &< 1 \\ C &< -1.06824\end{aligned}$$

where $\Omega = \sqrt{\frac{6.67259\times10^{-20}(m_e+m_m)}{r_{12}}}$, $\mu_1 = 6.67259 \times 10^{-20}m_e$, $\mu_2 = 6.67259 \times 10^{-20}m_m$, and $r_1 = r_2 = r_{12} = 384400$. To arrive at L_4 with a relative speed of 0, $C = -1.56824$. Since the true anomaly wasn't specified, I let $\nu = -\frac{\pi}{4}$. So the position of the spacecrafts burnout will be $\mathbf{r}_s = \langle -19.3098, -4651.35, 0\rangle$. For our trajectory, let $C = -1.21$. At this location and this instance, our velocity will be v_{b_o}.

$$\begin{aligned}v_{b_o} &= \sqrt{\Omega^2(x^2+y^2) + \frac{2\mu_1}{\sqrt{(x+\pi_2 r_{12})^2 + y^2}} + \frac{2\mu_2}{\sqrt{(x-\pi_2 r_{12})^2 + y^2}} + 2(-1.21)} \\ &= 10.8994415375 \text{ km/s}\end{aligned}$$

where x and y are the components of $\mathbf{r}_s$. Let the flight path angle be $\gamma = -14.02°$. The x and y components on $\mathbf{v}_0$ are

$$\begin{aligned}v_x &= v_{b_o}(\sin\gamma\cos\nu - \sin\nu\cos\gamma) \\ &= 7.76974\end{aligned}$$

$$\begin{aligned} v_y &= v_{b_o}(\sin\gamma\sin\nu + \cos\nu\cos\gamma) \\ &= 7.64389 \end{aligned}$$

Then we can write $\mathbf{v}_0 = \langle 7.76974, 7.64389, 0\rangle$.

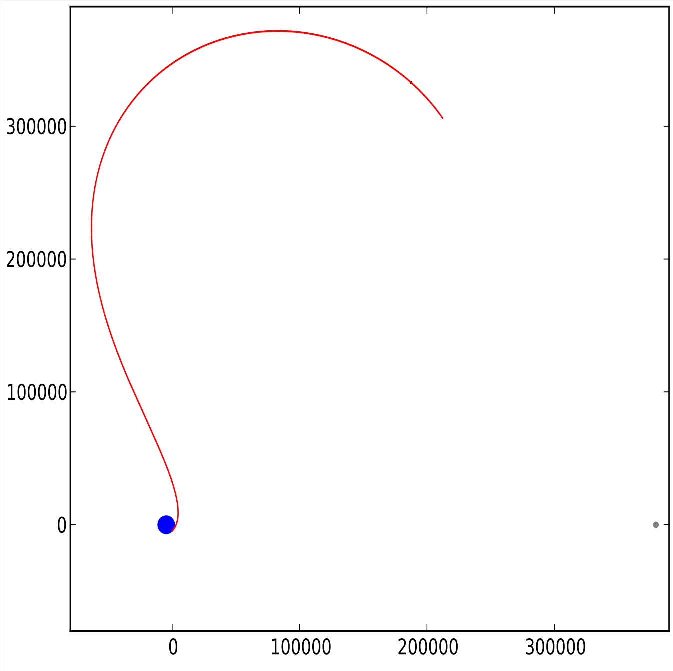


Figure 5.2.2: Flight path with the given conditions to L_4 of the Earth-Moon system.

At the position $\langle 187529.4904, 332900.6847, 0\rangle$, the associated velocity vector is $\langle 0.6428, -0.5506, 0\rangle$. That is, the speed at this location is

$$\sqrt{0.6428^2 + 0.5506^2} = 0.846455944534 \text{ km/s} < 1 \text{ km/s},$$

and the distance from L_4 is

$$\begin{aligned} \sqrt{(187529.4904 - 187529)^2 + (332900.6847 - 332900)^2} &= 0.846455944534 \text{ km} \\ &< 500 \text{ km}. \end{aligned}$$

For the exact calculations, see example6notes.py.

6 Orbital Position as a Function of Time

Recall that $h = r^2\dot{\nu}$ and $r = \frac{h}{\mu^2(1+e\cos(\nu))}$. Then

$$\frac{\mu^2}{h^3}\int_0^t dt' = \int_0^\nu \frac{d\nu'}{(1+e\cos(\nu'))^2} \tag{6.0.1}$$

where $t_0 = 0$ and $t(0) = \nu_0 = 0$. Integration of the RHS of equation (6.0.1) will depend on the value of e.

6.1 Circular Orbit

For $e = 0$, equation (6.0.1) becomes

$$\frac{\mu^2}{h^3}\int_0^t dt' = \int_0^\nu d\nu'$$

$$\frac{\mu^2}{h^3}t = \nu. \tag{6.1.1}$$

Solving for t in equation (6.1.1), we have $t = \frac{h^3\nu}{\mu^2}$. For a circular orbit, $r = \frac{h^2}{\mu}$. Then $r^{3/2} = \frac{h^3}{\mu^{3/2}}$. Now we can solve for h and substitute into t.

$$t = \frac{r^{3/2}\nu}{\sqrt{\mu}}$$

By Kepler's 3^{rd} Law, if $\nu = 2\pi$, then $t = T$, the orbital period. Therefore, $t = \frac{\nu}{2\pi}T$ so $\nu = 2\pi\frac{t}{T}$. Let $2\pi\frac{t}{T} = nt$ where n is the mean motion. Then $\nu(t) = nt$.

6.2 Elliptical Orbit

For $0 < e < 1$, we can try to integrate equation (6.0.1) or use Kepler's method of the inscribed ellipse in the circle.

6.2.1 Method 1: Kepler's Method

Let the area of the ellipse be $A_0 = \pi ab$, the green shaded area be A_1, and the blue triangular area be A_2. The equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If we solve for y, we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

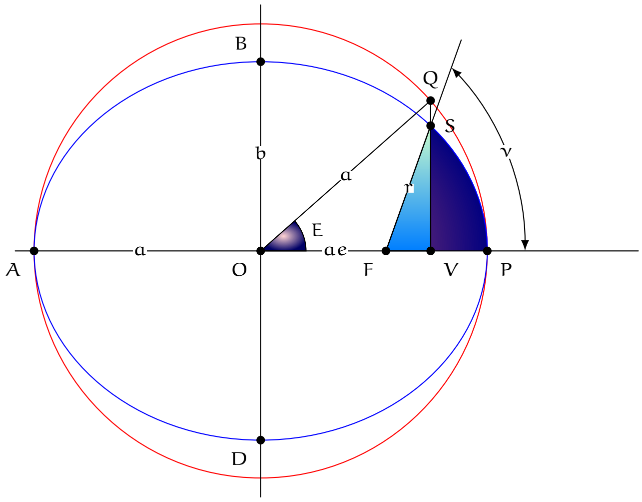


Figure 6.2.1: Ellipse inscribed in a circle.

Now, we can write $\frac{A_1}{A_0} = \frac{t-t_0}{T}$. The area of $A_1 = A_{SVP} - A_2$ where $A_2 = \frac{1}{2}$ base times height. The base can be found by taking $ae - a\cos(E) =$ base and the height is $b\sin(E) =$ height. Therefore, A_2 can be expressed as

$$A_2 = \frac{ab}{2}[e\sin(E) - \cos(E)\sin(E)].$$

Now let's consider the area A_{SVP}. We can write the area as $A_{SVP} = \frac{b}{a}A_{QVP}$ where

$$\begin{aligned} A_{QVP} &= A_{QOP} - A_{QOV} \\ &= \frac{1}{2}a^2E - \frac{1}{2}a^2\cos(E)\sin(E). \end{aligned}$$

Therefore, $A_{SVP} = \frac{ab}{2}\big[E - \sin(E)\cos(E)\big]$ and $A_1 = \frac{ab}{2}\big[E - e\sin(E)\big]$. Recall that $\frac{A_1}{A_0} = \frac{t-t_0}{T}$ so

$$\frac{t-t_0}{T} = \frac{1}{2\pi}\big[E - e\sin(E)\big]$$

where $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$.

$$\sqrt{\frac{\mu}{a^3}}(t - t_0) = E - e\sin(E) \tag{6.2.1}$$

where $n = \sqrt{\frac{\mu}{a^3}}$; that is, the mean motion $M_e = n(t - t_0) = E - e\sin(E)$ and E is the eccentric anomaly.

6.2.2 Method 2: Integration

We can always try to directly integrate the function.

$$\frac{\mu^2}{h^3}t = \int_0^{\nu} \frac{d\nu'}{(1+e\cos(\nu'))^2} \tag{6.2.2}$$

Consider

$$\frac{d}{d\nu'}\frac{\sin(\nu')}{1+e\cos(\nu')} = \frac{\cos(\nu')+e}{(1+e\cos(\nu'))^2}$$

Then

$$\frac{d}{d\nu'}\frac{e\sin(\nu')}{1+e\cos(\nu')} = \frac{1}{1+e\cos(\nu')} + \frac{e^2-1}{(1+e\cos(\nu'))^2}.$$

We can now isolate a somewhat easier integral.

$$\int_0^{\nu} \frac{d\nu'}{(1+e\cos(\nu'))^2} = \frac{e}{e^2-1}\frac{\sin(\nu)}{1+e\cos(\nu)} - \frac{1}{e^2-1}\int_0^{\nu}\frac{d\nu'}{1+e\cos(\nu')} \tag{6.2.3}$$

After integrating equation (6.2.3), we end up with

$$\int_0^{\nu} \frac{d\nu'}{(1+e\cos(\nu'))^2} = \frac{e}{e^2-1}\frac{\sin(\nu)}{1+e\cos(\nu)} - \frac{2}{(e^2-1)^{3/2}}\arctan\left[\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\nu}{2}\right)\right] = \frac{\mu^2}{h^3}t \tag{6.2.4}$$

and after simplifying equation (6.2.4), we have

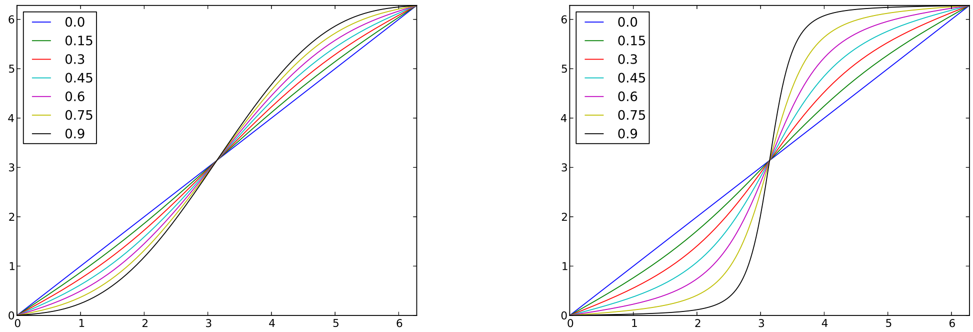
$$\frac{\mu^2}{h^3}(1-e^2)^{3/2}t = 2\arctan\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right)\right] - \frac{e\sqrt{1-e^2}\sin(\nu)}{1+e\cos(\nu)} = M_e. \tag{6.2.5}$$

Kepler's 3rd Law

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

That is, $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1-e^2}}\right)^3$. So $M_e = \left(\frac{2\pi}{T}\right)t = nt$. Referring back to figure 6.2.1, we have from the geometry that $a\cos(E) = ae + r\cos(\nu)$ where $r = \frac{p}{1+e\cos(\nu)} = \frac{a(1-e^2)}{1+e\cos(\nu)}$.

$$\begin{aligned} a\cos(E) &= ae + \frac{a(1-e^2)}{1+e\cos(\nu)} \\ \cos(E) &= \frac{e+\cos(\nu)}{1+e\cos(\nu)} \end{aligned}$$



(a) The mean anomaly as a function of the eccentric anomaly.

(b) The mean anomaly as a function of the true anomaly.

Figure 6.2.2: The mean anomaly as a function of the eccentric and true anomalies, respectively ($e < 1$).

Here we can solve for $\cos(\nu)$.

$$\cos(\nu) = \frac{e - \cos(E)}{e\cos(E) - 1}$$

Unfortunately, $\cos(\nu)$ is multi-valued for $\nu \in [0, 2\pi]$. Consider $\tan^2\left(\frac{E}{2}\right) = \frac{\sin^2\left(\frac{E}{2}\right)}{\cos^2\left(\frac{E}{2}\right)}$. Using the power rule for sine and cosine, we have

$$\tan^2\left(\frac{E}{2}\right) = \frac{1 - \cos(E)}{1 + \cos(E)}.$$

Next write $1 - \cos(E) = \frac{1+\cos(\nu)}{1+\cos(\nu)} - \frac{e+\cos(\nu)}{1+\cos(\nu)} = \frac{(1-e)(1-\cos(\nu))}{1+e\cos(\nu)}$ and $1 + \cos(E) = \frac{(1+e)(1+\cos(\nu))}{1+e\cos(\nu)}$. Then

$$\tan^2\left(\frac{E}{2}\right) = \frac{1-e}{1+e}\underbrace{\frac{1-\cos(\nu)}{1+\cos(\nu)}}_{\tan^2\left(\frac{\nu}{2}\right)}.$$

Thus, $E = 2\arctan\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right)\right]$ which is the first term in Kepler's method. As for the $\sin(E)$ term,

$$\begin{aligned} \sin(E) &= \sqrt{1 - \cos^2(E)} \\ &= \frac{\sqrt{1-e^2}\sin(\nu)}{1 + e\cos(\nu)} \end{aligned}$$

which is exactly what we needed.