Solutions to Roman's Advanced Linear Algebra

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Part I.

Basic Linear Algebra

1. Vector Spaces

1. Let V be a vector space over F. Prove that $0\boldsymbol{v} = \mathbf{0}$ and $r\mathbf{0} = \mathbf{0}$ for all $\boldsymbol{v} \in V$ and $r \in F$. Describe the different 0's in these equations. Prove that if $r\boldsymbol{v} = \mathbf{0}$, then $r = 0$ or $\boldsymbol{v} = \mathbf{0}$. Prove that $r\boldsymbol{v} = \boldsymbol{v}$ implies that $\boldsymbol{v} = \mathbf{0}$ or $r = 1$.

   With $0\boldsymbol{v}$, we scalar multiplication; that is, $0 \in F$ and $\boldsymbol{v} \in V$. Therefore, $0 \cdot v_i$ for $i = 1, 2, \ldots$ and where v_i is the i-th position of $\boldsymbol{v}$. Now, $0 \cdot v_i = 0$ for i so $0\boldsymbol{v} = \mathbf{0}$ where $\mathbf{0}$ is the zero vector. In the case of $r\mathbf{0}$ where $r \in F$ and $\mathbf{0} \in V$, we have for each i-th position of the zero vector a zero. Thus, $r \cdot 0_i \Rightarrow \mathbf{0}$ for each i again leaving us with the zero vector. In the first problem, where we have $0\boldsymbol{v}$, zero was a scalar in the field F; however, the solution to $0\boldsymbol{v} = \mathbf{0}$ was the zero vector in V. In second problem, where we have $r\mathbf{0}$, zero was a vector in V and the solution $r\mathbf{0} = \mathbf{0}$ was also the zero vector in V.

   Suppose on the contrary that $r\boldsymbol{v} = \mathbf{0}$, $r \neq 0$, and $\boldsymbol{v} \neq \mathbf{0}$. Since $r \neq 0$, we can divide out by r so $r\boldsymbol{v} = \mathbf{0} \iff \boldsymbol{v} = \mathbf{0}$. Thus, we have reached a contradiction; therefore, if $r\boldsymbol{v} = \mathbf{0}$, then either $r = 0$ or $\boldsymbol{v} = \mathbf{0}$.

   Let v_i be the i-th component of $\boldsymbol{v}$. Then $rv_i = v_i$ for all i. Now, suppose on the contrary that $r\boldsymbol{v} = \mathbf{0}$, $r \neq 1$, and $\boldsymbol{v} \neq \mathbf{0}$. Now, the equation $rv_i = v_i \iff r = 1$ or $v_i = 0$. Hence, we have reached a contradiction, and if $r\boldsymbol{v} = \boldsymbol{v}$, then either $r = 1$ or $\boldsymbol{v} = \mathbf{0}$.

2. Prove theorem 1.3 : The set $S(V)$ of all subspaces of a vector space V is a complete lattice under set inclusion, with smallest element $\{0\}$, largest element V, meet

$$\mathrm{glb}\{S_i : i \in K\} = \bigcap_{i \in K} S_i$$

   and join

$$\mathrm{lub}\{S_i : i \in K\} = \sum_{i \in K} S_i.$$

   We are given that the maximal and minimal elements of $S(V)$ are V and $\{0\}$, respectively. Therefore, we only need to show that each pair of elements has a meet and a join. If $S_i \not\subset S_j$ for all $i, j \in K$, then $\bigcap_i S_i = \varnothing$ so the meet is the empty set, $\{0\}$. Since $\varnothing \in S_i$ trivial for all $i \in K$, the set inclusion property is satisfied. For the join, we have $\sum_i S_i = \bigcup_i S_i$ so the join is the union of the sets. Now is $S_j \in \bigcap_i S_i$, then $S_j \subset \bigcap_i S_i$ so the set inclusion property is satisfied. Suppose $S_j \neq \varnothing$ and $S_j \subset S_i$ for some $i, j \in K$. Then $\bigcap_i S_i = S_j$ for S_j the smallest set of the union; that is, $S_j \subset S_1$, $S_j \subset S_2$, and so on where either $S_i \not\subset S_{i+1}$ or $S_i \subset S_{i+1}$ for $i \neq j$. Therefore, the meet is the smallest set $S_j = \bigcap_i S_i$. Again, the join is simple the union of all the sets in $\sum_i S_i$ which could be the maximal element V. Thus, $S(V)$ is a complete lattice.

3. (a) Find an abelian group V and a field F for which V is a vector space over F in at least two different ways, that is, there are two different definitions of scalar multiplication making V a vector space over F.

   (b) Find a vector space V over F and a subset S of V that is (1) a subspace of V and (2) a vector space using operations that differ from those of V.

4. Suppose that V is a vector space with basis $\mathcal{B} = \{b_i : i \in I\}$ and S is a subspace of V. Let $\{B_1, \ldots, B_k\}$ be a partition of $\mathcal{B}$. Then is it true that

$$S = \bigoplus_{i=1}^{k} (S \cap \langle B_i \rangle)$$

   What if $S \cap \langle B_i \rangle \neq \varnothing$ for all i?