Solutions to Roman's Advanced Linear Algebra

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Part I.

Basic Linear Algebra

1. Vector Spaces

1. Let V be a vector space over F. Prove that $0v = 0$ and $r0 = 0$ for all $v \in V$ and $r \in F$. Describe the different 0's in these equations. Prove that if $rv = 0$, then $r = 0$ or $v = 0$. Prove that $rv = v$ implies that $v = 0$ or $r = 1$.

   With $0v$, we scalar multiplication; that is, $0 \in F$ and $v \in V$. Therefore, $0 \cdot v_i$ for $i = 1, 2, \ldots$ and where v_i is the i-th position of v. Now, $0 \cdot v_i = 0$ for i so $0v = 0$ where 0 is the zero vector. In the case of $r0$ where $r \in F$ and $0 \in V$, we have for each i-th position of the zero vector a zero. Thus, $r \cdot 0_i \Rightarrow 0$ for each i again leaving us with the zero vector. In the first problem, where we have $0v$, zero was a scalar in the field F; however, the solution to $0v = 0$ was the zero vector in V. In second problem, where we have $r0$, zero was a vector in V and the solution $r0 = 0$ was also the zero vector in V.

   Suppose on the contrary that $rv = 0$, $r \neq 0$, and $v \neq 0$. Since $r \neq 0$, we can divide out by r so $rv = 0 \iff v = 0$. Thus, we have reached a contradiction; therefore, if $rv = 0$, then either $r = 0$ or $v = 0$.

2. Prove theorem 1.3 : The set $S(V)$ of all subspaces of a vector space V is a complete lattice under set inclusion, with smallest element $\{0\}$, largest element V, meet

$$\text{glb}\{S_i : i \in K\} = \bigcap_{i \in K} S_i$$

   and join

$$\text{lub}\{S_i : i \in K\} = \sum_{i \in K} S_i.$$

3. (a) Find an abelian group V and a field F for which V is a vector space over F in at least two different ways, that is, there are two different definitions of scalar multiplication making V a vector space over F.

   (b) Find a vector space V over F and a subset S of V that is (1) a subspace of V and (2) a vector space using operations that differ from those of V.

4. Suppose that V is a vector space with basis $\mathcal{B} = \{b_i : i \in I\}$ and S is a subspace of V. Let $\{B_1, \ldots, B_k\}$ be a partition of $\mathcal{B}$. Then is it true that

$$S = \bigoplus_{i=1}^{k} (S \cap \langle B_i \rangle)$$

   What if $S \cap \langle B_i \rangle \neq \varnothing$ for all i?