

SOLUTIONS TO SCHIFF'S THE LAPLACE TRANSFORM: THEORY AND  
APPLICATIONS

By:

DUSTIN SMITH



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# 1 Basic Principles

## 1.1 The Laplace Transform

1. From the definition of the Laplace transform compute  $\mathcal{L}\{f(t)\}(s)$  for

(a)  $f(t) = 4t$

The Laplace transform is defined as  $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt$ . Using the definition, we have

$$\mathcal{L}\{4t\}(s) = \int_0^\infty 4te^{-st}dt = 4 \lim_{\tau \rightarrow \infty} \int_0^\tau te^{-st}dt$$

We can now use integration by parts or we may notice that  $-\frac{\partial}{\partial s}e^{-st} = te^{-st}$ . Note that  $t$  and  $e^{-st}$  are entire functions in both the variable  $s$  and  $t$ ; that is,  $t$  and  $e^{-st}$  can be represented by a power series that converges everywhere in the complex plane with an infinite radius of convergence. In order to justify differentiation under the integral sign, the following theorem must be satisfied.

**Theorem:** Let  $f(s, t)$  be a function such that both  $f(s, t)$  and its partial derivatives are continuous in  $s$  and  $t$  in some region of the  $(s, t)$ -plane, including  $a(s) \leq t \leq b(s)$ ,  $s_0 \leq s \leq s_1$ . Also suppose that the functions  $a(s)$  and  $b(s)$  are both continuous and both have continuous derivatives for  $s_0 \leq s \leq s_1$ . Then for  $s_0 \leq s \leq s_1$

$$\frac{d}{ds} \int_{a(s)}^{b(s)} f(s, t)dt = f(s, b(s))b'(s) - f(s, a(s))a'(s) + \int_{a(s)}^{b(s)} f_s(s, t)dt.$$

Since  $f(s, t) = te^{-st}$  which is the product of two entire functions so is entire and  $t, e^{-st} \in C^\infty$ , the theorem is satisfied. Therefore, we can write

$$\begin{aligned} 4 \lim_{\tau \rightarrow \infty} \int_0^\tau te^{-st}dt &= -4 \frac{d}{ds} \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-st}dt \\ &= 4 \frac{d}{ds} \lim_{\tau \rightarrow \infty} \left. \frac{e^{-st}}{s} \right|_0^\tau \\ &= 4 \frac{d}{ds} \lim_{\tau \rightarrow \infty} \left( \frac{e^{-s\tau} - 1}{s} \right) \\ &= \frac{d}{ds} \frac{-4}{s} \\ \mathcal{L}\{4t\}(s) &= \frac{4}{s^2} \end{aligned}$$

(b)  $f(t) = e^{2t}$

For brevity, I am going to drop  $\lim_{\tau \rightarrow \infty} \int_0^\tau$  and just write  $\int_0^\infty$ .

$$\mathcal{L}\{e^{2t}\}(s) = \int_0^\infty e^{t(2-s)}dt = \left. \frac{e^{t(2-s)}}{2-s} \right|_0^\infty = \frac{1}{s-2}$$

A justifiable question would be why don't we have  $\infty$  since  $\lim_{t \rightarrow \infty} e^{t(2-s)} = \infty$ . If this was the case, we would have a divergent integral. In order for our integral to converge, we have to mind the exponential term. Let  $s = \sigma + i\omega$ . Then  $\exp[2t - \sigma] \exp[-i\omega]$ .

$$\left| \int_0^\infty \exp[(2-\sigma)t] \exp[-i\omega t]dt \right| \leq \int_0^\infty |\exp[(2-\sigma)t]| |\exp[-i\omega t]|dt$$

By Euler's formula,  $\exp[-i\omega] = \cos(\omega) - i\sin(\omega)$  so  $|e^{-i\omega}| = \sqrt{\cos^2(\omega) + \sin^2(\omega)} = 1$ . Thus, we have

$$\int_0^\infty \exp[t(2-\sigma)]dt$$

which converges when  $t(2-\sigma) < 0 \iff 2 < \sigma = \Re\{s\}$ . Therefore, since  $t(2-\sigma) < 0$ , we are justified in writing  $-t(\sigma-2)$  since  $\sigma-2 > 0$ . We then have  $\mathcal{L}\{e^{2t}\}(s) = \frac{1}{s-2}$ .

(c)  $f(t) = 2 \cos(3t)$

For this problem, we will need to use integration by parts twice. Let  $u = e^{-st}$  and  $dv = \cos(3t)dt$ . Then  $du = -se^{-st}$  and  $v = \sin(3t)/3$

$$\begin{aligned}\mathcal{L}\{2 \cos(3t)\}(s) &= 2 \int_0^{\infty} \cos(3t)e^{-st} dt \\ &= \frac{2e^{-st} \sin(3t)}{3} \Big|_0^{\infty} + \frac{2s}{3} \int_0^{\infty} \sin(3t)e^{-st} dt\end{aligned}$$

For the second integration by parts,  $u$  and  $du$  will remain the same but  $dv = \sin(3t)dt$  so  $v = -\cos(3t)/3$ .

$$\begin{aligned}&= \frac{2s}{3} \left[ \frac{-e^{-st} \cos(3t)}{3} \Big|_0^{\infty} - \frac{s}{3} \int_0^{\infty} \cos(3t)e^{-st} dt \right] \\ (1 + s^2/9)2 \int_0^{\infty} \cos(3t)e^{-st} dt &= \frac{2s}{9} \\ \mathcal{L}\{2 \cos(3t)\}(s) &= \frac{2s}{s^2 + 9}\end{aligned}$$

(d)  $f(t) = 1 - \cos(\omega t)$

By exercise 1 (c), we have

$$\mathcal{L}\{1 - \cos(\omega t)\}(s) = \int_0^{\infty} e^{-st} dt - \int_0^{\infty} \cos(\omega t)e^{-st} dt = \frac{1}{s} - \frac{s}{s^2 + \omega^2}$$

(e)  $f(t) = te^{2t}$

$$\mathcal{L}\{te^{2t}\} = -\frac{\partial}{\partial s} \int_0^{\infty} e^{t(2-s)} dt = -\frac{d}{ds} \frac{1}{s-2} = \frac{1}{(s-2)^2}$$

where we require that  $\Re\{s\} > 2$ .

(f)  $f(t) = e^t \sin(t)$

Again, we will use integration by parts where  $u = \sin(t)$  and  $dv = e^{t(1-s)}dt$ .

$$\begin{aligned}\mathcal{L}\{e^t \sin(t)\} &= \int_0^{\infty} \sin(t)e^{t(1-s)} dt \\ &= \frac{\sin(t)e^{t(1-s)}}{1-s} \Big|_0^{\infty} + \frac{1}{s-1} \int_0^{\infty} \cos(t)e^{t(1-s)} dt \\ &= \frac{1}{s-1} \left[ \frac{\cos(t)e^{t(1-s)}}{1-s} \Big|_0^{\infty} - \frac{1}{s-1} \int_0^{\infty} \sin(t)e^{t(1-s)} dt \right] \\ [1 + 1/(s+1)^2] \int_0^{\infty} \sin(t)e^{t(1-s)} dt &= \frac{1}{(s-1)^2} \\ \int_0^{\infty} \sin(t)e^{t(1-s)} dt &= \frac{1}{(s-1)^2 + 1}\end{aligned}$$

(g)  $f(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \int_0^a 0 \cdot e^{-st} dt + \int_a^{\infty} e^{-st} dt \\ &= \int_a^{\infty} e^{-st} dt \\ &= \frac{e^{-as}}{s}\end{aligned}$$

$$(h) \quad f(t) = \begin{cases} \sin(\omega t), & 0 < t < \pi/\omega \\ 0, & \pi/\omega \leq t \end{cases}$$

This is just another exercise in integration by parts.

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^{\pi/\omega} \sin(t\omega) e^{-st} dt \\ &= \frac{\omega(1 + e^{-s\pi/\omega})}{s^2 + \omega^2} \end{aligned}$$

$$(i) \quad f(t) = \begin{cases} 2, & t \leq 1 \\ e^t, & t > 1 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^1 e^{-st} dt + \int_1^\infty e^{t(1-s)} dt \\ &= \frac{2(1 - e^{-s})}{s} + \frac{e^{1-s}}{s-1} \end{aligned}$$

where we require that  $\Re\{s\} > 1$ .

2. Compute the Laplace transform of the function  $f(t)$  whose graph is given in figure 1.1.

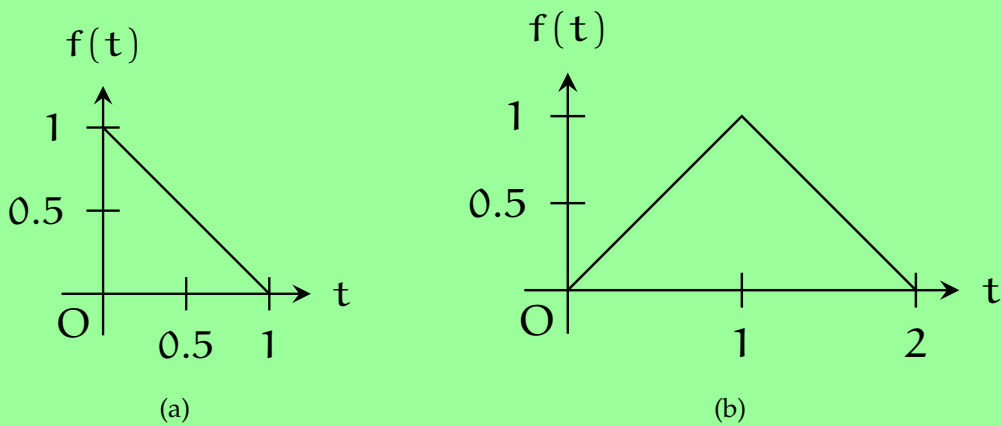


Figure 1.1: Plots of  $f(t)$  for problem two.

The first step to these two problems is to determine the the function  $f(t)$ . For figure 1.1a, we have

$$f(t) = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

and for figure 1.1b, we have

$$g(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

For figure 1.1a, the Laplace transform is

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^1 (1-t) e^{-st} dt \\ &= \frac{1 - e^{-s}}{s} + \frac{\partial}{\partial s} \int_0^1 e^{-st} dt \\ &= \frac{1 - e^{-s}}{s} + \frac{d}{ds} \frac{1 - e^{-s}}{s} \\ &= \frac{s - 1 + e^{-s}}{s^2} \end{aligned}$$

For figure 1.1b, the Laplace transform is

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= \int_0^1 te^{-st} dt + \int_1^2 (2-t)e^{-st} dt \\ &= \frac{e^{-2s}(e^s - 1)^2}{s^2}\end{aligned}$$

## 1.2 Convergence

1. Suppose that  $f$  is a continuous function on  $[0, \infty)$  and  $|f(t)| \leq M < \infty$  for  $0 \leq t < \infty$ .
  - (a) Show that the Laplace transform  $F(s) = \mathcal{L}\{f(t)\}(s)$  converges absolutely (and hence converges) for any  $s$  satisfying  $\Re\{s\} > 0$ .

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st} dt \quad (1.1)$$

In order for the Laplace transform to converge absolutely, we need to show that  $|\int_0^\infty f(t)e^{-st} dt| < \infty$ .

$$\begin{aligned}&= \left| \int_0^\infty f(t)e^{-st} dt \right| \\ &\leq M \int_0^\infty |e^{-st}| dt\end{aligned}$$

Now,  $s = \sigma + i\omega$  where  $\sigma, \omega \in \mathbb{R}$ . Then  $|e^{-st}| = e^{t\sigma}|e^{-it\omega}|$  since  $|e^{-t\sigma}| = \sqrt{e^{t^2\sigma^2}} = e^{t\sigma}$ . Recall the trigonometric identity  $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ . Then  $|e^{-it\omega}| = \sqrt{\Re\{e^{-it\omega}\}^2 + \Im\{e^{-it\omega}\}^2} = 1$ .

$$= M \int_0^\infty e^{t\sigma} dt \quad (1.2)$$

Equation (1.2) converges when  $e^{t\sigma}$  is bounded. That is,  $t\sigma < 0$ . From equation (1.1), we see that boundedness occurs when  $-st < 0$  so  $st > 0$ . Then  $-s < 0 \iff s > 0 \iff \Re\{s\} > 0$  where  $\Re\{s\} = \sigma$  since, by equation (1.2), boundedness was due to  $\sigma$ .

- (b) Show that  $\mathcal{L}\{f(t)\}$  converges uniformly if  $\Re\{s\} > x_0 > 0$ .
- (c) Show that  $F(s) = \mathcal{L}\{f(t)\}(s) \rightarrow 0$  as  $\Re\{s\} \rightarrow \infty$ .
2. Let  $f(t) = e^t$  on  $[0, \infty)$ .
  - (a) Show that  $F(s) = \mathcal{L}\{e^t\}(s)$  converges for  $\Re\{s\} > 1$ .
  - (b) Show that  $\mathcal{L}\{e^t\}(s)$  converges uniformly if  $\Re\{s\} > x_0 > 1$ .
  - (c) Show that  $F(s) = \mathcal{L}\{e^t\}(s) \rightarrow 0$  as  $\Re\{s\} \rightarrow \infty$ .
3. Show that the Laplace transform of the function  $f(t) = 1/t$ ,  $t > 0$  does not exist for any value of  $s$ .

## 1.3 Continuity Requirements

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