Solutions to Schiff's The Laplace Transform: Theory and Applications

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Contents

1 Basic Principles

1.1 The Laplace Transform

1. From the definition of the Laplace transform compute $\mathcal{L}\{f(t)\}(s)$ for

   (a) $f(t) = 4t$

   The Laplace transform is defined as $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt$. Using the definition, we have

$$\mathcal{L}\{4t\}(s) = \int_0^\infty 4te^{-st}dt = 4\lim_{\tau\to\infty}\int_0^\tau te^{-st}dt$$

   We can now use integration by parts or we may notice that $-\frac{\partial}{\partial s}e^{-st} = te^{-st}$. Note that t and e^{-st} are entire functions in both the variable s and t; that is, t and e^{-st} can be represented by a power series that converges everywhere in the complex plane with an infinite radius of convergence. In order to justify differentiation under the integral sign, the following theorem must be satisfied.

   Theorem: Let $f(s,t)$ be a function such that both $f(s,t)$ and its partial derivatives are continuous in s and t in some region of the (s,t)-plane, including $a(s) \leqslant t \leqslant b(s)$, $s_0 \leqslant s \leqslant s_1$. Also suppose that the functions $a(s)$ and $b(s)$ are both continuous and both have continuous derivatives for $s_0 \leqslant s \leqslant s_1$. Then for $s_0 \leqslant s \leqslant s_1$

$$\frac{d}{ds}\int_{a(s)}^{b(s)} f(s,t)dt = f(s,b(s))b'(s) - f(s,(a(s))a'(s) + \int_{a(s)}^{b(s)} f_s(s,t)dt.$$

   Since $f(s,t) = te^{-st}$ which is the product of two entire functions so is entire and $t, e^{-st} \in C^\infty$, the theorem is satisfied. Therefore, we can write

$$\begin{aligned}
4\lim_{\tau\to\infty}\int_0^\tau te^{-st}dt &= -4\frac{d}{ds}\lim_{\tau\to\infty}\int_0^\tau e^{-st}dt \\
&= 4\frac{d}{ds}\lim_{\tau\to\infty}\frac{e^{-st}}{s}\Big|_0^\tau \\
&= 4\frac{d}{ds}\lim_{\tau\to\infty}\left(\frac{e^{-s\tau}-1}{s}\right) \\
&= \frac{d}{ds}\frac{-4}{s} \\
\mathcal{L}\{4t\}(s) &= \frac{4}{s^2}
\end{aligned}$$

   (b) $f(t) = e^{2t}$

   For brevity, I am going to drop $\lim_{\tau\to\infty}\int_0^\tau$ and just write $\int_0^\infty$.

$$\mathcal{L}\{e^{2t}\}(s) = \int_0^\infty e^{t(2-s)}dt = \frac{e^{t(2-s)}}{2-s}\Big|_0^\infty = \frac{1}{s-2}$$

   A justifiable question would be why don't we have ∞ since $\lim_{t\to\infty} e^{t(2-s)} = \infty$. If this was the case, we would have a divergent integral. In order for our integral to converge, we have to mind the exponential term. Let $s = \sigma + i\omega$. Then $\exp[2t-\sigma]\exp[-i\omega]$.

$$\left|\int_0^\infty \exp[(2-\sigma)t]\exp[-it\omega]dt\right| \leqslant \int_0^\infty \big|\exp[(2-\sigma)t]\big|\big|\exp[-it\omega]\big|dt$$

   By Euler's formula, $\exp[-it\omega] = \cos(t\omega) - i\sin(t\omega)$ so $|e^{-i\omega}| = \sqrt{\cos^2(t\omega)+\sin^2(t\omega)} = 1$. Thus, we have

$$\int_0^\infty \exp[t(2-\sigma)]dt$$

   which converges when $t(2-\sigma) < 0 \iff 2 < \sigma = \Re\{s\}$. Therefore, since $t(2-\sigma) < 0$, we are justified in writing $-t(\sigma-2)$ since $\sigma - 2 > 0$. We then have $\mathcal{L}\{e^{2t}\}(s) = \frac{1}{s-2}$.

(c) $f(t) = 2\cos(3t)$

For this problem, we will need to use integration by parts twice. Let $u = e^{-st}$ and $dv = \cos(3t)dt$. Then $du = -se^{-st}$ and $v = \sin(3t)/3$

$$\begin{aligned}
\mathcal{L}\{2\cos(3t)\}(s) &= 2\int_0^\infty \cos(3t)e^{-st}dt \\
&= \cancelto{0}{\left.\frac{2e^{-st}\sin(3t)}{3}\right|_0^\infty} + \frac{2s}{3}\int_0^\infty \sin(3t)e^{-st}dt
\end{aligned}$$

For the second integration by parts, u and du will remain the same but $dv = \sin(3t)dt$ so $v = -\cos(3t)/3$.

$$\begin{aligned}
&= \frac{2s}{3}\left[\left.\frac{-e^{-st}\cos(3t)}{3}\right|_0^\infty - \frac{s}{3}\int_0^\infty \cos(3t)e^{-st}dt\right] \\
(1+s^2/9)2\int_0^\infty \cos(3t)e^{-st}dt &= \frac{2s}{9} \\
\mathcal{L}\{2\cos(3t)\}(s) &= \frac{2s}{s^2+9}
\end{aligned}$$

(d) $f(t) = 1 - \cos(\omega t)$

By exercise 1 (c), we have

$$\mathcal{L}\{1-\cos(\omega t)\}(s) = \int_0^\infty e^{-st}dt - \int_0^\infty \cos(\omega t)e^{-st}dt = \frac{1}{s} - \frac{s}{s^2+\omega^2}$$

(e) $f(t) = te^{2t}$

$$\mathcal{L}\{te^{2t}\} = -\frac{\partial}{\partial s}\int_0^\infty e^{t(2-s)}dt = -\frac{d}{ds}\frac{1}{s-2} = \frac{1}{(s-2)^2}$$

where we require that $\Re\{s\} > 2$.

(f) $f(t) = e^t\sin(t)$

Again, we will use integration by parts where $u = \sin(t)$ and $dv = e^{t(1-s)}dt$.

$$\begin{aligned}
\mathcal{L}\{e^t\sin(t)\} &= \int_0^\infty \sin(t)e^{t(1-s)}dt \\
&= \cancelto{0}{\left.\frac{\sin(t)e^{t(1-s)}}{1-s}\right|_0^\infty} + \frac{1}{s-1}\int_0^\infty \cos(t)e^{t(1-s)}dt \\
&= \frac{1}{s-1}\left[\left.\frac{\cos(t)e^{t(1-s)}}{1-s}\right|_0^\infty - \frac{1}{s-1}\int_0^\infty \sin(t)e^{t(1-s)}dt\right] \\
[1+1/(s+1)^2]\int_0^\infty \sin(t)e^{t(1-s)}dt &= \frac{1}{(s-1)^2} \\
\int_0^\infty \sin(t)e^{t(1-s)}dt &= \frac{1}{(s-1)^2+1}
\end{aligned}$$

(g) $f(t) = \begin{cases} 1, & t \geqslant a \\ 0, & t < a \end{cases}$

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \int_0^a 0\cdot e^{-st}dt + \int_a^\infty e^{-st}dt \\
&= \int_a^\infty e^{-st}dt \\
&= \frac{e^{-as}}{s}
\end{aligned}$$

(h) $f(t) = \begin{cases} \sin(\omega t), & 0 < t < \pi/\omega \\ 0, & \pi/\omega \leqslant t \end{cases}$

This is just another exercise in integration by parts.

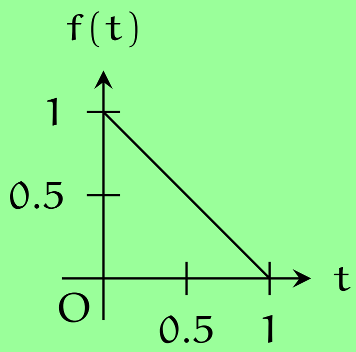
$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^{\pi/\omega} \sin(t\omega) e^{-st} dt \\ &= \frac{\omega(1 + e^{-s\pi/\omega})}{s^2 + \omega^2} \end{aligned}$$

(i) $f(t) = \begin{cases} 2, & t \leqslant 1 \\ e^t, & t > 1 \end{cases}$

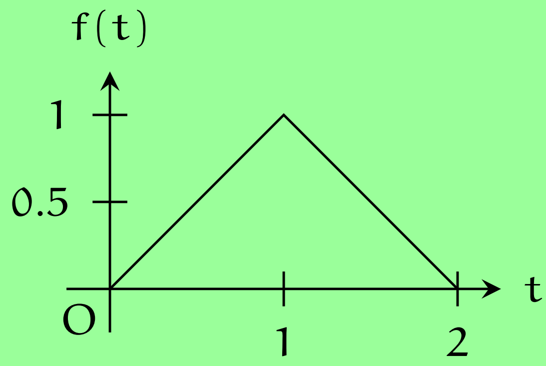
$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^1 e^{-st} dt + \int_1^\infty e^{t(1-s)} dt \\ &= \frac{2(1 - e^{-s})}{s} + \frac{e^{1-s}}{s-1} \end{aligned}$$

where we require that $\Re\{s\} > 1$.

2. Compute the Laplace transform of the function $f(t)$ whose graph is given in figure 1.1.



(a)



(b)

Figure 1.1: Plots of $f(t)$ for problem two.

The first step to these two problems is to determine the the function $f(t)$. For figure 1.1a, we have

$$f(t) = \begin{cases} 1 - t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

and for figure 1.1b, we have

$$g(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

For figure 1.1a, the Laplace transform is

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^1 (1 - t) e^{-st} dt \\ &= \frac{1 - e^{-s}}{s} + \frac{\partial}{\partial s} \int_0^1 e^{-st} dt \\ &= \frac{1 - e^{-s}}{s} + \frac{d}{ds} \frac{1 - e^{-s}}{s} \\ &= \frac{s - 1 + e^{-s}}{s^2} \end{aligned}$$

For figure 1.1b, the Laplace transform is

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt \\ &= \frac{e^{-2s}(e^s-1)^2}{s^2}\end{aligned}$$

1.2 Convergence

1. Suppose that f is a continuous function on $[0,\infty)$ and $|f(t)| \leqslant M < \infty$ for $0 \leqslant t < \infty$.

(a) Show that the Laplace transform $F(s) = \mathcal{L}\{f(t)\}(s)$ converges absolutely (and hence converges) for any s satisfying $\Re\{s\} > 0$.

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt \tag{1.1}$$

In order for the Laplace transform to converge absolutely, we need to show that $|\int_0^\infty f(t)e^{-st}dt| < \infty$.

$$\begin{aligned}&= \left|\int_0^\infty f(t)e^{-st}dt\right| \\ &\leqslant M\int_0^\infty |e^{-st}|dt\end{aligned}$$

Now, $s = \sigma + i\omega$ where $\sigma, \omega \in \mathbb{R}$. Then $|e^{-st}| = e^{t\sigma}|e^{-it\omega}|$ since $|e^{-t\sigma}| = \sqrt{e^{-2t\sigma}} = e^{-t\sigma}$. Recall the trigonometric identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Then $|e^{-it\omega}| = \sqrt{\Re\{e^{-it\omega}\}^2 + \Im\{e^{-it\omega}\}^2} = 1$.

$$= M\int_0^\infty e^{-t\sigma}dt \tag{1.2}$$

Equation (1.2) converges when $e^{-t\sigma}$ is bounded. That is, $-t\sigma < 0 \iff t\sigma > 0 \iff \sigma > 0$ since $t \in (0,\infty)$ so dividing out by t doesn't effect the inequality. Recall that $s = \sigma + i\omega$ so $\sigma > 0 \Rightarrow \Re\{s\} > 0$ in order for equation (1.1) to converge.

(b) Show that $\mathcal{L}\{f(t)\}(s)$ converges uniformly if $\Re\{s\} > a > 0$.

Let $\mathcal{L}\{f(t)\}(s) = F(s)$. If the Laplace transform converges uniformly, then for $\Re\{s\} > b$ and for $b > a$,

$$\sup_{s \geqslant b} |F_R(s) - F(s)| \to 0$$

as $R \to \infty$ for each $b > a$. Let $F_R(s) = \int_0^R f(t)e^{-st}dt$. Suppose $f(t)$ has exponential order; that is, $|f(t)| = Me^{at}$. Then

$$\begin{aligned}|F_R(s) - F(s)| &= \left|\int_R^\infty f(t)e^{-st}dt\right| \\ &\leqslant \int_R^\infty |f(t)|e^{-t\sigma}dt \\ &\leqslant M\int_R^\infty e^{-t(\sigma-a)}dt \\ &= \frac{M}{\sigma-a}e^{-R(\sigma-a)}\end{aligned} \tag{1.3}$$

Now for $\Re\{s\} \geqslant b > a$,

$$\sup_{\Re\{s\} \geqslant b} |F_R(s) - F(s)| \leqslant \frac{Me^{-R(b-a)}}{b-a} \to 0$$

as $R \to \infty$. Thus, $\mathcal{L}\{f(t)\}(s)$ converges uniformly for $\Re\{s\} > a$.

(c) Show that $F(s) = \mathcal{L}\{f(t)\}(s) \to 0$ as $\Re\{s\} \to \infty$.

From equation (1.3), we have

$$|F(s)| \leqslant \frac{Me^{-R(\Re\{s\}-a)}}{\Re\{s\} - a}$$

Now $R \in (0, \infty)$ so $R > 0$ and $\Re\{s\} > a > 0$ so $\Re\{s\} - a > 0$.

$$\begin{aligned} \lim_{\Re\{s\} \to \infty} \frac{Me^{-R(\Re\{s\}-a)}}{\Re\{s\} - a} &= \frac{M}{e^{\infty}\infty} \\ &= \frac{M}{\infty} \\ F(s) &\to 0 \end{aligned}$$

as $\Re\{s\} \to \infty$.

2. Let $f(t) = e^t$ on $[0, \infty)$.

   (a) Show that $F(s) = \mathcal{L}\{e^t\}(s)$ converges for $\Re\{s\} > 1$.

   $$\begin{aligned} \left| \int_0^\infty e^t e^{-st} dt \right| &\leqslant \int_0^\infty e^{-t(\sigma-1)} dt \\ &= \left. \frac{-e^{-t(\sigma-1)}}{\sigma - 1} \right|_0^\infty \end{aligned}$$

   In order for convergence, $-t(\sigma - 1) < 0 \iff \sigma > 1$. Since $s = \sigma + i\omega$, we have that $F(s)$ converges absolutely when $\sigma = \Re\{s\} > 1$.

   (b) Show that $\mathcal{L}\{e^t\}(s)$ converges uniformly if $\Re\{s\} > a > 1$.

   $$\begin{aligned} |F_R(s) - F(s)| &= \left| \int_R^\infty e^t e^{-st} dt \right| \\ &\leqslant \int_R^\infty e^{-t(\sigma-1)} dt \\ &= \frac{e^{-R(\sigma-1)}}{\sigma - 1} \end{aligned}$$

   For $\Re\{s\} \geqslant b > 1 > 0$,

   $$\sup_{\Re\{s\} \geqslant b} |F_R(s) - F(s)| \leqslant \frac{e^{-R(b-1)}}{b-1} \to 0$$

   as $R \to \infty$ since $b > 1 > 0 \iff b - 1 > 0 \iff b > 1$. Therefore, $\mathcal{L}\{e^t\}(s)$ converges uniformly when $\Re\{s\} > 1$.

   (c) Show that $F(s) = \mathcal{L}\{e^t\}(s) \to 0$ as $\Re\{s\} \to \infty$.

   This follows by the same reasoning of exercise 1 (c) with nothing new to gain from repeating the process.

3. Show that the Laplace transform of the function $f(t) = 1/t$, $t > 0$ does not exist for any value of s.

   Note the well known gamma function, $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$. Now, the Laplace transform of $f(t)$ is

   $$\mathcal{L}\{1/t\}(s) = \int_0^\infty \frac{e^{-st}}{t} dt$$

   Let $st = u$. Then $du = s dt$ and $1/t = s/u$. After making these substitutions, we have

   $$\int_0^\infty \frac{e^{-st}}{t} dt = s^2 \int_0^\infty u^{-1} e^{-u} du = \Gamma(0) \tag{1.4}$$

   By Euler's reflection formula, $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}$. Using Euler's reflection formula and equation (1.4), we have that

   $$\Gamma(0) = \lim_{z \to 0} \frac{\pi}{\Gamma(1)\sin(z\pi)}$$

   For n a positive integer, $\Gamma(n) = (n-1)!$ so $\Gamma(1) = 1$.

   $$\Gamma(0) = \lim_{z \to 0} \frac{\pi}{\sin(z\pi)} = \infty = \int_0^\infty \frac{e^{-st}}{t} dt = \mathcal{L}\{1/t\}(s)$$

   Therefore, the Laplace transform does not converge for $f(t) = 1/t$.

1.3 Continuity Requirements

Discuss the continuity of each of the following functions and locate any jump discontinuities.

1. $f(t) = \frac{1}{1+t}$

   The function $f(t)$ is continuous for $t \in \mathbb{R} \setminus \{-1\}$. Next, we need to determine what type of discontinuity we have $t_c = -1$.

   $$\lim_{t\to -1^-} \frac{1}{1+t} = -\infty$$
   $$\lim_{t\to -1^+} \frac{1}{1+t} = \infty$$

   Therefore, $f(t_c^-) \neq f(t_c^+)$ and neither are finite so we dont have jump or removable discontinuity.

2. $g(t) = t\sin(1/t)$ for $t \neq 0$.

   We need to determine what happens when $t = 0$. Now, sine is bounded by ± 1; that is, $-1 \leqslant \sin(1/t) \leqslant 1$ so $-t \leqslant t\sin(1/t) \leqslant t$. Since
   $$\lim_{t\to 0} -t = \lim_{t\to 0} t = 0,$$
   by the squeeze theorem, $\lim_{t\to 0} t\sin(1/t) = 0$. Thus, the left and right limits exists, are finite, and the same so we dont have a jump discontinuity at $t = 0$.

3. $h(t) = \begin{cases} t, & t \leqslant 1 \\ \frac{1}{1+t^2}, & t > 1 \end{cases}$

   The only value we need to check is when $t = 1$ since $t = \pm i \notin \mathbb{R}$. Then $\lim_{t\to 1^-} h(t) = 1$ and $\lim_{t\to 1^+} h(t) = 1/2$. Therefore, we the left right limits exists, finite, and not equal so we have a jump discontinuity at $t = 1$.

4. $i(t) = \begin{cases} \frac{\sinh(t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

5. $j(t) = \frac{1}{t}\sinh(1/t)$ for $t \neq 0$.

6. $k(t) = \begin{cases} \frac{1-e^{-t}}{t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$

7. $l(t) = \begin{cases} 1, & 2na \leqslant t < (2n+1)a \\ -1, & (2n+1)a \leqslant t < (2n+2)a \end{cases}$ for $a > 0$ and $n = 0, 1, \ldots$.

8. $m(t) = \lfloor \frac{t}{a} \rfloor$ for $t \geqslant 0$, $a > 0$ where $\lfloor x \rfloor$ is the greatest integer $\leqslant x$.