Numerical Modelling of Heat and Mass Transfer

– Construction of Dimensionless Groups –

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Abstract

This paper is intended as a supplement to the introduction to modelling of problems in heat and mass transfer. The construction of dimensionless numbers is discussed and illustrating examples are provided.

Contents

1 Construction of dimensionless groups based on a given differential equation

1.1 Steps of application

This method is mostly used when a governing differential equation is given. The dimensionless group will be constructed in the following steps:

1. Formulation of differential equation in appropriate form
2. Derivation of model equation from differential equation
3. Substitution of model variables
4. Division of differential equation by model equation
5. Choice of dimensionless group

1.2 Example: Non-isothermal fluid flow

1. Formulation of differential equation:

$$\underbrace{\frac{\partial}{\partial t}(\varrho\, h)}_{rate\ of\ change} = \underbrace{\mathrm{div}\,(\lambda\,\mathrm{grad}\,T)}_{conduction} - \underbrace{\mathrm{div}\,(\varrho\,\vec{v}\,h)}_{convection} + \underbrace{\dot{q}_{vol}}_{source} \tag{1}$$

Subtraction of the continuity equation

$$\frac{\partial \varrho}{\partial t} = \mathrm{div}\,(\,\varrho\,\vec{v}\,) \tag{2}$$

and substitution of the enthalpy $dh = c_p\, dT$ reduces the conservation equation to [1]

$$\underbrace{\varrho\,\frac{\partial}{\partial t}(c_p\,T)}_{rate\ of\ change} = \underbrace{\mathrm{div}\,(\lambda\,\mathrm{grad}\,T)}_{conduction} - \underbrace{\varrho\,\vec{v}\,\cdot\,\mathrm{grad}\,(c_p\,T)}_{convection} + \underbrace{\dot{q}_{vol}}_{source} \tag{3}$$

[1] $\mathrm{div}\,(\vec{v}\,u) = \mathrm{div}\,(\,\vec{v}\,)\,u + \vec{v}\,\cdot\,\mathrm{grad}\,u$

Now the equation will be applied to a steady problem in 1D space with constant material properties and without volume-specific source

$$\lambda \frac{d^2 T}{d x^2} = c_p \varrho v \frac{d T}{d x} \tag{4}$$

2. Derivation of model equation

$$\lambda_M \frac{d^2 T_M}{d x_M^2} = c_{p,M} \varrho_M v_M \frac{d T_M}{d x_M} \tag{5}$$

3. Substitution of model variables

The variables of the model equation will be replaced with the product of the variable i and the corresponding coefficient π_i, e.g.

$$x_M = x \pi_x \tag{6}$$

The model equation reads after substitution as [2]

$$(\lambda \pi_\lambda) \frac{d^2(T \pi_T)}{d(x \pi_x)^2} = (c_p \pi_{c_p}) (\varrho \pi_\varrho) (v \pi_v) \frac{d(T \pi_T)}{d(x \pi_x)} \tag{7}$$

$$\pi_\lambda \lambda \frac{\pi_T d^2T}{\pi_x^2 dx^2} = \pi_{c_p} \pi_\varrho \pi_v c_p \varrho v \frac{\pi_T dT}{\pi_x dx} \tag{8}$$

4. Division of differential equation (4) by model equation (5)

$$\pi_\lambda \frac{\pi_T}{\pi_x^2} = \pi_{c_p} \pi_\varrho \pi_v \frac{\pi_T}{\pi_x} \tag{9}$$

$$\frac{\pi_\lambda}{\pi_x} = \pi_{c_p} \pi_\varrho \pi_v \tag{10}$$

5. Choice of the dimensionless group Π

$$1 = \pi_v \pi_x \frac{\pi_{c_p} \pi_\varrho}{\pi_\lambda} \tag{11}$$

$$\Pi = v x \frac{c_p \varrho}{\lambda} \tag{12}$$

known as Peclet number

$$Pe = \frac{v l_{ref}}{a} \quad \text{with: } a = \frac{\lambda}{c_p \varrho} \tag{13}$$

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[2] Note: in the second order derivative is $d^2 (T \pi_T) = \pi_T d^2T$, but $d (x \pi_x)^2 = \pi_x^2 d^2 x$

2 Construction of Dimensionless Groups employing the Buckingham-Pi theorem

2.1 Steps of application

The Buckingham-Pi theorem can be used for the reduction of the number of parameters in situations where no governing differential equation is given. A matrix with the dimensional expressions of each variable

Variable	Symbol	Dimensions
length	l	L
mass	M	M
time	t	t
temperature	T	T

will be constructed in the following steps:

1. Dimensions of significant variables (number of involved variables $= n$)
2. Dimension matrix
3. Rank r of dimension matrix
4. Number of independent dimensionless parameters $i = n - r$
5. Choice of a core group (exclude variables whose effect should be isolated)
6. Formulation of dimensionless parameters π with exponents
7. Evaluation of the exponents

Step (5) is the creative part of this analysis. The achievement of physically meaningful dimensionless parameters depends on the proper choice of the core group.

2.2 Example: Isothermal flow of fluid external to solid body

1. Variables:

force	[M L / t^2]
velocity	v [L / t]
density	ϱ [M / L^3]
viscosity	μ [M / (L t)]
characteristic length of the body	L [L]
number of variables	$n = 5$

2. Dimensionless matrix represents the exponents of M, L, and t in the dimensional expression of each variable

	F	v	ϱ	μ	L
M	1	0	1	1	0
L	1	1	-3	-1	1
t	-2	-1	0	-1	0

3. Rank: number of columns in the largest nonzero determinant which can be formed $r = 3$

4. Number of dimensionless parameters: $i = n - r = 5 - 3 = 2$

5. Choice of core group of r variables (appears in every $\pi-$group, contains all fundamental dimensions): F and μ should be not in the core

$$\text{core} = (v, \varrho, L)$$

6. Dimensionsless parameters: π_1 and π_2 include variables of the core and F or μ, respectively:

$$\begin{aligned} \pi_1 &= F\, v^a \varrho^b L^c \\ \pi_2 &= \mu\, v^d \varrho^e L^f \end{aligned}$$

where a, b, c, d, e, and f are unknown exponents.

7. Evaluation of the exponents:

	$\pi_1 = F\ v^a \varrho^b L^c$	$\pi_2 = \mu\ v^d \varrho^e L^f$
	$\mathrm{M^0L^0t^0} = 1 = \frac{\mathrm{M\ L}}{\mathrm{t^2}} \left(\frac{\mathrm{L}}{\mathrm{t}}\right)^a \left(\frac{\mathrm{M}}{\mathrm{L^3}}\right)^b (\mathrm{L})^c$	$\mathrm{M^0L^0t^0} = 1 = \frac{\mathrm{M}}{\mathrm{L\ t}} \left(\frac{\mathrm{L}}{\mathrm{t}}\right)^d \left(\frac{\mathrm{M}}{\mathrm{L^3}}\right)^e (\mathrm{L})^f$
$M:$	$0 = 1 + b$	$0 = 1 + e$
$L:$	$0 = 1 + a - 3b + c$	$0 = -1 + d - 3e + f$
$t:$	$0 = -2 - a$	$0 = -1 - d$
	$a = -2,\ b = -1,\ c = -2$	$d = -1,\ e = -1,\ f = -1$
	$\pi_1 = F/(v^2\varrho L^2) \quad = Eu$	$\pi_2 = \mu/(v\varrho L) \quad = 1/Re$

2.3 Example: Heat transfer in forced convection

1. Variables:

velocity	v	[L / t]
density	ϱ	[M / L^3]
thermal conductivity	λ	[M L / (t^3 T)]
viscosity	η	[M / (L t)]
specific heat capacity (p=const)	c_p	[L^2 / (t^2 T)]
heat transfer coefficient	h	[M / (t^3 T)]
and diameter	D	[L]
number of variables		$n = 7$

2. Dimensionless matrix represents the exponents of M, L, t, and T in the dimensional expression of each variable

	v	ϱ	λ	η	c_p	h	D
M	1	1	1	1	0	1	0
L	1	-3	1	-1	2	0	1
t	0	0	-3	-1	-2	-3	0
T	0	0	-1	0	-1	-1	0

3. Rank: number of columns in the largest nonzero determinant which can be formed $r = 4$

4. Number of dimensionless parameters: $i = n - r = 7 - 4 = 3$

5. Choice of core group of r variables (appears in every $\pi-$group, contains all fundamental dimensions): h, η, and c_p should be not in the core

$$\text{core} = (v, D, \varrho, \lambda)$$

6. Dimensionless parameters: π_1, π_2, and π_3 include variables of the core and h, η, or c_p, respectively:

$$\begin{aligned} \pi_1 &= h\, v^a D^b \varrho^c \lambda^d \\ \pi_2 &= \eta\, v^e D^f \varrho^g \lambda^h \\ \pi_3 &= c_p\, v^i D^j \varrho^k \lambda^l \end{aligned} \tag{14}$$

where $a, b, \ldots, l$ are unknown exponents.

7. Evaluation of the exponents:

$$\pi_1 \;:\; \mathrm{M^0L^0t^0T^0} = \frac{\mathrm{M}}{\mathrm{t^3T}} \left(\frac{\mathrm{L}}{\mathrm{t}}\right)^a \mathrm{L}^b \left(\frac{\mathrm{M}}{\mathrm{L^3}}\right)^c \left(\frac{\mathrm{M\,L}}{\mathrm{t^3T}}\right)^d$$

$$\pi_2 \;:\; \mathrm{M^0L^0t^0T^0} = \frac{\mathrm{M}}{\mathrm{Lt}} \left(\frac{\mathrm{L}}{\mathrm{t}}\right)^e \mathrm{L}^f \left(\frac{\mathrm{M}}{\mathrm{L^3}}\right)^g \left(\frac{\mathrm{M\,L}}{\mathrm{t^3T}}\right)^h$$

$$\pi_3 \;:\; \mathrm{M^0L^0t^0T^0} = \frac{\mathrm{L^2}}{\mathrm{t^2T}} \left(\frac{\mathrm{L}}{\mathrm{t}}\right)^i \mathrm{L}^j \left(\frac{\mathrm{M}}{\mathrm{L^3}}\right)^k \left(\frac{\mathrm{M\,L}}{\mathrm{t^3T}}\right)^l$$

	$\pi_1 = h\, v^a D^b \varrho^c \lambda^d$	$\pi_2 = \eta\, v^e D^f \varrho^g \lambda^h$	$\pi_3 = c_p\, v^i D^j \varrho^k \lambda^l$
$M:$	$0 = 1 + c + d$	$0 = 1 + g + h$	$0 = k + l$
$L:$	$0 = a + b - 3c + d$	$0 = -1 + e + f - 3g + h$	$0 = 2 + i + j - 3k + l$
$t:$	$0 = -2 - a$	$0 = -1 - e - 3h$	$0 = -2 - i - 3l$
$T:$	$0 = -1 - d$	$0 = -h$	$0 = -1 - l$
	$a = 0, b = 1, c = 0, d = -1$	$e = f = g = -1, h = 0$	$i = j = k = 1, l = -1$
	$\pi_1 = hD/\lambda \quad = Nu$	$\pi_2 = \eta/(vD\varrho) \quad = 1/Re$	$\pi_3 = c_p v D \varrho/\lambda$

[3]

[3]Instead of π_3, the dimensionless Prandtl number $Pr = \pi_2\,\pi_3 = \nu/a$ with $a = \lambda/(c_p\,\varrho)$ and $\nu = \eta\,/\,\varrho$ is normally used. A typical application is the correlation of experimental data of heat transfer by forced convection expressed by the Nusselt number $Nu = Nu(\,Re,\, Pr\,)$