	1	2	3	4
	4	3 $\boxed{\text{D-H}}$	1	2
#	θ	d	a	α
0-1	θ_1	L_1	0	$-90°$
1-2	θ_2	0	L_2	0
2-3	θ_3	0	L_3	0

*$C90 = 0$
$-S90 = -1$

$Lc_n = \frac{1}{2} L_n$

$$T_0^1 = \underbrace{\begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rz_1} \begin{bmatrix} I & & & \\ & & & \\ & & & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & {}^*C90 & -S90 & 0 \\ 0 & S90 & C90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rx_1}$$

$$T_0^2 = \begin{bmatrix} Rz_1 Rx_1 & & & 0 \\ & & & 0 \\ & & & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rz_2} \begin{bmatrix} I & & & L_2 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 \begin{bmatrix} Rz_1 Rx_1 Rz_2 & & & x_1 \\ & & & x_2 \\ & & & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rz_3} \begin{bmatrix} I & & & L_3 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformations

Rotation Matrices

$$R_0^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1^0 = \begin{bmatrix} C_1 & 0 & S_1 \\ S_1 & 0 & -C_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_2^0 = \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 \\ S_1C_2 & -S_1S_2 & -C_1 \\ S_2 & C_2 & 0 \end{bmatrix}$$

$$R_3^0 = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 \\ S_{23} & C_{23} & 0 \end{bmatrix}$$

Translation vectors

To end of joint

$$d_{0j}^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$d_{1j}^0 = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

$$d_{2j}^0 = \begin{bmatrix} L_{c2}C1C2 \\ L_{c2}S1C2 \\ L_1 + L_2S2 \end{bmatrix}$$

$$d_{3j}^0 = \begin{bmatrix} C_1(L_{c3}C23 + L_2C2) \\ S_1(L_{c3}C23 + L_2C2) \\ L_1 + L_3S23 + L_2S2 \end{bmatrix}$$

To center of mass

$$d_{0c}^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$d_{1c}^0 = \begin{bmatrix} 0 \\ 0 \\ L_{c1} \end{bmatrix}$$

$$d_{2c}^0 = \begin{bmatrix} L_{c2}C1C2 \\ L_{c2}S1C2 \\ L_1 + L_{c2}S2 \end{bmatrix}$$

$$d_{3c}^0 = \begin{bmatrix} C1\cdot(L_{c3}C23 + L_2C2) \\ S1\cdot(L_{c3}C23 + L_2C2) \\ L1 + L_3S23 + L_2S2 \end{bmatrix}$$

Jacobians

$$J_1 = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}0\\0\\L_{c1}\end{bmatrix} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} \\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} \end{bmatrix} = \boxed{\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\1&0&0 \end{bmatrix}}$$

$$J_2 = \begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}L_{c2}C1C2\\L_{c2}S1C2\\L_1+L_{c2}S2\end{bmatrix} & \begin{bmatrix}C_1&0&S_1\\S_1&0&-C_1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} \times \left(\begin{bmatrix}L_{c2}C1C2\\L_{c2}S1C2\\L_1+L_{c2}S2\end{bmatrix} - \begin{bmatrix}0\\0\\L_1\end{bmatrix}\right) \\ \begin{matrix}0\\0\\1\end{matrix} & \begin{bmatrix}C_1&0&S_1\\S_1&0&-C_1\\0&1&0\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$\ldots \begin{matrix}0\\0\\0\\0\\0\\0\end{matrix} \Bigg] = \boxed{\begin{bmatrix} -L_{c2}C2S1 & -L_{c2}C1S2 & 0 \\ L_{c2}C1C2 & -L_{c2}S1S2 & 0 \\ 0 & L_{c2}C2 & 0 \\ 0 & S_1 & 0 \\ 0 & -C_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

$J_{vn} = J_n[1:3, :]$

$J_{\omega n} = J_n[4:6, :]$

$$J_3 = \left[\begin{array}{c:c}
\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}C1(L_{c3}C23+L_2C2)\\ S1(L_{c3}C23+L_2C2)\\ L1+L_3S23+L_2S2\end{bmatrix} & \begin{bmatrix}C1 & 0 & S1\\ S1 & 0 & -C1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} \times \left(\begin{bmatrix}C1(L_{c3}C23+L_2C2\\ S1(L_{c3}C23+L_2C2\\ L1+L_3S23+L_2S2\end{bmatrix}\right. \\
\begin{matrix}0\\0\\1\end{matrix} & \begin{bmatrix}C_1 & 0 & S_1\\ S_1 & 0 & -C_1\\ 0 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{array}\right.$$

$$\left.\begin{array}{c:c}
\left.\ldots - \begin{bmatrix}0\\0\\L_1\end{bmatrix}\right) & \begin{bmatrix}C_1C_2 & -C1S2 & S1\\ S1C2 & -S1S2 & -C1\\ S2 & C2 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} \times \left(\begin{bmatrix}C1(L_{c3}C23+L_2C2\\ S1(L_{c3}C23+L_2C2\\ L_1+L_3S23+L_2S2\end{bmatrix} - \begin{bmatrix}L_{c2}C1C2\\ L_{c2}S1C2\\ L_1+L_2S2\end{bmatrix}\right) \\
 & \begin{bmatrix}C_1C_{23} & -C_1S_{23} & S_1\\ S_1C_{23} & -S_1S_{23} & -C_1\\ S_{23} & C_{23} & 0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{array}\right]$$

$$\boxed{J_3 = \left[\begin{array}{c:c:c}
-S1(L_{c3}C23+L_2C2) & -C1(L_{c3}S23+L_2S2) & -L_{c3}S23C1\\
C1(L_{c3}C23+L_2C2) & -S1(L_{c3}S23+L_2S2) & -L_{c3}S23S1\\
0 & L_{c3}C23+L_2C2 & L_{c3}C23\\
0 & \begin{bmatrix}S_1\\ -C_1\\ 0\end{bmatrix} & \begin{bmatrix}S_1\\ -C_1\\ 0\end{bmatrix}\\
0 & & \\
1 & &
\end{array}\right]}$$

Jacobians

Rotation Matrices

Inertia

$$J_{v_{c1}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_{v_{c2}} = \begin{bmatrix} -L_{c2}C2S1 & -L_{c2}C1S2 & 0 \\ L_{c2}C1C2 & -L_{c2}S1S2 & 0 \\ 0 & L_{c2}C2 & 0 \end{bmatrix}$$

$$J_{v_{c3}} = \begin{bmatrix} -S1(L_{c3}C23+L_2C2) & -C1(L_{c3}S23+L_2S2) & -L_{c3}S23C1 \\ C1(L_{c3}C23+L_2C2) & -S1(L_{c3}S23+L_2S2) & -L_{c3}S23S1 \\ 0 & L_{c3}C23+L_2C2 & L_{c3}C23 \end{bmatrix}$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad J_{\omega_2} = \begin{bmatrix} 0 & S_1 & 0 \\ 0 & -C_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad J_{\omega_3} = \begin{bmatrix} 0 & S_1 & S_1 \\ 0 & -C_1 & -C_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{\omega 1} = \begin{bmatrix} C_1 & 0 & S_1 \\ S_1 & 0 & -C_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_{\omega 2} = \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 \\ S_1C_2 & -S_1S_2 & -C_1 \\ S_2 & C_2 & 0 \end{bmatrix}$$

$$R_{\omega 3} = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 \\ S_{23} & C_{23} & 0 \end{bmatrix}$$

$$R_{\omega n} = R_n^0$$

$$I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .083 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .083 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .33 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

KE

$$J_{\omega_1}^T R_{\omega_1} I_1 R_{\omega_1}^T J_{\omega_1} = \begin{bmatrix} I_{1yy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_{\omega_2}^T R_{\omega_2} I_2 R_{\omega_2}^T J_{\omega_2} = \begin{bmatrix} 0 & 0 & 1 \\ S_1 & -C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_1C_2 & -C_1C_2 & S_1 \\ S_1C_2 & -S_1S_2 & -C_1 \\ S_2 & C_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1C_2 & S_1C_2 & S_2 \\ -C_1C_2 & -S_1S_2 & C_2 \\ S_1 & -C_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & S_1 & 0 \\ 0 & -C_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{2yy} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_{\omega_3}^T R_{\omega_2} I_3 R_{\omega_3}^T J_{\omega_3} = \begin{bmatrix} 0 & 0 & 1 \\ S_1 & -C_1 & 0 \\ S_1 & -C_1 & 0 \end{bmatrix} \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 \\ S_{23} & C_{23} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{3yy} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C_1C_{23} & S_1C_{23} & S_{23} \\ -C_1S_{23} & -S_1S_{23} & C_{23} \\ S_1 & -C_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & S_1 & S_1 \\ 0 & -C_1 & -C_1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{3yy} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D(q) = m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} + m_3 J_{v_{c3}}^T J_{v_{c3}}$$

$$+ \begin{bmatrix} I_{1yy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} I_{2yy} + S2^2 - I_{2yy} S2^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} I_{3yy} + S23^2 - I_{3yy} S23^2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

KE

$$m_1 J_{v_{c1}}^T J_{v_{c1}} = 0$$

$$m_2 J_{v_{c2}}^T J_{v_{c2}} = m_2 \cdot \begin{bmatrix} -L_{c2} C2S1 & L_{c2} C1C2 & 0 \\ -L_{c2} C1S2 & -L_{c2} S1S2 & L_{c2} C2 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

$$\begin{bmatrix} -L_{c2} C2S1 & -L_{c2} C1S2 & 0 \\ L_{c2} C1C2 & -L_{c2} S1S2 & 0 \\ 0 & L_{c2} C2 & 0 \end{bmatrix} = \begin{bmatrix} -L_{c_2}^2 \cdot m_2 \cdot (S2-1) & 0 & 0 \\ 0 & L_c^2 \cdot 2m_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$m_3 J_{v_{c3}}^T J_{v_{c3}} = m_3 \cdot \begin{bmatrix} -S1(L_{c3} C23 + L_2 C2) & C1(L_{c3} C23 + L_2 C2) & 0 \\ -C1(L_{c3} S23 + L_2 S2) & -S1(L_{c3} S23 + L_2 S2) & L_{c3} C23 + L_2 C2 \\ -L_{c3} S23C1 & -L_{c3} S23S1 & L_{c3} C23 \end{bmatrix}$$

$$\begin{bmatrix} -S1(L_{c3} C23 + L_2 C2) & -C1(L_{c3} S23 + L_2 S2 & -L_{c3} S23C1 \\ C1(L_{c3} C23 + L_2 C2) & -S1(L_{c3} S23 + L_2 S2) & -L_{c3} S23S1 \\ 0 & L_{c3} C23 + L_2 C2 & L_{c3} C23 \end{bmatrix}$$

$$= \begin{bmatrix} m_3 (L_{c3} C23 + L_2 C2)^2 & 0 & 0 \\ 0 & m_3 (L_2^2 + 2C3 \cdot L_2 \cdot L_{c3} + L_{c3}^2) & L_{c3} m_3 (L_{c3} + L_2 C3) \\ 0 & L_{c3} m_3 (L_{c3} + L_2 C3) & L_{c3}^2 \cdot m_3 \end{bmatrix}$$

D(q) term

$$D(q) = \begin{bmatrix} -m_2 \cdot L_{c_2}^2 (S2 - 1) + m_3 (L_{c3} C23 + L_2 C2)^2 + I_{1yy} + I_{2yy} + S2^2 - I_{2yy} S2^2 + I_{3yy} + S23^2 - I_{3yy} S23^2 & \cdots \\ 0 & \\ 0 & \end{bmatrix}$$

column 1

column 2 + column 3

$$\cdots \begin{matrix} 0 & 0 \\ 2m_2 L_{c}^2 + m_3 (L_2^2 + 2C3 \cdot L_2 \cdot L_{c3} + L_{c3}^2) + 2 & L_{c3} m_3 (L_{c3} + L_2 C3) + 1 \\ L_{c3} m_3 (L_{c3} + L_2 C_3) + 1 & L_{c3}^2 \cdot m_3 + 1 \end{matrix} \Bigg]$$

C(q, q̇) term

$$C(q,\dot q) = \begin{bmatrix} -m_2 L_{c_2}^2 C2\, \dot q_2 + 2m_3 (L_{c3} C23 + L_2 C_2)(-L_{c3} S23 \dot q_1 \dot q_2 - L_2 S2 \dot q_2) + 2S2(-C2)\dot q_2 - 2I_{2yy} S2 \cdot (C2) \dot q_2 + \ldots \\ \ldots 2S23(C23)\dot q_2 \dot q_3 - 2I_{3yy} S23 (C23) \dot q_2 \dot q_3 \\ 0 \\ 0 \end{bmatrix}$$

column 1

$C(q,\dot q)$ term continued

$$C(q,\dot q) = \begin{array}{c} \\ \text{column 2} \end{array} \cdots \begin{bmatrix} 0 & 0 \\ -2m_3 S3\,\dot q_3 & -L_{c_3} m_3 L_2 S3\,\dot q_3 \\ -L_{c_3} m_3 L_2 S_3\,\dot q_3 & 0 \end{bmatrix}$$

$G(q)$ term

$$P = m_1 g L_{c_1} + m_2 g\,(L_1 + L_{c_2}\sin(q_2)) + m_3 g(L_1 + L_2\sin(q_2) + L_{c_3}\sin(q_3))$$

$$g_1(P) = 0$$

$$g_2(P) = m_2 g \cdot L_{c_2}\cos(q_2) + m_3 g L_2 \cos(q_2)$$

$$g_3(P) = m_3 g \cos(q_3)$$

$$G(q) = \begin{bmatrix} 0 \\ m_2 g \cdot L_{c_2}\cos(q_2) + m_3 g L_2 \cos(q_2) \\ m_3 g \cos(q_3) \end{bmatrix}$$

```matlab
%% HW2: Custom Inverse Dynamics Function
% David Yackzan

% function takes in three 3x1 matrices: position(q), velocity(dq), and
% acceleration(ddq) with values for each of the 3 actuators.
% Outputs a 3x1 matrix for the required torques of the three joints

function [Tau] = custom_inverse_dynamics(q, dq, ddq)
    % create variables for the input values
    q1 = q(1);
    q2 = q(2);
    q3 = q(3);
    dq1 = dq(1);
    dq2 = dq(2);
    dq3 = dq(3);
    ddq1 = ddq(1);
    ddq2 = ddq(2);
    ddq3 = ddq(3);

    % Create variables for our known values
    g = 9.81;
    m1 = 1;
    m2 = 2;
    m3 = 3;
    L1 = .4;
    L2 = 1;
    L3 = 1;
    Lc1 = L1/2;
    Lc2 = L2/2;
    Lc3 = L3/2;

    % Solve for Dq
    Dq = [(3*L2^2*cos(2*q2))/2 - (67*cos(2*q2 + 2*q3))/200 -
(917*cos(2*q2))/2000 + Lc2^2*cos(2*q2) + (3*Lc3^2*cos(2*q2 + 2*q3))/2 +
(3*L2^2)/2 + Lc2^2 + (3*Lc3^2)/2 + 3*L2*Lc3*cos(q3) + 3*L2*Lc3*cos(2*q2 + q3)
+ 2579/2000, 0, 0;...
          0, 3*L2^2 + 6*cos(q3)*L2*Lc3 + 2*Lc2^2 + 3*Lc3^2 + 2, 3*Lc3^2 +
3*L2*cos(q3)*Lc3 + 1;...
          0, 3*Lc3^2 + 3*L2*cos(q3)*Lc3 + 1, 3*Lc3^2 + 1]

    % Solve for C(q, dq)
    Cq_dq = [-(3/2)*L2^2*sin(2*q2)*(2*dq2) + (67/200)*sin(2*q2 +
2*q3)*(2*dq2)*(2*dq3) + (917/2000)*sin(2*q2)*(2*dq2) -
Lc2^2*sin(2*q2)*(2*dq2) - (3/2)*Lc3^2*sin(2*q2 + 2*q3)*(2*dq2)*(2*dq3) -
3*L2*Lc3*sin(q3)*dq3 - 3*L2*Lc3*sin(2*q2 + q3)*(2*dq2)*(dq3), 0 0;...
            0, -Lc3*m3*L2*sin(q3)*dq3, -Lc3*m3*L2*sin(q3)*dq3;...
            0, -Lc3*m3*L2*sin(q3)*dq3, 0]

    % Solve for G(q)
    Gq = [0; m2*g*Lc1*cos(q2)+m3*g*L2*cos(q2); m3*g*cos(q3)];

    % Solve for the output 3 variable array Tau
    Tau = Dq*ddq + Cq_dq*dq + Gq;
```