Spherical solutions for stars

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General Relativity I Presentations December 14th, 2015



Spherical stars

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General Relativity I Presentat

2015-12-12

Introduction

Spherical stars
—Introduction

Introduction

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Spherically symmetric coordinates

Spherical stars —Spherically symmetric coordinates

Spherically symmetric coordinates

General metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Metric on 2-sphere

$$dl^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) = r^2d\Omega^2$$

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Schutz (2009, p. 256)

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Spherical stars

Spherically symmetric coordinates

Two-sphere in flat spacetime

Two-sphere in flat spacetime

Schutz (2009, p. 256)

General metric

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Metric on 2-sphere

$$dl^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \equiv r^2 d\Omega^2$$

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Schutz (2009, p. 256) Daniel Wysocki (RIT)

Two-sphere in flat spacetime Spherical stars -Spherically symmetric coordinates $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ Two-sphere in flat spacetime $dl^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \equiv r^2 d\Omega^2$

Schutz (2009, p. 256)

Metric on 2-sphere

$$dl^2 = f(r', t)d\Omega^2$$

$$f(r',t) \equiv \tau$$

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Schutz (2009, pp. 256–257)

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-Spherically symmetric coordinates

Two-sphere in curved spacetime

Two-sphere in curved spacetime Schutz (2009, pp. 256-257)

Metric on 2-sphere

$$dl^2 = f(r', t)d\Omega^2$$

Relation to r

$$f(r',t) \equiv r^2$$

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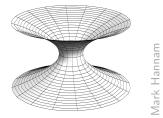
Schutz (2009, pp. 256–257)

Spherical stars —Spherically symmetric coordinates

__Two-sphere in curved spacetime



Meaning of r



- "curvature" or "area" coordinate
 - radius of curvature and area

Figure:

Surface with circular symmetry but no coordinate r = 0.

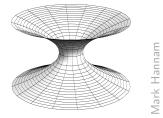
Schutz (2009, p. 257)

Spherical stars -Spherically symmetric coordinates

Meaning of r

 \sqsubseteq Meaning of r

Meaning of r



• "curvature" or "area" coordinate

- radius of curvature and area
- *not* proper distance from center
- r = const, t = const
 - $A = 4\pi r^2$
 - $C=2\pi r$

Figure: Surface

Surface with circular symmetry but no coordinate r = 0.

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Schutz (2009, p. 257)

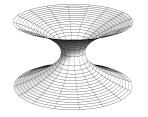
Spherical stars —Spherically symmetric coordinates

 \sqsubseteq Meaning of r



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Meaning of r



• "curvature" or "area" coordinate

- radius of curvature and area
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Figure: Surface

Surface with circular symmetry but no coordinate r = 0.

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Schutz (2009, p. 257)

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Spherical stars
—Spherically symmetric coordinates

 \sqsubseteq Meaning of r

Meaning of r

* "curvatum" or "uses" coordinate

*

Spherically symmetric spacetime

General metric

$$ds^{2} = q_{00} dt^{2} + 2q_{0r} dr dt + q_{rr} dr^{2} + r^{2} d\Omega^{2}$$

 $g_{00}, g_{0r},$ and g_{rr} functions of t and r



Schutz (2009, p. 258)

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Spherical stars
Spherically symmetric coordinates

_Spherically symmetric spacetime

Spherically symmetric spacetime $\frac{ds^2-g_0\cdot dt^2+2g_0\cdot dr\cdot dt+g_0\cdot dr^2+r^2d\Omega^2}{g_0,g_r\cdot and\ g_r\cdot functions\ of\ t$ and r

Spherical stars

-Static spacetimes

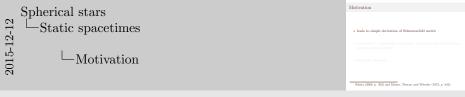
Static spacetimes

Static spacetimes



• generalizes to spherically symmetric, asymptotically flat Einstein vacuum field equations

• Birkhoff's theorem



• George David Birkhoff



• generalizes to spherically symmetric, asymptotically flat Einstein vacuum field equations

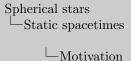


• George David Birkhoff



• generalizes to spherically symmetric, asymptotically flat Einstein vacuum field equations

• Birkhoff's theorem



Motivation · leads to simple derivation of Schwarzschild metri- generalizes to spherically symmetric, asymptotically flat Einstein · Birkhoff's theorem

Schutz (2009, p. 263) and Miener, Thorne, and Wheeler (1973, p. 843)

• George David Birkhoff



Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843) December 14th, 2015

Definition

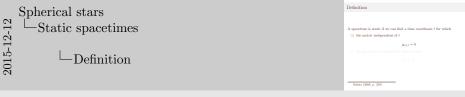
A spacetime is static if we can find a time coordinate t for which

(i) the metric independent of t

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \rightarrow -t$$



- ullet a spacetime which only satisfies the first condition is stationary
 - e.g. rotating star

Definition

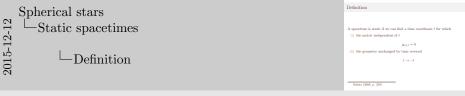
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• a spacetime which only satisfies the first condition is *stationary*

– e.g. rotating star

Time reversal

$$\Lambda: (t, x, y, z) \rightarrow (-t, x, y, z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} g_{\alpha\beta} = g_{\alpha\beta}$$

Transformation

$$\Lambda^{0}_{\ \bar{0}} = x^{0}_{\ ,\bar{0}} = -x^{0}_{\ ,0} = -1$$

$$\Lambda^{i}_{\ \bar{j}} = x^{i}_{\ ,\bar{j}} = x^{i}_{\ ,j} = \delta^{i}_{\ j}$$

$$\Lambda^{0}_{\ \bar{i}} = x^{0}_{\ ,\bar{i}} = x^{0}_{\ ,i} = 0$$

Metric

$$g_{\bar{0}\bar{0}} = (\Lambda^0_{\bar{0}})^2 g_{00} = g_{00}$$

$$g_{\bar{0}\bar{r}} = \Lambda^0_{\bar{0}} \Lambda^r_{\bar{r}} g_{0r} = -g_{0r}$$

$$g_{\bar{r}\bar{r}} = (\Lambda^r_{\bar{r}})^2 g_{rr} = g_{rr}$$

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Spherical stars
Static spacetimes

└─Time reversal



•
$$g_{\bar{0}\bar{r}} = -g_{0r} = 0$$

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Time reversal

$$\Lambda: (t, x, y, z) \rightarrow (-t, x, y, z)$$

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Spherical stars

Transformation

Schutz (2009, p. 258)

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$$\begin{split} & \Lambda^0_{\bar{0}} = x^0_{,\bar{0}} = -x^0_{,0} = -1 \\ & \Lambda^i_{\bar{j}} = x^i_{,\bar{j}} = x^i_{,j} = \delta^i_{\,j} \\ & \Lambda^0_{\bar{i}} = x^0_{\,\,\bar{i}} = x^0_{\,\,i} = 0 \end{split}$$

Metric

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Static spacetimes

__Time reversal



•
$$g_{\bar{0}\bar{r}} = -g_{0r} = 0$$

Time reversal

$$\Lambda: (t, x, y, z) \rightarrow (-t, x, y, z)$$

$$g_{ar{lpha}ar{eta}} = \Lambda^{lpha}_{ar{lpha}} \Lambda^{eta}_{\phantom{ar{eta}}ar{eta}} g_{lphaeta} = g_{lphaeta}$$

Transformation

$$\begin{split} & \Lambda^0_{\ \bar{0}} = x^0_{\ ,\bar{0}} = -x^0_{\ ,0} = -1 \\ & \Lambda^i_{\ \bar{j}} = x^i_{\ ,\bar{j}} = x^i_{\ ,j} = \delta^i_{\ j} \\ & \Lambda^0_{\ \bar{i}} = x^0_{\ ,\bar{i}} = x^0_{\ ,i} = 0 \end{split}$$

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Spherical stars
—Static spacetimes

└─Time reversal



•
$$g_{\bar{0}\bar{r}} = -g_{0r} = 0$$



Simplified metric

$$ds^{2} = g_{00} dt^{2} + g_{rr} dr^{2} + r^{2} d\Omega^{2}$$

Replacement

$$g_{00} \rightarrow -e^{2\Phi}$$
, $g_{rr} \rightarrow e^{2\Lambda}$, provided $g_{00} < 0 < g_{rr}$

Static spherically symmetric metric

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2$$

$$\lim_{r \to \infty} \Phi(r) = \lim_{r \to \infty} \Lambda(r) = 0$$

Schutz (2009, pp. 258–259)



Spherical stars
—Static spacetimes

The metric

dr' = gg(0r + gu dr' + r'dl'

Shat (2000, ys. 226-200)

The metric

- constraint $g_{00} < 0 < g_{rr}$ holds for stars but not black holes
- limits at infinity tell us that spacetime is asymptotically flat

Simplified metric

$$ds^{2} = q_{00} dt^{2} + q_{rr} dr^{2} + r^{2} d\Omega^{2}$$

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Spherical stars
—Static spacetimes



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Spherical stars
—Static spacetimes



- └─The metric
- constraint $g_{00} < 0 < g_{rr}$ holds for stars but not black holes
- ullet limits at infinity tell us that spacetime is asymptotically flat

General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$

$$G_{tt} = \frac{1}{r^2} e^{2\Phi} \frac{d}{dr} [r(1 - e^{-2\Lambda})],$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda} [\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r],$$

$$G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}$$

Schutz (2009, pp. 165, 260)



Spherical stars Static spacetimes

Einstein Tensor

 $G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$

Einstein Tensor

Schutz (2009, pp. 165, 260)

• $x' \equiv dx/dr$

Einstein tensor components

$$G_{tt} = \frac{1}{r^2} e^{2\Phi} \frac{d}{dr} [r(1 - e^{-2\Lambda})],$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda} [\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r],$$

$$G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}$$

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—Static spacetimes

LEinstein Tensor

record Function to some $G_{n,j} = R^{-1} - \frac{1}{2} \mu_{ij} M$ which is the order components $G_{n} = \frac{1}{12} 2^{n} \frac{d}{dr} [i(1 - e^{-i\phi})_{i}],$ $G_{n} = -\frac{1}{12} 2^{n} \frac{d}{dr} [i(1 - e^{-i\phi})_{i}],$ $G_{n} = -\frac{1}{2} e^{-i\phi} [1 - e^{-i\phi}],$ $G_{m} = e^{-i\phi} M [1 - e^{-i\phi}] e^{-i\phi} M - A/r/r],$ $G_{m} = m^{2} K G_{m}$ Solution (290), pp. 433.

Einstein Tensor

• $x' \equiv dx/dr$

Schutz (2009, pp. 165, 260)

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Static perfect fluid

Spherical stars

Static perfect fluid

Static perfect fluid

$$\vec{U} \cdot \vec{U} = -1$$
 (conservation law)

Solving for U

Solving for *IL*

$$H_0 = g_{00}H^0 = -e^{i\theta}$$

Schutz (2009, p. 260)

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—Static perfect fluid

└─Four-velocity



• "conservation law" is the conservation of four-momentum

$$g_{00}U^0U^0 = -1 \implies (U^0)^2 = (-g_{00})^{-1}$$

 $\implies U^0 = (-g_{00})^{-1/2}$
 $\implies U^0 = (e^{2\Phi})^{-1/2} = e^{-\Phi}$

$$\vec{U} \cdot \vec{U} = -1$$
 (conservation law)

Solving for U^0

$$q_{00}(U^0)^2 = -1 \implies U^0 = (-q_{00})^{-1/2} = e^{-\Phi}$$

Solving for *U*

$$H_0 = g_{00}H^0 = -e^6$$

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—Static perfect fluid

 $\sqsubseteq_{\text{Four-velocity}}$

Four-velocity $U^i = 0 \; (\text{data};) \qquad \bar{U} \cdot \bar{U} = -1 \; (\text{conservation law})$ Solves for I_0^i $g_0(U^0)^2 = -1 \; \Longrightarrow \; U^0 = (-g_0)^{-1/2} = e^{-\Phi}$ Solves (200, p. 200)

• "conservation law" is the conservation of four-momentum

$$g_{00}U^0U^0 = -1 \implies (U^0)^2 = (-g_{00})^{-1}$$

 $\implies U^0 = (-g_{00})^{-1/2}$
 $\implies U^0 = (e^{2\Phi})^{-1/2} = e^{-\Phi}$

Schutz (2009, p. 260)

Constraints

$$U^i = 0 \text{ (static)}$$
 $\vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$

Solving for U^0

$$g_{00}(U^0)^2 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

Solving for U_0

$$U_0 = q_{00}U^0 = -e^{\Phi}$$

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-Static perfect fluid Four-velocity

Spherical stars

Four-velocity $\chi_{00}(U^0)^2 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$ Schutz (2009, p. 200)

• "conservation law" is the conservation of four-momentum

$$g_{00}U^{0}U^{0} = -1 \implies (U^{0})^{2} = (-g_{00})^{-1}$$
$$\implies U^{0} = (-g_{00})^{-1/2}$$
$$\implies U^{0} = (e^{2\Phi})^{-1/2} = e^{-\Phi}$$

Schutz (2009, p. 260)

Components of $T_{\alpha\beta}$

$$T_{00} = (\rho + p)e^{2\Phi} + p(-e^{2\Phi}) = \rho e^{2\Phi}$$

$$T_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta; \quad T_{ii} = pg_{ii}$$

$$T_{\alpha\beta} = ne^{2\Lambda}; \quad T_{\alpha\beta} = nr^2; \quad T_{\alpha\beta} = nr^2 \sin^2 \theta = T_{\alpha\beta} \sin^2 \theta$$

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Spherical stars
Static perfect fluid

 $\sqsubseteq_{\text{Stress-energy tensor}}$

Stress-energy tensor $T_{c0} = (\rho + p)U_cU_f + pp_{c0}$ $T_{c0} = (\rho + p)U_cU_f + pp_$

- $T_{\alpha\beta} = 0$ because
 - the cross terms make $g_{\alpha\beta} = 0$ - T_{0i} makes one of the U's $U_i = 0$
- likewise, $T_{ii} = pg_{ii}$ because $U_iU_i = 0$

$T_{\alpha\beta}$ for perfect fluid

$$T_{\alpha\beta} = (\rho + p)U_{\alpha}U_{\beta} + pg_{\alpha\beta}$$

Components of $T_{\alpha\beta}$

$$T_{00} = (\rho + p)e^{2\Phi} + p(-e^{2\Phi}) = \rho e^{2\Phi}$$

$$T_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta; \quad T_{ii} = pg_{ii}$$

$$T_{rr} = pe^{2\Lambda}; \quad T_{\theta\theta} = pr^2; \quad T_{\phi\phi} = pr^2 \sin^2 \theta = T_{\theta\theta} \sin^2 \theta$$

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Spherical stars
—Static perfect fluid

 $\sqsubseteq_{\text{Stress-energy tensor}}$

Sites energy tensor
$$\begin{split} T_{ab} & \text{ for points find} \\ T_{ab} & = (\rho + p)U_aU_b + pp_{ab} \end{split}$$
 Component of $T_{ab} = (\rho + p)U_aU_b + pp_{ab}$ $T_{ba} = (\rho + p)^{2b} \cdot p(-L^{2b}) = \mu^{2b} \\ T_{ab} & = 0 \text{ for } \alpha \neq \beta \text{, } T_{bc} = pp_{a} \\ T_{ab} & = 0 \text{ for } \alpha \neq \beta \text{, } T_{bc} = pp_{a} \\ T_{cc} & = p^{2b}, \quad T_{cc} = p^{2}, \quad T_{cc} = p^{2}, \quad T_{cc} = p^{2}, \end{split}$

Schutz (2009, p. 200)

- $T_{\alpha\beta} = 0$ because
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- likewise, $T_{ii} = pg_{ii}$ because $U_iU_i = 0$

Local thermodynamic equilibrium

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

Spherical stars
—Static perfect fluid
—Equation of state



Conservation laws

$$T^{\alpha\beta}_{\ \ ;\beta}=0$$

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}$$

Spherical stars -Static perfect fluid

Equations of motion



$$T^{\alpha\beta}_{\ \ ;\beta} = 0$$

• symmetries make only non-trivial solution $\alpha = r$ TODO: prove

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)

Spherical stars -Static perfect fluid

Equations of motion



$$T^{\alpha\beta}_{\ \ ;\beta} = 0$$

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Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)

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Equations of motion



Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

$$m(r) \equiv \frac{1}{2}r(1 - e^{-2\Lambda})$$
 or $g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Schutz (2009, pp. 260–262)



Spherical stars Static perfect fluid

└─Mass function

- inspect (0,0) component of Einstein equations
- in Newtonian limit, m(r) is mass within radius r

$$m(r) = 4\pi \int_0^r (r')^2 \rho(r') dr'$$

Mass function

 $G_{00} = 8\pi T_{00} \implies \frac{1}{\omega^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}z} [r(1-e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$

• doesn't work in GR, because total energy is not localizable

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$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

m(r)

$$m(r) \equiv \frac{1}{2}r(1 - e^{-2\Lambda})$$
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- Spherical stars Static perfect fluid
 - └─Mass function



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$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

m(r)

$$m(r) \equiv \frac{1}{2}r(1 - e^{-2\Lambda})$$
 or $g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$

Relation to energy density

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Schutz (2009, pp. 260–262)



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Spherical stars

 $\sqsubseteq_{\mathrm{Mass}}$ function

Static perfect fluid

Mass function $G_{ab} = 8 T_{ab} = \frac{1}{r^2} e^{2 \theta} \frac{d}{dr} / (1 - e^{-2 \theta}) = 8 \sigma \mu e^{2 \theta}$ $m(r) = \frac{1}{2} (1 - e^{-2 \theta}) \text{ or } g_{ab} = e^{2 \theta} \left(1 - \frac{2 m(r)}{r} \right)^{-1}$ Thicknot to energy denotes $\frac{dm(r)}{dr} = 4 \pi^2 p$

- inspect (0,0) component of Einstein equations
- in Newtonian limit, m(r) is mass within radius r

$$m(r) = 4\pi \int_0^r (r')^2 \rho(r') dr'$$

• doesn't work in GR, because total energy is not localizable

$$G_{rr} = 8\pi T_{rr} \implies -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi pe^{2\Lambda}$$

$\Phi(r)$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 \eta}{r[r - 2m(r)]}$$



• inspect (r, r) component of Einstein equations

What is Φ called?!?!?!?!?



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$$G_{rr} = 8\pi T_{rr} \implies -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi pe^{2\Lambda}$$

$\Phi(r)$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

Spherical stars

Static perfect fluid

What is Φ called?!?!?!!?

What is Φ called?!?!?!!?

• inspect (r, r) component of Einstein equations

Exterior Geometry

Spherical stars -Exterior Geometry

Exterior Geometry

Schwarzschild metric I

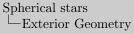
Condition

$$\rho = p = 0$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad m(r) = M$$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)} \qquad \Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

Schutz (2009, pp. 262–263)





Schwarzschild metric I

-Schwarzschild metric I

- the external conditions just state we are in a vaccuum
- breaks down when matter surrounds star
- M is a constant

$$\rho = p = 0$$

Consequences

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad m(r) \equiv M$$

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Spherical stars Exterior Geometry



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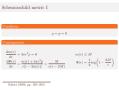
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Spherical stars Exterior Geometry

-Schwarzschild metric I



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Schutz (2009, pp. 262–263)

First two metric components

$$g_{tt} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$
 $g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1}$

$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 - \frac{2M}{r}\right)^{-1}dr^{2} + r^{2}d\Omega^{2}$$

Spherical stars

Exterior Geometry

Schwarzschild metric II

Schwarzschild metric II

Schutz (2009, pp. 258, 262-263)

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Schutz (2009, pp. 258, 262–263)

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Schwarzschild metric

Daniel Wysocki (RIT)

$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 - \frac{2M}{r}\right)^{-1}dr^{2} + r^{2}d\Omega^{2}$$

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Schutz (2009, pp. 258, 262–263)

Find two metric components $g_{ij}=-c^{2k}=-\left(1-\frac{2M}{r}\right) \qquad g_{ij}=c^{2k}=\left(1-\frac{2M}{r}\right)^{-1}$ Schwarzschild metric $\mathrm{d}s^2=-\left(1-\frac{2M}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2$

_Schwarzschild metric II

Schutz (2009, pp. 258, 262-263)

$$r \gg M$$

$$ds^{2} \approx -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 + \frac{2M}{r}\right)dr^{2} + r^{2}d\Omega^{2}$$

$$ds^2 \approx -\left(1 - \frac{2M}{R}\right) dt^2 + \left(1 + \frac{2M}{R}\right) (dx^2 + dy^2 + dz^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

Schutz (2009, pp. 263)



Spherical stars Exterior Geometry

Schutz (2009, pp. 263)

Far-field metric

• far-field metric of a star

Far-field metric

- mass M
- distance R

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$$r \gg M$$

Far-field Schwarzschild metric

$$ds^{2} \approx -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 + \frac{2M}{r}\right)dr^{2} + r^{2}d\Omega^{2}$$

Far-field Schwarzschild metric (Cartesian

$$ds^{2} \approx -\left(1 - \frac{2M}{R}\right)dt^{2} + \left(1 + \frac{2M}{R}\right)(dx^{2} + dy^{2} + dz^{2})$$

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Schutz (2009, pp. 263)



Spherical stars
—Exterior Geometry
—Far-field metric

Fas-field metric r>M fraction = r>M $ds^2 = -\left(1-\frac{2M}{r}\right)ds^2 + \left(1+\frac{2M}{r}\right)ds^2 + r^2d\Omega^2$ $ds^2 = -\left(1-\frac{2M}{r}\right)ds^2 + \left(1+\frac{2M}{r}\right)ds^2 + r^2d\Omega^2$

Schutz (2009, pp. 263)

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Schutz (2009, pp. 263) Spherical stars

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Far-field metric Spherical stars Exterior Geometry $ds^2 \approx -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + r^2d\Omega^2$ Far-field metric $ds^2 \approx -\left(1 - \frac{2M}{\nu}\right)dt^2 + \left(1 + \frac{2M}{\nu}\right)(dx^2 + dy^2 + dz^2)$ $R^2 \equiv x^2 + y^2 + z^2$

Spherical stars Exterior Geometry

□Birkhoff's Theorem

Birkhoff's Theorem Let the geometry of a given region of spacetime

Birkhoff and Langer (1923)

• George David Birkhoff

Birkhoff and Langer (1923)

Let the geometry of a given region of spacetime:

- be spherically symmetric

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Birkhoff's Theorem Spherical stars Exterior Geometry Let the geometry of a given region of spacetime Birkhoff's Theorem

Birkhoff and Langer (1923)

• George David Birkhoff

Let the geometry of a given region of spacetime:

- be spherically symmetric
- 2 be a solution to the Einstein field equations in vacuum.

Then that geometry is necessarily a piece of the Schwarzschild geometry.

(Proof given in Misner, Thorne, and Wheeler (1973, pp. 843–844))

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Spherical stars
—Exterior Geometry

☐Birkhoff's Theorem

Birkhoff's Theorem

Let the geometry of a given region of spacetime:

(a) be spherically symmetric

(b) be a solution to the Einstein field equations in vacuum.

Birkhoff and Loapet (1921)

• George David Birkhoff

Birkhoff and Langer (1923)

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Spherical stars

_Exterior Geometry

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Spherical stars Exterior Geometry

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Spherical stars

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2015-12-12

Interior structure

Spherical stars
—Interior structure

Interior structure

December 14th, 2015

$$\rho \neq 0; \quad p \neq 0$$

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)[m(r)+4\pi r^3 p]}{r[r-2m(r)]}$$

Spherical stars Interior structure

Schutz (2009, pp. 261-264)

Tolman-Oppenheimer-Volkov (T-O-V) equation

Tolman-Oppenheimer-Volkov (T-O-V) equation

- T-O-V equation coupled with dm/dr and $p(\rho)$
 - 3 equations
 - -3 unknowns (m, ρ, p)
 - $-\Phi(r)$ only intermediate variable

$$\rho \neq 0; \quad p \neq 0$$

Recall

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}$$

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T-O-V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)[m(r) + 4\pi r^3 p]}{r[r-2m(r)]}$$

Schutz (2009, pp. 261–264)



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Spherical stars
—Interior structure

—Tolman–Oppenheimer–Volkov (T–O–V) equation



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Spherical stars
_Interior structure

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Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho m(r)}{r^2}$$

Spherical stars
_Interior structure

 \sqsubseteq Newtonian hydrostatic equilibrium



• Newtonian limit reduces the T–O–V equation to HSE



Newtonian limit

$$p \ll \rho$$
; $4\pi r^3 p \ll m$; $m \ll r$

Equation of hydrostatic equilibrium

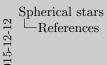
$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho m(r)}{r^2}$$

• Newtonian limit reduces the T–O–V equation to HSE



References

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