Spherical solutions for stars

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Rochester Institute of Technology

General Relativity I Presentations
December 14th, 2015

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Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- applications

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- Schwarzschild metric
- T–O–V equation
- applications

- I will model stars using GR assuming spherical symmetry
- I will derive the Schwarzschild metric and T–O–V equation
- finally I will relate these equations to modeling specific types of stars

Spherically symmetric coordinates

Spherically symmetric coordinates

- First we need to derive our coordinate system

Two-sphere in flat spacetime

General metric

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

Metric on 2-sphere

$$\mathrm{d}l^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \equiv r^2\mathrm{d}\Omega^2$$

Schutz (2009, p. 256)

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- we start with the simplest spherically symmetric coordinates
- 2-sphere in Minkowski space

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Two-sphere in curved spacetime

Metric on 2-sphere

$$\mathrm{d}l^2 = f(r', t)\mathrm{d}\Omega^2$$

Relation to r

$$f(r', t) \equiv r^2$$

Schutz (2009, pp. 256–257)

- generalize to 2-sphere in arbitrary curved spherically symmetric spacetime
- inclusion of curvature makes r^2 some function of r' and t

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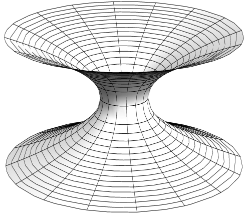
Meaning of r



Mark Hannam

Figure:
Surface with circular symmetry but no coordinate $r = 0$.

- *not* proper distance from center
- "curvature" or "area" coordinate
  - radius of curvature and area
- $r = \text{const}$, $t = \text{const}$
  - $A = 4\pi r^2$
  - $C = 2\pi r$

Schutz (2009, p. 257)

- r is not necessary the "distance from the center"
- it is merely a coordinate
- for instance, we may have a spacetime where the center is missing
  - example: wormhole spacetime
- surface of constant (r, t) is a two-sphere of area A and circumference C

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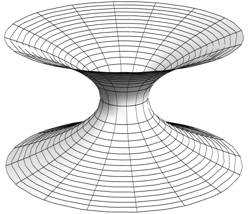

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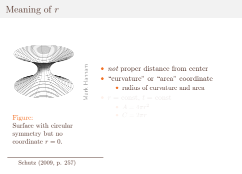

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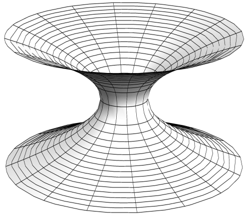

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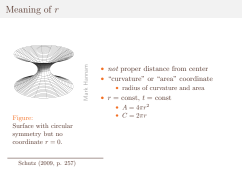

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Spherically symmetric spacetime

General metric

$$ds^2 = g_{00}\,dt^2 + 2g_{0r}\,dr\,dt + g_{rr}\,dr^2 + r^2 d\Omega^2$$

g_{00}, g_{0r}, and g_{rr} functions of t and r

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Schutz (2009, p. 258)

- now consider not only surface of 2-sphere, but whole spacetime
- now we have some unknown g_{00}, g_{rr}, and cross term g_{0r}
- cross terms g_{0i} for $i \in \{\theta, \phi\}$ are zero from symmetry
- need more constraints to say anything particular about them

Static spacetimes

Static spacetimes

- now I will impose the static constraint

Motivation

- leads to simple derivation of Schwarzschild metric
- unique solution to spherically symmetric, asymptotically flat Einstein vacuum field equations (Birkhoff's theorem)

Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843)

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  - it allows us to easily derive the Schwarzschild metric
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Definition

A spacetime is static if we can find a time coordinate t for which

(i) the metric independent of t

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \to -t$$

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- first condition is that the metric is independent of time
  - by itself, this condition is called “stationary”
- second condition is that metric unaffected by time reversal
- e.g. rotating stars are stationary but not static

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Time reversal

$$\mathbf{\Lambda} : (t, x, y, z) \to (-t, x, y, z)$$

$$g_{\bar\alpha\bar\beta} = \Lambda^{\alpha}{}_{\bar\alpha}\Lambda^{\beta}{}_{\bar\beta}\, g_{\alpha\beta} = g_{\alpha\beta}$$

Transformation

$$\Lambda^0{}_{\bar 0} = x^0{}_{,\bar 0} = -x^0{}_{,0} = -1$$
$$\Lambda^i{}_{\bar i} = x^i{}_{,\bar i} = x^i{}_{,i} = 1$$

Metric

$$g_{\bar 0\bar 0} = (\Lambda^0{}_{\bar 0})^2 g_{00} = g_{00}$$
$$g_{\bar r\bar r} = (\Lambda^r{}_{\bar r})^2 g_{rr} = g_{rr}$$
$$g_{\bar 0\bar r} = \Lambda^0{}_{\bar 0}\Lambda^r{}_{\bar r}\, g_{0r} = -g_{0r}$$

Schutz (2009, p. 258)

- now I use the static constraint to simplify the metric
- transformation
  - $(0, 0)$ term is $\mathrm{d}t/\mathrm{d}(-t)$
  - spatial terms are 1 if transformed to themselves
  - cross-terms are all zero, as coordinates independent of each other
- transformed metric
  - $(0, 0)$ term is unchanged, as -1 is squared
  - (r, r) term is unchanged, as transformation is 1
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The metric

Simplified metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

Replacement

$$g_{00} \to -e^{2\Phi}, \quad g_{rr} \to e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

Static spherically symmetric metric

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \Lambda(r) = 0$$

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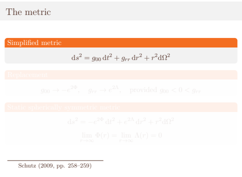

- now we simplify the metric, since the cross term is zero
- we assume g_{00} to be negative, and g_{rr} to be positive
  - signature is $(-,+,+,+)$
  - holds inside stars but not black holes
- limits at infinity tell us that spacetime is *asymptotically flat*
  - $\Phi = \Lambda = 0 \implies e^{2\Phi} = e^{2\Lambda} = 1$ and $\mathbf{g} = \eta$

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Einstein Tensor

General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Einstein tensor components

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r}[r(1 - e^{-2\Lambda})]$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

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- from that the Einstein tensor
- the derivation is involved, so we will just take them as is
- we're going to use some of these components later on
- $x' \equiv \mathrm{d}x/\mathrm{d}r$

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Static perfect fluid

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- stars are fluids – for simplicity we assume perfect
- thus we will impose additional constraints accordingly

Four-velocity

Constraints

$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

Solving for U^0

$$g_{00} U^0 U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

Solving for U_0

$$U_0 = g_{00} U^0 = -e^{\Phi}$$

Schutz (2009, p. 260)

- static fluid, so in MCRF three-velocity components all zero
- we find the only non-zero term, U^0, by relating to the dot product
- lower it with the metric, to use in next part

$$\begin{aligned} g_{00} U^0 U^0 = -1 &\implies (U^0)^2 = (-g_{00})^{-1} \\ &\implies U^0 = (-g_{00})^{-1/2} \\ &\implies U^0 = (e^{2\Phi})^{-1/2} = e^{-\Phi} \end{aligned}$$

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Stress–energy tensor

Stress-energy tensor for perfect fluid

$$T_{\alpha\beta} = (\rho + p) U_\alpha U_\beta + p g_{\alpha\beta}$$

Components of $T_{\alpha\beta}$

$T_{i\alpha} = p g_{i\alpha}$

$T_{\alpha\beta}$ is diagonal

Schutz (2009, p. 260)

- $T_{i\alpha} = p g_{i\alpha}$ because spatial components of U are zero
- $T_{\alpha\beta}$ is diagonal because of previous condition and $g_{\alpha\beta}$ is diagonal
- T_{00} requires a little algebra
- T_{ii} just need to multiply metric by p
- $T_{\phi\phi}$ can be written in terms of $T_{\theta\theta}$

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$$T_{00} = (\rho + p)e^{2\Phi} + p(-e^{2\Phi}) = \rho e^{2\Phi}$$

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Schutz (2009, p. 260)

Stress–energy tensor
Stress-energy tensor for perfect fluid
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Equation of state

Local thermodynamic equilibrium

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

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Local thermodynamic equilibrium
$p = p(\rho, S) \approx p(\rho)$
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- in a static fluid we have local thermodynamic equilibrium
- pressure a function of density and specific entropy
- specific entropy assumed negligibly small

Equations of motion

Conservation of 4-momentum

$$T^{\alpha\beta}{}_{;\beta} = 0$$

- symmetries make only non-trivial solution $\alpha = r$

Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)

- first equation follows from conservation of 4-momentum
- due to symmetry, the only non-trivial solution is for $\alpha = r$
- equation of motion for perfect fluid
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Mass function

Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r}[r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

$m(r)$

$$m(r) \equiv \frac{1}{2} r(1 - e^{-2\Lambda}) \quad \text{or} \quad g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

Relation to energy density

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

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- inspect $(0, 0)$ component of Einstein equations
- define the mass function, $m(r)$
- in Newtonian limit, $m(r)$ is mass within radius r

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- doesn't work in GR, because total energy is not localizable

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Exterior Geometry

Exterior Geometry

- until now, we've not considered whether we were inside or outside star
- properties inside different than outside (obviously)
- we're going to inspect both cases, starting with outside

Schwarzschild metric I

Condition

$$\rho = p = 0$$

Consequences

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho = 0 \qquad m(r) \equiv M$$

$$\frac{d\Phi(r)}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)} \qquad \Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

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- the external conditions just state we are in a vaccuum
  - breaks down when matter surrounds star
- $m(r)$ is constant, we call it M
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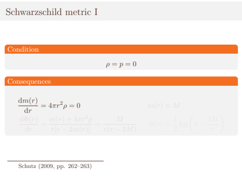

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Schwarzschild metric II

First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$

Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2$$

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Far-field metric

Condition

$$r \gg M$$

Far-field Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Far-field Schwarzschild metric (Cartesian)

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

Schutz (2009, pp. 263)

- far-field metric of a star (far away)
  - mass M
  - distance R

Far-field metric

Condition

$$r \gg M$$

Far-field Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Far-field Schwarzschild metric (Cartesian)

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Birkhoff's Theorem

If the geometry of a given region of spacetime is:

1. spherically symmetric
2. a solution to the Einstein field equations in vacuum

then that geometry is necessarily a subset of the Schwarzschild geometry.

(Proof given in Misner, Thorne, and Wheeler (1973, pp. 843–844))

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Interior structure

Interior structure

- now we look at the remaining, and most interesting regime
  - inside the star
- our assumptions from outside the star no longer hold

Tolman–Oppenheimer–Volkov (T–O–V) equation

Condition

$$\rho \neq 0 \quad p \neq 0$$

Recall

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

T–O–V equation

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System of coupled differential equations

T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

Mass function

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Equation of state

$$p = p(\rho)$$

Schutz (2009, pp. 261–262, 264)

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Schutz (2009, pp. 261–262, 264)

- T–O–V equation coupled with $\mathrm{d}m/\mathrm{d}r$ and $p(\rho)$
  - 3 equations
  - 3 unknowns (m, ρ, p)
  - $\Phi(r)$ only intermediate variable
- can integrate to find $m(r)$, $\rho(r)$, and $p(r)$

Newtonian hydrostatic equilibrium

Newtonian limit

$$p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll r$$

Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -\frac{\rho m(r)}{r^2}$$

Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3)

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Constant density solution I

Constraint

$$\rho \equiv \rho_0$$

Mass function

$$m(r) = \frac{4}{3}\pi\rho_0 \begin{cases} r^3, & r \le R, \\ R^3, & r \ge R. \end{cases}$$

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- because it is the simplest case, we are going to investigate a star of uniform density, $\rho(r) \equiv \rho_0$
  - this is unphysical
  - for instance, the speed of sound in such a star is infinite
  - neutron star density is *almost* uniform
  - also leads us to a result which holds for all stellar densities
- obtaining the mass function from the differential equation shown earlier is easy
  - equal to the density times the volume of the sphere enclosed by radius r inside
  - equal to the density times the volume of the entire star $(r = R)$ when outside
  - continuous at the boundary

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T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} = -\frac{4}{3}\pi r \frac{(\rho_0 + p)(\rho_0 + 3p)}{1 - \frac{8}{3} r^2 \rho_0}$$

Integrated from center to internal radius r

$$\frac{\rho_0 + 3p}{\rho_0 + p} = \frac{\rho_0 + 3p_c}{\rho_0 + p_c}\sqrt{1 - 2m/r}$$

Schutz (2009, pp. 264, 266-267)

Constant density solution II
Schutz (2009, pp. 264, 266-267)

- recall the T–O–V equation, which describes the interior of the star
- we can substitute $m(r)$ for $r \leq R$, to simplify it as shown
- this gives us a seperable differential equation
- we integrate the differential equation from the center ($r = 0$, $p = p_c$) to some radius ($r = r$, $p = p$)
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- now we have a relation between ρ_0, p, and $m(r)$ at a given r

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Constant density solution III

Radius R

$$R^2 = \frac{3}{8\pi\rho_0}\left[1 - \left(\frac{\rho + p_c}{\rho + 3p_c}\right)^2\right]$$

Central pressure p_c

$$p_c = \rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}$$

Limit on M/R

$$M/R \to 4/9 \implies p_c \to \infty$$

Schutz (2009, pp. 266-267, 269)

- at the surface, $r = R$ and $p = 0$
- can solve the previous equation for R
- from this, we can solve for p_c
  - this gives us an expression for the central pressure necessary
- we can see that this blows up when $M/R = 4/9$

$$3\sqrt{1 - 8/9} - 1 = 3\sqrt{1/9} - 1 = 1 - 0 = 0$$

- radius cannot be smaller than $(9/4)M$
  - less than the $2M$ needed for a black hole
- Buchdahl's theorem states that this is true in general for all stars
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Buchdahl's theorem

- even for non-constant density, $M/R < 4/9$
- intuitive explanation:

Carroll (2004, pp. 234)

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- restate $M/R < 4/9$ from Buchdahl's theorem
- give Carroll's intuitive explanation
  - if we assume there is a maximum sustainable density in nature
  - and we consider an object which fills a sphere with radius R
  - then the most massive possible object within that volume would have a uniform density
  - all other objects would need to have a lower density

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  - assume there *is* a maximum sustainable density, $(M/R)_{\max}$
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  - assume there is a maximum sustainable density, $(M/R)_{\max}$
  - consider an object of radius R
  - most massive possible object would have maximum density everywhere
  - all other sustainable objects have a lower M/R

Carroll (2004, pp. 234)

- restate $M/R < 4/9$ from Buchdahl's theorem
- give Carroll's intuitive explanation
  - if we assume there is a maximum sustainable density in nature
  - and we consider an object which fills a sphere with radius R
  - then the most massive possible object within that volume would have a uniform density
  - all other objects would need to have a lower density

Buchdahl's theorem

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Realistic stars

Realistic stars

- now we're going to have a brief overview of real stars

White dwarfs

- end-of-life for low mass stars
- held up by electron degeneracy pressure
- Newtonian structure accurate to 1%

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho m}{r^2}$$

- relativistic effects important on stability and pulsation for

$$10^8 \mathrm{g\,cm^{-3}} \lesssim \rho_c \lesssim 10^{8.4} \mathrm{g\,cm^{-3}}$$

Misner, Thorne, and Wheeler (1973, p. 627)

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- end-of-life form of lower mass stars like our Sun is as a white dwarf
- core left over after a star loses its outer shell as a planetary nebula
- nuclear fusion has halted, and only pressure of degenerate electron gas supports them
  - Pauli exclusion principle
- structure can be described by the equation of HSE to high accuracy
- relativistic effects come into play for central densities:
  - over $10^8 \mathrm{g\,cm^{-3}}$
  - up until the maximum

Neutron stars

- mass condensed further than white dwarf
- created in supernovae, or collapse of white dwarf
- protons and electrons combine to form neutrons and emitted neutrinos
- held up by neutron degeneracy pressure
- matter incredibly complex and possess many unknown properties

Schutz (2009, pp. 274–275)

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Schutz (2009, pp. 274–275)

- when a star condenses beyond a white dwarf, it may become a neutron star
- occurs in the aftermath of a supernova, or collapse of white dwarf
- compression beyond neutron star would form a black hole
- kinetic energy of electrons high
  - allows energy release when combined with a proton
  - energy carried away by neutrino, and neutron left behind

Rotating stars

Metric

$$ds^2 = -e^{2\nu}\,dt + e^{2\psi}(d\phi - \omega\,dt)^2 + e^{2\mu}(dr^2 + r^2\,d\theta^2),$$

where ν, ψ, ω, and μ are functions of r and θ

- can still assume perfect fluid to high accuracy

Stergioulas (2003, p. 8)

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where ν, ψ, ω, and μ are functions of r and θ
• can still assume perfect fluid to high accuracy
Stergioulas (2003, p. 8)

- much more complicated when we allow for rotation
- metric no longer static
  - addition of cross terms between t and ϕ
  - metric dependence on θ in addition to r

Pulsars

- rapidly rotating neutron stars
- magnetic field produces electromagnetic radiation
- pulses of radio waves observed with the right orientation

Misner, Thorne, and Wheeler (1973, p. 628)

Pulsars
- rapidly rotating neutron stars
- magnetic field produces electromagnetic radiation
- pulses of radio waves observed with the right orientation

Misner, Thorne, and Wheeler (1973, p. 628)

- pulsars are rapidly rotating neutron stars
- they have a strong magnetic field which causes emission of light
- magnetic poles may be offset from axis of rotation
- if observed from right angle, see pulses of radio light, like lighthouse

References

References

- You made it to the end!

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Bonus slides

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Equations of motion

$$
\begin{aligned}
T^{\alpha\beta}{}_{;\beta} &= 0, \quad T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta} \\
T^{r\beta}{}_{;\beta} &= (\rho + p) U^\beta U^r{}_{;\beta} + g^{rr} p_{,r} = 0 \\
&= (\rho + p) U^\beta U^\lambda \Gamma^r{}_{\lambda\beta} + e^{-2\Lambda} p_{,r} = 0 \\
&= (\rho + p) (U^0)^2 \Gamma^r{}_{00} + e^{-2\Lambda} p_{,r} = 0 \\
&= (\rho + p) (e^{-2\Phi}) (e^{-2\Lambda} e^{2\Phi} \Phi_{,r}) + e^{-2\Lambda} p_{,r} = 0 \\
-\frac{\mathrm{d}p}{\mathrm{d}r} &= (\rho + p) \frac{\mathrm{d}\Phi}{\mathrm{d}r}
\end{aligned}
$$

Schutz (2009, pp. 101, 261)

2015-12-14

Spherical stars
└Bonus slides
└Equations of motion

Equations of motion

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&= (\rho + p) U^\beta U^\lambda \Gamma^r{}_{\lambda\beta} + e^{-2\Lambda} p_{,r} = 0 \\
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