

## Chapter 2

# Vector analysis in special relativity

### 2.1 Definition of a vector

### 2.2 Vector algebra

### 2.3 The four-velocity

An object's four velocity, denoted  $\vec{U}$ , is the vector tangent to its world line, with unit length. This means it extends one unit in time, and zero in space, so it is timelike.

For an *accelerated* particle (which we have not considered up to now), we may not be able to define an inertial frame, but we *can* define a **momentarily comoving reference frame** (MCRF) which, as the name suggests, moves with the same velocity as the observer for an infinitesimal period of time. We can therefore construct a continuous sequence of MCRFs for any object. If an object has MCRF  $\mathcal{O}$ , then its four-velocity is *defined* to be the basis vector  $\vec{e}_0$ .

### 2.4 The four-momentum

Analogous to the three-momentum, we define the four-momentum to be

$$\vec{p} = m\vec{U}. \tag{Schutz 2.19}$$

It has components

$$\vec{p} \rightarrow_{\mathcal{O}} (E, p^1, p^2, p^3). \tag{Schutz 2.20}$$

Calling  $p^0$  “ $E$ ” is no accident, it is in fact the energy. There is an interesting consequence to this: since vectors are invariant with respect to reference frame, but vector components are not, this means that the four-momentum does not change in different reference frames, but the energy *does*. One example would be

the doppler effect, which causes the color (or energy) of a photon to shift depending on the radial velocity of the source and observer.

## 2.5 Scalar product

$$\vec{A} \cdot \vec{B} = -(A^0 B^0) + (A^1 B^1) + (A^2 B^2) + (A^3 B^3)$$

## 2.6 Applications

## 2.7 Photons

$\vec{x} \cdot \vec{x} = 0$ , so we cannot define  $\vec{U}$  for photons. We can, however, define  $\vec{p}$ . Since  $\vec{p} \cdot \vec{p} = -m^2$ , and photons are massless, we have  $\vec{p} \cdot \vec{p} = 0$ .

## 2.8 Further reading

## 2.9 Exercises

**2** Identify the free and dummy indices in the following equations, and write equivalent expressions with different indices. Also, write how many equations are represented by each expression.

*Note, I will express the set of free indices by  $\mathcal{F}$  and the set of dummy indices as  $\mathcal{D}$ , and I will use the original index names.*

(a)  $A^\alpha B_\beta = 5 \implies A^\beta B_\alpha = 5$  (16 equations,  $\mathcal{F} = \{\alpha, \beta\}$ ,  $\mathcal{D} = \emptyset$ )

(b)  $A^\mu = \Lambda^\mu_\nu A^\nu \implies A^\bar{\nu} = \Lambda^\bar{\nu}_\mu A^\mu$  (4 equations,  $\mathcal{F} = \{\bar{\mu}\}$ ,  $\mathcal{D} = \{\nu\}$ ).

(c)  $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha} \implies T^{\eta\phi\theta} A_\phi C_\theta{}^\zeta = D^{\zeta\eta}$  (16 equations,  $\mathcal{F} = \{\alpha, \gamma\}$ ,  $\mathcal{D} = \{\mu, \lambda\}$ )

(d)  $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = G_{\mu\nu} \implies R_{\chi\epsilon} - \frac{1}{2}g_{\chi\epsilon} = G_{\chi\epsilon}$  (16 equations,  $\mathcal{F} = \{\mu, \nu\}$ ,  $\mathcal{D} = \emptyset$ )

**4** Given vectors  $\vec{A} \rightarrow_{\mathcal{O}} (5, -1, 0, 1)$  and  $\vec{B} \rightarrow_{\mathcal{O}} (-2, 1, 1, -6)$ , find the components in  $\mathcal{O}$  of

(a)  $-6\vec{A} \rightarrow_{\mathcal{O}} (-30, 6, 0, -6)$

(b)  $3\vec{A} + \vec{B} \rightarrow_{\mathcal{O}} (13, -2, 1, -3)$

(c)  $-6\vec{A} + 3\vec{B} \rightarrow_{\mathcal{O}} (-36, 9, 3, -24)$

**6** Draw a spacetime diagram from  $\mathcal{O}$ 's reference frame. There are two other frames,  $\bar{\mathcal{O}}$  and  $\bar{\bar{\mathcal{O}}}$ , which are each moving with velocity 0.6 in the  $+x$  direction from each respective frame. Plot each frame's basis vectors, as observed by  $\mathcal{O}$ .

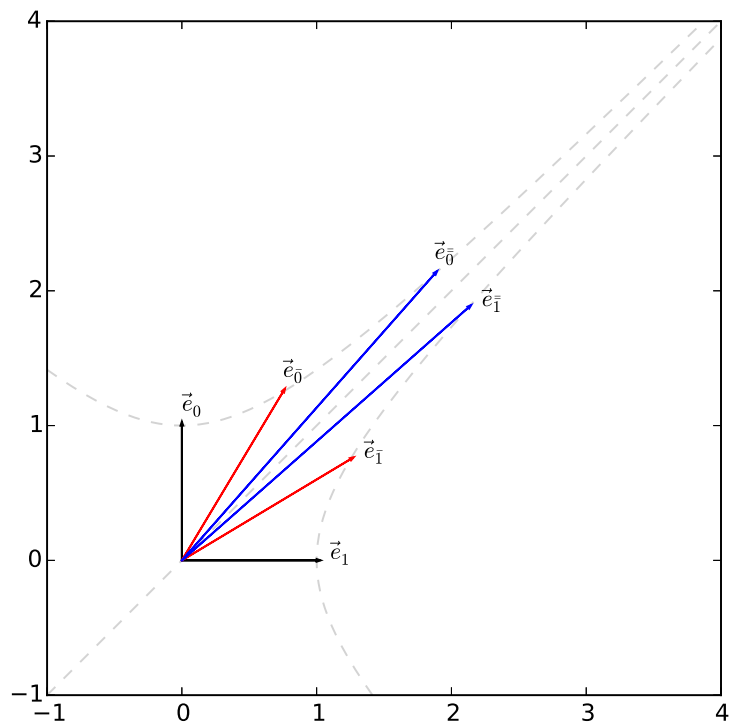


Figure 2.1: Exercise 6

See Figure 2.1.

**9** Prove, by writing out all the terms that

$$\sum_{\bar{\alpha}=0}^3 \left( \sum_{\beta=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right) = \sum_{\beta=0}^3 \left( \sum_{\bar{\alpha}=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right)$$

$$\begin{aligned}
\sum_{\bar{\alpha}=0}^3 \left( \sum_{\beta=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right) &= \sum_{\bar{\alpha}=0}^3 \left( \Lambda^{\bar{\alpha}}_0 A^0 \vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}_1 A^1 \vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}_2 A^2 \vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}_3 A^3 \vec{e}_{\bar{\alpha}} \right) \\
&= \Lambda^{\bar{0}}_0 A^0 \vec{e}_{\bar{0}} + \Lambda^{\bar{0}}_1 A^1 \vec{e}_{\bar{0}} + \Lambda^{\bar{0}}_2 A^2 \vec{e}_{\bar{0}} + \Lambda^{\bar{0}}_3 A^3 \vec{e}_{\bar{0}} \\
&\quad + \Lambda^{\bar{1}}_0 A^0 \vec{e}_{\bar{1}} + \Lambda^{\bar{1}}_1 A^1 \vec{e}_{\bar{1}} + \Lambda^{\bar{1}}_2 A^2 \vec{e}_{\bar{1}} + \Lambda^{\bar{1}}_3 A^3 \vec{e}_{\bar{1}} \\
&\quad + \Lambda^{\bar{2}}_0 A^0 \vec{e}_{\bar{2}} + \Lambda^{\bar{2}}_1 A^1 \vec{e}_{\bar{2}} + \Lambda^{\bar{2}}_2 A^2 \vec{e}_{\bar{2}} + \Lambda^{\bar{2}}_3 A^3 \vec{e}_{\bar{2}} \\
&\quad + \Lambda^{\bar{3}}_0 A^0 \vec{e}_{\bar{3}} + \Lambda^{\bar{3}}_1 A^1 \vec{e}_{\bar{3}} + \Lambda^{\bar{3}}_2 A^2 \vec{e}_{\bar{3}} + \Lambda^{\bar{3}}_3 A^3 \vec{e}_{\bar{3}} \\
&= \Lambda^{\bar{0}}_0 A^0 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_0 A^0 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_0 A^0 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_0 A^0 \vec{e}_{\bar{3}} \\
&\quad + \Lambda^{\bar{0}}_1 A^1 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_1 A^1 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_1 A^1 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_1 A^1 \vec{e}_{\bar{3}} \\
&\quad + \Lambda^{\bar{0}}_2 A^2 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_2 A^2 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_2 A^2 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_2 A^2 \vec{e}_{\bar{3}} \\
&\quad + \Lambda^{\bar{0}}_3 A^3 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_3 A^3 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_3 A^3 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_3 A^3 \vec{e}_{\bar{3}} \\
&= \sum_{\beta=0}^3 \left( \Lambda^{\bar{0}}_{\beta} A^{\beta} \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_{\beta} A^{\beta} \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_{\beta} A^{\beta} \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_{\beta} A^{\beta} \vec{e}_{\bar{3}} \right) \\
&= \sum_{\beta=0}^3 \left( \sum_{\bar{\alpha}=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right)
\end{aligned}$$

**11** Let  $\Lambda^{\bar{\alpha}}_{\beta}$  be the matrix of the Lorentz transformation from  $\mathcal{O}$  to  $\bar{\mathcal{O}}$ , given in Equation 1.12. Let  $\vec{A}$  be an arbitrary vector with components  $(A^0, A^1, A^2, A^3)$  in frame  $\mathcal{O}$ .

(a) Write down the matrix of  $\Lambda^{\nu}_{\bar{\mu}}(-v)$ .

Intuitively, it should appear the same as  $\Lambda^{\bar{\alpha}}_{\beta}$ , but with the negative signs removed. More rigorously, it is given by the matrix inverse of  $\Lambda^{\bar{\alpha}}_{\beta}$ , as their product should be the identity matrix. I have used a computer algebra system (Wolfram Alpha) to take the inverse of this matrix symbolically, confirming my suspicion:

$$\Lambda^{\nu}_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Find  $A^{\bar{\alpha}}$  for all  $\bar{\alpha}$ .

$$A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} A^{\beta}$$

$$A^{\bar{0}} = \gamma(A^0 - vA^1)$$

$$A^{\bar{1}} = \gamma(A^1 - vA^0)$$

$$A^{\bar{2}} = A^2$$

$$A^{\bar{3}} = A^3$$

- (c) Verify Equation 2.18 by performing the sum for all values of  $\nu$  and  $\alpha$ .

To simplify things, I do this via matrix multiplication

$$\begin{aligned}
 \Lambda^{\bar{\alpha}}_{\beta}(v)\Lambda^{\nu}_{\bar{\mu}}(-v) &= \begin{pmatrix} \gamma^2 - v^2\gamma^2 & v\gamma^2 - v\gamma^2 & 0 & 0 \\ v\gamma^2 - v\gamma^2 & \gamma^2 - v^2\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \gamma^2(1 - v^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1 - v^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\nu}_{\alpha}
 \end{aligned}$$

- (d) Write down the Lorentz transformation matrix from  $\bar{\mathcal{O}}$  to  $\mathcal{O}$ , justifying each term.

It should just be  $\Lambda^{\nu}_{\bar{\mu}}(-v)$ . I'm not sure what else to say at this point.

- (e) Using the result from part (d), find  $A^{\beta}$  from  $A^{\bar{\alpha}}$ . How does this relate to Equation 2.18?

$$\begin{aligned}
 \Lambda^{\beta}_{\bar{\alpha}}A^{\bar{\alpha}} &= \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(A^0 - vA^1) \\ \gamma(A^1 - vA^0) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma^2(A^0 - vA^1) + v\gamma^2(A^1 - vA^0) + 0 + 0 \\ v\gamma^2(A^0 - vA^1) + \gamma^2(A^1 - vA^0) + 0 + 0 \\ A^2 \\ A^3 \end{pmatrix} \\
 &= \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) + A^1(v\gamma^2 - v\gamma^2) \\ A^0(v\gamma^2 - v^2\gamma^2) + A^1(\gamma^2 - v\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) \\ A^1(\gamma^2 - v^2\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = A^{\beta}
 \end{aligned}$$

Since  $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta}(v)$ , this goes to show that  $\Lambda^{\nu}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}_{\alpha}(-v)A^{\alpha} = A^{\nu} \implies \Lambda^{\nu}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}_{\alpha}(-v) = \delta^{\nu}_{\alpha}$ .

- (f) Verify in the same manner as (c) that

$$\Lambda^{\nu}_{\bar{\beta}}(v)\Lambda^{\bar{\alpha}}_{\nu}(-v) = \delta^{\bar{\alpha}}_{\bar{\beta}}$$

My matrix multiplication approach will just give me the same result as before. Perhaps another approach was intended?

(g) Establish that

$$\begin{aligned}\vec{e}_\alpha &= \Lambda^\beta_\alpha \vec{e}_\beta = \Lambda^\beta_\alpha \Lambda^\nu_\beta \vec{e}_\nu = \delta^\nu_\alpha \vec{e}_\nu \\ A^\beta &= \Lambda^\beta_\alpha A^\alpha = \Lambda^\beta_\alpha \Lambda^\alpha_\mu A^\mu = \delta^\beta_\mu A^\mu\end{aligned}$$

**14** The following matrix gives a Lorentz transformation from  $\mathcal{O}$  to  $\bar{\mathcal{O}}$ :

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a) What is the velocity of  $\bar{\mathcal{O}}$  relative to  $\mathcal{O}$ ?

This would correspond to a Lorentz boost along the  $z$ -axis, meaning

$$\Lambda^{\bar{\alpha}}_{\beta}(v) = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix},$$

and thus we have  $\gamma = 1.25$  and  $-v\gamma = 0.75$ . Solving for  $v$ , we get

$$-v\gamma = \frac{3}{4} \implies v = -\frac{3}{4\gamma} = -\frac{3 \cdot 4}{4 \cdot 5} = -\frac{3}{5}.$$

So  $\bar{\mathcal{O}}$  is moving with speed 0.6 relative to the  $-z$ -axis of  $\mathcal{O}$ .

(b) What is the inverse matrix to the given one?

Numerically, it comes out to be

$$\begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix},$$

which makes sense, when you consider that the inverse matrix should be a Lorentz transformation with the velocity negated.

(c) Find the components in  $\mathcal{O}$  of  $\vec{A} \rightarrow_{\mathcal{O}} (1, 2, 0, 0)$ .

$$\vec{A} \rightarrow_{\mathcal{O}} \begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2 \\ 0 \\ -0.75 \end{pmatrix}$$

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(a) Compute the four-velocity components in  $\mathcal{O}$  of a particle whose speed is  $v$  in the  $+x$ -direction relative to  $\mathcal{O}$ , using the Lorentz transformation.

$$\vec{U} = \vec{e}_0$$

$$U^\alpha = \Lambda^\alpha_{\bar{\beta}} (\vec{e}_0)^{\bar{\beta}} = \Lambda^\alpha_0,$$

$$U^0 = \gamma$$

$$U^1 = v\gamma$$

$$U^2 = U^3 = 0$$

(b) Generalize to arbitrary velocities  $\mathbf{v}$ , where  $|v| < 1$ .

$$\Lambda^\alpha_{\bar{\beta}}(\mathbf{v}) = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

$$U^0 = \gamma \quad U^1 = \gamma v_x \quad U^2 = \gamma v_y \quad U^3 = \gamma v_z$$

(c) Use this result to express  $\mathbf{v}$  as a function of the components  $\{U^\alpha\}$ .

$$\mathbf{v} = v_x \vec{e}_1 + v_y \vec{e}_2 + v_z \vec{e}_3$$

$$v_i = \frac{U^i}{\gamma}$$

$$\mathbf{v} = \frac{1}{\gamma} U^i \vec{e}_i$$

(d) Find the three-velocity  $\mathbf{v}$  of a particle with four-velocity components  $(2, 1, 1, 1)$ .

$$U^0 = \gamma = 2, \text{ and } U^i = 1, \text{ so}$$

$$\mathbf{v} = \frac{1}{2} \vec{e}_i$$

## 17

**Not sure how to approach this problem.**

- (a) Prove that any timelike vector  $\vec{U}$  for which  $U^0 > 0$  and  $\vec{U} \cdot \vec{U} = -1$  is the four-velocity of *some* world line.
- (b) Use this to prove that for any timelike vector  $\vec{V}$  there is a Lorentz frame in which the  $\vec{V}$  has zero spatial components.

**19** A body is uniformly accelerated if the four-vector  $\vec{a}$  has constant spatial direction and magnitude,  $\vec{a} \cdot \vec{a} = \alpha^2 \geq 0$ .

- (a) Show that this implies the components of  $\vec{a}$  in the body's MCRF are all constant, and that these are equivalent to the Galilean "acceleration".

We normalize the vector  $\vec{a}$  by dividing each of its terms by the magnitude of the vector, so

$$\frac{a^\lambda}{\alpha}.$$

Since  $\alpha$  is constant, and also the *direction* is constant, this means that the above expression is *also* constant, as the normalized components tell you about the direction. If we multiply a constant by a constant, we should still get a constant, so we multiply the above expression by  $\alpha$ , getting  $a^\lambda$  to be constant.

In the MCRF of an object,  $d\tau = dt$ , and so we can write

$$\vec{a} = \frac{d\vec{U}}{dt} = \left(0, \frac{dU^1}{dt}, \frac{dU^2}{dt}, \frac{dU^3}{dt}\right),$$

which is analogous to the Galilean acceleration.

- (b) A body is uniformly accelerated with  $\alpha = 10 \text{ m/s}^2$ . It starts from rest, and falls for a time  $t$ . Find its speed as a function of  $t$ , and find the time to reach  $v = 0.999$ .



$$\begin{aligned}
\vec{U}_{\text{MCRF}} &\rightarrow (1, 0, 0, 0) \\
&\rightarrow (\gamma, \gamma v, 0, 0) \\
\frac{d\vec{U}}{d\tau}_{\text{MCRF}} &\rightarrow (0, \alpha, 0, 0) \\
&\rightarrow (\gamma, \gamma \alpha, 0, 0) \\
U^x &= \int_0^t \frac{dU^x}{d\tau} d\tau = \int_0^t \gamma \alpha \frac{dt}{\gamma} = \int_0^t \alpha dt = \alpha t \\
&= \gamma v = \frac{v}{\sqrt{1-v^2}} \\
v^2 &= (\alpha t)^2 (1-v^2) = (\alpha t)^2 - (\alpha t v)^2 \\
v^2 (1 + (\alpha t)^2) &= (\alpha t)^2 \\
v^2 &= \frac{(\alpha t)^2}{1 + (\alpha t)^2} \implies v = \sqrt{\frac{(\alpha t)^2}{1 + (\alpha t)^2}}
\end{aligned}$$

To find the time to reach  $v = 0.999$ , we go back to the expression  $\gamma v = \alpha t$ , solve for  $t$ , and substitute for  $v$  and  $\alpha$ . Note that in natural units,  $\alpha = 10 \text{ m/s}^2 c^{-2} \approx 1.11 \times 10^{-16} \text{ m}^{-1}$

$$t = \frac{v}{\alpha \sqrt{1-v^2}} = \frac{0.999}{1.11 \times 10^{-16} \text{ m}^{-1} \sqrt{1-0.999^2}} \approx 2.01 \times 10^{17} \text{ m}.$$

**24** Show that a positron and electron cannot annihilate to form a single photon, but they can annihilate to form two photons.

We consider the center of momentum frame, where  $\sum \vec{p}_{(i)} \rightarrow_{\text{CM}} (E_{\text{total}}, 0, 0, 0)$ . Without loss of generality, we assume that the velocities of the two particles are equal and opposite, such that

$$\vec{p}_{e^+} \rightarrow_{\text{CM}} m_e(\gamma, \gamma v, 0, 0), \quad \vec{p}_{e^-} \rightarrow_{\text{CM}} m_e(\gamma, -\gamma v, 0, 0).$$

The photon they create will have to have a momentum of  $\vec{p}_{\gamma, \text{single}} \rightarrow_{\text{CM}} (h\nu, h\nu, 0, 0)$ . By conservation of four-momentum, we have

$$\begin{aligned}
\vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma, \text{single}} \\
(\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= \vec{p}_{\gamma, \text{single}} \cdot \vec{p}_{\gamma, \text{single}} \\
(\vec{p}_{e^+} \cdot \vec{p}_{e^+}) + (\vec{p}_{e^-} \cdot \vec{p}_{e^-}) + (\vec{p}_{e^+} \cdot \vec{p}_{e^-}) &= 0 \\
-m_e^2 - m_e^2 - m_e^2 &= 0 \implies m_e = 0!
\end{aligned}$$

Since we know that  $m_e$  is in fact non-zero, this cannot possibly happen.

Now consider the scenario wherein two photons are created, moving in opposite directions. Then they would have momenta:  $\vec{p}_{\gamma, 1} \rightarrow_{\text{CM}} (h\nu, h\nu, 0, 0)$  and  $\vec{p}_{\gamma, 2} \rightarrow_{\text{CM}} (h\nu, -h\nu, 0, 0)$ . Invoking conservation of four-

momentum as before, we get

$$\begin{aligned}
 \vec{p}_{e+} + \vec{p}_{e-} &= \vec{p}_{\gamma,1} + \vec{p}_{\gamma,2} \\
 (\vec{p}_{e+} + \vec{p}_{e-}) \cdot (\vec{p}_{e+} + \vec{p}_{e-}) &= (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \cdot (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \\
 -3m_e^2 &= (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,1}) + (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,2}) + (\vec{p}_{\gamma,2} \cdot \vec{p}_{\gamma,2}) \\
 &= 0 + (-h^2\nu^2 - h^2\nu^2) + 0 = -2h^2\nu^2,
 \end{aligned}$$

so we end up with  $3m_e^2 = 2h^2\nu^2$ , meaning two photons are produced with  $E^2 = \frac{3}{2}m_e^2$ , which is entirely reasonable.

## 25

(a) Consider a frame  $\bar{\mathcal{O}}$  moving with a speed  $v$  along the  $x$ -axis of  $\mathcal{O}$ . Now consider a photon moving at an angle  $\theta$  from  $\mathcal{O}$ 's  $x$ -axis. Find the ratio of its frequency in  $\bar{\mathcal{O}}$  and in  $\mathcal{O}$ .

We must first construct the particle's four-momentum. In the case where the photon was moving along the  $x$ -axis (see Section 2.7), it had been found that the four-momentum was

$$\vec{p} \xrightarrow{\mathcal{O}} (E, E, 0, 0),$$

as this satisfied

$$\vec{p} \cdot \vec{p} = -E^2 + E^2 = 0. \quad (\text{Schutz 2.37})$$

Now that the photon is moving at an angle  $\theta$  from the  $x$ -axis, we need to redistribute the 3-momentum accordingly. No specification was given as photon's angle in the  $y$ - or  $z$ -axis, so without loss of generality, I assume it is constrained to the  $x$ - $y$  plane. This means we can write the four-momentum as

$$\vec{p} \xrightarrow{\mathcal{O}} (E, E \cos \theta, E \sin \theta, 0),$$

which you can easily confirm satisfies  $\vec{p} \cdot \vec{p} = 0$ .

Now we may apply the Lorentz transformation  $\Lambda_{\alpha}^{\bar{0}}(v)$  to find the photon's energy as observed by  $\bar{\mathcal{O}}$ , and from that the frequency.

$$\begin{aligned}
 p^{\bar{0}} = \bar{E} &= \Lambda_{\alpha}^{\bar{0}} p^{\alpha} = \gamma p^0 - v\gamma p^1 + 0 + 0 = \gamma E - v\gamma E \cos \theta \\
 \implies h\bar{\nu} &= \gamma h\nu - v\gamma h\nu \cos \theta \\
 \implies \frac{\bar{\nu}}{\nu} &= \gamma - v\gamma \cos \theta = \frac{1 - v \cos \theta}{\sqrt{1 - v^2}}
 \end{aligned}$$

(b) Even when the photon moves perpendicular to the  $x$ -axis ( $\theta = \pi/2$ ) there is a frequency shift. This is the *transverse Doppler shift*, which is a result of time dilation. At which angle  $\theta$  must the photon move such that there is no Doppler shift between  $\mathcal{O}$  and  $\bar{\mathcal{O}}$ ?

To do this, we simply set  $\bar{\nu}/\nu = 1$ , and solve for  $\theta$ .

$$\begin{aligned} 1 &= \frac{1 - v \cos \theta}{\sqrt{1 - v^2}} \implies \cos \theta = 1 - \sqrt{1 - v^2} \\ \implies \theta &= \pm \arccos(1 - \sqrt{1 - v^2}) \end{aligned}$$

(c) Now use Equations 2.35 and 2.38 to find  $\bar{\nu}/\nu$ .

Recall that  $\vec{U} \rightarrow_{\mathcal{O}} (\gamma, v\gamma, 0, 0)$ . Using Equation 2.35 we have

$$\begin{aligned} \bar{E} &= h\bar{\nu} = -(E, E \cos \theta, E \sin \theta, 0) \cdot (\gamma, v\gamma, 0, 0) \\ &= -(-(E\gamma) + E\gamma v \cos \theta) = E\gamma(1 - v \cos \theta) = h\nu\gamma(1 - v \cos \theta) \\ \frac{\bar{\nu}}{\nu} &= \frac{1 - v \cos \theta}{\sqrt{1 - v^2}} \end{aligned}$$

**26** Calculate the energy required to accelerate a particle of rest mass  $m > 0$  from speed  $v$  to speed  $v + \delta v$  ( $\delta v \ll v$ ), to first order in  $\delta v$ . Show that it would take infinite energy to accelerate to  $c$ .

From the four-momentum we have  $E_v = m\gamma$ , and from that

$$E_{v+\delta v} = \frac{m}{\sqrt{1 - (v + \delta v)^2}}.$$

If we do a Taylor expansion on  $(1 - (v + \delta v)^2)^{-1/2}$  we get

$$\frac{1}{\sqrt{1 - v^2}} + \frac{v \delta v}{(1 - v^2)^{3/2}} + \mathcal{O}(v^2),$$

so

$$\begin{aligned} E_{v+\delta v} &\approx \frac{m}{\sqrt{1 - v^2}} + \frac{mv \delta v}{(1 - v^2)^{3/2}} \\ \Delta E &= E_{v+\delta v} - E_v \approx \frac{mv \delta v}{(1 - v^2)^{3/2}} = m\gamma^3 v \delta v. \end{aligned}$$

As  $v \rightarrow c$ ,  $\gamma \rightarrow \infty$  and therefore  $\Delta E \rightarrow \infty$ .

**30** A rocket ship has four-velocity  $\vec{U} \rightarrow_{\mathcal{O}} (2, 1, 1, 1)$ , and it passes a cosmic ray with four-momentum  $\vec{p} \rightarrow_{\mathcal{O}} (300, 299, 0, 0) \times 10^{-27} \text{kg}$ . Compute the energy of the ray as measured by the rocket, using two different methods.

(a) Find the Lorentz transformation from  $\mathcal{O}$  to the rocket's MCRF, and from that find the components  $p^{\bar{\alpha}}$ .

The Lorentz transformation for a boost in the  $x$ ,  $y$ , and  $z$  directions is given by

$$\Lambda^{\bar{\beta}}_{\alpha} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

If we write out the terms of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

then we are left with a system of equations

$$1 = \gamma(2 + v_x + v_y + v_z),$$

$$0 = \gamma(2v_x + 1),$$

$$0 = \gamma(2v_y + 1),$$

$$0 = \gamma(2v_z + 1).$$

Since  $\gamma$  may never be zero, we divide the last 3 terms by  $\gamma$  to obtain

$$2v_i + 1 = 0 \implies v_i = -\frac{1}{2},$$

and plugging into the first equation gives  $\gamma = 2$ . From this we see that our Lorentz transformation matrix is

$$\Lambda^{\bar{\beta}}_{\alpha} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

Now to find the energy as observed by the rocket, we need to find  $\bar{E} = p^{\bar{0}}$

$$\begin{aligned} p^{\bar{0}} &= \Lambda^{\bar{0}}_{\alpha} p^{\alpha} = 2p^0 - p^1 - p^2 - p^3 \\ &= (2 \cdot 300 - 1 \cdot 299 - 1 \cdot 0 - 1 \cdot 0) \times 10^{-27} \text{kg} = 3.01 \times 10^{-25} \text{kg} = \bar{E} \end{aligned}$$

(b) Use Schutz's Equation 2.35.

$$\begin{aligned} \bar{E} &= -\vec{p} \cdot \vec{U}_{\text{obs}} = -(-(300 \cdot 2) + (299 \cdot 1) + (0 \cdot 1) + (0 \cdot 1)) \times 10^{-27} \text{kg} \\ &= 3.01 \times 10^{-25} \text{kg} \end{aligned}$$

(c) Which is quicker? Why?

Using Equation 2.35 was *much* quicker, as it was derived to handle this special case.

**32** Consider a particle with charge  $e$  and mass  $m$ , which begins at rest, but scatters a photon with frequency  $\nu_i$  (Compton scattering). The photon comes off at an angle  $\theta$  from the direction of the initial photon's path. Use conservation of four-momentum to find the scattered photon's frequency,  $\nu_f$ .

We will invoke: conservation of four-momentum and  $\vec{p} \cdot \vec{p} = -m^2$ .  $\vec{p}_i$  and  $\vec{p}_f$  denote the initial and final

photon, and  $\vec{p}_e$  and  $\vec{p}_{e'}$  denote the electron before and after collision.

$$\vec{p}_i \xrightarrow{\mathcal{O}} (E_i, E_i, 0, 0)$$

$$\vec{p}_e \xrightarrow{\mathcal{O}} (m, 0, 0, 0)$$

$$\vec{p}_f \xrightarrow{\mathcal{O}} (E_f, E_f \cos \theta, E_f \sin \theta, 0)$$

$$\vec{p}_i + \vec{p}_e = \vec{p}_f + \vec{p}_{e'}$$

$$\vec{p}_{e'} = \vec{p}_i + \vec{p}_e - \vec{p}_f$$

$$\vec{p}_{e'} \cdot \vec{p}_{e'} = (\vec{p}_i + \vec{p}_e - \vec{p}_f) \cdot (\vec{p}_i + \vec{p}_e - \vec{p}_f)$$

$$\begin{aligned} -m^2 &= \vec{p}_i \cdot \vec{p}_i + \vec{p}_e \cdot \vec{p}_e + \vec{p}_f \cdot \vec{p}_f + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\ &= 0 - m^2 + 0 + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \end{aligned}$$

$$\begin{aligned} 0 &= \vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f \\ &= -E_i m - (-E_i E_f + E_i E_f \cos \theta) + E_f m \\ &= m(E_f - E_i) + E_i E_f (1 - \cos \theta) \end{aligned}$$

$$m(E_i - E_f) = E_i E_f (1 - \cos \theta)$$

$$mh(\nu_i - \nu_f) = h^2 \nu_i \nu_f (1 - \cos \theta)$$

$$\frac{\nu_i - \nu_f}{\nu_i \nu_f} = h \frac{1 - \cos \theta}{m}$$

$$\frac{1}{\nu_f} - \frac{1}{\nu_i} = h \frac{1 - \cos \theta}{m}$$

$$\frac{1}{\nu_f} = \frac{1}{\nu_i} + h \frac{1 - \cos \theta}{m}$$