

Chapter 3

Tensor analysis in special relativity

3.3 The $(^0_1)$ tensors: one-forms

The symbol $\tilde{\cdot}$ is used to denote a one-form, as $\vec{\cdot}$ is used to denote a vector. So \tilde{p} is a one-form, or a type $(^0_1)$ tensor.

Normal one-forms

Let \mathcal{S} be some surface.

$\forall \vec{V}$ tangent to \mathcal{S} , $\tilde{p}(\vec{V}) = 0 \implies \tilde{p}$ is normal to \mathcal{S} .

Furthermore, if \mathcal{S} is a *closed* surface & \tilde{p} is normal to \mathcal{S} & $\forall \vec{U}$ pointing outwards from \mathcal{S} , $\tilde{p}(\vec{U}) > 0 \implies \tilde{p}$ is an outward normal one-form.

3.5 Metric as a mapping of vectors into one-forms

Normal vectors and unit normal one-forms

\vec{V} is normal to a surface if \tilde{V} is normal to the surface. They are said to be *unit normal* if their magnitude is ± 1 , so $\vec{V}^2 = \tilde{V}^2 = \pm 1$.

- A time-like unit normal has magnitude -1
- A space-like unit normal has magnitude $+1$
- A null normal cannot be a unit normal, because $\vec{V}^2 = \tilde{V}^2 = 0$

3.10 Exercises

(a)

$$\begin{aligned}\tilde{p}(A^\alpha \vec{e}_\alpha) &= A^\alpha \tilde{p}(\vec{e}_\alpha) = \tilde{p}(A^0 \vec{e}_0 + A^1 \vec{e}_1 + A^2 \vec{e}_2 + A^3 \vec{e}_3) \\ &= A^0 \tilde{p}(\vec{e}_0) + A^1 \tilde{p}(\vec{e}_1) + A^2 \tilde{p}(\vec{e}_2) + A^3 \tilde{p}(\vec{e}_3) = A^\alpha \tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha \in \mathbb{R}\end{aligned}$$

(b)

$$\tilde{p} \xrightarrow{\mathcal{O}} (-1, 1, 2, 0)$$

$$\vec{A} \xrightarrow{\mathcal{O}} (2, 1, 0, -1)$$

$$\vec{B} \xrightarrow{\mathcal{O}} (0, 2, 0, 0)$$

$$\tilde{p}(\vec{A}) = -2 + 1 + 0 + 0 = -1$$

$$\tilde{p}(\vec{B}) = 0 + 2 + 0 + 0 = 2$$

$$\tilde{p}(\vec{A} - 3\vec{B}) = \tilde{p}(\vec{A}) - 3\tilde{p}(\vec{B}) = -1 - 3 \cdot 2 = -7$$

4 Given the following vectors

$$\begin{array}{ll}\vec{A} \xrightarrow{\mathcal{O}} (2, 1, 1, 0) & \vec{B} \xrightarrow{\mathcal{O}} (1, 2, 0, 0) \\ \vec{C} \xrightarrow{\mathcal{O}} (0, 0, 1, 1) & \vec{D} \xrightarrow{\mathcal{O}} (-3, 2, 0, 0)\end{array}$$

(Note that all parts were done with the assistance of `numpy`.)

(a) Show that they are linearly independent.

We do this by constructing a matrix, \mathbf{X} , whose columns correspond to the four vectors. If the determinant of \mathbf{X} is non-zero, then that means the vectors are linearly independent.

$$\det(\mathbf{X}) = \det \begin{pmatrix} 2 & 1 & 0 & -3 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -8$$

(b) Find the components of \tilde{p} if

$$\tilde{p}(\vec{A}) = 1, \quad \tilde{p}(\vec{B}) = -1, \quad \tilde{p}(\vec{C}) = -1, \quad \tilde{p}(\vec{D}) = 0$$

We do this by observing that $\tilde{p} = A^\alpha p_\alpha$, and so we have a system of four equations, which we can write in

matrix form as

$$\begin{aligned} \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \\ \vec{D} \end{pmatrix} \tilde{p} &= \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \\ \implies \tilde{p} &= \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \\ \vec{D} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ +\frac{15}{8} \\ -\frac{23}{8} \end{pmatrix}. \end{aligned}$$

(c) Find $\tilde{p}(\vec{E})$, where $\vec{E} \rightarrow_{\mathcal{O}} (1, 1, 0, 0)$.

$$\tilde{p}(\vec{E}) = p_\alpha E^\alpha = -\frac{5}{8}$$

(d) Determine whether $\tilde{p}, \tilde{q}, \tilde{r}$, and \tilde{s} are linearly independent.

We do this by first setting up a system of equations for each of \tilde{q}, \tilde{r} , and \tilde{s} , as was done for \tilde{p} , and solving. I will refer to the matrix whose rows were $\vec{A}, \vec{B}, \vec{C}$, and \vec{D} as \mathbf{X} .

$$\begin{aligned} \mathbf{X}\tilde{q} &= \begin{pmatrix} +0 \\ +0 \\ +1 \\ -1 \end{pmatrix} & \mathbf{X}\tilde{r} &= \begin{pmatrix} +2 \\ +0 \\ +0 \\ +0 \end{pmatrix} & \mathbf{X}\tilde{s} &= \begin{pmatrix} -1 \\ -1 \\ +0 \\ +0 \end{pmatrix} \\ \tilde{q} &= \begin{pmatrix} +\frac{1}{4} \\ -\frac{1}{8} \\ -\frac{3}{8} \\ +\frac{11}{8} \end{pmatrix} & \tilde{r} &= \begin{pmatrix} +0 \\ +0 \\ +2 \\ +2 \end{pmatrix} & \tilde{s} &= \begin{pmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ -\frac{1}{8} \\ +\frac{1}{8} \end{pmatrix} \end{aligned}$$

Now if the matrix whose columns are comprised of $\tilde{p}, \tilde{q}, \tilde{r}$, and \tilde{s} has a non-zero determinant, then the four covectors must be linearly independent.

$$\det \begin{pmatrix} \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} \end{pmatrix} = \frac{1}{4},$$

and so they are indeed linearly independent.

(a) Show that $\tilde{p} \neq \tilde{p}(\vec{e}_\alpha)\tilde{\lambda}^\alpha$ for arbitrary \tilde{p} .

Let us choose $\tilde{p} \rightarrow_{\mathcal{O}} (0, 1, e, \pi)$, as a counter-example.

$$\begin{aligned} p_\alpha \tilde{\lambda}^\alpha &\xrightarrow{\mathcal{O}} 0 \cdot (1, 1, 0, 0) + 1 \cdot (1, -1, 0, 0) + e \cdot (0, 0, 1, -1) + \pi \cdot (0, 0, 1, 1) \\ &\xrightarrow{\mathcal{O}} (1, -1, e + \pi, 0) \not\rightarrow \tilde{p} \end{aligned}$$

(b) $\tilde{p} \rightarrow_{\mathcal{O}} (1, 1, 1, 1)$. Find l_α such that

$$\tilde{p} = l_\alpha \tilde{\lambda}^\alpha$$

We may do this with a simple matrix inversion. We define $\mathbf{\Lambda}$ to be the matrix whose rows are formed by $\tilde{\lambda}^\alpha$.

$$\mathbf{\Lambda}l = p \implies l = \mathbf{\Lambda}^{-1}p = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

8 Draw the basis one-forms \tilde{dt} and \tilde{dx} of frame \mathcal{O} .

They are

$$\begin{aligned} \tilde{dt} &\xrightarrow{\mathcal{O}} (1, 0, 0, 0), \\ \tilde{dx} &\xrightarrow{\mathcal{O}} (0, 1, 0, 0), \end{aligned}$$

and they are shown in Figure 3.1.

9 At the points \mathcal{P} and \mathcal{Q} , estimate the components of the gradient \tilde{dT} .

Recall that $\tilde{dT} \rightarrow_{\mathcal{O}} \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right)$, and so $\Delta T = \tilde{dT}_\alpha x^\alpha = \tilde{dT}_x \Delta x + \tilde{dT}_y \Delta y$.

Now if we move only in the x direction from one of the points, we move some distance Δx , change our temperature by Δt , and $\Delta y = 0$. Likewise for a movement in the y direction. Thus we can say

$$\begin{aligned} \Delta T &= \tilde{dT}_x \Delta x & \Delta T &= \tilde{dT}_y \Delta y \\ \tilde{dT}_x &= \frac{\Delta T}{\Delta x} & \tilde{dT}_y &= \frac{\Delta T}{\Delta y} \end{aligned}$$

In Figure 3.2, from \mathcal{P} I move a distance $\Delta x = 0.5$, which causes a temperature change of $\Delta T = -7$, giving $\tilde{dT}_x = -14$. Then I move a distance $\Delta y = 0.5$ and get the same temperature change of $\Delta T = -7$, and so I conclude that at point \mathcal{P} , $\tilde{dT} \rightarrow_{\mathcal{O}} (-14, -14)$.

At \mathcal{Q} , we are in a flat region where $T = 0$. If we move any non-zero distance Δx or Δy , so long as it does not cross the $T = 0$ isotherm, we have a $\Delta T = 0$, and thus $\tilde{dT}_p \rightarrow_{\mathcal{O}} (0, 0)$.

13 Prove that \tilde{df} is normal to surfaces of constant f .

If we move some small distance $\Delta x^\alpha = \epsilon$, then there will be no change in the value of f , and thus we can say $\partial f / \partial x^\alpha = 0$, so

$$\tilde{df} = \frac{\partial f}{\partial x^\alpha} \tilde{dx}^\alpha = 0 \tilde{dx}^\alpha = 0.$$

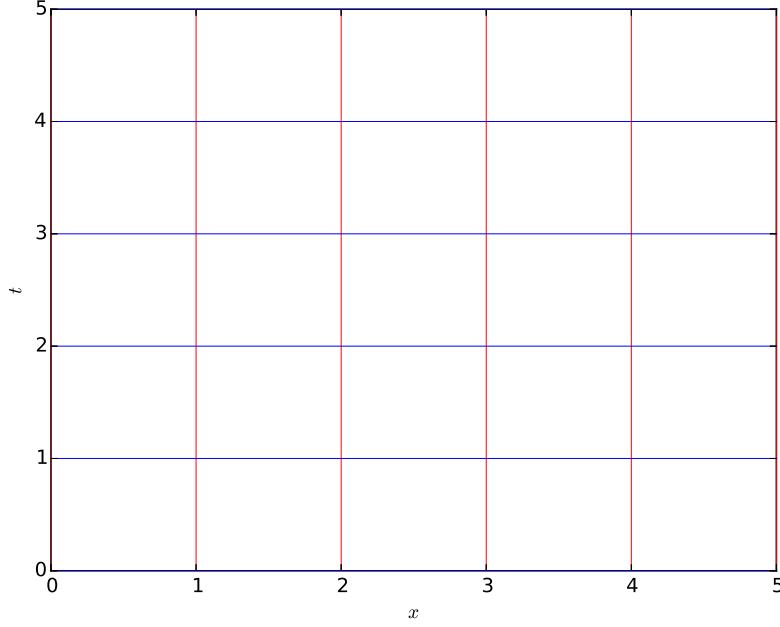


Figure 3.1: Problem 8: Basis one-forms of \mathcal{O} . \tilde{dt} is given in blue and \tilde{dx} in red.

Since \tilde{df} is defined to be normal to a surface if it is zero on every tangent vector, we have shown that \tilde{df} is normal to any surface of constant f .

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$$\tilde{p} \xrightarrow{\mathcal{O}} (1, 1, 0, 0)$$

$$\tilde{q} \xrightarrow{\mathcal{O}} (-1, 0, 1, 0)$$

Prove by giving two vectors \vec{A} and \vec{B} as arguments that $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

$$\begin{aligned} (\tilde{p} \otimes \tilde{q})(\vec{A}, \vec{B}) &= \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = A^\alpha p_\alpha B^\beta q_\beta = (A^0 + A^1)(-B^0 + B^2), \\ &= -A^0 B^0 + A^0 B^2 - A^1 B^0 + A^1 B^2 \\ (\tilde{q} \otimes \tilde{p})(\vec{A}, \vec{B}) &= \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = A^\alpha q_\alpha B^\beta p_\beta = (-A^0 + A^2)(B^0 + B^1) \\ &= -A^0 B^0 - A^0 B^1 + A^2 B^0 + A^2 B^1, \end{aligned}$$

And so we see that \otimes is not commutative.

The components of the outer product of two tensors are given by the products of the components of the

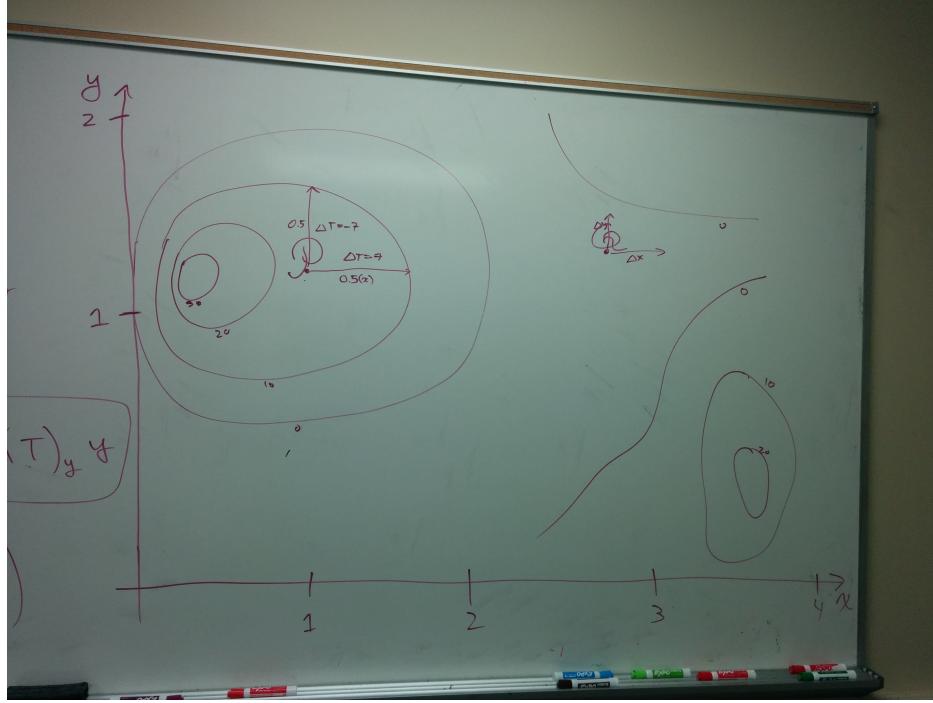


Figure 3.2: Problem 9: Isotherms.

individual tensors. Thus we can write the components as a 4×4 matrix.

$$(\tilde{p} \otimes \tilde{q})_{\alpha\beta} = p_\alpha q_\beta = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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(a) Find the one-forms mapped by \mathbf{g} from

$$\begin{aligned} \vec{A} \xrightarrow{\mathcal{O}} (1, 0, -1, 0), & \quad \vec{B} \xrightarrow{\mathcal{O}} (0, 1, 1, 0), \\ \vec{C} \xrightarrow{\mathcal{O}} (-1, 0, -1, 0), & \quad \vec{D} \xrightarrow{\mathcal{O}} (0, 0, 1, 1). \end{aligned}$$

In general,

$$\vec{V} \xrightarrow{\mathcal{O}} (V^0, V^1, V^2, V^3) \implies \tilde{V} = \mathbf{g}\vec{V} \xrightarrow{\mathcal{O}} (-V^0, V^1, V^2, V^3),$$

and so

$$\begin{aligned} \tilde{A} \xrightarrow{\mathcal{O}} (-1, 0, -1, 0), & \quad \tilde{B} \xrightarrow{\mathcal{O}} (0, 1, 1, 0), \\ \tilde{C} \xrightarrow{\mathcal{O}} (1, 0, -1, 0), & \quad \tilde{D} \xrightarrow{\mathcal{O}} (0, 0, 1, 1). \end{aligned}$$

(b) Find the vectors mapped by \mathbf{g} from

$$\begin{aligned}\tilde{p} &\xrightarrow{\mathcal{O}} (3, 0, -1, -1), & \tilde{q} &\xrightarrow{\mathcal{O}} (1, -1, 1, 1), \\ \tilde{r} &\xrightarrow{\mathcal{O}} (0, -5, -1, 0), & \tilde{s} &\xrightarrow{\mathcal{O}} (-2, 1, 0, 0).\end{aligned}$$

By using the inverse tensor in reverse, we have the same effect as before, of negating the first component

$$\begin{aligned}\vec{p} &\xrightarrow{\mathcal{O}} (-3, 0, -1, -1), & \vec{q} &\xrightarrow{\mathcal{O}} (-1, -1, 1, 1), \\ \vec{r} &\xrightarrow{\mathcal{O}} (0, -5, -1, 0), & \vec{s} &\xrightarrow{\mathcal{O}} (2, 1, 0, 0).\end{aligned}$$

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In Euclidean 3-space, vectors and covectors are usually treated as the same, because they transform the same. We will now prove this.

(a) Show that $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} A^{\beta}$ and $P_{\bar{\beta}} = \Lambda^{\alpha}_{\bar{\beta}} P_{\alpha}$ are the same transformations if $\{\Lambda^{\alpha}_{\bar{\beta}}\}$ is equal to the transpose of its inverse.

We can write that last statement as

$$\Lambda^{\alpha}_{\bar{\beta}} = ((\Lambda^{\alpha}_{\bar{\beta}})^{-1})^T$$

and we know that

$$(\Lambda^{\alpha}_{\bar{\beta}})^{-1} = \Lambda^{\bar{\beta}}_{\alpha},$$

and also we know that the Lorentz transformation is symmetric, and so

$$(\Lambda^{\bar{\beta}}_{\alpha})^T = \Lambda^{\bar{\beta}}_{\alpha},$$

which leads us to conclude that $\Lambda^{\alpha}_{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\alpha}$, meaning the two transformations are the same.

(b) The metric has components $\{\delta_{ij}\}$. Prove that transformations between Cartesian coordinate systems must satisfy

$$\delta_{\bar{i}\bar{j}} = \Lambda^k_{\bar{i}} \Lambda^l_{\bar{j}} \delta_{kl},$$

and that this implies that $\Lambda^k_{\bar{i}}$ is an orthogonal matrix.

$$\delta_{\bar{i}\bar{j}} = \mathbf{g}(\vec{e}_{\bar{i}}, \vec{e}_{\bar{j}}) = \mathbf{g}(\Lambda^k_{\bar{i}} \vec{e}_k, \Lambda^l_{\bar{j}} \vec{e}_l) = \Lambda^k_{\bar{i}} \Lambda^l_{\bar{j}} \mathbf{g}(\vec{e}_k, \vec{e}_l) = \Lambda^k_{\bar{i}} \Lambda^l_{\bar{j}} \delta_{kl}$$

Now show it is orthogonal

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(a) A region of the t - x plane is bounded by lines $t = 0$, $t = 1$, $x = 0$, and $x = 1$. Within the plane, find the unit outward normal 1-forms and their vectors for each boundary line.

I define unit outward normals as follows:

Let \mathcal{S} be a closed surface. If, for each \vec{V} tangent to \mathcal{S} , we have $\tilde{p}(\vec{V}) = 0$, then \tilde{p} is normal to \mathcal{S} .

In addition, if, for each \vec{U} which points outwards from the surface, we have $\tilde{p}(\vec{U}) > 0$, then \tilde{p} is an outward

normal.

Furthermore, if $\tilde{p}^2 = \pm 1$, then it is a unit outward normal.

For the problem at hand, I define the region inside the four lines to be *Inside*, and the region outside to be *Outside*. For each of the four lines, I draw a vector \vec{V} tangent (parallel) to the line, and \vec{U} pointing outwards (See Figure 3.3).

It helps to look at $t = 0$ and $t = 1$ together, and likewise for x , so I will start with t . We start with an arbitrary $\tilde{p} \rightarrow_{\mathcal{O}} (p_0, p_1)$, and $\vec{V} \rightarrow_{\mathcal{O}} (0, V^1)$, where $V^1 \neq 0$.

$$\tilde{p}(\vec{V}) = p_0 \cdot 0 + p_1 V^1 = 0 \implies p_1 = 0,$$

so $\tilde{p} \rightarrow_{\mathcal{O}} (p_0, 0)$ is a normal 1-form to both lines. Now we find the corresponding *unit* normal, by taking

$$\tilde{p}^2 = \pm 1 = -(p_0)^2 \implies \tilde{p}^2 = -1 \text{ & } p_0 = \pm 1.$$

Whether we choose p_0 to be positive or negative now depends on the line we are looking at, and which direction is outward. For $t = 0$, we have a vector $\vec{U} = (-U^0, U^1)$, where $U^0 > 0$.

$$\tilde{p}(\vec{U}) = p_0(-U^0) + 0 \cdot U^1 > 0 \implies -p_0 U^0 > 0 \implies p_0 < 0,$$

so for $t = 0$ we have $\tilde{p} \rightarrow_{\mathcal{O}} (-1, 0)$, and likewise for $t = 1$ we have $\tilde{p} \rightarrow_{\mathcal{O}} (1, 0)$. To get the associated *vectors*, we apply the metric $\eta^{\alpha\beta}$, giving us $\vec{p} \rightarrow_{\mathcal{O}} (1, 0)$ for $t = 0$ and $\vec{p} \rightarrow_{\mathcal{O}} (-1, 0)$ for $t = 1$.

For $x = 0$ and $x = 1$, we instead have $\vec{V} \rightarrow_{\mathcal{O}} (V^0, 0)$, and following the same steps as before, we conclude that: for $x = 0$, $\tilde{p} \rightarrow_{\mathcal{O}} (0, -1)$, $\vec{p} \rightarrow_{\mathcal{O}} (0, -1)$, and for $x = 1$, $\tilde{p} \rightarrow_{\mathcal{O}} (0, 1)$, $\vec{p} \rightarrow_{\mathcal{O}} (0, 1)$.

Figure 3.3: Problem 21.a

(b) Let another region be bounded by the set of points $\{(1, 0), (1, 1), (2, 1)\}$. Find an outward normal for the null boundary and the associated vector.

23

(a) Prove that the set of all $\binom{M}{N}$ tensors forms a vector space, V .

Let T be the set of all $\binom{M}{N}$ tensors, $\mathbf{s}, \mathbf{p}, \mathbf{q} \in T$, $\vec{A} \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. For T to be a vector space, we must define the operations of addition, and scalar multiplication (amongst others).

Addition:

$$\mathbf{s} = \mathbf{p} + \mathbf{q} \implies \mathbf{s}(\vec{A}) = \mathbf{p}(\vec{A}) + \mathbf{q}(\vec{A})$$

Scalar Multiplication:

$$\mathbf{r} = \alpha \mathbf{p} \implies \mathbf{r}(\vec{A}) = \alpha \mathbf{p}(\vec{A})$$

(b)

Prove that a basis for T is

$$\{\vec{e}_\alpha \otimes \dots \otimes \vec{e}_\gamma \otimes \tilde{\omega}^\mu \otimes \dots \otimes \tilde{\omega}^\lambda\}$$

Still working on it**24** Given:

$$M^{\alpha\beta} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(a) Find:

(i)

$$M^{(\alpha\beta)} \rightarrow \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{pmatrix}; \quad M^{[\alpha\beta]} \rightarrow \begin{pmatrix} 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} & 0 \end{pmatrix}$$

(ii)

$$M^\alpha{}_\beta = \eta_{\beta\mu} M^{\alpha\mu} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iii)

$$M_\alpha{}^\beta = \eta_{\alpha\mu} M^{\mu\beta} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iv)

$$M_{\alpha\beta} = \eta_{\beta\mu} M_\alpha{}^\mu \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(b) Does it make sense to separate the $\binom{1}{1}$ tensor with components $M^\alpha{}_\beta$ into symmetric and antisymmetric parts?

No, it would not make sense. For one, the notation for (anti)symmetric tensors do not even allow one to write it sensibly ($M^{(\alpha\beta)}$). More importantly, one argument refers to vectors, and the other to covectors, so it does not make sense to switch them.

(c)

$$\eta^{\alpha}_{\beta} = \eta^{\alpha\mu}\eta_{\beta\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\alpha}_{\beta}$$

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Still working on it

(33)

34 Define double-null coordinates $u = t - x$, $v = t + x$ in Minkowski space.

- (a) Let \vec{e}_u be the vector connecting the (u, v, y, t) coordinates $(0, 0, 0, 0)$ and $(1, 0, 0, 0)$, and let \vec{e}_v be the vector connecting $(0, 0, 0, 0)$ and $(0, 1, 0, 0)$. Find \vec{e}_u and \vec{e}_v in terms of \vec{e}_t and \vec{e}_x , and plot the basis vectors in a spacetime diagram of the t - x plane.

$$u = t - x = 0 \implies t = +x$$

$$v = t + x = 0 \implies t = -x$$

$$u = t - x = 1 \implies t = 1 + x$$

$$v = t + x = 1 \implies t = 1 - x$$

We draw the vectors \vec{e}_u and \vec{e}_v in Figure 3.4, such that they point from the appropriate points of intersection on these lines of constant u and v . From this it is obvious that $\vec{e}_v + \vec{e}_u = \vec{e}_t$, and that $\vec{e}_v - \vec{e}_u = \vec{e}_x$, or likewise $\vec{e}_v = \vec{e}_t - \vec{e}_u$ and $\vec{e}_u = \vec{e}_v - \vec{e}_x$. This is a system of 2 equations with two unknowns.

$$\begin{aligned} \vec{e}_v &= \vec{e}_t - \vec{e}_u + \vec{e}_x \implies &\vec{e}_v &= \frac{1}{2}(\vec{e}_t + \vec{e}_x), \\ \vec{e}_u &= \frac{1}{2}(\vec{e}_t + \vec{e}_x) - \vec{e}_x \implies &\vec{e}_u &= \frac{1}{2}(\vec{e}_t - \vec{e}_x). \end{aligned}$$

- (b) Show that \vec{e}_{α} , $\alpha \in \{u, v, y, z\}$ form a basis for vectors in Minkowski space.

$$\begin{aligned} \vec{A} &= A^{\alpha}\vec{e}_{\alpha} = A^u\vec{e}_u + A^v\vec{e}_v + A^y\vec{e}_y + A^z\vec{e}_z \\ &= \frac{A^u}{2}(\vec{e}_t - \vec{e}_x) + \frac{A^v}{2}(\vec{e}_t + \vec{e}_x) + A^y\vec{e}_y + A^z\vec{e}_z \\ &= \frac{1}{2}(A^v + A^u)\vec{e}_t + \frac{1}{2}(A^v - A^u)\vec{e}_x + A^y\vec{e}_y + A^z\vec{e}_z \end{aligned}$$

If we let $A^t = \frac{1}{2}(A^v + A^u)$ and $A^x = \frac{1}{2}(A^v - A^u)$, then

$$\vec{A} = A^{\alpha}\vec{e}_{\alpha} = A^t\vec{e}_t + A^x\vec{e}_x + A^y\vec{e}_y + A^z\vec{e}_z$$

- (c) Find the components of the metric tensor, \mathbf{g} in this new basis.

To make this concise, we will begin with some definitions. Let $w \in \{u, v\}$, and $q \in \{y, z\}$. We also define

$$\lambda(w) \equiv \begin{cases} -1, & \text{if } w = u, \\ +1, & \text{if } w = v. \end{cases}$$

It follows that

$$\vec{e}_w = \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x).$$

Now we can show that

$$\begin{aligned} g_{ww} &= \vec{e}_w \cdot \vec{e}_w = \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 2\lambda(\vec{e}_t \cdot \vec{e}_x) + \lambda^2(\vec{e}_x \cdot \vec{e}_x)] \\ &= \frac{1}{4}(-1 + 2\lambda \cdot 0 + 1 \cdot 1) = 0, \end{aligned}$$

so $g_{uu} = g_{vv} = 0$.

For the u and v cross terms, we have

$$\begin{aligned} g_{uv} = g_{vu} &= \vec{e}_u \cdot \vec{e}_v = \frac{1}{2}(\vec{e}_t - \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 0 \cdot \vec{e}_t \cdot \vec{e}_x - \vec{e}_x \cdot \vec{e}_x] \\ &= \frac{1}{4}(-1 + 0 - 1) = -\frac{1}{2} \end{aligned}$$

For the w with y and z cross terms we have

$$\begin{aligned} g_{wq} &= \vec{e}_w \cdot \vec{e}_q = \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \vec{e}_q \\ &= \frac{1}{2}[\vec{e}_t \cdot \vec{e}_t + \lambda \vec{e}_x \cdot \vec{e}_x] \\ &= 0 \end{aligned}$$

so $g_{uy} = g_{vy} = g_{uz} = g_{vz} = 0$. We also already know $g_{yy} = g_{zz} = 1$, and $g_{yz} = g_{zy} = 0$, so we can write the components of the metric tensor in this new coordinate system as

$$g_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) Show that \vec{e}_u and \vec{e}_v are null, but not orthogonal.

$$\vec{e}_u \cdot \vec{e}_u = g_{uu} = 0 \implies \vec{e}_u \text{ is null}$$

$$\vec{e}_v \cdot \vec{e}_v = g_{vv} = 0 \implies \vec{e}_v \text{ is null}$$

$$\vec{e}_u \cdot \vec{e}_v = g_{uv} = -\frac{1}{2} \neq 0 \implies \vec{e}_u \text{ and } \vec{e}_v \text{ are not orthogonal.}$$

(e) Compute the four one-forms \tilde{du} , \tilde{dv} , $\mathbf{g}(\vec{e}_u,)$, and $\mathbf{g}(\vec{e}_v,)$ in terms of \tilde{dt} and \tilde{dx} .

$$\tilde{d}\phi \rightarrow_{\mathcal{O}} \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right),$$

so

$$\begin{aligned} \tilde{dt} &\rightarrow_{\mathcal{O}} (1, 0, 0, 0), & \tilde{dx} &\rightarrow_{\mathcal{O}} (0, 1, 0, 0), \\ \tilde{du} &\rightarrow_{\mathcal{O}} \frac{1}{2}(1, -1, 0, 0), & \tilde{du} &\rightarrow_{\mathcal{O}} \frac{1}{2}(1, 1, 0, 0), \end{aligned}$$

from which it is obvious that

$$\tilde{du} = \frac{1}{2}(\tilde{dt} - \tilde{dx}), \quad \tilde{dv} = \frac{1}{2}(\tilde{dt} + \tilde{dx}).$$

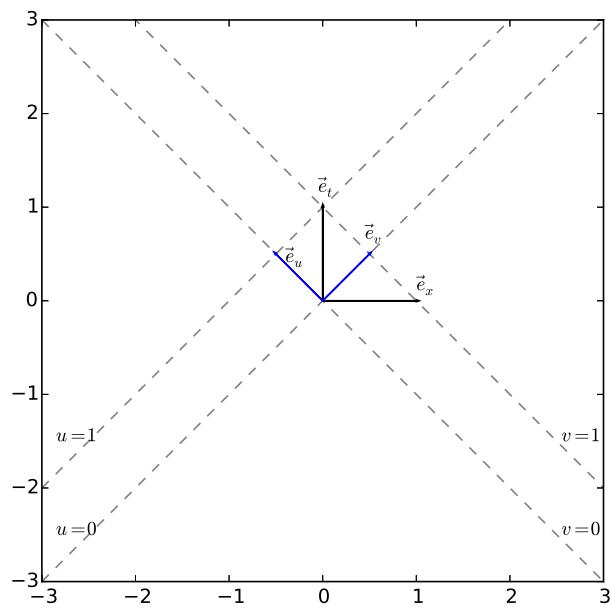


Figure 3.4: Problem 34a: Spacetime diagram of double-null coordinate basis vectors in t - x plane.