Spherical solutions for stars

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General Relativity I Presentations
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Introduction

Spherically symmetric coordinates

Two-sphere in flat spacetime

General metric

$$ds^2 = -\,dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Metric on 2-sphere

$$dl^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \equiv r^2 d\Omega^2$$

Schutz (2009, p. 256)

Two-sphere in flat spacetime

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Schutz (2009, p. 256)

Two-sphere in curved spacetime

Metric on 2-sphere

$$\mathrm{d}l^2 = f(r', t)\mathrm{d}\Omega^2$$

Relation to r

$$f(r', t) \equiv r^2$$

Schutz (2009, pp. 256–257)

Two-sphere in curved spacetime

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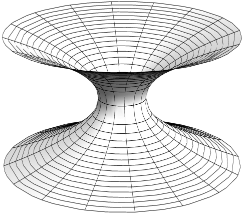
Meaning of r



Mark Hannam

Figure:
Surface with circular symmetry but no coordinate $r = 0$.

- "curvature" or "area" coordinate
  - radius of curvature and area
- *not* proper distance from center
- $r = \text{const}$, $t = \text{const}$
  - $A = 4\pi r^2$
  - $C = 2\pi r$

Schutz (2009, p. 257)

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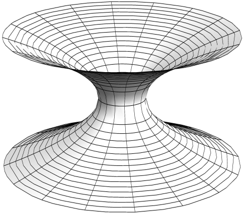

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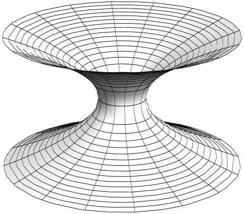

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Schutz (2009, p. 257)

Spherically symmetric spacetime

General metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + 2g_{0r}\,\mathrm{d}r\,\mathrm{d}t + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

g_{00}, g_{0r}, and g_{rr} functions of t and r

Schutz (2009, p. 258)

Static spacetimes

Motivation

- leads to simple derivation of Schwarzschild metric
- generalizes to spherically symmetric, asymptotically flat Einstein vacuum field equations
- Birkhoff's theorem

Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843)

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Definition

A spacetime is static if we can find a time coordinate t for which

(i) the metric independent of t

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \to -t$$

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Time reversal

$$\mathbf{\Lambda} : (t, x, y, z) \to (-t, x, y, z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} g_{\alpha\beta} = g_{\alpha\beta}$$

Transformation

$$\Lambda^{0}{}_{\bar{0}} = x^{0}{}_{,\bar{0}} = -x^{0}{}_{,0} = -1$$
$$\Lambda^{i}{}_{\bar{j}} = x^{i}{}_{,\bar{j}} = x^{i}{}_{,j} = \delta^{i}{}_{j}$$
$$\Lambda^{0}{}_{\bar{i}} = x^{0}{}_{,\bar{i}} = x^{0}{}_{,i} = 0$$

Metric

$$g_{\bar{0}\bar{0}} = (\Lambda^{0}{}_{\bar{0}})^2 g_{00} = g_{00}$$
$$g_{\bar{0}\bar{r}} = \Lambda^{0}{}_{\bar{0}} \Lambda^{r}{}_{\bar{r}} g_{0r} = -g_{0r}$$
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Schutz (2009, p. 258)

The metric

Simplified metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Replacement

$$g_{00} \to -e^{2\Phi}, \quad g_{rr} \to e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

Static spherically symmetric metric

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\lim_{r\to\infty}\Phi(r) = \lim_{r\to\infty}\Lambda(r) = 0$$

Schutz (2009, pp. 258–259)

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Einstein Tensor

General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Einstein tensor components

$$G_{tt} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})],$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda} [\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r],$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

Schutz (2009, pp. 165, 260)

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Schutz (2009, pp. 165, 260)

Static perfect fluid

Four-velocity

Constraints

$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

Solving for U^0

$$g_{00}(U^0)^2 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

Solving for U_0

$$U_0 = g_{00} U^0 = -e^{\Phi}$$

Schutz (2009, p. 260)

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Stress–energy tensor

$T_{\alpha\beta}$ for perfect fluid

$$T_{\alpha\beta} = (\rho + p) U_\alpha U_\beta + p g_{\alpha\beta}$$

Components of $T_{\alpha\beta}$

$$T_{00} = (\rho + p) e^{2\Phi} + p(-e^{2\Phi}) = \rho e^{2\Phi}$$

$$T_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta; \quad T_{ii} = p g_{ii}$$

$$T_{rr} = p e^{2\Lambda}; \quad T_{\theta\theta} = p r^2; \quad T_{\phi\phi} = p r^2 \sin^2\theta = T_{\theta\theta} \sin^2\theta$$

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Equation of state

Local thermodynamic equilibrium

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

Schutz (2009, p. 261)

Equations of motion

Conservation laws

$$T^{\alpha\beta}{}_{;\beta} = 0$$

- symmetries make only non-trivial solution $\alpha = r$
  TODO: prove

Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)

Equations of motion

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Schutz (2009, pp. 175, 261)

Equation of motion (continued)

TODO
Show 10.31

Exterior Geometry

Schwarzschild metric

TODO

Birkhoff's Theorem

Let the geometry of a given region of spacetime:

1. be spherically symmetric
2. be a solution to the Einstein field equations in vacuum.

Then that geometry is necessarily a piece of the Schwarzschild geometry.

(Proof given in Misner, Thorne, and Wheeler (1973, pp. 843–844))

Birkhoff and Langer (1923)

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References

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