

# Spherical solutions for stars

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General Relativity I Presentations

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# Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- applications

# Spherically symmetric coordinates



# Two-sphere in flat spacetime

## General metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

## Metric on 2-sphere

$$dl^2 = r^2(d\theta^2 + \sin^2 \theta d\phi^2) \equiv r^2 d\Omega^2$$

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## Relation to $r$

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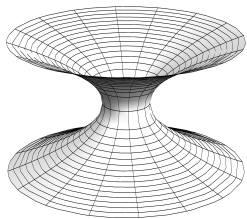
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# Meaning of $r$



Mark Hannam

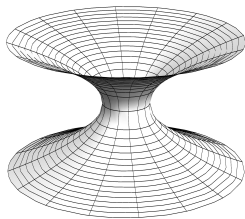
Figure:

Surface with circular symmetry but no coordinate  $r = 0$ .

- *not* proper distance from center
- “curvature” or “area” coordinate
  - radius of curvature and area
- $r = \text{const}, t = \text{const}$ 
  - $A = 4\pi r^2$
  - $C = 2\pi r$



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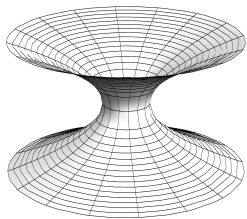
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Schutz (2009, p. 257)

# Spherically symmetric spacetime

## General metric

$$ds^2 = g_{00} dt^2 + 2g_{0r} dr dt + g_{rr} dr^2 + r^2 d\Omega^2$$

$g_{00}$ ,  $g_{0r}$ , and  $g_{rr}$  functions of  $t$  and  $r$

# Static spacetimes



# Motivation

- leads to simple derivation of Schwarzschild metric
- unique solution to spherically symmetric, asymptotically flat Einstein vacuum field equations (Birkhoff's theorem)

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Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843)

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# Definition

A spacetime is static if we can find a time coordinate  $t$  for which

(i) the metric independent of  $t$

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \rightarrow -t$$

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# Time reversal

$$\Lambda : (t, x, y, z) \rightarrow (-t, x, y, z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}_{\bar{\alpha}} \Lambda^{\beta}_{\bar{\beta}} g_{\alpha\beta} = g_{\alpha\beta}$$

## Transformation

$$\Lambda^0_{\bar{0}} = x^0_{,\bar{0}} = -x^0_{,0} = -1$$

$$\Lambda^i_{\bar{i}} = x^i_{,\bar{i}} = x^i_{,i} = 1$$

## Metric

$$g_{\bar{0}\bar{0}} = (\Lambda^0_{\bar{0}})^2 g_{00} = g_{00}$$

$$g_{\bar{r}\bar{r}} = (\Lambda^r_{\bar{r}})^2 g_{rr} = g_{rr}$$

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# The metric

## Simplified metric

$$ds^2 = g_{00} dt^2 + g_{rr} dr^2 + r^2 d\Omega^2$$

## Replacement

$$g_{00} \rightarrow -e^{2\Phi}, \quad g_{rr} \rightarrow e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

## Static spherically symmetric metric

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2$$

$$\lim_{r \rightarrow \infty} \Phi(r) = \lim_{r \rightarrow \infty} \Lambda(r) = 0$$

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# Einstein Tensor

## General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$

## Einstein tensor components

$$G_{00} = \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}[r(1 - e^{-2\Lambda})]$$

$$G_{rr} = -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi'$$

$$G_{\theta\theta} = r^2e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

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# Static perfect fluid



# Four-velocity

## Constraints

$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

## Solving for $U^0$

$$g_{00}U^0U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

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# Stress-energy tensor

## Stress-energy tensor for perfect fluid

$$T_{\alpha\beta} = (\rho + p)U_{\alpha}U_{\beta} + pg_{\alpha\beta}$$

## Components of $T_{\alpha\beta}$

$$T_{00} = \rho g_{00}$$

$T_{\alpha\beta}$  is diagonal

$$T_{00} = (\rho + p)U^0U_0 + p(-1) = -\rho$$

$$T_{11} = (\rho + p)U^1U_1 + p(1) = p$$

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# Equation of state

## Local thermodynamic equilibrium

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

# Equations of motion

## Conservation of 4-momentum

$$T^{\alpha\beta}_{;\beta} = 0$$

- symmetries make only non-trivial solution  $\alpha = r$   
**TODO:** prove

## Equation of motion

$$(\rho + p) \frac{d\Phi}{dr} = -\frac{d\rho}{dr}$$

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# Mass function

## Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{d}{dr} [r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

$$m(r)$$

$$m(r) \equiv \frac{1}{2} r (1 - e^{-2\Lambda}) \quad \text{or} \quad g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

## Relation to energy density

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho$$

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$$\frac{d\Phi(r)}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

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# Exterior Geometry





# Schwarzschild metric I

## Condition

$$\rho = p = 0$$

## Consequences

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho = 0$$

$$m(r) \equiv M$$

$$\frac{u'(r)}{u(r)} = \frac{2\alpha(r) + 4\pi r^3 \rho}{r(r - 2\alpha(r))} = \frac{2\alpha}{r(r - 2\alpha)} \quad \Rightarrow \quad \ln u(r) = \frac{1}{2} \ln \left( \frac{r - 2\alpha(r)}{r} \right)$$

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# Schwarzschild metric II

## First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$

## Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

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# Far-field metric

## Condition

$$r \gg M$$

## Schwarzschild metric

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## Far-field Schwarzschild metric (Cartesian)

$$ds^2 \approx - \left( 1 - \frac{2M}{R} \right) dt^2 + \left( 1 + \frac{2M}{R} \right) (dx^2 + dy^2 + dz^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

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# Birkhoff's Theorem

If the geometry of a given region of spacetime is:

- ① spherically symmetric
- ② a solution to the Einstein field equations in vacuum

then that geometry is necessarily a subset of the Schwarzschild geometry.

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# Interior structure



# Tolman–Oppenheimer–Volkov (T–O–V) equation

## Condition

$$\rho \neq 0 \quad p \neq 0$$

## Recall

$$(\rho + p) \frac{d\Phi}{dr} = - \frac{dp}{dr}$$

$$\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

## T–O–V equation

$$\frac{dp}{dr} = - \frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

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$$\frac{dp}{dr} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

# System of coupled differential equations

## T–O–V equation

$$\frac{dp}{dr} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

## Mass function

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho$$

## Equation of state

$$p = p(\rho)$$

# Newtonian hydrostatic equilibrium

## Newtonian limit

$$p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll r$$

## Equation of hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -\frac{\rho m(r)}{r^2}$$

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Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3)



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