Chapter 3

Tensor analysis in special relativity

3.3 The $\binom{0}{1}$ tensors: one-forms

The symbol $\tilde{}$ is used to denote a one-form, as $\vec{}$ is used to denote a vector. So $\tilde{p}$ is a one-form, or a type $\binom{0}{1}$ tensor.

Normal one-forms

Let $\mathcal{S}$ be some surface.
$\forall \vec{V}$ tangent to $\mathcal{S}$, $\tilde{p}(\vec{V}) = 0 \implies \tilde{p}$ is normal to $\mathcal{S}$.
Furthermore, if $\mathcal{S}$ is a *closed* surface & $\tilde{p}$ is normal to $\mathcal{S}$ & $\forall \vec{U}$ pointing outwards from $\mathcal{S}$, $\tilde{p}(\vec{U}) > 0 \implies \tilde{p}$ is an outward normal one-form.

3.5 Metric as a mapping of vectors into one-forms

Normal vectors and unit normal one-forms

$\vec{V}$ is normal to a surface if $\tilde{V}$ is normal to the surface. They are said to be *unit normal* if their magnitude is ± 1, so $\vec{V}^2 = \tilde{V}^2 = \pm 1$.

- A time-like unit normal has magnitude -1
- A space-like unit normal has magnitude $+1$
- A null normal cannot be a unit normal, because $\vec{V}^2 = \tilde{V}^2 = 0$

3.10 Exercises

3

(a)

$$
\begin{aligned}
\tilde{p}(A^\alpha \vec{e}_\alpha) &= A^\alpha \tilde{p}(\vec{e}_\alpha) = \tilde{p}(A^0 \vec{e}_0 + A^1 \vec{e}_1 + A^2 \vec{e}_2 + A^3 \vec{e}_3) \\
&= A^0 \tilde{p}(\vec{e}_0) + A^1 \tilde{p}(\vec{e}_1 + A^2 \tilde{p}(\vec{e}_2) + A^3 \tilde{p}(\vec{e}_3 = A^\alpha \tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha \in \mathbb{R}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\tilde{p} &\underset{\mathcal{O}}{\rightarrow} (-1, 1, 2, 0) \\
\vec{A} &\underset{\mathcal{O}}{\rightarrow} (2, 1, 0, -1) \\
\vec{B} &\underset{\mathcal{O}}{\rightarrow} (0, 2, 0, 0)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{p}(\vec{A}) &= -2 + 1 + 0 + 0 = -1 \\
\tilde{p}(\vec{B}) &= 0 + 2 + 0 + 0 = 2 \\
\tilde{p}(\vec{A} - 3\vec{B}) &= \tilde{p}(\vec{A}) - 3\tilde{p}(\vec{B}) = -1 - 3 \cdot 2 = -7
\end{aligned}
$$

4 Given the following vectors

$$
\vec{A} \underset{\mathcal{O}}{\rightarrow} (2, 1, 1, 0) \qquad \vec{B} \underset{\mathcal{O}}{\rightarrow} (1, 2, 0, 0)
$$

$$
\vec{C} \underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1) \qquad \vec{D} \underset{\mathcal{O}}{\rightarrow} (-3, 2, 0, 0)
$$

(Note that all parts were done with the assistance of `numpy`.)

(a) Show that they are linearly independent.

We do this by constructing a matrix, $\mathbf{X}$, whose columns correspond to the four vectors. If the determinant of $\mathbf{X}$ is non-zero, then that means the vectors are linearly independent.

$$
\det(\mathbf{X}) = \det \begin{pmatrix} 2 & 1 & 0 & -3 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -8
$$

(b) Find the components of $\tilde{p}$ if

$$
\tilde{p}(\vec{A}) = 1, \quad \tilde{p}(\vec{B}) = -1, \quad \tilde{p}(\vec{C}) = -1, \quad \tilde{p}(\vec{D}) = 0
$$

We do this by observing that $\tilde{p} = A^\alpha p_\alpha$, and so we have a system of four equations, which we can write in matrix form as

$$\begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\\ \vec{D}\end{pmatrix}\tilde{p} = \begin{pmatrix}1\\ -1\\ -1\\ 0\end{pmatrix}$$

$$\implies \tilde{p} = \begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\\ \vec{D}\end{pmatrix}^{-1}\begin{pmatrix}1\\ -1\\ -1\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{4}\\ -\frac{3}{8}\\ +\frac{15}{8}\\ -\frac{23}{8}\end{pmatrix}.$$

(c) Find $\tilde{p}(\vec{E})$, where $\vec{E} \to_{\mathcal{O}} (1, 1, 0, 0)$.

$$\tilde{p}(\vec{E}) = p_\alpha E^\alpha = -\frac{5}{8}$$

(d) Determine whether $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$ are linearly independent.

We do this by first setting up a system of equations for each of $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$, as was done for $\tilde{p}$, and solving. I will refer to the matrix whose rows were $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ as $\mathbf{X}$.

$$\mathbf{X}\tilde{q} = \begin{pmatrix}+0\\ +0\\ +1\\ -1\end{pmatrix} \qquad \mathbf{X}\tilde{r} = \begin{pmatrix}+2\\ +0\\ +0\\ +0\end{pmatrix} \qquad \mathbf{X}\tilde{s} = \begin{pmatrix}-1\\ -1\\ +0\\ +0\end{pmatrix}$$

$$\tilde{q} = \begin{pmatrix}+\frac{1}{4}\\ -\frac{1}{8}\\ -\frac{3}{8}\\ +\frac{11}{8}\end{pmatrix} \qquad \tilde{r} = \begin{pmatrix}+0\\ +0\\ +2\\ +2\end{pmatrix} \qquad \tilde{s} = \begin{pmatrix}-\frac{1}{4}\\ -\frac{3}{8}\\ -\frac{1}{8}\\ +\frac{1}{8}\end{pmatrix}$$

Now if the matrix whose columns are comprised of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$ has a non-zero determinant, then the four covectors must be linearly independent.

$$\det\begin{pmatrix}\tilde{p} & \tilde{q} & \tilde{r} & \tilde{s}\end{pmatrix} = \frac{1}{4},$$

and so they are indeed linearly independent.

6

(a) Show that $\tilde{p} \neq \tilde{p}(\vec{e}_\alpha)\tilde{\lambda}^\alpha$ for arbitrary $\tilde{p}$.

Let us choose $\tilde{p} \to_{\mathcal{O}} (0, 1, e, \pi)$, as a counter-example.

$$\begin{aligned} p_\alpha \tilde{\lambda}^\alpha &\underset{\mathcal{O}}{\to} 0 \cdot (1, 1, 0, 0) + 1 \cdot (1, -1, 0, 0) + e \cdot (0, 0, 1, -1) + \pi \cdot (0, 0, 1, 1) \\ &\underset{\mathcal{O}}{\to} (1, -1, e + \pi, 0) \underset{\mathcal{O}}{\not\to} \tilde{p} \end{aligned}$$

(b) $\tilde{p} \to_{\mathcal{O}} (1, 1, 1, 1)$. Find l_α such that

$$\tilde{p} = l_\alpha \tilde{\lambda}^\alpha$$

We may do this with a simple matrix inversion. We define $\mathbf{\Lambda}$ to be the matrix whose rows are formed by $\tilde{\lambda}^\alpha$.

$$\mathbf{\Lambda} l = p \implies l = \mathbf{\Lambda}^{-1} p = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

8 Draw the basis one-forms $\tilde{\mathrm{d}}t$ and $\tilde{\mathrm{d}}x$ of frame $\mathcal{O}$.

They are

$$\begin{aligned} \tilde{\mathrm{d}}t &\underset{\mathcal{O}}{\to} (1, 0, 0, 0), \\ \tilde{\mathrm{d}}x &\underset{\mathcal{O}}{\to} (0, 1, 0, 0), \end{aligned}$$

and they are shown in Figure 3.1.

9 At the points $\mathcal{P}$ and $\mathcal{Q}$, estimate the components of the gradient $\tilde{\mathrm{d}}T$.

Recall that $\tilde{\mathrm{d}}T \to_{\mathcal{O}} \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$, and so $\Delta T = \tilde{\mathrm{d}}T_\alpha x^\alpha = \tilde{\mathrm{d}}T_x \Delta x + \tilde{\mathrm{d}}T_y \Delta y$.

Now if we move only in the x direction from one of the points, we move some distance Δx, change our temperature by Δt, and $\Delta y = 0$. Likewise for a movement in the y direction. Thus we can say

$$\begin{aligned} \Delta T &= \tilde{\mathrm{d}}T_x \Delta x & \Delta T &= \tilde{\mathrm{d}}T_y \Delta y \\ \tilde{\mathrm{d}}T_x &= \frac{\Delta T}{\Delta x} & \tilde{\mathrm{d}}T_y &= \frac{\Delta T}{\Delta y} \end{aligned}$$

In Figure 3.2, from $\mathcal{P}$ I move a distance $\Delta x = 0.5$, which causes a temperature change of $\Delta T = -7$, giving $\tilde{\mathrm{d}}T_x = -14$. Then I move a distance $\Delta y = 0.5$ and get the same temperature change of $\Delta T = -7$, and so I conclude that at point $\mathcal{P}$, $\tilde{\mathrm{d}}T \to_{\mathcal{O}} (-14, -14)$.

At $\mathcal{Q}$, we are in a flat region where $T = 0$. If we move any non-zero distance Δx or Δy, so long as it does not cross the $T = 0$ isotherm, we have a $\Delta T = 0$, and thus $\tilde{\mathrm{d}}Tp \to_{\mathcal{O}} (0, 0)$.

13 Prove that $\tilde{\mathrm{d}}f$ is normal to surfaces of constant f.

If we move some small distance $\Delta x^\alpha = \epsilon$, then there will be no change in the value of f, and thus we can say $\partial f / \partial x^\alpha = 0$, so

$$\tilde{\mathrm{d}}f = \frac{\partial f}{\partial x^\alpha} \tilde{\mathrm{d}}x^\alpha = 0 \tilde{\mathrm{d}}x^\alpha = 0.$$

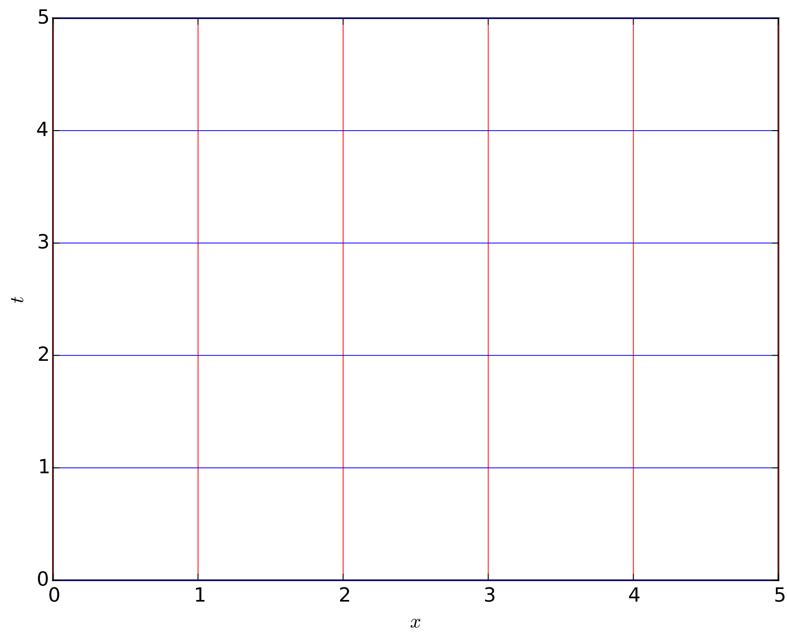

Figure 3.1: Problem 8: Basis one-forms of $\mathcal{O}$. $\tilde{\mathrm{d}}t$ is given in blue and $\tilde{\mathrm{d}}x$ in red.

Since $\tilde{\mathrm{d}}f$ is defined to be normal to a surface if it is zero on every tangent vector, we have shown that $\tilde{\mathrm{d}}f$ is normal to any surface of constant f.

14

$$\tilde{p} \underset{\mathcal{O}}{\rightarrow} (1,1,0,0) \qquad\qquad \tilde{q} \underset{\mathcal{O}}{\rightarrow} (-1,0,1,0)$$

Prove by giving two vectors $\vec{A}$ and $\vec{B}$ as arguments that $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

$$\begin{aligned}
(\tilde{p} \otimes \tilde{q})(\vec{A}, \vec{B}) &= \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = A^\alpha p_\alpha B^\beta q_\beta = (A^0 + A^1)(-B^0 + B^2), \\
&= -A^0B^0 + A^0B^2 - A^1B^0 + A^1B^2 \\
(\tilde{q} \otimes \tilde{p})(\vec{A}, \vec{B}) &= \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = A^\alpha q_\alpha B^\beta p_\beta = (-A^0 + A^2)(B^0 + B^1) \\
&= -A^0B^0 - A^0B^1 + A^2B^0 + A^2B^1,
\end{aligned}$$

And so we see that $\otimes$ is not commutative.

The components of the outer product of two tensors are given by the products of the components of the

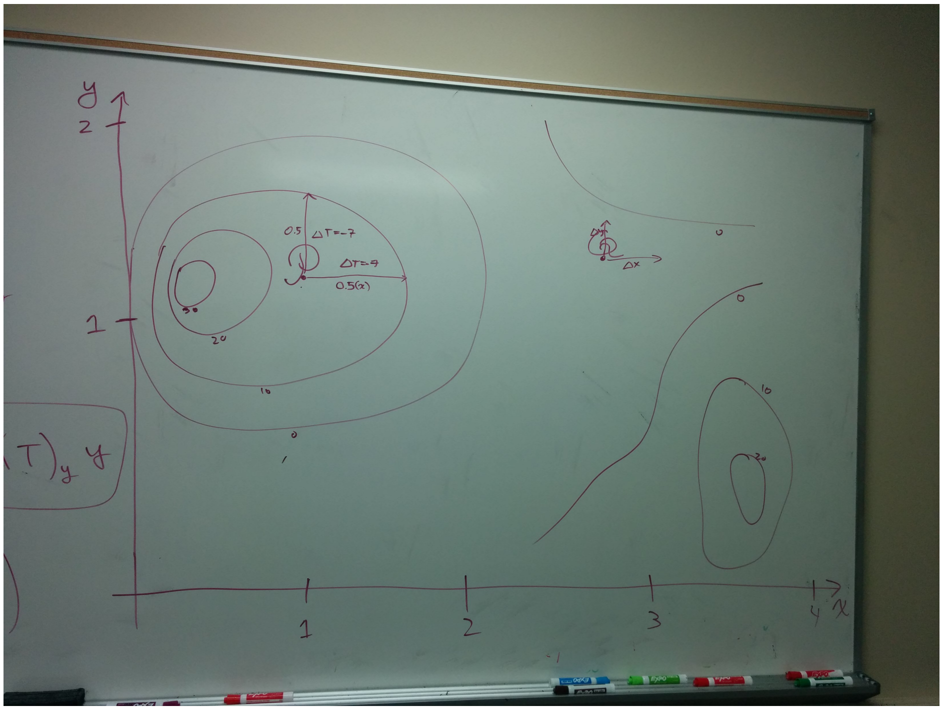

Figure 3.2: Problem 9: Isotherms.

individual tensors. Thus we can write the components as a 4×4 matrix.

$$(\tilde{p} \otimes \tilde{q})_{\alpha\beta} = p_\alpha q_\beta = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

18

(a) Find the one-forms mapped by $\mathbf{g}$ from

$$\vec{A} \underset{\mathcal{O}}{\rightarrow} (1, 0, -1, 0), \qquad \vec{B} \underset{\mathcal{O}}{\rightarrow} (0, 1, 1, 0),$$
$$\vec{C} \underset{\mathcal{O}}{\rightarrow} (-1, 0, -1, 0), \qquad \vec{D} \underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1).$$

In general,

$$\vec{V} \underset{\mathcal{O}}{\rightarrow} (V^0, V^1, V^2, V^3) \implies \tilde{V} = \mathbf{g}\vec{V} \underset{\mathcal{O}}{\rightarrow} (-V^0, V^1, V^2, V^3),$$

and so

$$\tilde{A} \underset{\mathcal{O}}{\rightarrow} (-1, 0, -1, 0), \qquad \tilde{B} \underset{\mathcal{O}}{\rightarrow} (0, 1, 1, 0),$$
$$\tilde{C} \underset{\mathcal{O}}{\rightarrow} (1, 0, -1, 0), \qquad \tilde{D} \underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1).$$

(b) Find the vectors mapped by $\mathbf{g}$ from

$$\tilde{p} \underset{\mathcal{O}}{\rightarrow} (3, 0, -1, -1), \qquad \tilde{q} \underset{\mathcal{O}}{\rightarrow} (1, -1, 1, 1),$$
$$\tilde{r} \underset{\mathcal{O}}{\rightarrow} (0, -5, -1, 0), \qquad \tilde{s} \underset{\mathcal{O}}{\rightarrow} (-2, 1, 0, 0).$$

By using the inverse tensor in reverse, we have the same effect as before, of negating the first component

$$\vec{p} \underset{\mathcal{O}}{\rightarrow} (-3, 0, -1, -1), \qquad \vec{q} \underset{\mathcal{O}}{\rightarrow} (-1, -1, 1, 1),$$
$$\vec{r} \underset{\mathcal{O}}{\rightarrow} (0, -5, -1, 0), \qquad \vec{s} \underset{\mathcal{O}}{\rightarrow} (2, 1, 0, 0).$$

20

In Euclidean 3-space, vectors and covectors are usually treated as the same, because they transform the same. We will now prove this.

(a) Show that $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} A^{\beta}$ and $P_{\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\beta}} P_{\alpha}$ are the same transformations if $\{\Lambda^{\alpha}{}_{\bar{\beta}}\}$ is equal to the transpose of its inverse.

We can write that last statement as

$$\Lambda^{\alpha}{}_{\bar{\beta}} = ((\Lambda^{\alpha}{}_{\bar{\beta}})^{-1})^{T}$$

and we know that

$$(\Lambda^{\alpha}{}_{\bar{\beta}})^{-1} = \Lambda^{\bar{\beta}}{}_{\alpha},$$

and also we know that the Lorentz transformation is symmetric, and so

$$(\Lambda^{\bar{\beta}}{}_{\alpha})^{T} = \Lambda^{\bar{\beta}}{}_{\alpha},$$

which leads us to conclude that $\Lambda^{\alpha}{}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}$, meaning the two transformations are the same.

(b) The metric has components $\{\delta_{ij}\}$. Prove that transformations between Cartesian coordinate systems must satisfy

$$\delta_{\bar{i}\bar{j}} = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \delta_{kl},$$

and that this implies that $\Lambda^{k}{}_{\bar{i}}$ is an orthogonal matrix.

$$\delta_{\bar{i}\bar{j}} = \mathbf{g}(\vec{e}_{\bar{i}}, \vec{e}_{\bar{j}}) = \mathbf{g}(\Lambda^{k}{}_{\bar{i}} \vec{e}_{k}, \Lambda^{l}{}_{\bar{j}} \vec{e}_{j}) = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \mathbf{g}(\vec{e}_{k}, \vec{e}_{j}) = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \delta_{kl}$$

Now show it is orthogonal

21

(a) A region of the t–x plane is bounded by lines $t = 0$, $t = 1$, $x = 0$, and $x = 1$. Within the plane, find the unit outward normal 1-forms and their vectors for each boundary line.

I define unit outward normals as follows:

Let $\mathcal{S}$ be a closed surface. If, for each $\vec{V}$ tangent to $\mathcal{S}$, we have $\tilde{p}(\vec{V}) = 0$, then $\tilde{p}$ is normal to $\mathcal{S}$.

In addition, if, for each $\vec{U}$ which points outwards from the surface, we have $\tilde{p}(\vec{U}) > 0$, then $\tilde{p}$ is an outward normal.

Furthermore, if $\tilde{p}^2 = \pm 1$, then it is a unit outward normal.

For the problem at hand, I define the region inside the four lines to be *Inside*, and the region outside to be *Outside*. For each of the four lines, I draw a vector $\vec{V}$ tangent (parallel) to the line, and $\vec{U}$ pointing outwards (See Figure 3.3).

It helps to look at $t = 0$ and $t = 1$ together, and likewise for x, so I will start with t. We start with an arbitrary $\tilde{p} \to_{\mathcal{O}} (p_0, p_1)$, and $\vec{V} \to_{\mathcal{O}} (0, V^1)$, where $V^1 \neq 0$.

$$\tilde{p}(\vec{V}) = p_0 \cdot 0 + p_1 V^1 = 0 \implies p_1 = 0,$$

so $\tilde{p} \to_{\mathcal{O}} (p_0, 0)$ is a normal 1-form to both lines. Now we find the corresponding *unit* normal, by taking

$$\tilde{p}^2 = \pm 1 = -(p_0)^2 \implies \tilde{p}^2 = -1 \;\&\; p_0 = \pm 1.$$

Whether we choose p_0 to be positive or negative now depends on the line we are looking at, and which direction is outward. For $t = 0$, we have a vector $\vec{U} = (-U^0, U^1)$, where $U^0 > 0$.

$$\tilde{p}(\vec{U}) = p_0(-U^0) + 0 \cdot U^1 > 0 \implies -p_0 U^0 > 0 \implies p_0 < 0,$$

so for $t = 0$ we have $\tilde{p} \to_{\mathcal{O}} (-1, 0)$, and likewise for $t = 1$ we have $\tilde{p} \to_{\mathcal{O}} (1, 0)$. To get the associated *vectors*, we apply the metric $\eta^{\alpha\beta}$, giving us $\vec{p} \to_{\mathcal{O}} (1, 0)$ for $t = 0$ and $\vec{p} \to_{\mathcal{O}} (-1, 0)$ for $t = 1$.

For $x = 0$ and $x = 1$, we instead have $\vec{V} \to_{\mathcal{O}} (V^0, 0)$, and following the same steps as before, we conclude that: for $x = 0$, $\tilde{p} \to_{\mathcal{O}} (0, -1)$, $\vec{p} \to_{\mathcal{O}} (0, -1)$, and for $x = 1$, $\tilde{p} \to_{\mathcal{O}} (0, 1)$, $\vec{p} \to_{\mathcal{O}} (0, 1)$.

Figure 3.3: Problem 21.a

(b) Let another region be bounded by the set of points $\{(1, 0), (1, 1), (2, 1)\}$. Find an outward normal for the null boundary and the associated vector.

23

(a) Prove that the set of all $\binom{M}{N}$ tensors forms a vector space, V.

Let T be the set of all $\binom{M}{N}$ tensors, $\mathbf{s}, \mathbf{p}, \mathbf{q} \in T$, $\vec{A} \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. For T to be a vector space, we must define the operations of addition, and scalar multiplication (amongst others).

Addition:

$$\mathbf{s} = \mathbf{p} + \mathbf{q} \implies \mathbf{s}(\vec{A}) = \mathbf{p}(\vec{A}) + \mathbf{q}(\vec{A})$$

Scalar Multiplication:

$$\mathbf{r} = \alpha \mathbf{p} \implies \mathbf{r}(\vec{A}) = \alpha \mathbf{p}(\vec{A})$$

(b)

Prove that a basis for T is

$$\{\vec{e}_\alpha \otimes \ldots \otimes \vec{e}_\gamma \otimes \tilde{\omega}^\mu \otimes \ldots \otimes \tilde{\omega}^\lambda\}$$

Still working on it

24 Given:

$$M^{\alpha\beta} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(a) Find:

(i)

$$M^{(\alpha\beta)} \rightarrow \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{pmatrix}; \quad M^{[\alpha\beta]} \rightarrow \begin{pmatrix} 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} & 0 \end{pmatrix}$$

(ii)

$$M^{\alpha}{}_{\beta} = \eta_{\beta\mu} M^{\alpha\mu} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iii)

$$M_{\alpha}{}^{\beta} = \eta_{\alpha\mu} M^{\mu\beta} \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iv)

$$M_{\alpha\beta} = \eta_{\beta\mu} M_{\alpha}{}^{\mu} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(b) Does it make sense to separate the $\binom{1}{1}$ tensor with components $M^{\alpha}{}_{\beta}$ into symmetric and antisymmetric parts?

No, it would not make sense. For one, the notation for (anti)symmetric tensors do not even allow one to write it sensibly ($M^{(\alpha}{}_{\beta)}$). More importantly, one argument refers to vectors, and the other to covectors, so it does not make sense to switch them.

(c)

$$\eta^{\alpha}{}_{\beta} = \eta^{\alpha\mu}\eta_{\beta\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\alpha}{}_{\beta}$$

31

Still working on it

(**33**)

34 Define double-null coordinates $u = t - x$, $v = t + x$ in Minkowski space.

(a) Let $\vec{e}_u$ be the vector connecting the (u, v, y, t) coordinates $(0,0,0,0)$ and $(1,0,0,0)$, and let $\vec{e}_v$ be the vector connecting $(0,0,0,0)$ and $(0,1,0,0)$. Find $\vec{e}_u$ and $\vec{e}_v$ in terms of $\vec{e}_t$ and $\vec{e}_x$, and plot the basis vectors in a spacetime diagram of the t–x plane.

$$u = t - x = 0 \implies t = +x \qquad v = t + x = 0 \implies t = -x$$
$$u = t - x = 1 \implies t = 1 + x \qquad v = t + x = 1 \implies t = 1 - x$$

We draw the vectors $\vec{e}_u$ and $\vec{e}_v$ in Figure 3.4, such that they point from the appropriate points of intersection on these lines of constant u and v. From this it is obvious that $\vec{e}_v + \vec{e}_u = \vec{e}_t$, and that $\vec{e}_v - \vec{e}_u = \vec{e}_x$, or likewise $\vec{e}_v = \vec{e}_t - \vec{e}_u$ and $\vec{e}_u = \vec{e}_v - \vec{e}_x$. This is a system of 2 equations with two unknowns.

$$\vec{e}_v = \vec{e}_t - \vec{e}_v + \vec{e}_x \implies \qquad \vec{e}_v = \frac{1}{2}(\vec{e}_t + \vec{e}_x),$$
$$\vec{e}_u = \frac{1}{2}(\vec{e}_t + \vec{e}_x) - \vec{e}_x \implies \qquad \vec{e}_u = \frac{1}{2}(\vec{e}_t - \vec{e}_x).$$

(b) Show that $\vec{e}_\alpha$, $\alpha \in \{u, v, y, z\}$ form a basis for vectors in Minkowski space.

$$\begin{aligned} \vec{A} = A^\alpha \vec{e}_\alpha &= A^u \vec{e}_u + A^v \vec{e}_v + A^y \vec{e}_y + A^z \vec{e}_z \\ &= \frac{A^u}{2}(\vec{e}_t - \vec{e}_x) + \frac{A^v}{2}(\vec{e}_t + \vec{e}_x) + A^y \vec{e}_y + A^z \vec{e}_z \\ &= \frac{1}{2}(A^v + A^u)\vec{e}_t + \frac{1}{2}(A^v - A^u)\vec{e}_x + A^y \vec{e}_y + A^z \vec{e}_z \end{aligned}$$

If we let $A^t = \frac{1}{2}(A^v + A^u)$ and $A^x = \frac{1}{2}(A^v - A^u)$, then

$$\vec{A} = A^\alpha \vec{e}_\alpha = A^t \vec{e}_t + A^x \vec{e}_x + A^y \vec{e}_y + A^z \vec{e}_z$$

(c) Find the components of the metric tensor, $\mathbf{g}$ in this new basis.

To make this concise, we will begin with some definitions. Let $w \in \{u, v\}$, and $q \in \{y, z\}$. We also define

$$\lambda(w) \equiv \begin{cases} -1, & \text{if } w = u, \\ +1, & \text{if } w = v. \end{cases}$$

It follows that

$$\vec{e}_w = \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x).$$

Now we can show that

$$\begin{aligned} g_{ww} = \vec{e}_w \cdot \vec{e}_w &= \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 2\lambda(\vec{e}_t \cdot \vec{e}_x) + \lambda^2(\vec{e}_x \cdot \vec{e}_x)] \\ &= \frac{1}{4}(-1 + 2\lambda \cdot 0 + 1 \cdot 1) = 0, \end{aligned}$$

so $g_{uu} = g_{vv} = 0$.

For the u and v cross terms, we have

$$\begin{aligned} g_{uv} = g_{vu} = \vec{e}_u \cdot \vec{e}_v &= \frac{1}{2}(\vec{e}_t - \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 0 \cdot \vec{e}_t \cdot \vec{e}_x - \vec{e}_x \cdot \vec{e}_x] \\ &= \frac{1}{4}(-1 + 0 - 1) = -\frac{1}{2} \end{aligned}$$

For the w with y and z cross terms we have

$$\begin{aligned} g_{wq} = \vec{e}_w \cdot \vec{e}_q &= \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \vec{e}_q \\ &= \frac{1}{2}[\vec{e}_t \cdot \vec{e}_t + \lambda \vec{e}_x \cdot \vec{e}_x] \\ &= 0 \end{aligned}$$

so $g_{uy} = g_{vy} = g_{uz} = g_{vz} = 0$. We also already know $g_{yy} = g_{zz} = 1$, and $g_{yz} = g_{zy} = 0$, so we can write the components of the metric tensor in this new coordinate system as

$$g_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) Show that $\vec{e}_u$ and $\vec{e}_v$ are null, but not orthogonal.

$$
\begin{aligned}
\vec{e}_u \cdot \vec{e}_u &= g_{uu} = 0 \implies \vec{e}_u \text{ is null} \\
\vec{e}_v \cdot \vec{e}_v &= g_{vv} = 0 \implies \vec{e}_v \text{ is null} \\
\vec{e}_u \cdot \vec{e}_v &= g_{uv} = -\frac{1}{2} \neq 0 \implies \vec{e}_u \text{ and } \vec{e}_v \text{ are not orthogonal.}
\end{aligned}
$$

(e) Compute the four one-forms $\tilde{\mathrm{d}}u$, $\tilde{\mathrm{d}}v$, $\mathbf{g}(\vec{e}_u,)$, and $\mathbf{g}(\vec{e}_v,)$ in terms of $\tilde{\mathrm{d}}t$ and $\tilde{\mathrm{d}}x$.

$$\tilde{\mathrm{d}}\phi \to_{\mathcal{O}} \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right),$$

so

$$
\begin{aligned}
\tilde{\mathrm{d}}t &\to_{\mathcal{O}} (1,0,0,0), & \qquad \tilde{\mathrm{d}}x &\to_{\mathcal{O}} (0,1,0,0), \\
\tilde{\mathrm{d}}u &\to_{\mathcal{O}} \frac{1}{2}(1,-1,0,0), & \qquad \tilde{\mathrm{d}}u &\to_{\mathcal{O}} \frac{1}{2}(1,1,0,0),
\end{aligned}
$$

from which it is obvious that

$$\tilde{\mathrm{d}}u = \frac{1}{2}(\tilde{\mathrm{d}}t - \tilde{\mathrm{d}}x), \qquad \tilde{\mathrm{d}}v = \frac{1}{2}(\tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x).$$

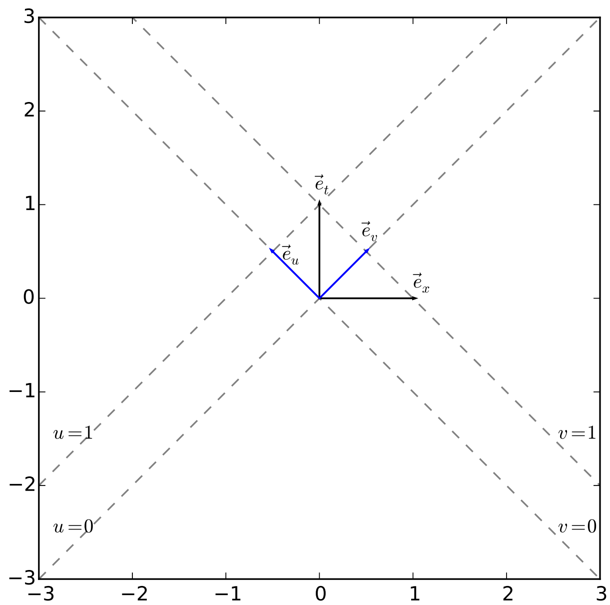

Figure 3.4: Problem 34a: Spacetime diagram of double-null coordinate basis vectors in t–x plane.