

Chapter 2

Vector analysis in special relativity

2.1 Definition of a vector

2.2 Vector algebra

2.3 The four-velocity

An object's four velocity, denoted \vec{U} , is the vector tangent to its world line, with unit length. This means it extends one unit in time, and zero in space, so it is timelike.

For an *accelerated* particle (which we have not considered up to now), we may not be able to define an inertial frame, but we *can* define a **momentarily comoving reference frame** (MCRF) which, as the name suggests, moves with the same velocity as the observer for an infinitesimal period of time. We can therefore construct a continuous sequence of MCRFs for any object. If an object has MCRF \mathcal{O} , then its four-velocity is *defined* to be the basis vector \vec{e}_0 .

2.4 The four-momentum

Analogous to the three-momentum, we define the four-momentum to be

$$\vec{p} = m\vec{U}. \tag{Schutz 2.19}$$

It has components

$$\vec{p} \rightarrow_{\mathcal{O}} (E, p^1, p^2, p^3). \tag{Schutz 2.20}$$

Calling p^0 “ E ” is no accident, it is in fact the energy. There is an interesting consequence to this: since vectors are invariant with respect to reference frame, but vector components are not, this means that the four-momentum does not change in different reference frames, but the energy *does*. One example would be

the doppler effect, which causes the color (or energy) of a photon to shift depending on the radial velocity of the source and observer.

2.5 Scalar product

$$\vec{A} \cdot \vec{B} = -(A^0 B^0) + (A^1 B^1) + (A^2 B^2) + (A^3 B^3)$$

2.6 Applications

2.7 Photons

$\vec{x} \cdot \vec{x} = 0$, so we cannot define \vec{U} for photons. We can, however, define \vec{p} . Since $\vec{p} \cdot \vec{p} = -m^2$, and photons are massless, we have $\vec{p} \cdot \vec{p} = 0$.

2.8 Further reading

2.9 Exercises

2 Identify the free and dummy indices in the following equations, and write equivalent expressions with different indices. Also, write how many equations are represented by each expression.

Note, I will express the set of free indices by \mathcal{F} and the set of dummy indices as \mathcal{D} , and I will use the original index names.

(a) $A^\alpha B_\beta = 5 \implies A^\beta B_\alpha = 5$ (16 equations, $\mathcal{F} = \{\alpha, \beta\}$, $\mathcal{D} = \emptyset$)

(b) $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu} A^\nu \implies A^{\bar{\nu}} = \Lambda^{\bar{\nu}}_{\mu} A^\mu$ (4 equations, $\mathcal{F} = \{\bar{\mu}\}$, $\mathcal{D} = \{\nu\}$).

(c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha} \implies T^{\eta\phi\theta} A_\phi C_\theta{}^\zeta = D^{\zeta\eta}$ (16 equations, $\mathcal{F} = \{\alpha, \gamma\}$, $\mathcal{D} = \{\mu, \lambda\}$)

(d) $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = G_{\mu\nu} \implies R_{\chi\epsilon} - \frac{1}{2}g_{\chi\epsilon} = G_{\chi\epsilon}$ (16 equations, $\mathcal{F} = \{\mu, \nu\}$, $\mathcal{D} = \emptyset$)

4 Given vectors $\vec{A} \rightarrow_{\mathcal{O}} (5, -1, 0, 1)$ and $\vec{B} \rightarrow_{\mathcal{O}} (-2, 1, 1, -6)$, find the components in \mathcal{O} of

(a) $-6\vec{A} \rightarrow_{\mathcal{O}} (-30, 6, 0, -6)$

(b) $3\vec{A} + \vec{B} \rightarrow_{\mathcal{O}} (13, -2, 1, -3)$

(c) $-6\vec{A} + 3\vec{B} \rightarrow_{\mathcal{O}} (-36, 9, 3, -24)$

6 Draw a spacetime diagram from \mathcal{O} 's reference frame. There are two other frames, $\bar{\mathcal{O}}$ and $\bar{\bar{\mathcal{O}}}$, which are each moving with velocity 0.6 in the $+x$ direction from each respective frame. Plot each frame's basis vectors, as observed by \mathcal{O} .

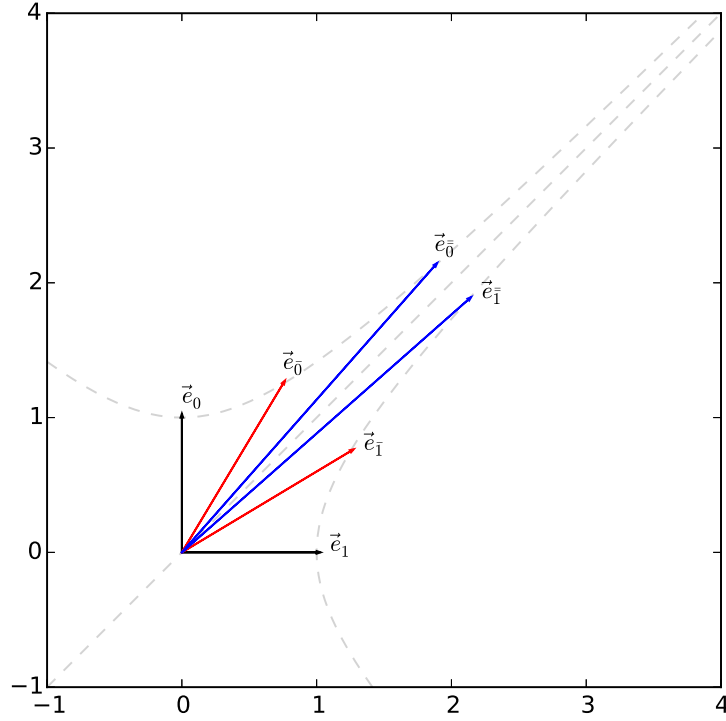


Figure 2.1: Exercise 6

See Figure 2.1.

9 Prove, by writing out all the terms that

$$\sum_{\bar{\alpha}=0}^3 \left(\sum_{\beta=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right) = \sum_{\beta=0}^3 \left(\sum_{\bar{\alpha}=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right)$$

$$\begin{aligned} \sum_{\bar{\alpha}=0}^3 \left(\sum_{\beta=0}^3 \Lambda^{\bar{\alpha}}_{\beta} A^{\beta} \vec{e}_{\bar{\alpha}} \right) &= \sum_{\bar{\alpha}=0}^3 \left(\Lambda^{\bar{\alpha}}_0 A^0 \vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}_1 A^1 \vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}_2 A^2 \vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}_3 A^3 \vec{e}_{\bar{\alpha}} \right) \\ &= \Lambda^{\bar{0}}_0 A^0 \vec{e}_{\bar{0}} + \Lambda^{\bar{0}}_1 A^1 \vec{e}_{\bar{0}} + \Lambda^{\bar{0}}_2 A^2 \vec{e}_{\bar{0}} + \Lambda^{\bar{0}}_3 A^3 \vec{e}_{\bar{0}} \\ &\quad + \Lambda^{\bar{1}}_0 A^0 \vec{e}_{\bar{1}} + \Lambda^{\bar{1}}_1 A^1 \vec{e}_{\bar{1}} + \Lambda^{\bar{1}}_2 A^2 \vec{e}_{\bar{1}} + \Lambda^{\bar{1}}_3 A^3 \vec{e}_{\bar{1}} \\ &\quad + \Lambda^{\bar{2}}_0 A^0 \vec{e}_{\bar{2}} + \Lambda^{\bar{2}}_1 A^1 \vec{e}_{\bar{2}} + \Lambda^{\bar{2}}_2 A^2 \vec{e}_{\bar{2}} + \Lambda^{\bar{2}}_3 A^3 \vec{e}_{\bar{2}} \\ &\quad + \Lambda^{\bar{3}}_0 A^0 \vec{e}_{\bar{3}} + \Lambda^{\bar{3}}_1 A^1 \vec{e}_{\bar{3}} + \Lambda^{\bar{3}}_2 A^2 \vec{e}_{\bar{3}} + \Lambda^{\bar{3}}_3 A^3 \vec{e}_{\bar{3}} \\ &= \Lambda^{\bar{0}}_0 A^0 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_0 A^0 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_0 A^0 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_0 A^0 \vec{e}_{\bar{3}} \\ &\quad + \Lambda^{\bar{0}}_1 A^1 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_1 A^1 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_1 A^1 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_1 A^1 \vec{e}_{\bar{3}} \\ &\quad + \Lambda^{\bar{0}}_2 A^2 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}_2 A^2 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}_2 A^2 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}_2 A^2 \vec{e}_{\bar{3}} \end{aligned}$$

$$\begin{aligned}
& + \Lambda^{\bar{0}}_3 A^3 \vec{e}_0 + \Lambda^{\bar{1}}_3 A^3 \vec{e}_1 + \Lambda^{\bar{2}}_3 A^3 \vec{e}_2 + \Lambda^{\bar{3}}_3 A^3 \vec{e}_3 \\
& = \sum_{\beta=0}^3 \left(\Lambda^{\bar{0}}_\beta A^\beta \vec{e}_0 + \Lambda^{\bar{1}}_\beta A^\beta \vec{e}_1 + \Lambda^{\bar{2}}_\beta A^\beta \vec{e}_2 + \Lambda^{\bar{3}}_\beta A^\beta \vec{e}_3 \right) \\
& = \sum_{\beta=0}^3 \left(\sum_{\bar{\alpha}=0}^3 \Lambda^{\bar{\alpha}}_\beta A^\beta \vec{e}_{\bar{\alpha}} \right)
\end{aligned}$$

11 Let $\Lambda^{\bar{\alpha}}_\beta$ be the matrix of the Lorentz transformation from \mathcal{O} to $\bar{\mathcal{O}}$, given in Equation 1.12. Let \vec{A} be an arbitrary vector with components (A^0, A^1, A^2, A^3) in frame \mathcal{O} .

(a) Write down the matrix of $\Lambda^\nu_{\bar{\mu}}(-v)$.

Intuitively, it should appear the same as $\Lambda^{\bar{\alpha}}_\beta$, but with the negative signs removed. More rigorously, it is given by the matrix inverse of $\Lambda^{\bar{\alpha}}_\beta$, as their product should be the identity matrix. I have used a computer algebra system (Wolfram Alpha) to take the inverse of this matrix symbolically, confirming my suspicion:

$$\Lambda^\nu_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Find $A^{\bar{\alpha}}$ for all $\bar{\alpha}$.

$$\begin{aligned}
A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}_\beta A^\beta \\
A^{\bar{0}} &= \gamma(A^0 - vA^1) \\
A^{\bar{1}} &= \gamma(A^1 - vA^0) \\
A^{\bar{2}} &= A^2 \\
A^{\bar{3}} &= A^3
\end{aligned}$$

(c) Verify Equation 2.18 by performing the sum for all values of ν and α .

To simplify things, I do this via matrix multiplication

$$\Lambda^{\bar{\alpha}}_\beta(v) \Lambda^\nu_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma^2 - v^2\gamma^2 & v\gamma^2 - v\gamma^2 & 0 & 0 \\ v\gamma^2 - v\gamma^2 & \gamma^2 - v^2\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} \gamma^2(1-v^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-v^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^\nu_\alpha
\end{aligned}$$

(d) Write down the Lorentz transformation matrix from $\bar{\mathcal{O}}$ to \mathcal{O} , justifying each term.

It should just be $\Lambda^\nu_{\bar{\mu}}(-v)$. I'm not sure what else to say at this point.

(e) Using the result from part (d), find A^β from $A^{\bar{\alpha}}$. How does this relate to Equation 2.18?

$$\begin{aligned}
\Lambda^\beta_{\bar{\alpha}} A^{\bar{\alpha}} &= \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(A^0 - vA^1) \\ \gamma(A^1 - vA^0) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma^2(A^0 - vA^1) + v\gamma^2(A^1 - vA^0) + 0 + 0 \\ v\gamma^2(A^0 - vA^1) + \gamma^2(A^1 - vA^0) + 0 + 0 \\ A^2 \\ A^3 \end{pmatrix} \\
&= \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) + A^1(v\gamma^2 - v\gamma^2) \\ A^0(v\gamma^2 - v^2\gamma^2) + A^1(\gamma^2 - v\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) \\ A^1(\gamma^2 - v^2\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = A^\beta
\end{aligned}$$

Since $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_\beta(v)$, this goes to show that $\Lambda^\nu_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}_\alpha(-v)A^\alpha = A^\nu \implies \Lambda^\nu_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}_\alpha(-v) = \delta^\nu_\alpha$.

(f) Verify in the same manner as (c) that

$$\Lambda^\nu_{\bar{\beta}}(v)\Lambda^{\bar{\alpha}}_\nu(-v) = \delta^{\bar{\alpha}}_{\bar{\beta}}$$

My matrix multiplication approach will just give me the same result as before. Perhaps another approach was intended?

(g) Establish that

$$\begin{aligned}
\vec{e}_\alpha &= \Lambda^{\bar{\beta}}_\alpha \vec{e}_{\bar{\beta}} = \Lambda^{\bar{\beta}}_\alpha \Lambda^\nu_{\bar{\beta}} \vec{e}_\nu = \delta^\nu_\alpha \vec{e}_\nu \\
A^{\bar{\beta}} &= \Lambda^{\bar{\beta}}_\alpha A^\alpha = \Lambda^{\bar{\beta}}_\alpha \Lambda^\alpha_{\bar{\mu}} A^{\bar{\mu}} = \delta^{\bar{\beta}}_{\bar{\mu}} A^{\bar{\mu}}
\end{aligned}$$

14 The following matrix gives a Lorentz transformation from \mathcal{O} to $\bar{\mathcal{O}}$:

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a) What is the velocity of $\bar{\mathcal{O}}$ relative to \mathcal{O} ?

This would correspond to a Lorentz boost along the z -axis, meaning

$$\Lambda_{\bar{\alpha}\beta}(v) = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix},$$

and thus we have $\gamma = 1.25$ and $-v\gamma = 0.75$. Solving for v , we get

$$-v\gamma = \frac{3}{4} \implies v = -\frac{3}{4\gamma} = -\frac{3 \cdot 4}{4 \cdot 5} = -\frac{3}{5}.$$

So $\bar{\mathcal{O}}$ is moving with speed 0.6 relative to the $-z$ -axis of \mathcal{O} .

(b) What is the inverse matrix to the given one?

Numerically, it comes out to be

$$\begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix},$$

which makes sense, when you consider that the inverse matrix should be a Lorentz transformation with the velocity negated.

(c) Find the components in \mathcal{O} of $\vec{A} \rightarrow_{\bar{\mathcal{O}}} (1, 2, 0, 0)$.

$$\vec{A} \xrightarrow{\mathcal{O}} \begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2 \\ 0 \\ -0.75 \end{pmatrix}$$

15

(a) Compute the four-velocity components in \mathcal{O} of a particle whose speed is v in the $+x$ -direction relative

to \mathcal{O} , using the Lorentz transformation.

$$\begin{aligned}\vec{U} &= \vec{e}_{\bar{0}} \\ U^\alpha &= \Lambda^\alpha_{\bar{\beta}}(\vec{e}_{\bar{0}})^{\bar{\beta}} = \Lambda^\alpha_{\bar{0}}, \\ U^0 &= \gamma \\ U^1 &= v\gamma \\ U^2 &= U^3 = 0\end{aligned}$$

(b) Generalize to arbitrary velocities \mathbf{v} , where $|v| < 1$.

$$\Lambda^\alpha_{\bar{\beta}}(\mathbf{v}) = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

$$U^0 = \gamma \quad U^1 = \gamma v_x \quad U^2 = \gamma v_y \quad U^3 = \gamma v_z$$

(c) Use this result to express \mathbf{v} as a function of the components $\{U^\alpha\}$.

$$\begin{aligned}\mathbf{v} &= v_x \vec{e}_1 + v_y \vec{e}_2 + v_z \vec{e}_3 \\ v_i &= \frac{U^i}{\gamma} \\ \mathbf{v} &= \frac{1}{\gamma} U^i \vec{e}_i\end{aligned}$$

(d) Find the three-velocity \mathbf{v} of a particle with four-velocity components $(2, 1, 1, 1)$.

$$U^0 = \gamma = 2, \text{ and } U^i = 1, \text{ so}$$

$$\mathbf{v} = \frac{1}{2} \vec{e}_i$$

17

Not sure how to approach this problem.

(a) Prove that any timelike vector \vec{U} for which $U^0 > 0$ and $\vec{U} \cdot \vec{U} = -1$ is the four-velocity of *some* world line.

(b) Use this to prove that for any timelike vector \vec{V} there is a Lorentz frame in which the \vec{V} has zero spatial components.

19 A body is uniformly accelerated if the four-vector \vec{a} has constant spatial direction and magnitude, $\vec{a} \cdot \vec{a} = \alpha^2 \geq 0$.

(a) Show that this implies the components of \vec{a} in the body's MCRF are all constant, and that these are equivalent to the Galilean “acceleration”.

We normalize the vector \vec{a} by dividing each of its terms by the magnitude of the vector, so

$$\frac{a^\lambda}{\alpha}.$$

Since α is constant, and also the *direction* is constant, this means that the above expression is *also* constant, as the normalized components tell you about the direction. If we multiply a constant by a constant, we should still get a constant, so we multiply the above expression by α , getting a^λ to be constant.

In the MCRF of an object, $d\tau = dt$, and so we can write

$$\vec{a} = \frac{d\vec{U}}{dt} = \left(0, \frac{dU^1}{dt}, \frac{dU^2}{dt}, \frac{dU^3}{dt}\right),$$

which is analogous to the Galilean acceleration.

(b) A body is uniformly accelerated with $\alpha = 10 \text{ m/s}^2$. It starts from rest, and falls for a time t . Find its speed as a function of t , and find the time to reach $v = 0.999$.

$$\begin{aligned} \vec{U} &\xrightarrow{\text{MCRF}} (1, 0, 0, 0) \\ &\xrightarrow{\mathcal{O}} (\gamma, \gamma v, 0, 0) \\ \frac{d\vec{U}}{d\tau} &\xrightarrow{\text{MCRF}} (0, \alpha, 0, 0) \\ &\xrightarrow{\mathcal{O}} (\gamma, \gamma\alpha, 0, 0) \\ U^x &= \int_0^t \frac{dU^x}{d\tau} d\tau = \int_0^t \gamma\alpha \frac{dt}{\gamma} = \int_0^t \alpha dt = \alpha t \\ &= \gamma v = \frac{v}{\sqrt{1-v^2}} \\ v^2 &= (\alpha t)^2 (1-v^2) = (\alpha t)^2 - (\alpha t v)^2 \\ v^2 (1 + (\alpha t)^2) &= (\alpha t)^2 \\ v^2 &= \frac{(\alpha t)^2}{1 + (\alpha t)^2} \implies v = \sqrt{\frac{(\alpha t)^2}{1 + (\alpha t)^2}} \end{aligned}$$

To find the time to reach $v = 0.999$, we go back to the expression $\gamma v = \alpha t$, solve for t , and substitute for v and α . Note that in natural units, $\alpha = 10 \text{ m/s}^2 c^{-2} \approx 1.11 \times 10^{-16} \text{ m}^{-1}$

$$t = \frac{v}{\alpha \sqrt{1-v^2}} = \frac{0.999}{1.11 \times 10^{-16} \text{ m}^{-1} \sqrt{1-0.999^2}} \approx 2.01 \times 10^{17} \text{ m}.$$

24 Show that a positron and electron cannot annihilate to form a single photon, but they can annihilate to form two photons.

We consider the center of momentum frame, where $\sum \vec{p}_{(i)} \rightarrow_{\text{CM}} (E_{\text{total}}, 0, 0, 0)$. Without loss of generality,

we assume that the velocities of the two particles are equal and opposite, such that

$$\vec{p}_{e+} \rightarrow_{\text{CM}} m_e(\gamma, \gamma v, 0, 0), \quad \vec{p}_{e-} \rightarrow_{\text{CM}} m_e(\gamma, -\gamma v, 0, 0).$$

The photon they create will have to have a momentum of $\vec{p}_{\gamma, \text{single}} \rightarrow_{\text{CM}} (h\nu, h\nu, 0, 0)$. By conservation of four-momentum, we have

$$\begin{aligned} \vec{p}_{e+} + \vec{p}_{e-} &= \vec{p}_{\gamma, \text{single}} \\ (\vec{p}_{e+} + \vec{p}_{e-}) \cdot (\vec{p}_{e+} + \vec{p}_{e-}) &= \vec{p}_{\gamma, \text{single}} \cdot \vec{p}_{\gamma, \text{single}} \\ (\vec{p}_{e+} \cdot \vec{p}_{e+}) + (\vec{p}_{e-} \cdot \vec{p}_{e-}) + (\vec{p}_{e+} \cdot \vec{p}_{e-}) &= 0 \\ -m_e^2 - m_e^2 - m_e^2 &= 0 \implies m_e = 0! \end{aligned}$$

Since we know that m_e is in fact non-zero, this cannot possibly happen.

Now consider the scenario wherein two photons are created, moving in opposite directions. Then they would have momenta: $\vec{p}_{\gamma, 1} \rightarrow_{\text{CM}} (h\nu, h\nu, 0, 0)$ and $\vec{p}_{\gamma, 2} \rightarrow_{\text{CM}} (h\nu, -h\nu, 0, 0)$. Invoking conservation of four-momentum as before, we get

$$\begin{aligned} \vec{p}_{e+} + \vec{p}_{e-} &= \vec{p}_{\gamma, 1} + \vec{p}_{\gamma, 2} \\ (\vec{p}_{e+} + \vec{p}_{e-}) \cdot (\vec{p}_{e+} + \vec{p}_{e-}) &= (\vec{p}_{\gamma, 1} + \vec{p}_{\gamma, 2}) \cdot (\vec{p}_{\gamma, 1} + \vec{p}_{\gamma, 2}) \\ -3m_e^2 &= (\vec{p}_{\gamma, 1} \cdot \vec{p}_{\gamma, 1}) + (\vec{p}_{\gamma, 1} \cdot \vec{p}_{\gamma, 2}) + (\vec{p}_{\gamma, 2} \cdot \vec{p}_{\gamma, 2}) \\ &= 0 + (-h^2\nu^2 - h^2\nu^2) + 0 = -2h^2\nu^2, \end{aligned}$$

so we end up with $3m_e^2 = 2h^2\nu^2$, meaning two photons are produced with $E^2 = \frac{3}{2}m_e^2$, which is entirely reasonable.

25

(a) Consider a frame $\bar{\mathcal{O}}$ moving with a speed v along the x -axis of \mathcal{O} . Now consider a photon moving at an angle θ from \mathcal{O} 's x -axis. Find the ratio of its frequency in $\bar{\mathcal{O}}$ and in \mathcal{O} .

We must first construct the particle's four-momentum. In the case where the photon was moving along the x -axis (see Section 2.7), it had been found that the four-momentum was

$$\vec{p} \xrightarrow{\mathcal{O}} (E, E, 0, 0),$$

as this satisfied

$$\vec{p} \cdot \vec{p} = -E^2 + E^2 = 0. \quad (\text{Schutz 2.37})$$

Now that the photon is moving at an angle θ from the x -axis, we need to redistribute the 3-momentum accordingly. No specification was given as photon's angle in the y - or z -axis, so without loss of generality, I assume it is constrained to the x - y plane. This means we can write the four-momentum as

$$\vec{p} \xrightarrow{\mathcal{O}} (E, E \cos \theta, E \sin \theta, 0),$$

which you can easily confirm satisfies $\vec{p} \cdot \vec{p} = 0$.

Now we may apply the Lorentz transformation $\Lambda_{\alpha}^{\bar{0}}(v)$ to find the photon's energy as observed by $\bar{\mathcal{O}}$, and from that the frequency.

$$\begin{aligned} p^{\bar{0}} &= \bar{E} = \Lambda_{\alpha}^{\bar{0}} p^{\alpha} = \gamma p^0 - v\gamma p^1 + 0 + 0 = \gamma E - v\gamma E \cos \theta \\ \implies h\bar{\nu} &= \gamma h\nu - v\gamma h\nu \cos \theta \\ \implies \frac{\bar{\nu}}{\nu} &= \gamma - v\gamma \cos \theta = \frac{1 - v \cos \theta}{\sqrt{1 - v^2}} \end{aligned}$$

(b) Even when the photon moves perpendicular to the x -axis ($\theta = \pi/2$) there is a frequency shift. This is the *transverse Doppler shift*, which is a result of time dilation. At which angle θ must the photon move such that there is no Doppler shift between \mathcal{O} and $\bar{\mathcal{O}}$?

To do this, we simply set $\bar{\nu}/\nu = 1$, and solve for θ .

$$\begin{aligned} 1 &= \frac{1 - v \cos \theta}{\sqrt{1 - v^2}} \implies \cos \theta = 1 - \sqrt{1 - v^2} \\ \implies \theta &= \pm \arccos(1 - \sqrt{1 - v^2}) \end{aligned}$$

(c) Now use Equations 2.35 and 2.38 to find $\bar{\nu}/\nu$.

Recall that $\vec{U} \rightarrow_{\mathcal{O}} (\gamma, v\gamma, 0, 0)$. Using Equation 2.35 we have

$$\begin{aligned} \bar{E} &= h\bar{\nu} = -(E, E \cos \theta, E \sin \theta, 0) \cdot (\gamma, v\gamma, 0, 0) \\ &= -(-(E\gamma) + E\gamma v \cos \theta) = E\gamma(1 - v \cos \theta) = h\nu\gamma(1 - v \cos \theta) \\ \frac{\bar{\nu}}{\nu} &= \frac{1 - v \cos \theta}{\sqrt{1 - v^2}} \end{aligned}$$

26 Calculate the energy required to accelerate a particle of rest mass $m > 0$ from speed v to speed $v + \delta v$ ($\delta v \ll v$), to first order in δv . Show that it would take infinite energy to accelerate to c .

From the four-momentum we have $E_v = m\gamma$, and from that

$$E_{v+\delta v} = \frac{m}{\sqrt{1 - (v + \delta v)^2}}.$$

If we do a Taylor expansion on $(1 - (v + \delta v)^2)^{-1/2}$ we get

$$\frac{1}{\sqrt{1 - v^2}} + \frac{v \delta v}{(1 - v^2)^{3/2}} + \mathcal{O}(v^2),$$

so

$$\begin{aligned} E_{v+\delta v} &\approx \frac{m}{\sqrt{1 - v^2}} + \frac{mv \delta v}{(1 - v^2)^{3/2}} \\ \Delta E &= E_{v+\delta v} - E_v \approx \frac{mv \delta v}{(1 - v^2)^{3/2}} = m\gamma^3 v \delta v. \end{aligned}$$

As $v \rightarrow c$, $\gamma \rightarrow \infty$ and therefore $\Delta E \rightarrow \infty$.

30 A rocket ship has four-velocity $\vec{U} \rightarrow_{\mathcal{O}} (2, 1, 1, 1)$, and it passes a cosmic ray with four-momentum $\vec{p} \rightarrow_{\mathcal{O}} (300, 299, 0, 0) \times 10^{-27} \text{kg}$. Compute the energy of the ray as measured by the rocket, using two different methods.

(a) Find the Lorentz transformation from \mathcal{O} to the rocket's MCRF, and from that find the components $p^{\bar{\alpha}}$.

The Lorentz transformation for a boost in the x , y , and z directions is given by

$$\Lambda^{\bar{\beta}}_{\alpha} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

If we write out the terms of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

then we are left with a system of equations

$$1 = \gamma(2 + v_x + v_y + v_z),$$

$$0 = \gamma(2v_x + 1),$$

$$0 = \gamma(2v_y + 1),$$

$$0 = \gamma(2v_z + 1).$$

Since γ may never be zero, we divide the last 3 terms by γ to obtain

$$2v_i + 1 = 0 \implies v_i = -\frac{1}{2},$$

and plugging into the first equation gives $\gamma = 2$. From this we see that our Lorentz transformation matrix is

$$\Lambda^{\bar{\beta}}_{\alpha} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

Now to find the energy as observed by the rocket, we need to find $\bar{E} = p^{\bar{0}}$

$$\begin{aligned} p^{\bar{0}} &= \Lambda^{\bar{0}}_{\alpha} p^{\alpha} = 2p^0 - p^1 - p^2 - p^3 \\ &= (2 \cdot 300 - 1 \cdot 299 - 1 \cdot 0 - 1 \cdot 0) \times 10^{-27} \text{kg} = 3.01 \times 10^{-25} \text{kg} = \bar{E} \end{aligned}$$

(b) Use Schutz's Equation 2.35.

$$\begin{aligned}\bar{E} &= -\vec{p} \cdot \vec{U}_{\text{obs}} = -(-(300 \cdot 2) + (299 \cdot 1) + (0 \cdot 1) + (0 \cdot 1)) \times 10^{-27} \text{kg} \\ &= 3.01 \times 10^{-25} \text{kg}\end{aligned}$$

(c) Which is quicker? Why?

Using Equation 2.35 was *much* quicker, as it was derived to handle this special case.

32 Consider a particle with charge e and mass m , which begins at rest, but scatters a photon with frequency ν_i (Compton scattering). The photon comes off at an angle θ from the direction of the initial photon's path. Use conservation of four-momentum to find the scattered photon's frequency, ν_f .

We will invoke: conservation of four-momentum and $\vec{p} \cdot \vec{p} = -m^2$. \vec{p}_i and \vec{p}_f denote the initial and final photon, and \vec{p}_e and $\vec{p}_{e'}$ denote the electron before and after collision.

$$\begin{aligned}\vec{p}_i &\xrightarrow{\mathcal{O}} (E_i, E_i, 0, 0) \\ \vec{p}_e &\xrightarrow{\mathcal{O}} (m, 0, 0, 0) \\ \vec{p}_f &\xrightarrow{\mathcal{O}} (E_f, E_f \cos \theta, E_f \sin \theta, 0) \\ \vec{p}_i + \vec{p}_e &= \vec{p}_f + \vec{p}_{e'} \\ \vec{p}_{e'} &= \vec{p}_i + \vec{p}_e - \vec{p}_f \\ \vec{p}_{e'} \cdot \vec{p}_{e'} &= (\vec{p}_i + \vec{p}_e - \vec{p}_f) \cdot (\vec{p}_i + \vec{p}_e - \vec{p}_f) \\ -m^2 &= \vec{p}_i \cdot \vec{p}_i + \vec{p}_e \cdot \vec{p}_e + \vec{p}_f \cdot \vec{p}_f + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\ &= 0 - m^2 + 0 + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\ 0 &= \vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f \\ &= -E_i m - (-E_i E_f + E_i E_f \cos \theta) + E_f m \\ &= m(E_f - E_i) + E_i E_f (1 - \cos \theta) \\ m(E_i - E_f) &= E_i E_f (1 - \cos \theta) \\ mh(\nu_i - \nu_f) &= h^2 \nu_i \nu_f (1 - \cos \theta) \\ \frac{\nu_i - \nu_f}{\nu_i \nu_f} &= h \frac{1 - \cos \theta}{m} \\ \frac{1}{\nu_f} - \frac{1}{\nu_i} &= h \frac{1 - \cos \theta}{m} \\ \frac{1}{\nu_f} &= \frac{1}{\nu_i} + h \frac{1 - \cos \theta}{m}\end{aligned}$$