Spherical solutions for stars

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General Relativity I Presentations
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Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- applications

Spherically symmetric coordinates

Two-sphere in flat spacetime

General metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

Metric on 2-sphere

$$\mathrm{d}l^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \equiv r^2\mathrm{d}\Omega^2$$

Schutz (2009, p. 256)

Two-sphere in flat spacetime

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Two-sphere in curved spacetime

Metric on 2-sphere

$$\mathrm{d}l^2 = f(r', t)\mathrm{d}\Omega^2$$

Relation to r

$$f(r', t) \equiv r^2$$

Schutz (2009, pp. 256–257)

Two-sphere in curved spacetime

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Meaning of r

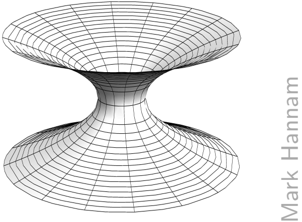

Figure:
Surface with circular symmetry but no coordinate $r = 0$.

- *not* proper distance from center
- "curvature" or "area" coordinate
  - radius of curvature and area
- $r = \text{const}$, $t = \text{const}$
  - $A = 4\pi r^2$
  - $C = 2\pi r$

Schutz (2009, p. 257)

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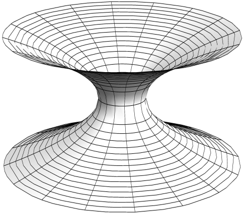

Mark Hannam

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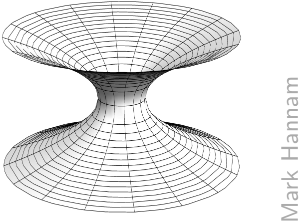

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Spherically symmetric spacetime

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$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + 2g_{0r}\,\mathrm{d}r\,\mathrm{d}t + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

g_{00}, g_{0r}, and g_{rr} functions of t and r

Schutz (2009, p. 258)

Static spacetimes

Motivation

- leads to simple derivation of Schwarzschild metric

- unique solution to spherically symmetric, asymptotically flat Einstein vacuum field equations (Birkhoff's theorem)

Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843)

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Definition

A spacetime is static if we can find a time coordinate t for which

(i) the metric independent of t

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \to -t$$

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Time reversal

$$\mathbf{\Lambda} : (t, x, y, z) \to (-t, x, y, z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} g_{\alpha\beta} = g_{\alpha\beta}$$

Transformation

$$\Lambda^{0}{}_{\bar{0}} = x^{0}{}_{,\bar{0}} = -x^{0}{}_{,0} = -1$$
$$\Lambda^{i}{}_{\bar{i}} = x^{i}{}_{,\bar{i}} = x^{i}{}_{,i} = 1$$

Metric

$$g_{\bar{0}\bar{0}} = (\Lambda^{0}{}_{\bar{0}})^2 g_{00} = g_{00}$$
$$g_{\bar{r}\bar{r}} = (\Lambda^{r}{}_{\bar{r}})^2 g_{rr} = g_{rr}$$
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The metric

Simplified metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Replacement

$$g_{00} \to -e^{2\Phi}, \quad g_{rr} \to e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

Static spherically symmetric metric

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \Lambda(r) = 0$$

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Einstein Tensor

General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Einstein tensor components

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r}[r(1 - e^{-2\Lambda})]$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

Schutz (2009, pp. 165, 260)

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Schutz (2009, pp. 165, 260)

Static perfect fluid

Four-velocity

Constraints

$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

Solving for U^0

$$g_{00} U^0 U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

Solving for U_0

$$U_0 = g_{00} U^0 = -e^{\Phi}$$

Schutz (2009, p. 260)

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Stress–energy tensor

Stress-energy tensor for perfect fluid

$$T_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + p g_{\alpha\beta}$$

Components of $T_{\alpha\beta}$

$$T_{i\alpha} = p g_{i\alpha}$$

$T_{\alpha\beta}$ is diagonal

$$T_{00} = (\rho + p)e^{2\Phi} + p(-e^{2\Phi}) = \rho e^{2\Phi}$$

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Schutz (2009, p. 260)

Equation of state

Local thermodynamic equilibrium

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

Schutz (2009, p. 261)

Equations of motion

Conservation of 4-momentum

$$T^{\alpha\beta}{}_{;\beta} = 0$$

- symmetries make only non-trivial solution $\alpha = r$

Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)

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Schutz (2009, pp. 175, 261)

Mass function

Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r}[r(1 - e^{-2\Lambda})] = 8\pi\rho e^{2\Phi}$$

$m(r)$

$$m(r) \equiv \frac{1}{2} r (1 - e^{-2\Lambda}) \quad \text{or} \quad g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

Relation to energy density

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Schutz (2009, pp. 260–262)

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$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

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Exterior Geometry

Schwarzschild metric I

Condition

$$\rho = p = 0$$

Consequences

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad m(r) \equiv M$$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)} \qquad \Phi(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right)$$

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Schwarzschild metric II

First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$

Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

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Schwarzschild metric II

First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$

Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Schutz (2009, pp. 258, 262–263)

Far-field metric

Condition

$$r \gg M$$

Far-field Schwarzschild metric

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Far-field Schwarzschild metric (Cartesian)

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

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Schutz (2009, pp. 263)

Birkhoff's Theorem

If the geometry of a given region of spacetime is:

1. spherically symmetric
2. a solution to the Einstein field equations in vacuum

then that geometry is necessarily a subset of the Schwarzschild geometry.

(Proof given in Misner, Thorne, and Wheeler (1973, pp. 843–844))

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Interior structure

Tolman–Oppenheimer–Volkov (T–O–V) equation

Condition

$$\rho \neq 0 \quad p \neq 0$$

Recall

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

Schutz (2009, pp. 261–264)

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Schutz (2009, pp. 261–264)

System of coupled differential equations

T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

Mass function

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Equation of state

$$p = p(\rho)$$

Schutz (2009, pp. 261–262, 264)

Newtonian hydrostatic equilibrium

Newtonian limit

$$p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll r$$

Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -\frac{\rho m(r)}{r^2}$$

Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3)

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Constant density solution I

Constraint

$$\rho \equiv \rho_0$$

Mass function

$$m(r) = \frac{4}{3}\pi\rho_0 \begin{cases} r^3, & r \leq R, \\ R^3, & r \geq R. \end{cases}$$

Schutz (2009, pp. 266-267)

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Schutz (2009, pp. 266-267)

Constant density solution II

T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} = -\frac{4}{3}\pi r \frac{(\rho_0 + p)(\rho_0 + 3p)}{1 - \frac{8}{3} r^2 \rho_0}$$

Integrated from center to internal radius r

$$\frac{\rho_0 + 3p}{\rho_0 + p} = \frac{\rho_0 + 3p_c}{\rho_0 + p_c} \sqrt{1 - 2m/r}$$

Schutz (2009, pp. 264, 266-267)

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Schutz (2009, pp. 264, 266-267)

Constant density solution III

Radius R

$$R^2 = \frac{3}{8\pi\rho_0}\left[1 - \left(\frac{\rho + p_c}{\rho + 3p_c}\right)^2\right]$$

Central pressure p_c

$$p_c = \rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}$$

Limit on M/R

$$M/R \to 4/9 \implies p_c \to \infty$$

Schutz (2009, pp. 266-267, 269)

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Schutz (2009, pp. 266-267, 269)

Buchdahl's theorem

- even for non-constant density, $M/R < 4/9$
- intuitive explanation:
  - assume there is a maximum sustainable density, $(M/R)_{max}$

Carroll (2004, pp. 234)

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- even for non-constant density, $M/R < 4/9$
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  - assume there *is* a maximum sustainable density, $(M/R)_{\text{max}}$
  - consider an object of radius R
  - most massive possible object would have maximum density everywhere
  - all other sustainable objects have a lower M/R

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Carroll (2004, pp. 234)

Realistic stars

White dwarfs

- end-of-life for low mass stars
- held up by electron degeneracy pressure
- Newtonian structure accurate to 1%

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho m}{r^2}$$

- relativistic effects important on stability and pulsation for

$$10^8 \mathrm{g\,cm^{-3}} \lesssim \rho_c \lesssim 10^{8.4} \mathrm{g\,cm^{-3}}$$

Misner, Thorne, and Wheeler (1973, p. 627)

Neutron stars

- mass condensed further than white dwarf
- created in supernovae, or collapse of white dwarf
- protons and electrons combine to form neutrons and emitted neutrinos
- held up by neutron degeneracy pressure
- matter incredibly complex and possess many unknown properties

Schutz (2009, pp. 274–275)

Rotating stars

Metric

$$ds^2 = -e^{2\nu}\,dt + e^{2\psi}(d\phi - \omega\,dt)^2 + e^{2\mu}(dr^2 + r^2\,d\theta^2),$$

where ν, ψ, ω, and μ are functions of r and θ

- can still assume perfect fluid to high accuracy

Stergioulas (2003, p. 8)

Pulsars

- rapidly rotating neutron stars
- magnetic field produces electromagnetic radiation
- pulses of radio waves observed with the right orientation

Misner, Thorne, and Wheeler (1973, p. 628)

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Bonus slides

Equations of motion

$$
\begin{aligned}
T^{\alpha\beta}{}_{;\beta} &= 0, \quad T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta} \\
T^{r\beta}{}_{;\beta} &= (\rho + p) U^\beta U^r{}_{;\beta} + g^{rr} p_{,r} = 0 \\
&= (\rho + p) U^\beta U^\lambda \Gamma^r{}_{\lambda\beta} + e^{-2\Lambda} p_{,r} = 0 \\
&= (\rho + p) (U^0)^2 \Gamma^r{}_{00} + e^{-2\Lambda} p_{,r} = 0 \\
&= (\rho + p)(e^{-2\Phi})(e^{-2\Lambda} e^{2\Phi} \Phi_{,r}) + e^{-2\Lambda} p_{,r} = 0 \\
-\frac{\mathrm{d}p}{\mathrm{d}r} &= (\rho + p) \frac{\mathrm{d}\Phi}{\mathrm{d}r}
\end{aligned}
$$

Schutz (2009, pp. 101, 261)