Introduction to General Relativity (ASTP-760) Notes

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Fall 2015

Contents

Chapter 1

Special relativity

1.1 Fundamental principles of special relativity (SR) theory

Special relativity can be summarized by two fundamental postulates:

1. The principle of relativity (Galileo), which states that no experiment may measure the absolute velocity of an observer.
2. The universality of the speed of light (Einstein), which states that the speed of light is constant when measured from any inertial reference frame.

1.2 Definition of an inertial observer in SR

When we say "observer", what we really mean is a coordinate system. Thus an inertial observer is a coordinate system that meets the following 3 criteria:

1. The distance between two spatial points P_1 and P_2 is independent of time.
2. Time is synchronized and moves at the same rate at all spatial points.
3. At any constant time, space is Euclidean.

It follows from these criteria that the observer must be **unaccelerated**.

1.3 New units

The speed of light, c, is approximately $3.00 \times 10^8\,\mathrm{ms}^{-1}$ in SI units. However, these units predate relativity, and are very inconvenient. Life becomes easier if we define our units around c, such that $c \equiv 1$.

This can be done by repurposing the meter as a measure of time as well. We thereby define the meter as "the time it takes light to travel 1 meter". Thus the speed of light becomes

$$c = \frac{1\,\mathrm{m}}{1\,\mathrm{m}}.$$

Indeed, it turns out in SR that time is most conveniently measured in distance ($c = 3.00 \times 10^{10}\,\mathrm{cm}$), and in GR mass is as well ($G/c^{-2} = 7.425 \times 10^{-29}\,\mathrm{cm\,g^{-1}}$).

1.4 Spacetime diagrams

1.5 Construction of the coordinates used by another observer

1.6 Invariance of the interval

For two nearby events, we can define the **invariant interval**, defining a 4D Minskowski spacetime:

$$\mathrm{d}s^2 = -(c\,\mathrm{d}t)^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,,$$

or when we set $c \equiv 1$:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,. \qquad \text{(Schutz 1.1)}$$

This notation can be simplified be defining

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathrm{d}s^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

When we want to find $\mathrm{d}\bar{s}^2$, we can consider the fact that each of its components, $\mathrm{d}\bar{x}^\alpha$, is a linear combination of the components of $\mathrm{d}s^2$,

$$\mathrm{d}\bar{x}^\alpha = \sum_{\beta=0}^{3} a_{\alpha\beta} x^\beta.$$

Now, when we consider the square of $\mathrm{d}\bar{x}^\alpha$, the cross terms make it a quadratic function. Since the sum of four quadratics (the four $\mathrm{d}\bar{x}^\alpha$'s) is also a quadratic, we can write $\mathrm{d}\bar{s}^2$ as

$$\mathrm{d}\bar{s} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} M_{\alpha\beta}(\mathrm{d}x^\alpha)(\mathrm{d}x^\beta) \qquad \text{(Schutz 1.2)}$$

If are talking about light, $\mathrm{d}s^2 = 0$, and so we can say

$$\mathrm{d}s^2 = 0 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 \implies \mathrm{d}t = \mathrm{d}r$$

Now by looking at Exercise 8 in Section 1.14, we see that

$$\begin{aligned}\mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &\quad + 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &\quad + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \end{aligned} \qquad \text{(Schutz 1.3)}$$

where

$$M_{0i} = 0 \qquad \text{(Schutz 1.4a)}$$

and

$$M_{ij} = -(M_{00})\delta_{ij}, \qquad \text{(Schutz 1.4b)}$$

where δ_{ij} is the Kronecker delta.

1.7 Invariant hyperbolae

1.8 Particularly important results

1.9 The Lorentz transformation

1.10 The velocity-composition law

1.11 Paradoxes and physical intuition

1.12 Further reading

1.13 Appendix: The twin 'paradox' dissected

Consider two twins, Joe and Ed. Joe goes off in a straight line traveling at a speed of $(24/25)c$ for 7 years, as measured on his clock, then instantaneously reverses and returns at half the speed. Ed remains at home. When they return, what is the difference in ages between Joe and Ed?

$\tau_1 = 7\,\mathrm{yr}$. $t_1 = \tau_1\gamma_1$, where $\gamma_1 = \left[1 - \left(\frac{24}{25}\right)^2\right]^{-1/2}$. So $t_1 = 25\,\mathrm{yr}$.

$t_2 = 2t_1 = 50\,\mathrm{yr}$.

$\tau_2 = t_2\gamma_2^{-1}$, where $\gamma_1 = \left[1 - \left(\frac{12}{25}\right)^2\right]^{-1/2}$. So $\tau_2 = 2\sqrt{481}\text{yr} \approx 44\,\text{yr}$. Finally, $\tau = \tau_1 + \tau_2 \approx 51\,\text{yr}$, and $t = t_1 + t_2 = 75\,\text{yr}$, so Ed ages $t - \tau \approx 24\,\text{years}$ more than Joe.

1.14 Exercises

1 Convert the following to units in which $c = 1$, expressing everything in terms of m and kg.

(Note that $c = 1 \implies 1 \approx 3 \times 10^8\,\text{m}\,\text{s}^{-1} \approx (3 \times 10^8)^{-1}\text{m}^{-1}\,\text{s}$

(a) 10 J

$$\begin{aligned}10\,\text{J} = 10\,\text{N}\,\text{m} = 10\,\text{kg}\,\text{m}^2\,\text{s}^{-2} &\approx 10\,\text{kg}\,\text{m}^2\,\text{s}^{-2} \cdot ((3 \times 10^8)^{-1}\text{m}^{-1}\,\text{s})^2 \\ &\approx 10\,\text{kg}(3 \times 10^8)^{-2} = 10\,\text{kg}\left(\frac{1}{9} \times 10^{-16}\right) \approx 1.11 \times 10^{-16}\,\text{kg}\end{aligned}$$

(b) 100 W

$$\begin{aligned}100\,\text{W} = 100\,\text{kg}\,\text{m}^2\,\text{s}^{-3} &\approx 100\,\text{kg}\,\text{m}^2\,\text{s}^{-3} \cdot ((3 \times 10^8)^{-1}\text{m}^{-1}\,\text{s})^3 \\ &\approx 100\,\text{kg}\,\text{m}^{-1}(3^{-3} \times 10^{-24}) = \frac{100}{27} \times 10^{-24}\text{kg}\,\text{m}^{-1} \approx 3.7 \times 10^{-24}\,\text{kg}\,\text{m}^{-1}\end{aligned}$$

2 Convert the following from natural units ($c = 1$) to SI units:

(a) A velocity $v = 10^{-2}$.

$$v = 10^{-2} = 10^{-2}c = 10^{-2}3 \times 10^8\,\text{m}\,\text{s}^{-1} = 3 \times 10^6\,\text{m}\,\text{s}^{-1}$$

(b) Pressure $P = 10^{19}\text{kg}\,\text{m}^{-3}$.

$$\begin{aligned}P = 10^{19}\text{kg}\,\text{m}^{-3} &\approx 10^{19}\text{kg}\,\text{m}^{-3}(3 \times 10^8\,\text{m}\,\text{s}^{-1})^2 \\ &\approx 10^{19}\text{kg}\,\text{m}^{-3}(9 \times 10^{16}\,\text{m}^2\,\text{s}^{-2}) = 9 \times 10^{35}\,\text{N}\,\text{m}^2\end{aligned}$$

3 Draw the t and x axes of the spacetime coordinates of an observer $\mathcal{O}$ and then draw:

(a) The world line of $\mathcal{O}$'s clock at $x = 1\,\text{m}$.

4 Write out all the terms of the following sums, substituting the coordinate names (t, x, y, z) for (x^0, x^1, x^2, x^3):

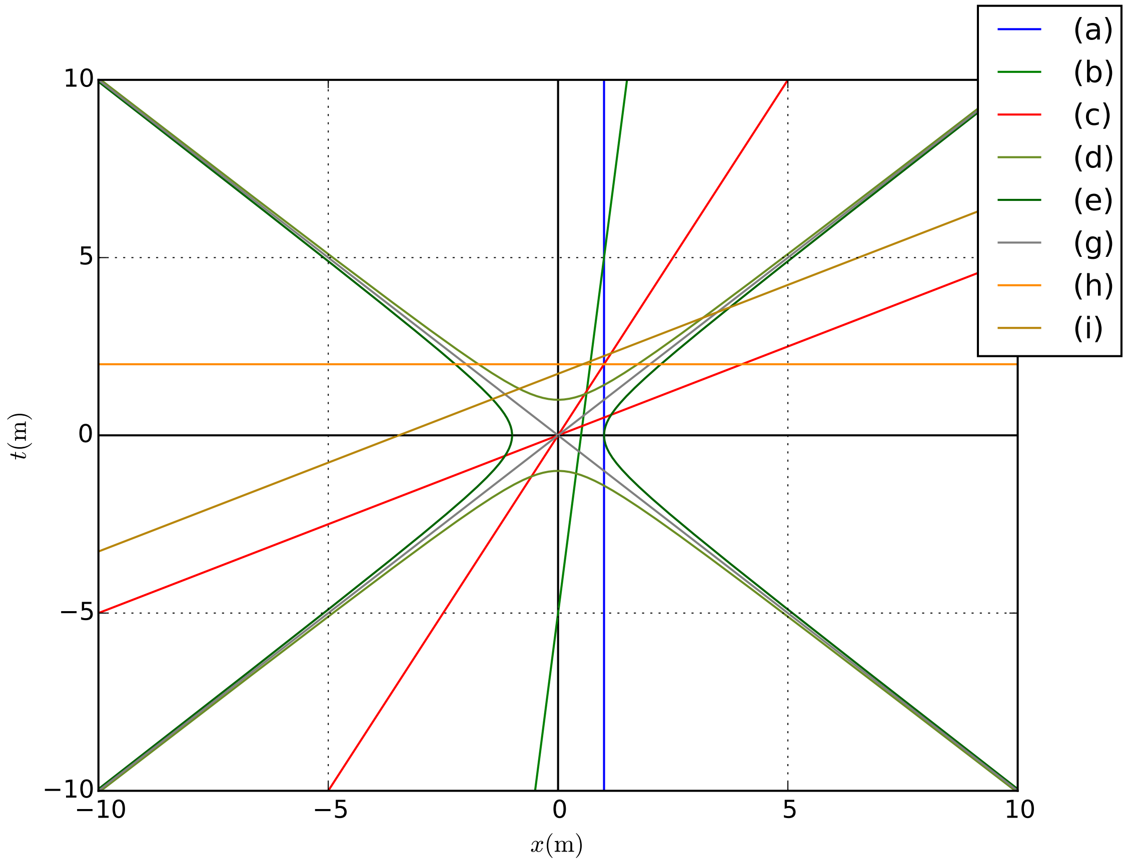
(a) $\sum_{\alpha=0}^{3} V_\alpha\,\mathrm{d}x^\alpha = V_0\,\mathrm{d}t + V_1\,\mathrm{d}x + V_2\,\mathrm{d}y + V_3\,\mathrm{d}z$.

(b) $\sum_{i=1}^{3}(\mathrm{d}x^i)^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2$.

5

(a) Use the spacetime diagram of an observer $\mathcal{O}$ to describe the following experiment performed by $\mathcal{O}$. Two bursts of particles of speed $v = 0.5$ are emitted from $x = 0$ at $t = -2\,\text{m}$, one traveling in the $+x$ direction and the other in the $-x$ direction. These encounter detectors located at $x = \pm 2\,\text{m}$. After a delay of 0.5 m of time, the detectors send signals back to $x = 0$ at speed $v = 0.75$.

See figure below



Exercise 3

(b) The signals arrive back at $x = 0$ at the same event. (Make sure your spacetime diagram shows this!) From this the experimenter concludes that the particle detectors did indeed send out their signals simultaneously, since he knows they are equal distances from $x = 0$. Explain why this conclusion is valid.

Assuming he knows the signals traveled with equal speeds, and the detectors are an equal distance away, then they must have been emitted simultaneously, in order for them to arrive at $x = 0$ simultaneously.

(c) A second observer $\bar{\mathcal{O}}$ moves with speed $v = 0.75$ in the $-x$ direction relative to $\mathcal{O}$. Draw the spacetime diagram of $\bar{\mathcal{O}}$ and in it depict the experiment performed by $\mathcal{O}$. Does $\bar{\mathcal{O}}$ conclude that particle detectors sent out their signals simultaneously? If not, which signal was sent first.

See the diagram below. On it, I have drawn lines $\bar{t}_{\text{left}}$ and $\bar{t}_{\text{right}}$ (note that they are parallel to the $\bar{x}$ axis). As you can see from the plot, the left emission occurs *before* the right emission.

(d)

Using $\mathcal{O}$, the distance is

$$\Delta s^2 = \Delta x^2 = 16\,\text{m}^2.$$

Using $\bar{\mathcal{O}}$, we first need to find $\bar{x}_{\{a,b\}}$ and $\bar{t}_{\{a,b\}}$. We use the Lorentz transformation to do this.

$$\bar{t} = \gamma(t - vx)$$
$$\bar{x} = \gamma(x - vt)$$

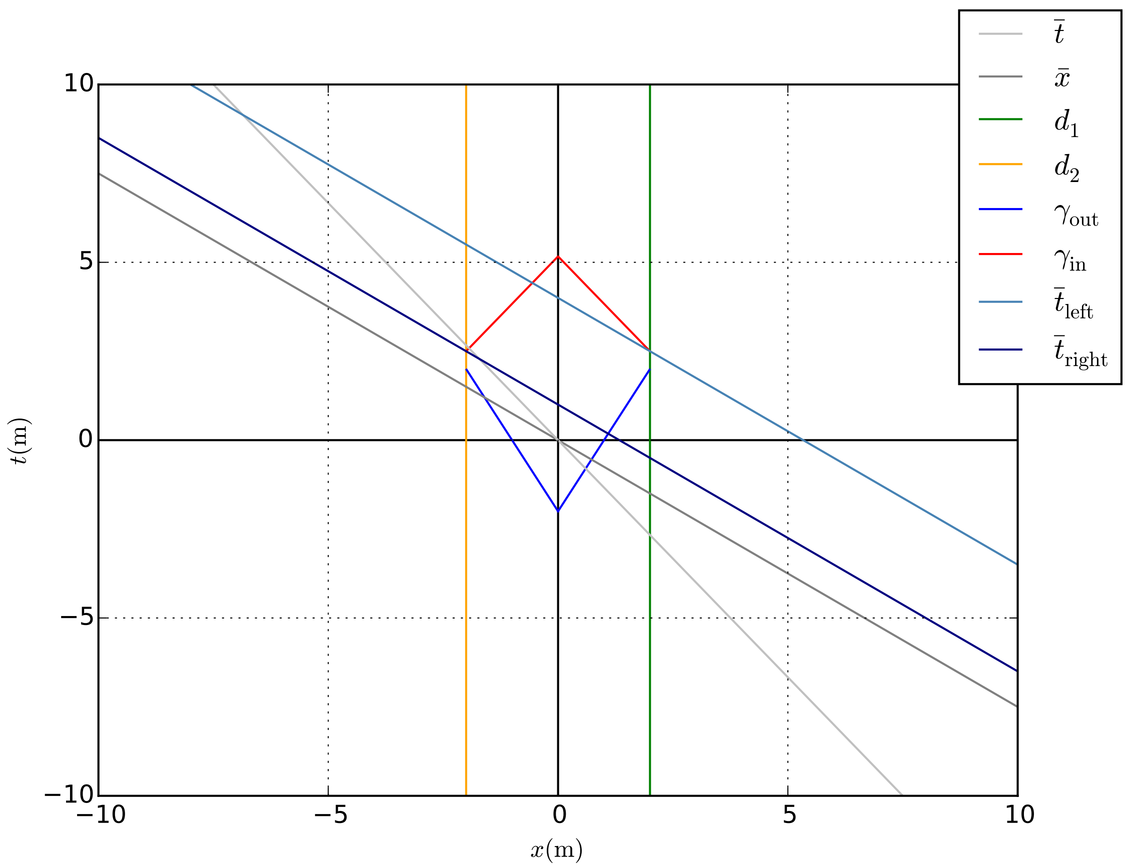
Using this, we find

$$\bar{t}_a = \frac{16\sqrt{7}}{7} \qquad \bar{t}_b = \frac{4\sqrt{7}}{7}$$
$$\bar{x}_a = \frac{-31\sqrt{7}}{14} \qquad \bar{x}_b = \frac{\sqrt{7}}{14}$$

This gives us a distance of

$$\Delta\bar{s}^2 = -(\Delta\bar{t})^2 + (\Delta\bar{x})^2 = 16\,\mathrm{m}^2,$$

which is of course what we expect.



Exercise 5

6 Show that Equation (Schutz 1.2) contains only $M_{\alpha\beta} + M_{\beta\alpha}$ when $\alpha \neq \beta$, not $M_{\alpha\beta}$ and $M_{\beta\alpha}$ independently. Argue that this enables us to set $M_{\alpha\beta} = M_{\beta\alpha}$ without loss of generality.

When we expand the summation in (Schutz 1.2), there is no point where

$$\mathrm{d}\bar{s}^2 = \ldots + M_{\alpha\alpha}(\mathrm{d}x^\alpha)^2 + M_{\alpha\alpha}(\mathrm{d}x^\alpha)^2 + \ldots$$

occurs, because a double summation only contains $M_{\alpha\alpha}$ once. If it did, we could absorb the two $M_{\alpha\beta}$ terms into a single one. Therefore we can assert the first point.

Now we consider the second point. If we expand the summation, assuming now that an $M_{\alpha\beta}$ and $M_{\beta\alpha}$ term only occur when $\alpha \neq \beta$, then we see

$$\begin{aligned}
\mathrm{d}\bar{s}^2 &= \ldots + M_{\alpha\beta}(\mathrm{d}x^\alpha)(\mathrm{d}x^\beta) + M_{\beta\alpha}(\mathrm{d}x^\beta)(\mathrm{d}x^\alpha) + \ldots \\
&= \ldots + (M_{\alpha\beta} + M_{\beta\alpha})\Big[(\mathrm{d}x^\alpha)(\mathrm{d}x^\beta)\Big] + \ldots \\
&= \ldots + \mathbf{X}\Big[(\mathrm{d}x^\alpha)(\mathrm{d}x^\beta)\Big] + \ldots .
\end{aligned}$$

Now, what really matters in this summation is the value of $\mathbf{X} = M_{\alpha\beta} + M_{\beta\alpha}$, not the individual values of $M_{\alpha\beta}$ and $M_{\beta\alpha}$. Therefore we can *choose*, without loss of generality, $M_{\alpha\beta} = M_{\beta\alpha} = \mathbf{X}/2$, thereby asserting the second point.

7 In the discussion leading up to Equation (Schutz 1.2), assume that the coordinates of $\bar{\mathcal{O}}$ are given as the following linear combinations of those $\mathcal{O}$:

$$\begin{aligned}
\bar{t} &= \alpha t + \beta x, \\
\bar{x} &= \mu t + \nu x, \\
\bar{y} &= a y, \\
\bar{z} &= b z,
\end{aligned}$$

where α, β, μ, ν, a, and b may be functions of the velocity $\vec{v}$ of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$, but they do not depend on the coordinates. Find the values of $M_{\alpha\beta}$ of Equation (Schutz 1.2).

$$
\begin{aligned}
\mathrm{d}\bar{s}^2 &= -(\mathrm{d}\bar{t})^2 + (\mathrm{d}\bar{x})^2 + (\mathrm{d}\bar{y})^2 + (\mathrm{d}\bar{z})^2 \\
&= -(\alpha\,\mathrm{d}t + \beta\,\mathrm{d}x)^2 + (\mu\,\mathrm{d}t + \nu\,\mathrm{d}x)^2 + (a\,\mathrm{d}y)^2 + (b\,\mathrm{d}z)^2 \\
&= -\alpha^2\,\mathrm{d}t^2 - \alpha\beta\,\mathrm{d}t\,\mathrm{d}x - \beta^2\,\mathrm{d}x^2 + \mu^2\,\mathrm{d}t^2 + \mu\nu\,\mathrm{d}t\,\mathrm{d}x + \nu^2\,\mathrm{d}x^2 + a^2\,\mathrm{d}y^2 + b^2\,\mathrm{d}z^2 \\
&= (\mu^2 - \alpha^2)\,\mathrm{d}t^2 + (\mu\nu - \alpha\beta)\,\mathrm{d}t\,\mathrm{d}x + (\nu^2 - \beta^2)\,\mathrm{d}x^2 + a^2\,\mathrm{d}y^2 + b^2\,\mathrm{d}z^2 \\
M_{00} &= \mu^2 - \alpha^2 \\
M_{01} &= M_{10} = \frac{\mu\nu - \alpha\beta}{2} \\
M_{11} &= \nu^2 - \beta^2 \\
M_{22} &= a^2 \\
M_{33} &= b^2,
\end{aligned}
$$

and all other $M_{\alpha\beta} = 0$.

8

(a) Derive Equation (Schutz 1.3) from (Schutz 1.2) for general $M_{\alpha\beta}$.

Equation (Schutz 1.3) is just an expansion of the summation in (Schutz 1.2).

We start by taking out the $\mathrm{d}t^2$ term, which corresponds to $\alpha = \beta = 0$, which gives us

$$\mathrm{d}\bar{s}^2 = M_{00}(\mathrm{d}t)^2 + \ldots,$$

now we use the equivalence of $\mathrm{d}t$ and $\mathrm{d}r$ to make the substitution

$$\mathrm{d}\bar{s}^2 = M_{00}(\mathrm{d}r)^2 + \ldots.$$

For the middle terms, we use the fact that $M_{\alpha\beta} = M_{\beta\alpha}$, and look at only the terms where *one* of α and β is zero. The symmetry means we can write $M_{0i} = M_{i0}$, and pull out a 2 because there are twice as many terms, giving us

$$
\begin{aligned}
\mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\
&\quad + 2\left(\sum_{i=1}^{3} M_{0i}(\mathrm{d}x^i)(\mathrm{d}t)\right) \\
&\quad + \ldots.
\end{aligned}
$$

Now we use the equivalence of $\mathrm{d}t$ and $\mathrm{d}r$ once again, and pull the term out of the sum, giving us

$$\begin{aligned}\mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &\quad + 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &\quad + \ldots .\end{aligned}$$

Finally, we simply include the terms which have not yet been accounted for, which are all the *spacial-only* terms, which arrives us back at Equation (Schutz 1.3):

$$\begin{aligned}\mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &\quad + 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &\quad + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,.\end{aligned}$$

(b) Since $\mathrm{d}\bar{s}^2 = 0$ in Equation (Schutz 1.3), for *any* $\mathrm{d}x^i$, replace $\mathrm{d}x^i$ with $-\,\mathrm{d}x^i$, and subtract that result from the original equation. This will establish that $M_{0i} = 0$.

$$\begin{aligned}\mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &\quad - 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &\quad + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,.\end{aligned}$$

$$\begin{aligned}\mathrm{d}\bar{s}^2 - \mathrm{d}\bar{s}^2 = 0 &= \cancel{0\,M_{00}(\mathrm{d}r)^2} \\ &\quad + 4\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &\quad + \cancel{0\sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}\,.\end{aligned}$$

$$0 = \cancel{4}\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\cancel{\mathrm{d}r}$$

Now there are two possibilities. In one case, $\mathrm{d}x^i \equiv 0$, but that is a trivial solution and in general is not true.

The other case is that $M_{0i} \equiv 0$, which means we can simplify Equation (Schutz 1.3) to

$$\begin{aligned} \mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \end{aligned}$$

(c) Use the result of part (b) with $\mathrm{d}\bar{s}^2 = 0$ to establish Equation (Schutz 1.4b).

$$\begin{aligned} \mathrm{d}\bar{s}^2 = 0 &= M_{00}(\mathrm{d}r)^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ \implies -M_{00}(\mathrm{d}r)^2 &= \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \end{aligned}$$

now if we expand $(\mathrm{d}r)^2$, we see that there can only be non-zero M_{ij} when $i = j$, and so

$$\begin{aligned} -M_{00}\Big((\mathrm{d}x^2) + (\mathrm{d}y^2) + (\mathrm{d}z^2)\Big) &= \sum_{i=1}^{3} M_{ii}(\mathrm{d}x^i)^2 \\ \implies -(M_{00})\delta_{ij} &= M_{ij}, \end{aligned}$$

which is simply Equation (Schutz 1.4b).

9 Explain why the line $\mathcal{PL}$ in Figure 1.7 is drawn in the manner described in the text.

10 For the pairs of events whose coordinates (t, x, y, z) in some frame are given below, classify their separations as timelike, spacelike, or null.

(a) $(0, 0, 0, 0)$ and $(-1, 1, 0, 0)$:

$$\mathrm{d}s^2 = -(0+1)^2 + (0-1)^2 + (0-0)^2 + (0-0)^2 = -1 + 1 + 0 + 0 = 0 \implies \text{null}$$

(b) $(1, 1, -1, 0)$ and $(-1, 1, 0, 2)$:

$$\mathrm{d}s^2 = -(1+1)^2 + (1-1)^2 + (-1-0)^2 + (0-2)^2 = -4 + 0 + 1 + 4 = 1 \implies \text{spacelike}$$

(c) $(6, 0, 1, 0)$ and $(5, 0, 1, 0)$:

$$\mathrm{d}s^2 = -(6-5)^2 + (0-0)^2 + (1-1)^2 + (0-0)^2 = -1 + 0 + 0 + 0 = -1 \implies \text{timelike}$$

(d) $(-1, 1, -1, 1)$ and $(4, 1, -1, 6)$:

$$\mathrm{d}s^2 = -(-1-4)^2 + (1-1)^2 + (-1+1)^2 + (1-6)^2 = -25 + 0 + 0 + 25 = 0 \implies \text{null}$$

11 Show that the hyperbolae $-t^2 + x^2 = a^2$ and $-t^2 + x^2 = -b^2$ are asymptotic to the lines $t = \pm x$, regardless of a and b.

We will generalize a and $-b$ with a new constant, $\alpha \in \mathbb{R}$, and so we have: $-t^2 + x^2 = \alpha^2$. Now if we solve for t, we get $t = \pm\sqrt{x^2 - \alpha^2}$.

Now take the limit of t as $x \to \infty$ (or $-\infty$, they are equivalent since x is real and squared), which gives us:

$$\lim_{x\to\infty} t = \lim_{x\to\infty} \pm\sqrt{x^2-\alpha^2} = \pm\sqrt{x^2} = \pm x.$$

Note that we dropped the α^2 term in the limit, as it was being subtracted from a number approaching infinity, and was therefore negligible.

12

(a) Use the fact that the tangent to the hyperbola $\mathcal{DB}$ in Figure 1.14 is the line of simultaneity for $\bar{\mathcal{O}}$ to show that the time interval $\mathcal{AE}$ is shorter than the time recorded on $\bar{\mathcal{O}}$'s clock as it moved from $\mathcal{A}$ to $\mathcal{B}$.

If we look at the figure, we see that $\mathcal{AD}$ and $\mathcal{AB}$ lie along the same hyperbola. This means that when $\mathcal{O}$ measures $\mathrm{d}t = \mathcal{AD}$, and $\bar{\mathcal{O}}$ measures $\mathrm{d}\bar{t} = \mathcal{AB}$, the two measurements are the same. Since $\mathrm{d}t = \mathcal{AE}$ is clearly shorter than $\mathrm{d}t = \mathcal{AD}$, then $\mathrm{d}t = \mathcal{AD} < \mathrm{d}\bar{t} = \mathcal{AB}$.

(b) Calculate that

$$(\mathrm{d}s^2)_{\mathcal{AC}} = (1-v^2)(\mathrm{d}s^2)_{\mathcal{AB}}$$

$$\begin{aligned}
(\mathrm{d}s^2)_{\mathcal{AC}} &= -(\mathrm{d}t)^2_{\mathcal{AC}} \\
(\mathrm{d}s^2)_{\mathcal{AB}} &= (\mathrm{d}\bar{s}^2)_{\mathcal{AB}} \\
&= -(\mathrm{d}\bar{t})^2_{\mathcal{AB}} \\
&= -(\gamma(\mathrm{d}t - v\,\mathrm{d}x))^2 = -(\gamma(\mathrm{d}t - v\cdot 0))^2 = -(\gamma\,\mathrm{d}t)^2 = \gamma^2[-(\mathrm{d}t)^2] \\
&= \gamma^2(\mathrm{d}s^2)_{\mathcal{AC}} = \frac{(\mathrm{d}s^2)_{\mathcal{AC}}}{1-v^2} \\
\implies (\mathrm{d}s^2)_{\mathcal{AC}} &= (1-v^2)(\mathrm{d}s^2)_{\mathcal{AB}}
\end{aligned}$$

13 The Half-life of the elementary particle called the π-meson (or pion) is $2.5\times10^{-8}\,\mathrm{s}$ when the pion is at rest relative to the observer measuring its decay time. Show, by the principle of relativity, that pions moving at speed $v = 0.999$ must have a half-life of $5.6\times10^{-7}\,\mathrm{s}$, as measured by an observer at rest.

$$\mathrm{d}t = \gamma\,\mathrm{d}\bar{t} = \frac{2.5\times10^{-8}\,\mathrm{s}}{\sqrt{1-0.999^2}} \approx 5.59\times10^{-7}\,\mathrm{s}$$

14 Suppose the velocity $\mathbf{v}$ of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$ is small, $|\mathbf{v}| \ll 1$. Show that the time dilation, Lorentz contraction, and velocity-addition formulae can be approximated by respectively:

(a) $\mathrm{d}t \approx (1+\frac{1}{2}v^2)\,\mathrm{d}\bar{t}$

$$\gamma = (1-v^2)^{-1/2} = \sum_{k=0}^{\infty}\binom{-1/2}{k}x^k = 1 + (-1/2)(-v^2) + \frac{(-1/2)(-1/2-1)}{2!}(-v^2)^2 + \ldots \approx 1 + \frac{1}{2}v^2$$

$$\mathrm{d}t = \gamma\,\mathrm{d}\bar{t} \approx \left(1+\frac{1}{2}v^2\right)\mathrm{d}\bar{t}$$

(b) $\mathrm{d}x \approx (1-\frac{1}{2}v^2)\,\mathrm{d}\bar{x}$

$$\begin{aligned}
\gamma^{-1} &= (1-v^2)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + (1/2)(-v^2) + \frac{(1/2)(1/2-1)}{2!}(-v^2)^2 + \ldots \approx 1 - \frac{1}{2}v^2 \\
\mathrm{d}x &= \gamma^{-1}\,\mathrm{d}\bar{x} \approx \left(1 - \frac{1}{2}v^2\right)\mathrm{d}\bar{x}
\end{aligned}$$

(c) $W' \approx W + v - Wv(W+v)$ (with $|W| \ll 1$ as well)

$$\begin{aligned}
W' &= \frac{W+v}{1+Wv} = (W+v)(1+Wv)^{-1} \\
(1+Wv)^{-1} &= \sum_{k=0}^{\infty} \binom{-1}{k}(Wv)^k = 1 - Wv + \frac{1}{2}\cdot 1(1+1)(Wv)^2 - \frac{1}{6}\cdot 1(1+1)(1+2)(Wv)^3 + \ldots \\
&\approx 1 - Wv + (Wv)^2 \\
W' &\approx (W+v)(1 - Wv + (Wv)^2) = W + v - Wv(W+v) + (Wv)^2(W+v) \\
&\approx W + v - Wv(W+v)
\end{aligned}$$

What are the relative errors in these approximations when $|\mathbf{v}| = W = 0.1$?

TODO

15 Suppose that the velocity $\mathbf{v}$ of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$ is nearly that of light, $|\mathbf{v}| = 1 - \varepsilon$, $0 < \varepsilon \ll 1$. Show that the same formulae of Exercise 14 become

(a) $\mathrm{d}t \approx \mathrm{d}\bar{t}/\sqrt{2\varepsilon}$

$$\begin{aligned}
v = 1 - \varepsilon &\implies v^2 = (1-\varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2 \\
\implies 1 - v^2 &= 1 - (1 - 2\varepsilon + \varepsilon^2) = 2\varepsilon - \varepsilon^2 = 2\varepsilon\left(1 - \frac{\varepsilon}{2}\right) \\
\gamma &= (1-v^2)^{-1/2} = \left(2\varepsilon\left(1 - \frac{\varepsilon}{2}\right)\right)^{-1/2} = \frac{1}{\sqrt{2\varepsilon}}\left(1 - \frac{\varepsilon}{2}\right)^{-1/2} \approx \frac{1}{\sqrt{2\varepsilon}} \\
\mathrm{d}t &= \gamma\,\mathrm{d}\bar{t} \approx \frac{\mathrm{d}\bar{t}}{\sqrt{2\varepsilon}}
\end{aligned}$$

(b) $\mathrm{d}x \approx \mathrm{d}\bar{x}\,\sqrt{2\varepsilon}$

$$\begin{aligned}
v = 1 - \varepsilon &\implies v^2 = (1-\varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2 \\
\implies 1 - v^2 &= 1 - (1 - 2\varepsilon + \varepsilon^2) = 2\varepsilon - \varepsilon^2 = 2\varepsilon\left(1 - \frac{\varepsilon}{2}\right) \\
\gamma^{-1/2} &= (1-v^2)^{1/2} = \left(2\varepsilon\left(1 - \frac{\varepsilon}{2}\right)\right)^{1/2} = \sqrt{2\varepsilon}\left(1 - \frac{\varepsilon}{2}\right)^{1/2} \approx \sqrt{2\varepsilon} \\
\mathrm{d}x &= \gamma^{-1}\,\mathrm{d}\bar{x} \approx \mathrm{d}\bar{t}\,\sqrt{2\varepsilon}
\end{aligned}$$

(c) $W' \approx 1 - \varepsilon(1-W)/(1+W)$

TODO

What are the relative errors on these approximations when $\varepsilon = 0.1$ and $W = 0.9$?

TODO

16 Use the Lorentz transformation, Equation 1.12, to derive (a) the time dilation, and (b) the Lorentz contraction formulae. Do this by identifying pairs of events where the separations (in time or space) are to be compared, and then using the Lorentz transformation to accomplish the algebra that the invariant hyperb b olae had been used for in the text.

(a) To derive the time dilation formula, we choose two events that occur at $x = c$, and times t_1 and t_2. Thus, from $\mathcal{O}$'s frame, the time elapsed between these two events is $\Delta t = t_2 - t_1$, and the distance between them is $\Delta x = 0$. Another observer, $\bar{\mathcal{O}}$, moves with some velocity v relative to $\mathcal{O}$. As it passes through the lines $t = t_1$ and $t = t_2$, its clock moves forward by a time $\Delta\tau = \bar{t}_2 - \bar{t}_1$. We now use the Lorentz transformation to write $\Delta\tau$ in terms of $\mathcal{O}$'s coordinates.

$$\begin{aligned}\Delta\tau = \bar{t}_2 - \bar{t}_1 &= \gamma\big[(t_2 - vx_2) - (t_1 - vx_1)\big] = \gamma\big[(t_2 - t_1) + (vx_1 - vx_2)\big] \\ &= \gamma[\Delta t + v\Delta x] = \gamma[\Delta t + v \cdot 0] \\ &= \gamma\Delta t\end{aligned}$$

and thus we have arrived at the formula for time dilation.

(b) To derive the Lorentz contraction formula, we take a slightly different approach. In the $\mathcal{O}$ frame, a stick lies parallel to x, such that its length $\ell = x_2 - x_1$. In this frame, the world lines of the two ends of the stick form vertical lines. Another observer, $\bar{\mathcal{O}}$, moves with a velocity v, relative to $\mathcal{O}$. Two events, $\mathcal{A}$ and $\mathcal{B}$ occur on either end of the stick, such that $\bar{\mathcal{O}}$ observes the two events to be simultaneous. Thus, from the $\bar{\mathcal{O}}$ frame, the events are located a distance $\Delta\bar{x} = \bar{\ell}$ apart, and $\Delta\bar{t} = 0$. However, from the $\mathcal{O}$ frame, the events occur a distance $\Delta x = \ell$ apart, and a time separation $\Delta t \neq 0$.

$$\begin{aligned}\ell = x_2 - x_1 &= \gamma\big[(\bar{x}_2 - v\bar{t}_2) - (\bar{x}_1 - v\bar{t}_1)\big] = \gamma\big[(\bar{x}_2 - \bar{x}_1) + v(\bar{t}_1 - \bar{t}_2)\big] = \gamma\bar{\ell} \\ \implies \bar{\ell} &= \frac{\ell}{\gamma}\end{aligned}$$

17 A lightweight pole, 20 m long, lies on the ground next to a barn 15 m long. An Olympic athlete picks up the pole, carries it far away, and runs with it toward the end of the barn at a speed 0.8. His friend remains at rest, standing by the door of the barn. Attempt all parts of this question, even if you can't answer some.

(a) How long does the friend measure the pole to be, as it approaches the barn?

We use the Lorentz contraction equation to find the length the friend measures.

$$\bar{\ell} = \ell/\gamma = \ell\sqrt{1 - v^2} = 20\,\mathrm{m}\sqrt{1 - 0.8^2} = 12\,\mathrm{m}$$

(b) The barn door is initially open and, immediately after the runner and pole are entirely inside the barn, the friend shuts the door. How long after the door is shut does the front of the pole hit the other end of the barn, as measured by the friend? Compute the interval between the events of shutting the door and hitting the wall. Is it spacelike, timelike, or null?

From the runner's point of view, we must consider the length contraction of the barn

(c) In the reference frame of the runner, what is the length of the barn and the pole?

(d) Does the runner believe that the pole is entirely inside the barn when its front hits the end of the barn? Can you explain why?

(e) After the collision, the pole and runner come to rest relative to the barn. From the friend's point of view, the 20 m pole is now inside a 15 m barn, since the barn door was shut before the pole stopped. How is this possible? Alternatively, from the runner's point of view, the collision should have stopped the pole *before* the door closed, so the door could not be closed at all. Was or was not the door closed with the pole inside?

(f) Draw a spacetime diagram from the friend's point of view and use it to illustrate and justify all your conclusions.

18

(a) The Einstein velocity-addition law, Equation 1.13, has a simpler form if we introduce the concept of the *velocity parameter* u, defined by the equation

$$v = \tanh u.$$

Notice that for $-\infty < u < \infty$, the velocity is confined to the acceptable limits $-1 < v < 1$. Show that if

$$v = \tanh u$$

and

$$w = \tanh U,$$

then Equation 1.13 implies

$$w' = \tanh(u + U).$$

This means that velocity parameters add linearly.

There exists an identity:

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

If we simply use $x = u$ and $y = U$, then we arrive at

$$\tanh(u + U) = \frac{\tanh(u) + \tanh(U)}{1 + \tanh(u)\tanh(U)} = w'$$

(b) Use this to solve the following problem. A star measures a second star to be moving away at speed $v = 0.9$. The second star measures a third to be receding in the same direction at 0.9. Similarly, the third measures a fourth, and so on, up to some large number N of stars. What is the velocity of the Nth star relative to the first? Give an exact answer and an approximation useful for large N.

Let w^N be the velocity of the Nth star relative to the original star, which we will call star 0. We will use an induction proof to find an expression for w^N. The base case is trivial, $w^0 = 0$, as the star does not move relative to itself. For the next case, $w^1 = v$, we still aren't really doing velocity addition, so we will skip to the w^2 case, where things get interesting, though we will later show that the general expression holds for w^0 and w^1.

For w^2, we simply use the Einstein velocity-addition law:

$$w^2 = \tanh(u + U) = \tanh\Big(\tanh^{-1} v + \tanh^{-1} w^1\Big) = \tanh\Big(2 \tanh^{-1} v\Big).$$

Now I will prove that this is one instance of a general expression, that $w^N = \tanh\Big(N \tanh^{-1} v\Big)$.

$$\begin{aligned}
&w^N = \tanh\Big(N \tanh^{-1} v\Big) \\
\implies &\tanh^{-1} w^N = N \tanh^{-1} v \\
\implies &\tanh^{-1} w^N + \tanh^{-1} v = N \tanh^{-1} v + \tanh^{-1} v \\
\implies &\tanh^{-1} w^N + \tanh^{-1} v = (N+1) \tanh^{-1} v \\
\implies &\tanh\Big(\tanh^{-1} w^N + \tanh^{-1} v\Big) = \tanh\Big((N+1) \tanh^{-1} v\Big) \\
\implies &w^{N+1} = \tanh\Big((N+1) \tanh^{-1} v\Big).
\end{aligned}$$

If you can believe the last step, then this is proof that it works for all N. The last step is saying that, if we have a star N, moving away from star 0 at a speed w^N, and another star $N + 1$, moving away from star N at a speed v, then star $N + 1$ as observed from star 0 is given by the Einstein velocity-addition law, meaning we can rewrite that expression as w^{N+1}.

Now I'd like to go back and show that this works for $N = 0$ and $N = 1$. For $N = 1$, we get

$$w^1 = \tanh\Big(1 \tanh^{-1} v\Big) = v,$$

which is what we would expect, and for $N = 0$, we get

$$w^0 = \tanh\Big(0 \tanh^{-1} v\Big) = 0,$$

which we also expect. So the general expression,

$$w^N = \tanh\Big(N \tanh^{-1} v\Big),$$

holds true for all non-negative integers N. We can also write this more elegantly as

$$w^N = \tanh(Nu).$$

Now we want to consider the behaviour at large N. We first write tanh in its exponential form, as

$$w^N = \frac{1-\exp(-2Nu)}{1+\exp(-2Nu)}.$$

When N is very large, then the exponential in the bottom term goes to zero, allowing us to rewrite it as

$$w^N \approx 1-\exp(-2Nu).$$

We can go a step further. Since $v = 0.9$, $u \approx 1.47$, which we can neglect for large N, and so we finally arrive at

$$w^N \approx 1-\exp(-2N).$$

19

(a) Using the velocity parameter (u) introduced in Exercise 18, show that the Lorentz transformation equations, Equation 1.12, can be put in the form

$$\begin{aligned} \bar{t} &= t\cosh u - x\sinh u & \qquad \bar{y} &= y \\ \bar{x} &= -t\sinh u + x\cosh u & \qquad \bar{z} &= z \end{aligned}$$

We start by putting γ in terms of u.

$$\gamma = (1-v^2)^{-1/2} = (1-\tanh^2 u)^{-1/2} = \frac{1}{\operatorname{sech} u} = \cosh u.$$

Now we can substitute this into the Lorentz transformation equations

$$\begin{aligned} \bar{t} &= \gamma(t - vx) = \cosh u(t - x\tanh u) = t\cosh u - x\sinh u \\ \bar{x} &= \gamma(x - vt) = \cosh u(x - t\tanh u) = x\cosh u - t\sinh u \end{aligned}$$

(b) Use the identity $\cosh^2 u - \sinh^2 u = 1$ to demonstrate the invariance of the interval from these equations.

$$\begin{aligned} \mathrm{d}s^2 &= -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ \mathrm{d}\bar{s}^2 &= -(\mathrm{d}t\cosh u - \mathrm{d}x\sinh u)^2 + (\mathrm{d}x\cosh u - \mathrm{d}t\sinh u)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -\Big(\mathrm{d}t^2\cosh^2 u - \cancel{\mathrm{d}x\,\mathrm{d}t\sinh u\cosh u} + \mathrm{d}x^2\sinh^2 u\Big) \\ &\quad + \Big(\mathrm{d}x^2\cosh^2 u - \cancel{\mathrm{d}t\,\mathrm{d}x\sinh u\cosh u} + \mathrm{d}t^2\sinh^2 u\Big) + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -\,\mathrm{d}t^2\cancel{\Big(\cosh^2 u - \sinh^2 u\Big)} + \mathrm{d}x^2\cancel{\Big(\cosh^2 u - \sinh^2 u\Big)} + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= \mathrm{d}s^2 \end{aligned}$$

(c) Draw as many parallels as you can between the geometry of spacetime and ordinary two-dimensional

Euclidean geometry, where the coordinate transformation analogous to the Lorentz transformation is

$$\bar{x} = +x\cos\theta + y\sin\theta,$$
$$\bar{y} = -x\sin\theta + y\cos\theta.$$

What is the analog of the interval? Of the invariant hyperbolae?

The analog of the interval would be

$$\begin{aligned}
\mathrm{d}\bar{r}^2 = \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 &= (\mathrm{d}x\cos\theta + \mathrm{d}y\sin\theta)^2 + (\mathrm{d}y\cos\theta - \mathrm{d}x\sin\theta)^2 + \\
&= \mathrm{d}x^2\cos^2\theta + \cancel{2\,\mathrm{d}x\,\mathrm{d}y\sin\theta\cos\theta} + \mathrm{d}y^2\sin^2\theta \\
&+ \mathrm{d}y^2\cos^2\theta - \cancel{2\,\mathrm{d}x\,\mathrm{d}y\sin\theta\cos\theta} + \mathrm{d}x^2\sin^2\theta \\
&= \mathrm{d}x^2(\sin^2\theta + \cos^2\theta) + \mathrm{d}y^2(\sin^2\theta + \cos^2\theta) \\
&= \mathrm{d}x^2 + \mathrm{d}y^2
\end{aligned}$$

The analog of the invariant hyperbola would be the invariant circle, as $\bar{x}$ and $\bar{y}$ are both equations of a circle.

20 Write the Lorentz transformation equations in matrix form.

$$\begin{aligned}
\bar{t} &= \gamma(t - vx) & \bar{t} &= \gamma t - \gamma v x + 0y + 0z \\
\bar{x} &= \gamma(x - vt) & \bar{x} &= -\gamma v t + \gamma x + 0y + 0z \\
\bar{y} &= y & \bar{y} &= y \\
\bar{z} &= z & \bar{z} &= z
\end{aligned}$$

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

21

(a) Show that if the two events are timelike separated, there is a Lorentz frame in which they occur at the same point, i.e. at the same spatial coordinate values.

If the two events are timelike separated, then it must be possible to have an object with a worldline which crosses the two points, as it is inside the light cone. If such an object exists, then we can draw a Lorentz frame for it, so its time axis, $\bar{t}$ is that line, meaning $\bar{x} = 0$ for both events.

(b) Similarly, if the two events are spacelike separated, there is a Lorentz frame in which they occur simultaneously.

If the two events are spacelike separated, then it must be possible to draw a coordinate frame where $\bar{x}$ has slope v in $\mathcal{O}$'s frame. This means that $\bar{t} = 0$ for both events, and so they are simultaneous.

Chapter 2

Vector analysis in special relativity

2.1 Definition of a vector

2.2 Vector algebra

2.3 The four-velocity

An object's four velocity, denoted $\vec{U}$, is the vector tangent to its world line, with unit length. This means it extends one unit in time, and zero in space, so it is timelike.

For an *accelerated* particle (which we have not considered up to now), we may not be able to define an inertial frame, but we *can* define a **momentarily comoving reference frame** (MCRF) which, as the name suggests, moves with the same velocity as the observer for an infinitesimal period of time. We can therefore construct a continuous sequence of MCRFs for any object. If an object has MCRF $\mathcal{O}$, then its four-velocity is *defined* to be the basis vector $\vec{e}_0$.

2.4 The four-momentum

Analogous to the three-momentum, we define the four-momentum to be

$$\vec{p} = m\vec{U}. \tag{Schutz 2.19}$$

It has components

$$\vec{p} \underset{\mathcal{O}}{\rightarrow} (E, p^1, p^2, p^3). \tag{Schutz 2.20}$$

Calling p^0 "E" is no accident, it is in fact the energy. There is an interesting consequence to this: since vectors are invariant with respect to reference frame, but vector components are not, this means that the four-momentum does not change in different reference frames, but the energy *does*. One example would be the doppler effect, which causes the color (or energy) of a photon to shift depending on the radial velocity of the source and observer.

2.5 Scalar product

$$\vec{A} \cdot \vec{B} = -(A^0 B^0) + (A^1 B^1) + (A^2 B^2) + (A^3 B^3)$$

2.6 Applications

2.7 Photons

$\vec{x} \cdot \vec{x} = 0$, so we cannot define $\vec{U}$ for photons. We can, however, define $\vec{p}$. Since $\vec{p} \cdot \vec{p} = -m^2$, and photons are massless, we have $\vec{p} \cdot \vec{p} = 0$.

2.8 Further reading

2.9 Exercises

2 Identify the free and dummy indices in the following equations, and write equivalent expressions with different indices. Also, write how many equations are represented by each expression.
Note, I will express the set of free indices by $\mathcal{F}$ and the set of dummy indices as $\mathcal{D}$, and I will use the original index names.

(a) $A^\alpha B_\beta = 5 \implies A^\beta B_\alpha = 5$ (16 equations, $\mathcal{F} = \{\alpha, \beta\}$, $\mathcal{D} = \emptyset$)

(b) $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_\nu A^\nu \implies A^{\bar{\nu}} = \Lambda^{\bar{\nu}}{}_\mu A^\mu$ (4 equations, $\mathcal{F} = \{\bar{\mu}\}$, $\mathcal{D} = \{\nu\}$).

(c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha} \implies T^{\eta\phi\theta} A_\phi C_\theta{}^\zeta = D^{\zeta\eta}$ (16 equations, $\mathcal{F} = \{\alpha, \gamma\}$, $\mathcal{D} = \{\mu, \lambda\}$)

(d) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = G_{\mu\nu} \implies R_{\chi\epsilon} - \frac{1}{2} g_{\chi\epsilon} = G_{\chi\epsilon}$ (16 equations, $\mathcal{F} = \{\mu, \nu\}$, $\mathcal{D} = \emptyset$)

4 Given vectors $\vec{A} \to_{\mathcal{O}} (5, -1, 0, 1)$ and $\vec{B} \to_{\mathcal{O}} (-2, 1, 1, -6)$, find the components in $\mathcal{O}$ of

(a) $-6\vec{A} \to_{\mathcal{O}} (-30, 6, 0, -6)$

(b) $3\vec{A} + \vec{B} \to_{\mathcal{O}} (13, -2, 1, -3)$

(c) $-6\vec{A} + 3\vec{B} \to_{\mathcal{O}} (-36, 9, 3, -24)$

6 Draw a spacetime diagram from $\mathcal{O}$'s reference frame. There are two other frames, $\bar{\mathcal{O}}$ and $\bar{\bar{\mathcal{O}}}$, which are each moving with velocity 0.6 in the $+x$ direction from each respective frame. Plot each frame's basis vectors, as observed by $\mathcal{O}$.

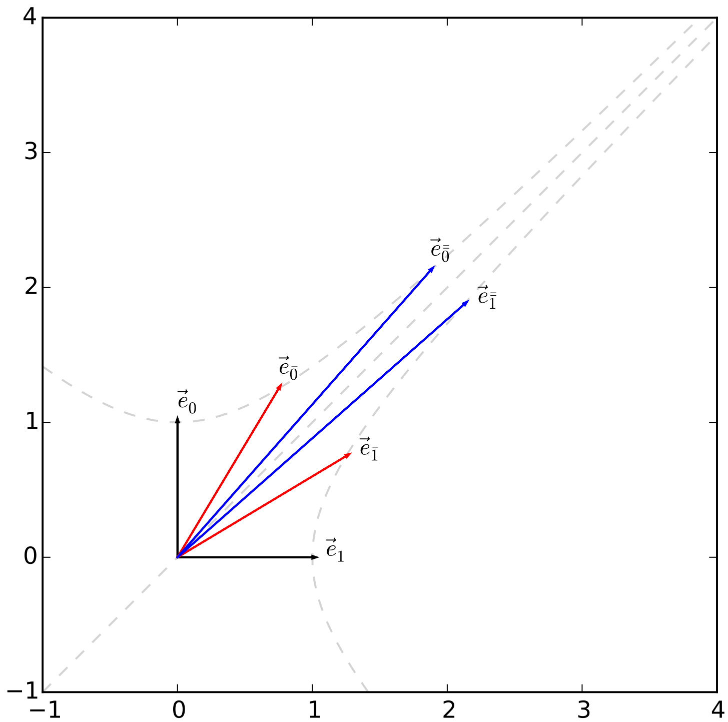


Figure 2.1: Exercise 6

See Figure 2.1.

9 Prove, by writing out all the terms that

$$\sum_{\bar{\alpha}=0}^{3}\left(\sum_{\beta=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)=\sum_{\beta=0}^{3}\left(\sum_{\bar{\alpha}=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)$$

$$\begin{aligned}
\sum_{\bar{\alpha}=0}^{3}\left(\sum_{\beta=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right) &= \sum_{\bar{\alpha}=0}^{3}\left(\Lambda^{\bar{\alpha}}{}_{0}A^{0}\vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}{}_{1}A^{1}\vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}{}_{2}A^{2}\vec{e}_{\bar{\alpha}} + \Lambda^{\bar{\alpha}}{}_{3}A^{3}\vec{e}_{\bar{\alpha}}\right) \\
&= \Lambda^{\bar{0}}{}_{0}A^{0}\vec{e}_{\bar{0}} + \Lambda^{\bar{0}}{}_{1}A^{1}\vec{e}_{\bar{0}} + \Lambda^{\bar{0}}{}_{2}A^{2}\vec{e}_{\bar{0}} + \Lambda^{\bar{0}}{}_{3}A^{3}\vec{e}_{\bar{0}} \\
&+ \Lambda^{\bar{1}}{}_{0}A^{0}\vec{e}_{\bar{1}} + \Lambda^{\bar{1}}{}_{1}A^{1}\vec{e}_{\bar{1}} + \Lambda^{\bar{1}}{}_{2}A^{2}\vec{e}_{\bar{1}} + \Lambda^{\bar{1}}{}_{3}A^{3}\vec{e}_{\bar{1}} \\
&+ \Lambda^{\bar{2}}{}_{0}A^{0}\vec{e}_{\bar{2}} + \Lambda^{\bar{2}}{}_{1}A^{1}\vec{e}_{\bar{2}} + \Lambda^{\bar{2}}{}_{2}A^{2}\vec{e}_{\bar{2}} + \Lambda^{\bar{2}}{}_{3}A^{3}\vec{e}_{\bar{2}} \\
&+ \Lambda^{\bar{3}}{}_{0}A^{0}\vec{e}_{\bar{3}} + \Lambda^{\bar{3}}{}_{1}A^{1}\vec{e}_{\bar{3}} + \Lambda^{\bar{3}}{}_{2}A^{2}\vec{e}_{\bar{3}} + \Lambda^{\bar{3}}{}_{3}A^{3}\vec{e}_{\bar{3}} \\
&= \Lambda^{\bar{0}}{}_{0}A^{0}\vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_{0}A^{0}\vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_{0}A^{0}\vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_{0}A^{0}\vec{e}_{\bar{3}} \\
&+ \Lambda^{\bar{0}}{}_{1}A^{1}\vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_{1}A^{1}\vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_{1}A^{1}\vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_{1}A^{1}\vec{e}_{\bar{3}} \\
&+ \Lambda^{\bar{0}}{}_{2}A^{2}\vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_{2}A^{2}\vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_{2}A^{2}\vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_{2}A^{2}\vec{e}_{\bar{3}} \\
&+ \Lambda^{\bar{0}}{}_{3}A^{3}\vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_{3}A^{3}\vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_{3}A^{3}\vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_{3}A^{3}\vec{e}_{\bar{3}} \\
&= \sum_{\beta=0}^{3}\left(\Lambda^{\bar{0}}{}_{\beta}A^{\beta}\vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_{\beta}A^{\beta}\vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_{\beta}A^{\beta}\vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_{\beta}A^{\beta}\vec{e}_{\bar{3}}\right) \\
&= \sum_{\beta=0}^{3}\left(\sum_{\bar{\alpha}=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)
\end{aligned}$$

11 Let $\Lambda^{\bar{\alpha}}{}_{\beta}$ be the matrix of the Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$, given in Equation 1.12. Let $\vec{A}$ be an arbitrary vector with components (A^0, A^1, A^2, A^3) in frame $\mathcal{O}$.

(a) Write down the matrix of $\Lambda^{\nu}{}_{\bar{\mu}}(-v)$.

Intuitively, it should appear the same as $\Lambda^{\bar{\alpha}}{}_{\beta}$, but with the negative signs removed. More rigorously, it is given by the matrix inverse of $\Lambda^{\bar{\alpha}}{}_{\beta}$, as their product should be the identity matrix. I have used a computer algebra system (Wolfram Alpha) to take the inverse of this matrix symbolically, confirming my suspicion:

$$\Lambda^{\nu}{}_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Find $A^{\bar{\alpha}}$ for all $\bar{\alpha}$.

$$\begin{aligned}
A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta} \\
A^{\bar{0}} &= \gamma(A^0 - vA^1) \\
A^{\bar{1}} &= \gamma(A^1 - vA^0) \\
A^{\bar{2}} &= A^2 \\
A^{\bar{3}} &= A^3
\end{aligned}$$

(c) Verify Equation 2.18 by performing the sum for all values of ν and α.

To simplify things, I do this via matrix multiplication

$$\begin{aligned}
\Lambda^{\bar{\alpha}}{}_{\beta}(v)\Lambda^{\nu}{}_{\bar{\mu}}(-v) &= \begin{pmatrix} \gamma^2 - v^2\gamma^2 & v\gamma^2 - v\gamma^2 & 0 & 0 \\ v\gamma^2 - v\gamma^2 & \gamma^2 - v^2\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2(1-v^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-v^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\nu}{}_{\alpha}
\end{aligned}$$

(d) Write down the Lorentz transformation matrix from $\bar{\mathcal{O}}$ to $\mathcal{O}$, justifying each term.

It should just be $\Lambda^{\nu}{}_{\bar{\mu}}(-v)$. I'm not sure what else to say at this point.

(e) Using the result from part (d), find A^{β} from $A^{\bar{\alpha}}$. How does this relate to Equation 2.18?

$$\begin{aligned}
\Lambda^{\beta}{}_{\bar{\alpha}}A^{\bar{\alpha}} &= \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma(A^0 - vA^1) \\ \gamma(A^1 - vA^0) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma^2(A^0 - vA^1) + v\gamma^2(A^1 - vA^0) + 0 + 0 \\ v\gamma^2(A^0 - vA^1) + \gamma^2(A^1 - vA^0) + 0 + 0 \\ A^2 \\ A^3 \end{pmatrix} \\
&= \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) + A^1(v\gamma^2 - v\gamma^2) \\ A^0(v\gamma^2 - v^2\gamma^2) + A^1(\gamma^2 - v\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) \\ A^1(\gamma^2 - v^2\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = A^{\beta}
\end{aligned}$$

Since $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta}(v)$, this goes to show that $\Lambda^{\nu}{}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}{}_{\alpha}(-v)A^{\lambda} = A^{\lambda} \implies \Lambda^{\nu}{}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}{}_{\alpha}(-v) = \delta^{\nu}{}_{\alpha}$.

(f) Verify in the same manner as (c) that

$$\Lambda^{\nu}{}_{\bar{\beta}}(v)\Lambda^{\bar{\alpha}}{}_{\nu}(-v) = \delta^{\bar{\alpha}}{}_{\bar{\beta}}$$

My matrix multiplication approach will just give me the same result as before. Perhaps another approach was intended?

(g) Establish that

$$\vec{e}_\alpha = \Lambda^{\bar{\beta}}{}_\alpha \vec{e}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_\alpha \Lambda^\nu{}_{\bar{\beta}} \vec{e}_\nu = \delta^\nu{}_\alpha \vec{e}_\nu$$
$$A^{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_\alpha A^\alpha = \Lambda^{\bar{\beta}}{}_\alpha \Lambda^\alpha{}_{\bar{\mu}} A^{\bar{\mu}} = \delta^{\bar{\beta}}{}_{\bar{\mu}} A^{\bar{\mu}}$$

14 The following matrix gives a Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$:

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a) What is the velocity of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$?

This would correspond to a Lorentz boost along the z-axis, meaning

$$\Lambda^{\bar{\alpha}}{}_\beta(v) = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix},$$

and thus we have $\gamma = 1.25$ and $-v\gamma = 0.75$. Solving for v, we get

$$-v\gamma = \frac{3}{4} \implies v = -\frac{3}{4\gamma} = -\frac{3 \cdot 4}{4 \cdot 5} = -\frac{3}{5}.$$

So $\bar{\mathcal{O}}$ is moving with speed 0.6 relative to the $-z$-axis of $\mathcal{O}$.

(b) What is the inverse matrix to the given one?

Numerically, it comes out to be

$$\begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix},$$

which makes sense, when you consider that the inverse matrix should be a Lorentz transformation with the velocity negated.

(c) Find the components in $\mathcal{O}$ of $\vec{A} \to_{\bar{\mathcal{O}}} (1,2,0,0)$.

$$\vec{A} \underset{\mathcal{O}}{\to} \begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2 \\ 0 \\ -0.75 \end{pmatrix}$$

15

(a) Compute the four-velocity components in $\mathcal{O}$ of a particle whose speed is v in the $+x$-direction relative to $\mathcal{O}$, using the Lorentz transformation.

$$\begin{aligned} \vec{U} &= \vec{e}_{\bar{0}} \\ U^\alpha &= \Lambda^\alpha{}_{\bar{\beta}} (\vec{e}_{\bar{0}})^{\bar{\beta}} = \Lambda^\alpha_{\bar{0}}, \\ U^0 &= \gamma \\ U^1 &= v\gamma \\ U^2 &= U^3 = 0 \end{aligned}$$

(b) Generalize to arbitrary velocities $\boldsymbol{v}$, where $|v| < 1$.

$$\Lambda^\alpha{}_{\bar{\beta}}(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

$$U^0 = \gamma \quad U^1 = \gamma v_x \quad U^2 = \gamma v_y \quad U^3 = \gamma v_z$$

(c) Use this result to express $\boldsymbol{v}$ as a function of the components $\{U^\alpha\}$.

$$\begin{aligned} \boldsymbol{v} &= v_x \vec{e}_1 + v_y \vec{e}_2 + v_z \vec{e}_3 \\ v_i &= \frac{U^i}{\gamma} \\ \boldsymbol{v} &= \frac{1}{\gamma} U^i \vec{e}_i \end{aligned}$$

(d) Find the three-velocity $\boldsymbol{v}$ of a particle with four-velocity components $(2,1,1,1)$.

$U^0 = \gamma = 2$, and $U^i = 1$, so

$$\boldsymbol{v} = \frac{1}{2} \vec{e}_i$$

17

Not sure how to approach this problem.

(a) Prove that any timelike vector $\vec{U}$ for which $U^0 > 0$ and $\vec{U} \cdot \vec{U} = -1$ is the four-velocity of *some* world line.

(b) Use this to prove that for any timelike vector $\vec{V}$ there is a Lorentz frame in which the $\vec{V}$ has zero spatial components.

19 A body is uniformly accelerated if the four-vector $\vec{a}$ has constant spatial direction and magnitude, $\vec{a} \cdot \vec{a} = \alpha^2 \geqslant 0$.

(a) Show that this implies the components of $\vec{a}$ in the body's MCRF are all constant, and that these are equivalent to the Galilean "acceleration".

We normalize the vector $\vec{a}$ by dividing each of its terms by the magnitude of the vector, so

$$\frac{a^\lambda}{\alpha}.$$

Since α is constant, and also the *direction* is constant, this means that the above expression is *also* constant, as the normalized components tell you about the direction. If we multiply a constant by a constant, we should still get a constant, so we multiply the above expression by α, getting a^λ to be constant.

In the MCRF of an object, $\mathrm{d}\tau = \mathrm{d}t$, and so we can write

$$\vec{a} = \frac{\mathrm{d}\vec{U}}{\mathrm{d}t} = \left(0, \frac{\mathrm{d}U^1}{\mathrm{d}t}, \frac{\mathrm{d}U^2}{\mathrm{d}t}, \frac{\mathrm{d}U^3}{\mathrm{d}t}\right),$$

which is analogous to the Galilean acceleration.

(b) A body is uniformly accelerated with $\alpha = 10\,\mathrm{m/s^2}$. It starts from rest, and falls for a time t. Find its speed as a function of t, and find the time to reach $v = 0.999$.

$$
\begin{aligned}
\vec{U} &\underset{\text{MCRF}}{\rightarrow} (1,0,0,0) \\
&\underset{\mathcal{O}}{\rightarrow} (\gamma, \gamma v, 0, 0) \\
\frac{\mathrm{d}\vec{U}}{\mathrm{d}\tau} &\underset{\text{MCRF}}{\rightarrow} (0,\alpha,0,0) \\
&\underset{\mathcal{O}}{\rightarrow} (\gamma, \gamma\alpha, 0, 0) \\
U^x &= \int_0^t \frac{\mathrm{d}U^x}{\mathrm{d}\tau}\mathrm{d}\tau = \int_0^t \gamma\alpha\frac{\mathrm{d}t}{\gamma} = \int_0^t \alpha\,\mathrm{d}t = \alpha t \\
&= \gamma v = \frac{v}{\sqrt{1-v^2}} \\
v^2 &= (\alpha t)^2(1-v^2) = (\alpha t)^2 - (\alpha t v)^2 \\
v^2(1+(\alpha t)^2) &= (\alpha t)^2 \\
v^2 &= \frac{(\alpha t)^2}{1+(\alpha t)^2} \implies v = \sqrt{\frac{(\alpha t)^2}{1+(\alpha t)^2}}
\end{aligned}
$$

To find the time to reach $v = 0.999$, we go back to the expression $\gamma v = \alpha t$, solve for t, and substitute for v and α. Note that in natural units, $\alpha = 10\,\mathrm{m/s^2}c^{-2} \approx 1.11 \times 10^{-16}\,\mathrm{m}^{-1}$

$$t = \frac{v}{\alpha\sqrt{1-v^2}} = \frac{0.999}{1.11 \times 10^{-16}\,\mathrm{m}^{-1}\sqrt{1-0.999^2}} \approx 2.01 \times 10^{17}\,\mathrm{m}.$$

24 Show that a positron and electron cannot annihilate to form a single photon, but they can annihilate to form two photons.

We consider the center of momentum frame, where $\sum \vec{p}_{(i)} \rightarrow_{\text{CM}} (E_{\text{total}}, 0, 0, 0)$. Without loss of generality, we assume that the velocities of the two particles are equal and opposite, such that

$$\vec{p}_{e^+} \rightarrow_{\text{CM}} m_e(\gamma, \gamma v, 0, 0), \qquad \vec{p}_{e^-} \rightarrow_{\text{CM}} m_e(\gamma, -\gamma v, 0, 0).$$

The photon they create will have to have a momentum of $\vec{p}_{\gamma,\text{single}} \rightarrow_{\text{CM}} (h\nu, h\nu, 0, 0)$. By conservation of four-momentum, we have

$$
\begin{aligned}
\vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma,\text{single}} \\
(\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= \vec{p}_{\gamma,\text{single}} \cdot \vec{p}_{\gamma,\text{single}} \\
(\vec{p}_{e^+} \cdot \vec{p}_{e^+}) + (\vec{p}_{e^-} \cdot \vec{p}_{e^-}) + (\vec{p}_{e^+} \cdot \vec{p}_{e^-}) &= 0 \\
-m_e^2 - m_e^2 - m_e^2 &= 0 \implies m_e = 0!
\end{aligned}
$$

Since we know that m_e is in fact non-zero, this cannot possibly happen.

Now consider the scenario wherein two photons are created, moving in opposite directions. Then they would have momenta: $\vec{p}_{\gamma,1} \rightarrow_{\text{CM}} (h\nu, h\nu, 0, 0)$ and $\vec{p}_{\gamma,2} \rightarrow_{\text{CM}} (h\nu, -h\nu, 0, 0)$. Invoking conservation of four-momentum as before, we get

$$\begin{aligned}
\vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma,1} + \vec{p}_{\gamma,2} \\
(\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \cdot (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \\
-3m_e^2 &= (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,1}) + (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,2}) + (\vec{p}_{\gamma,2} \cdot \vec{p}_{\gamma,2}) \\
&= 0 + (-h^2\nu^2 - h^2\nu^2) + 0 = -2h^2\nu^2,
\end{aligned}$$

so we end up with $3m_e^2 = 2h^2\nu^2$, meaning two photons are produced with $E^2 = \frac{3}{2}m_e^2$, which is entirely reasonable.

25

(a) Consider a frame $\bar{\mathcal{O}}$ moving with a speed v along the x-axis of $\mathcal{O}$. Now consider a photon moving at an angle θ from $\mathcal{O}$'s x-axis. Find the ratio of its frequency in $\bar{\mathcal{O}}$ and in $\mathcal{O}$.

We must first construct the particle's four-momentum. In the case where the photon was moving along the x-axis (see Section 2.7), it had been found that the four-momentum was

$$\vec{p} \underset{\mathcal{O}}{\to} (E, E, 0, 0),$$

as this satisfied

$$\vec{p} \cdot \vec{p} = -E^2 + E^2 = 0. \tag{Schutz 2.37}$$

Now that the photon is moving at an angle θ from the x-axis, we need to redistribute the 3-momentum accordingly. No specification was given as photon's angle in the y- or z-axis, so without loss of generality, I assume it is constrained to the x-y plane. This means we can write the four-momentum as

$$\vec{p} \underset{\mathcal{O}}{\to} (E, E\cos\theta, E\sin\theta, 0),$$

which you can easily confirm satisfies $\vec{p} \cdot \vec{p} = 0$.

Now we may apply the Lorentz transformation ${\Lambda^{\bar{0}}}_{\alpha}(v)$ to find the photon's energy as observed by $\bar{\mathcal{O}}$, and from that the frequency.

$$\begin{aligned}
& p^{\bar{0}} = \bar{E} = {\Lambda^{\bar{0}}}_{\alpha} p^{\alpha} = \gamma p^0 - v\gamma p^1 + 0 + 0 = \gamma E - v\gamma E\cos\theta \\
\implies\ & h\bar{\nu} = \gamma h\nu - v\gamma h\nu\cos\theta \\
\implies\ & \frac{\bar{\nu}}{\nu} = \gamma - v\gamma\cos\theta = \frac{1 - v\cos\theta}{\sqrt{1 - v^2}}
\end{aligned}$$

(b) Even when the photon moves perpendicular to the x-axis ($\theta = \pi/2$) there is a frequency shift. This is the *transverse Doppler shift*, which is a result of time dilation. At which angle θ must the photon move such that there is no Doppler shift between $\mathcal{O}$ and $\bar{\mathcal{O}}$?

To do this, we simply set $\bar{\nu}/\nu = 1$, and solve for θ.

$$1 = \frac{1 - v\cos\theta}{\sqrt{1-v^2}} \implies \cos\theta = 1 - \sqrt{1-v^2}$$
$$\implies \theta = \pm\arccos\left(1 - \sqrt{1-v^2}\right)$$

(c) Now use Equations 2.35 and 2.38 to find $\bar{\nu}/\nu$.

Recall that $\vec{U} \to_{\mathcal{O}} (\gamma, v\gamma, 0, 0)$. Using Equation 2.35 we have

$$\begin{aligned}\bar{E} = h\bar{\nu} &= -(E, E\cos\theta, E\sin\theta, 0)\cdot(\gamma, v\gamma, 0, 0)\\ &= -(-(E\gamma) + E\gamma v\cos\theta) = E\gamma(1 - v\cos\theta) = h\nu\gamma(1 - v\cos\theta)\\ \frac{\bar{\nu}}{\nu} &= \frac{1 - v\cos\theta}{\sqrt{1-v^2}}\end{aligned}$$

26 Calculate the energy required to accelerate a particle of rest mass $m > 0$ from speed v to speed $v + \delta v$ ($\delta v \ll v$), to first order in δv. Show that it would take infinite energy to accelerate to c.

From the four-momentum we have $E_v = m\gamma$, and from that

$$E_{v+\delta v} = \frac{m}{\sqrt{1-(v+\delta v)^2}}.$$

If we do a Taylor expansion on $(1-(v+\delta v)^2)^{-1/2}$ we get

$$\frac{1}{\sqrt{1-v^2}} + \frac{v\,\delta v}{(1-v^2)^{3/2}} + \mathcal{O}\left(v^2\right),$$

so

$$\begin{aligned}E_{v+\delta v} &\approx \frac{m}{\sqrt{1-v^2}} + \frac{mv\,\delta v}{(1-v^2)^{3/2}}\\ \Delta E = E_{v+\delta v} - E_v &\approx \frac{mv\,\delta v}{(1-v^2)^{3/2}} = m\gamma^3 v\,\delta v\,.\end{aligned}$$

As $v \to c$, $\gamma \to \infty$ and therefore $\Delta E \to \infty$.

30 A rocket ship has four-velocity $\vec{U} \to_{\mathcal{O}} (2, 1, 1, 1)$, and it passes a cosmic ray with four-momentum $\vec{p} \to \mathcal{O}(300, 299, 0, 0) \times 10^{-27}$kg. Compute the energy of the ray as measured by the rocket, using two different methods.

(a) Find the Lorentz transformation from $\mathcal{O}$ to the rocket's MCRF, and from that find the components $p^{\bar{\alpha}}$.

The Lorentz transformation for a boost in the x, y, and z directions is given by

$$\Lambda^{\bar{\beta}}{}_{\alpha} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

If we write out the terms of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

then we are left with a system of equations

$$\begin{aligned} 1 &= \gamma(2 + v_x + v_y + v_z), \\ 0 &= \gamma(2v_x + 1), \\ 0 &= \gamma(2v_y + 1), \\ 0 &= \gamma(2v_z + 1). \end{aligned}$$

Since γ may never be zero, we divide the last 3 terms by γ to obtain

$$2v_i + 1 = 0 \implies v_i = -\frac{1}{2},$$

and plugging into the first equation gives $\gamma = 2$. From this we see that our Lorentz transformation matrix is

$$\Lambda^{\bar{\beta}}{}_{\alpha} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

Now to find the energy as observed by the rocket, we need to find $\bar{E} = p^{\bar{0}}$

$$\begin{aligned} p^{\bar{0}} &= \Lambda^{\bar{0}}{}_{\alpha} p^{\alpha} = 2p^0 - p^1 - p^2 - p^3 \\ &= (2 \cdot 300 - 1 \cdot 299 - 1 \cdot 0 - 1 \cdot 0) \times 10^{-27}\text{kg} = 3.01 \times 10^{-25}\,\text{kg} = \bar{E} \end{aligned}$$

(b) Use Schutz's Equation 2.35.

$$\begin{aligned} \bar{E} &= -\vec{p} \cdot \vec{U}_{\text{obs}} = -(-(300 \cdot 2) + (299 \cdot 1) + (0 \cdot 1) + (0 \cdot 1)) \times 10^{-27}\text{kg} \\ &= 3.01 \times 10^{-25}\,\text{kg} \end{aligned}$$

(c) Which is quicker? Why?

Using Equation 2.35 was *much* quicker, as it was derived to handle this special case.

32 Consider a particle with charge e and mass m, which begins at rest, but scatters a photon with frequency ν_i (Compton scattering). The photon comes off at an angle θ from the direction of the initial photon's path. Use conservation of four-momentum to find the scattered photon's frequency, ν_f.

We will invoke: conservation of four-momentum and $\vec{p} \cdot \vec{p} = -m^2$. $\vec{p}_i$ and $\vec{p}_f$ denote the initial and final photon, and $\vec{p}_e$ and $\vec{p}_{e'}$ denote the electron before and after collision.

$$
\begin{aligned}
\vec{p}_i &\underset{\mathcal{O}}{\rightarrow} (E_i, E_i, 0, 0) \\
\vec{p}_e &\underset{\mathcal{O}}{\rightarrow} (m, 0, 0, 0) \\
\vec{p}_f &\underset{\mathcal{O}}{\rightarrow} (E_f, E_f \cos\theta, E_f \sin\theta, 0) \\
\vec{p}_i + \vec{p}_e &= \vec{p}_f + \vec{p}_{e'} \\
\vec{p}_{e'} &= \vec{p}_i + \vec{p}_e - \vec{p}_f \\
\vec{p}_{e'} \cdot \vec{p}_{e'} &= (\vec{p}_i + \vec{p}_e - \vec{p}_f) \cdot (\vec{p}_i + \vec{p}_e - \vec{p}_f) \\
-m^2 &= \vec{p}_i \cdot \vec{p}_i + \vec{p}_e \cdot \vec{p}_e + \vec{p}_f \cdot \vec{p}_f + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\
&= 0 - m^2 + 0 + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\
0 &= \vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f \\
&= -E_i m - (-E_i E_f + E_i E_f \cos\theta) + E_f m \\
&= m(E_f - E_i) + E_i E_f (1 - \cos\theta) \\
m(E_i - E_f) &= E_i E_f (1 - \cos\theta) \\
mh(\nu_i - \nu_f) &= h^2 \nu_i \nu_f (1 - \cos\theta) \\
\frac{\nu_i - \nu_f}{\nu_i \nu_f} &= h \frac{1 - \cos\theta}{m} \\
\frac{1}{\nu_f} - \frac{1}{\nu_i} &= h \frac{1 - \cos\theta}{m} \\
\frac{1}{\nu_f} &= \frac{1}{\nu_i} + h \frac{1 - \cos\theta}{m}
\end{aligned}
$$

Chapter 3

Tensor analysis in special relativity

3.3 The $\binom{0}{1}$ tensors: one-forms

The symbol $\tilde{}$ is used to denote a one-form, as $\vec{}$ is used to denote a vector. So $\tilde{p}$ is a one-form, or a type $\binom{0}{1}$ tensor.

Normal one-forms

Let $\mathcal{S}$ be some surface.
$\forall\vec{V}$ tangent to $\mathcal{S}$, $\tilde{p}(\vec{V}) = 0 \implies \tilde{p}$ is normal to $\mathcal{S}$.
Furthermore, if $\mathcal{S}$ is a *closed* surface & $\tilde{p}$ is normal to $\mathcal{S}$ & $\forall\vec{U}$ pointing outwards from $\mathcal{S}$, $\tilde{p}(\vec{U}) > 0 \implies \tilde{p}$ is an outward normal one-form.

3.5 Metric as a mapping of vectors into one-forms

Normal vectors and unit normal one-forms

$\vec{V}$ is normal to a surface if $\tilde{V}$ is normal to the surface. They are said to be *unit normal* if their magnitude is ±1, so $\vec{V}^2 = \tilde{V}^2 = \pm1$.

- A time-like unit normal has magnitude -1
- A space-like unit normal has magnitude $+1$
- A null normal cannot be a unit normal, because $\vec{V}^2 = \tilde{V}^2 = 0$

3.10 Exercises

3

(a)

$$\begin{aligned}\tilde{p}(A^\alpha \vec{e}_\alpha) &= A^\alpha \tilde{p}(\vec{e}_\alpha) = \tilde{p}(A^0 \vec{e}_0 + A^1 \vec{e}_1 + A^2 \vec{e}_2 + A^3 \vec{e}_3) \\ &= A^0 \tilde{p}(\vec{e}_0) + A^1 \tilde{p}(\vec{e}_1 + A^2 \tilde{p}(\vec{e}_2) + A^3 \tilde{p}(\vec{e}_3 = A^\alpha \tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha \in \mathbb{R}\end{aligned}$$

(b)

$$\begin{aligned}\tilde{p} &\underset{\mathcal{O}}{\rightarrow} (-1, 1, 2, 0) \\ \vec{A} &\underset{\mathcal{O}}{\rightarrow} (2, 1, 0, -1) \\ \vec{B} &\underset{\mathcal{O}}{\rightarrow} (0, 2, 0, 0)\end{aligned}$$

$$\begin{aligned}\tilde{p}(\vec{A}) &= -2 + 1 + 0 + 0 = -1 \\ \tilde{p}(\vec{B}) &= 0 + 2 + 0 + 0 = 2 \\ \tilde{p}(\vec{A} - 3\vec{B}) &= \tilde{p}(\vec{A}) - 3\tilde{p}(\vec{B}) = -1 - 3 \cdot 2 = -7\end{aligned}$$

4 Given the following vectors

$$\begin{aligned}\vec{A} &\underset{\mathcal{O}}{\rightarrow} (2, 1, 1, 0) & \vec{B} &\underset{\mathcal{O}}{\rightarrow} (1, 2, 0, 0) \\ \vec{C} &\underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1) & \vec{D} &\underset{\mathcal{O}}{\rightarrow} (-3, 2, 0, 0)\end{aligned}$$

(Note that all parts were done with the assistance of `numpy`.)

(a) Show that they are linearly independent.

We do this by constructing a matrix, $\mathbf{X}$, whose columns correspond to the four vectors. If the determinant of $\mathbf{X}$ is non-zero, then that means the vectors are linearly independent.

$$\det(\mathbf{X}) = \det \begin{pmatrix} 2 & 1 & 0 & -3 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -8$$

(b) Find the components of $\tilde{p}$ if

$$\tilde{p}(\vec{A}) = 1, \quad \tilde{p}(\vec{B}) = -1, \quad \tilde{p}(\vec{C}) = -1, \quad \tilde{p}(\vec{D}) = 0$$

We do this by observing that $\tilde{p} = A^\alpha p_\alpha$, and so we have a system of four equations, which we can write in matrix form as

$$\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \\ \vec{D} \end{pmatrix} \tilde{p} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

$$\implies \tilde{p} = \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \\ \vec{D} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ +\frac{15}{8} \\ -\frac{23}{8} \end{pmatrix}.$$

(c) Find $\tilde{p}(\vec{E})$, where $\vec{E} \to_{\mathcal{O}} (1, 1, 0, 0)$.

$$\tilde{p}(\vec{E}) = p_\alpha E^\alpha = -\frac{5}{8}$$

(d) Determine whether $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$ are linearly independent.

We do this by first setting up a system of equations for each of $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$, as was done for $\tilde{p}$, and solving. I will refer to the matrix whose rows were $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ as $\mathbf{X}$.

$$\mathbf{X}\tilde{q} = \begin{pmatrix} +0 \\ +0 \\ +1 \\ -1 \end{pmatrix} \qquad \mathbf{X}\tilde{r} = \begin{pmatrix} +2 \\ +0 \\ +0 \\ +0 \end{pmatrix} \qquad \mathbf{X}\tilde{s} = \begin{pmatrix} -1 \\ -1 \\ +0 \\ +0 \end{pmatrix}$$

$$\tilde{q} = \begin{pmatrix} +\frac{1}{4} \\ -\frac{1}{8} \\ -\frac{3}{8} \\ +\frac{11}{8} \end{pmatrix} \qquad \tilde{r} = \begin{pmatrix} +0 \\ +0 \\ +2 \\ +2 \end{pmatrix} \qquad \tilde{s} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{3}{8} \\ -\frac{1}{8} \\ +\frac{1}{8} \end{pmatrix}$$

Now if the matrix whose columns are comprised of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$ has a non-zero determinant, then the four covectors must be linearly independent.

$$\det \begin{pmatrix} \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} \end{pmatrix} = \frac{1}{4},$$

and so they are indeed linearly independent.

6

(a) Show that $\tilde{p} \neq \tilde{p}(\vec{e}_\alpha)\tilde{\lambda}^\alpha$ for arbitrary $\tilde{p}$.

Let us choose $\tilde{p} \to_\mathcal{O} (0, 1, e, \pi)$, as a counter-example.

$$p_\alpha \tilde{\lambda}^\alpha \underset{\mathcal{O}}{\to} 0 \cdot (1,1,0,0) + 1 \cdot (1,-1,0,0) + e \cdot (0,0,1,-1) + \pi \cdot (0,0,1,1)$$
$$\underset{\mathcal{O}}{\to} (1, -1, e+\pi, 0) \underset{\mathcal{O}}{\not\to} \tilde{p}$$

(b) $\tilde{p} \to_\mathcal{O} (1,1,1,1)$. Find l_α such that

$$\tilde{p} = l_\alpha \tilde{\lambda}^\alpha$$

We may do this with a simple matrix inversion. We define $\mathbf{\Lambda}$ to be the matrix whose rows are formed by $\tilde{\lambda}^\alpha$.

$$\mathbf{\Lambda} l = p \implies l = \mathbf{\Lambda}^{-1} p = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

8 Draw the basis one-forms $\tilde{\mathrm{d}}t$ and $\tilde{\mathrm{d}}x$ of frame $\mathcal{O}$.

They are

$$\tilde{\mathrm{d}}t \underset{\mathcal{O}}{\to} (1,0,0,0),$$
$$\tilde{\mathrm{d}}x \underset{\mathcal{O}}{\to} (0,1,0,0),$$

and they are shown in Figure 3.1.

9 At the points $\mathcal{P}$ and $\mathcal{Q}$, estimate the components of the gradient $\tilde{\mathrm{d}}T$.

Recall that $\tilde{\mathrm{d}}T \to_\mathcal{O} \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$, and so $\Delta T = \tilde{\mathrm{d}}T_\alpha x^\alpha = \tilde{\mathrm{d}}T_x \Delta x + \tilde{\mathrm{d}}T_y \Delta y$.

Now if we move only in the x direction from one of the points, we move some distance Δx, change our temperature by Δt, and $\Delta y = 0$. Likewise for a movement in the y direction. Thus we can say

$$\Delta T = \tilde{\mathrm{d}}T_x \Delta x \qquad \Delta T = \tilde{\mathrm{d}}T_y \Delta y$$
$$\tilde{\mathrm{d}}T_x = \frac{\Delta T}{\Delta x} \qquad \tilde{\mathrm{d}}T_y = \frac{\Delta T}{\Delta y}$$

In Figure 3.2, from $\mathcal{P}$ I move a distance $\Delta x = 0.5$, which causes a temperature change of $\Delta T = -7$, giving $\tilde{\mathrm{d}}T_x = -14$. Then I move a distance $\Delta y = 0.5$ and get the same temperature change of $\Delta T = -7$, and so I conclude that at point $\mathcal{P}$, $\tilde{\mathrm{d}}T \to_\mathcal{O} (-14, -14)$.

At $\mathcal{Q}$, we are in a flat region where $T = 0$. If we move any non-zero distance Δx or Δy, so long as it does not cross the $T = 0$ isotherm, we have a $\Delta T = 0$, and thus $\tilde{\mathrm{d}}Tp \to_\mathcal{O} (0, 0)$.

13 Prove that $\tilde{\mathrm{d}}f$ is normal to surfaces of constant f.

If we move some small distance $\Delta x^\alpha = \epsilon$, then there will be no change in the value of f, and thus we can say $\partial f / \partial x^\alpha = 0$, so

$$\tilde{\mathrm{d}}f = \frac{\partial f}{\partial x^\alpha} \tilde{\mathrm{d}}x^\alpha = 0 \tilde{\mathrm{d}}x^\alpha = 0.$$

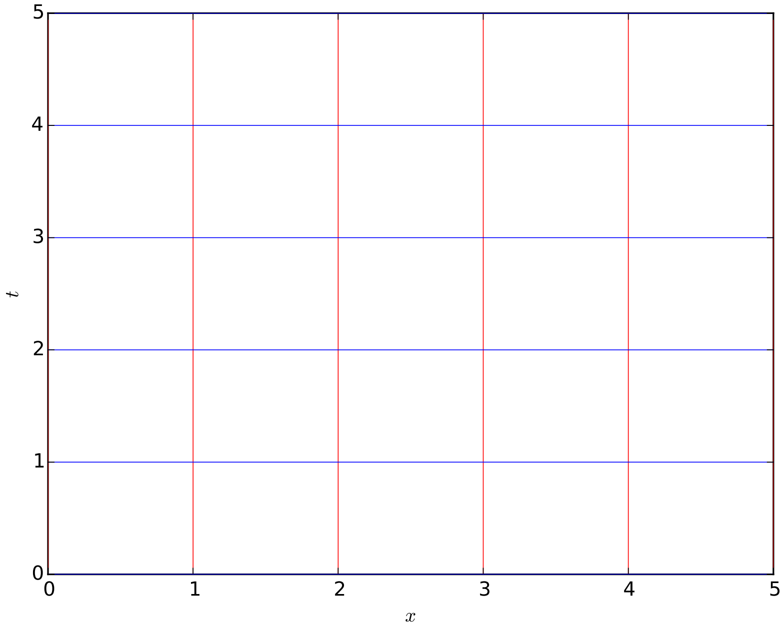


Figure 3.1: Problem 8: Basis one-forms of $\mathcal{O}$. $\tilde{\mathrm{d}}t$ is given in blue and $\tilde{\mathrm{d}}x$ in red.

Since $\tilde{\mathrm{d}}f$ is defined to be normal to a surface if it is zero on every tangent vector, we have shown that $\tilde{\mathrm{d}}f$ is normal to any surface of constant f.

14

$$\tilde{p} \underset{\mathcal{O}}{\rightarrow} (1,1,0,0) \qquad\qquad \tilde{q} \underset{\mathcal{O}}{\rightarrow} (-1,0,1,0)$$

Prove by giving two vectors $\vec{A}$ and $\vec{B}$ as arguments that $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

$$\begin{aligned}
(\tilde{p} \otimes \tilde{q})(\vec{A}, \vec{B}) &= \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = A^\alpha p_\alpha B^\beta q_\beta = (A^0 + A^1)(-B^0 + B^2), \\
&= -A^0 B^0 + A^0 B^2 - A^1 B^0 + A^1 B^2 \\
(\tilde{q} \otimes \tilde{p})(\vec{A}, \vec{B}) &= \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = A^\alpha q_\alpha B^\beta p_\beta = (-A^0 + A^2)(B^0 + B^1) \\
&= -A^0 B^0 - A^0 B^1 + A^2 B^0 + A^2 B^1,
\end{aligned}$$

And so we see that $\otimes$ is not commutative.

The components of the outer product of two tensors are given by the products of the components of the

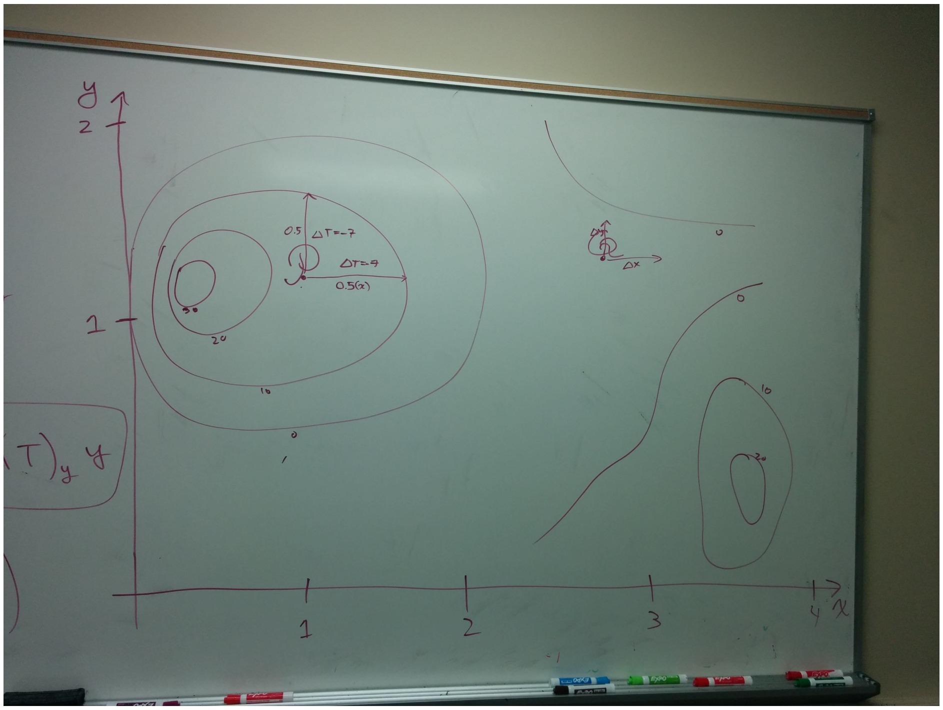


Figure 3.2: Problem 9: Isotherms.

individual tensors. Thus we can write the components as a 4×4 matrix.

$$(\tilde{p} \otimes \tilde{q})_{\alpha\beta} = p_\alpha q_\beta = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

18

(a) Find the one-forms mapped by $\mathbf{g}$ from

$$\vec{A} \underset{\mathcal{O}}{\to} (1, 0, -1, 0), \qquad \vec{B} \underset{\mathcal{O}}{\to} (0, 1, 1, 0),$$
$$\vec{C} \underset{\mathcal{O}}{\to} (-1, 0, -1, 0), \qquad \vec{D} \underset{\mathcal{O}}{\to} (0, 0, 1, 1).$$

In general,

$$\vec{V} \underset{\mathcal{O}}{\to} (V^0, V^1, V^2, V^3) \implies \tilde{V} = \mathbf{g}\vec{V} \underset{\mathcal{O}}{\to} (-V^0, V^1, V^2, V^3),$$

and so

$$\tilde{A} \underset{\mathcal{O}}{\to} (-1, 0, -1, 0), \qquad \tilde{B} \underset{\mathcal{O}}{\to} (0, 1, 1, 0),$$
$$\tilde{C} \underset{\mathcal{O}}{\to} (1, 0, -1, 0), \qquad \tilde{D} \underset{\mathcal{O}}{\to} (0, 0, 1, 1).$$

(b) Find the vectors mapped by $\mathbf{g}$ from

$$\tilde{p} \underset{\mathcal{O}}{\rightarrow} (3, 0, -1, -1), \qquad \tilde{q} \underset{\mathcal{O}}{\rightarrow} (1, -1, 1, 1),$$

$$\tilde{r} \underset{\mathcal{O}}{\rightarrow} (0, -5, -1, 0), \qquad \tilde{s} \underset{\mathcal{O}}{\rightarrow} (-2, 1, 0, 0).$$

By using the inverse tensor in reverse, we have the same effect as before, of negating the first component

$$\vec{p} \underset{\mathcal{O}}{\rightarrow} (-3, 0, -1, -1), \qquad \vec{q} \underset{\mathcal{O}}{\rightarrow} (-1, -1, 1, 1),$$

$$\vec{r} \underset{\mathcal{O}}{\rightarrow} (0, -5, -1, 0), \qquad \vec{s} \underset{\mathcal{O}}{\rightarrow} (2, 1, 0, 0).$$

20

In Euclidean 3-space, vectors and covectors are usually treated as the same, because they transform the same. We will now prove this.

(a) Show that $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} A^{\beta}$ and $P_{\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\beta}} P_{\alpha}$ are the same transformations if $\{\Lambda^{\alpha}{}_{\bar{\beta}}\}$ is equal to the transpose of its inverse.

We can write that last statement as

$$\Lambda^{\alpha}{}_{\bar{\beta}} = ((\Lambda^{\alpha}{}_{\bar{\beta}})^{-1})^T$$

and we know that

$$(\Lambda^{\alpha}{}_{\bar{\beta}})^{-1} = \Lambda^{\bar{\beta}}{}_{\alpha},$$

and also we know that the Lorentz transformation is symmetric, and so

$$(\Lambda^{\bar{\beta}}{}_{\alpha})^T = \Lambda^{\bar{\beta}}{}_{\alpha},$$

which leads us to conclude that $\Lambda^{\alpha}{}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}$, meaning the two transformations are the same.

(b) The metric has components $\{\delta_{ij}\}$. Prove that transformations between Cartesian coordinate systems must satisfy

$$\delta_{\bar{i}\bar{j}} = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \delta_{kl},$$

and that this implies that $\Lambda^{k}{}_{\bar{i}}$ is an orthogonal matrix.

$$\delta_{\bar{i}\bar{j}} = \mathbf{g}(\vec{e}_{\bar{i}}, \vec{e}_{\bar{j}}) = \mathbf{g}(\Lambda^{k}{}_{\bar{i}} \vec{e}_k, \Lambda^{l}{}_{\bar{j}} \vec{e}_j) = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \mathbf{g}(\vec{e}_k, \vec{e}_j) = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \delta_{kl}$$

Now show it is orthogonal

21

(a) A region of the t–x plane is bounded by lines $t = 0$, $t = 1$, $x = 0$, and $x = 1$. Within the plane, find the unit outward normal 1-forms and their vectors for each boundary line.

I define unit outward normals as follows:

Let $\mathcal{S}$ be a closed surface. If, for each $\vec{V}$ tangent to $\mathcal{S}$, we have $\tilde{p}(\vec{V}) = 0$, then $\tilde{p}$ is normal to $\mathcal{S}$.

In addition, if, for each $\vec{U}$ which points outwards from the surface, we have $\tilde{p}(\vec{U}) > 0$, then $\tilde{p}$ is an outward normal.

Furthermore, if $\tilde{p}^2 = \pm 1$, then it is a unit outward normal.

For the problem at hand, I define the region inside the four lines to be *Inside*, and the region outside to be *Outside*. For each of the four lines, I draw a vector $\vec{V}$ tangent (parallel) to the line, and $\vec{U}$ pointing outwards (See Figure 3.3).

It helps to look at $t = 0$ and $t = 1$ together, and likewise for x, so I will start with t. We start with an arbitrary $\tilde{p} \to_{\mathcal{O}} (p_0, p_1)$, and $\vec{V} \to_{\mathcal{O}} (0, V^1)$, where $V^1 \neq 0$.

$$\tilde{p}(\vec{V}) = p_0 \cdot 0 + p_1 V^1 = 0 \implies p_1 = 0,$$

so $\tilde{p} \to_{\mathcal{O}} (p_0, 0)$ is a normal 1-form to both lines. Now we find the corresponding *unit* normal, by taking

$$\tilde{p}^2 = \pm 1 = -(p_0)^2 \implies \tilde{p}^2 = -1 \;\&\; p_0 = \pm 1.$$

Whether we choose p_0 to be positive or negative now depends on the line we are looking at, and which direction is outward. For $t = 0$, we have a vector $\vec{U} = (-U^0, U^1)$, where $U^0 > 0$.

$$\tilde{p}(\vec{U}) = p_0(-U^0) + 0 \cdot U^1 > 0 \implies -p_0 U^0 > 0 \implies p_0 < 0,$$

so for $t = 0$ we have $\tilde{p} \to_{\mathcal{O}} (-1, 0)$, and likewise for $t = 1$ we have $\tilde{p} \to_{\mathcal{O}} (1, 0)$. To get the associated *vectors*, we apply the metric $\eta^{\alpha\beta}$, giving us $\vec{p} \to_{\mathcal{O}} (1, 0)$ for $t = 0$ and $\vec{p} \to_{\mathcal{O}} (-1, 0)$ for $t = 1$.

For $x = 0$ and $x = 1$, we instead have $\vec{V} \to_{\mathcal{O}} (V^0, 0)$, and following the same steps as before, we conclude that: for $x = 0$, $\tilde{p} \to_{\mathcal{O}} (0, -1)$, $\vec{p} \to_{\mathcal{O}} (0, -1)$, and for $x = 1$, $\tilde{p} \to_{\mathcal{O}} (0, 1)$, $\vec{p} \to_{\mathcal{O}} (0, 1)$.

Figure 3.3: Problem 21.a

(b) Let another region be bounded by the set of points $\{(1, 0), (1, 1), (2, 1)\}$. Find an outward normal for the null boundary and the associated vector.

23

(a) Prove that the set of all $\binom{M}{N}$ tensors forms a vector space, V.

Let T be the set of all $\binom{M}{N}$ tensors, $\mathbf{s}, \mathbf{p}, \mathbf{q} \in T$, $\vec{A} \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. For T to be a vector space, we must define the operations of addition, and scalar multiplication (amongst others).

Addition:

$$\mathbf{s} = \mathbf{p} + \mathbf{q} \implies \mathbf{s}(\vec{A}) = \mathbf{p}(\vec{A}) + \mathbf{q}(\vec{A})$$

Scalar Multiplication:

$$\mathbf{r} = \alpha \mathbf{p} \implies \mathbf{r}(\vec{A}) = \alpha \mathbf{p}(\vec{A})$$

(b)

Prove that a basis for T is

$$\{\vec{e}_\alpha \otimes \ldots \otimes \vec{e}_\gamma \otimes \tilde{\omega}^\mu \otimes \ldots \otimes \tilde{\omega}^\lambda\}$$

Still working on it

24 Given:

$$M^{\alpha\beta} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(a) Find:

(i)

$$M^{(\alpha\beta)} \to \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{pmatrix}; \quad M^{[\alpha\beta]} \to \begin{pmatrix} 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} & 0 \end{pmatrix}$$

(ii)

$$M^{\alpha}{}_{\beta} = \eta_{\beta\mu} M^{\alpha\mu} \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iii)

$$M_{\alpha}{}^{\beta} = \eta_{\alpha\mu} M^{\mu\beta} \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iv)

$$M_{\alpha\beta} = \eta_{\beta\mu} M_{\alpha}{}^{\mu} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(b) Does it make sense to separate the $\binom{1}{1}$ tensor with components $M^{\alpha}{}_{\beta}$ into symmetric and antisymmetric parts?

No, it would not make sense. For one, the notation for (anti)symmetric tensors do not even allow one to write it sensibly ($M^{(\alpha}{}_{\beta)}$). More importantly, one argument refers to vectors, and the other to covectors, so it does not make sense to switch them.

(c)

$$\eta^{\alpha}{}_{\beta} = \eta^{\alpha\mu}\eta_{\beta\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\alpha}{}_{\beta}$$

31

Still working on it

(**33**)

34 Define double-null coordinates $u = t - x$, $v = t + x$ in Minkowski space.

(a) Let $\vec{e}_u$ be the vector connecting the (u, v, y, t) coordinates $(0, 0, 0, 0)$ and $(1, 0, 0, 0)$, and let $\vec{e}_v$ be the vector connecting $(0, 0, 0, 0)$ and $(0, 1, 0, 0)$. Find $\vec{e}_u$ and $\vec{e}_v$ in terms of $\vec{e}_t$ and $\vec{e}_x$, and plot the basis vectors in a spacetime diagram of the t–x plane.

$$u = t - x = 0 \implies t = +x \qquad v = t + x = 0 \implies t = -x$$
$$u = t - x = 1 \implies t = 1 + x \qquad v = t + x = 1 \implies t = 1 - x$$

We draw the vectors $\vec{e}_u$ and $\vec{e}_v$ in Figure 3.4, such that they point from the appropriate points of intersection on these lines of constant u and v. From this it is obvious that $\vec{e}_v + \vec{e}_u = \vec{e}_t$, and that $\vec{e}_v - \vec{e}_u = \vec{e}_x$, or likewise $\vec{e}_v = \vec{e}_t - \vec{e}_u$ and $\vec{e}_u = \vec{e}_v - \vec{e}_x$. This is a system of 2 equations with two unknowns.

$$\vec{e}_v = \vec{e}_t - \vec{e}_v + \vec{e}_x \implies \qquad \vec{e}_v = \frac{1}{2}(\vec{e}_t + \vec{e}_x),$$
$$\vec{e}_u = \frac{1}{2}(\vec{e}_t + \vec{e}_x) - \vec{e}_x \implies \qquad \vec{e}_u = \frac{1}{2}(\vec{e}_t - \vec{e}_x).$$

(b) Show that $\vec{e}_\alpha$, $\alpha \in \{u, v, y, z\}$ form a basis for vectors in Minkowski space.

$$\begin{aligned} \vec{A} = A^\alpha \vec{e}_\alpha &= A^u \vec{e}_u + A^v \vec{e}_v + A^y \vec{e}_y + A^z \vec{e}_z \\ &= \frac{A^u}{2}(\vec{e}_t - \vec{e}_x) + \frac{A^v}{2}(\vec{e}_t + \vec{e}_x) + A^y \vec{e}_y + A^z \vec{e}_z \\ &= \frac{1}{2}(A^v + A^u)\vec{e}_t + \frac{1}{2}(A^v - A^u)\vec{e}_x + A^y \vec{e}_y + A^z \vec{e}_z \end{aligned}$$

If we let $A^t = \frac{1}{2}(A^v + A^u)$ and $A^x = \frac{1}{2}(A^v - A^u)$, then

$$\vec{A} = A^\alpha \vec{e}_\alpha = A^t \vec{e}_t + A^x \vec{e}_x + A^y \vec{e}_y + A^z \vec{e}_z$$

(c) Find the components of the metric tensor, $\mathbf{g}$ in this new basis.

To make this concise, we will begin with some definitions. Let $w \in \{u, v\}$, and $q \in \{y, z\}$. We also define

$$\lambda(w) \equiv \begin{cases} -1, & \text{if } w = u, \\ +1, & \text{if } w = v. \end{cases}$$

It follows that

$$\vec{e}_w = \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x).$$

Now we can show that

$$\begin{aligned} g_{ww} = \vec{e}_w \cdot \vec{e}_w &= \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 2\lambda(\vec{e}_t \cdot \vec{e}_x) + \lambda^2(\vec{e}_x \cdot \vec{e}_x)] \\ &= \frac{1}{4}(-1 + 2\lambda \cdot 0 + 1 \cdot 1) = 0, \end{aligned}$$

so $g_{uu} = g_{vv} = 0$.

For the u and v cross terms, we have

$$\begin{aligned} g_{uv} = g_{vu} = \vec{e}_u \cdot \vec{e}_v &= \frac{1}{2}(\vec{e}_t - \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 0 \cdot \vec{e}_t \cdot \vec{e}_x - \vec{e}_x \cdot \vec{e}_x] \\ &= \frac{1}{4}(-1 + 0 - 1) = -\frac{1}{2} \end{aligned}$$

For the w with y and z cross terms we have

$$\begin{aligned} g_{wq} = \vec{e}_w \cdot \vec{e}_q &= \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \vec{e}_q \\ &= \frac{1}{2}[\vec{e}_t \cdot \vec{e}_t + \lambda \vec{e}_x \cdot \vec{e}_x] \\ &= 0 \end{aligned}$$

so $g_{uy} = g_{vy} = g_{uz} = g_{vz} = 0$. We also already know $g_{yy} = g_{zz} = 1$, and $g_{yz} = g_{zy} = 0$, so we can write the components of the metric tensor in this new coordinate system as

$$g_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) Show that $\vec{e}_u$ and $\vec{e}_v$ are null, but not orthogonal.

$$\vec{e}_u \cdot \vec{e}_u = g_{uu} = 0 \implies \vec{e}_u \text{ is null}$$

$$\vec{e}_v \cdot \vec{e}_v = g_{vv} = 0 \implies \vec{e}_v \text{ is null}$$

$$\vec{e}_u \cdot \vec{e}_v = g_{uv} = -\frac{1}{2} \neq 0 \implies \vec{e}_u \text{ and } \vec{e}_v \text{ are not orthogonal.}$$

(e) Compute the four one-forms $\tilde{\mathrm{d}}u$, $\tilde{\mathrm{d}}v$, $\mathbf{g}(\vec{e}_u,)$, and $\mathbf{g}(\vec{e}_v,)$ in terms of $\tilde{\mathrm{d}}t$ and $\tilde{\mathrm{d}}x$.

$$\tilde{\mathrm{d}}\phi \to_{\mathcal{O}} \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

so

$$\tilde{\mathrm{d}}t \to_{\mathcal{O}} (1,0,0,0), \qquad \tilde{\mathrm{d}}x \to_{\mathcal{O}} (0,1,0,0),$$

$$\tilde{\mathrm{d}}u \to_{\mathcal{O}} \frac{1}{2}(1,-1,0,0), \qquad \tilde{\mathrm{d}}u \to_{\mathcal{O}} \frac{1}{2}(1,1,0,0),$$

from which it is obvious that

$$\tilde{\mathrm{d}}u = \frac{1}{2}(\tilde{\mathrm{d}}t - \tilde{\mathrm{d}}x), \qquad \tilde{\mathrm{d}}v = \frac{1}{2}(\tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x).$$

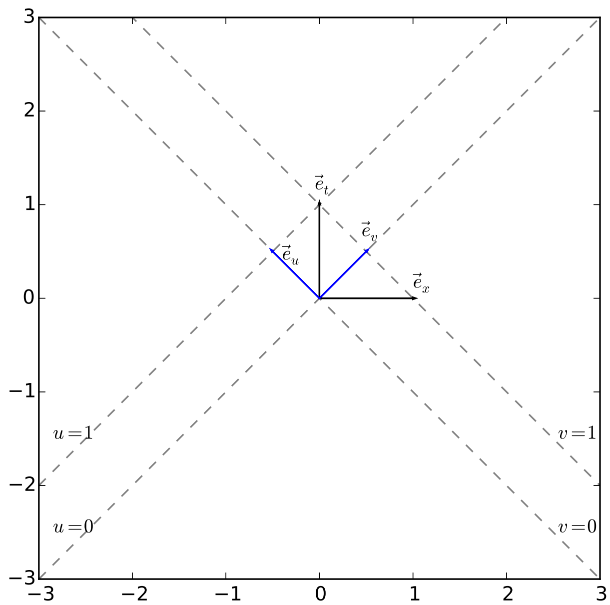


Figure 3.4: Problem 34a: Spacetime diagram of double-null coordinate basis vectors in t–x plane.