Spherical solutions for stars

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General Relativity I Presentations
December 14th, 2015

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Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- applications

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- Schwarzschild metric
- T–O–V equation
- applications

- I will model stars using GR assuming spherical symmetry
- I will derive the Schwarzschild metric and T–O–V equation
- finally I will relate these equations to modeling specific types of stars

Spherically symmetric coordinates

Spherically symmetric coordinates

- First we need to derive our coordinate system

Two-sphere in flat spacetime

General metric

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Metric on 2-sphere

$$\mathrm{d}l^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \equiv r^2 \mathrm{d}\Omega^2$$

Schutz (2009, p. 256)

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- we start with the simplest spherically symmetric coordinates
- 2-sphere in Minkowski space

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Two-sphere in curved spacetime

Metric on 2-sphere

$$\mathrm{d}l^2 = f(r', t)\mathrm{d}\Omega^2$$

Relation to r

$$f(r', t) \equiv r^2$$

Schutz (2009, pp. 256–257)

- generalize to 2-sphere in arbitrary curved spherically symmetric spacetime
- inclusion of curvature makes r^2 some function of r' and t

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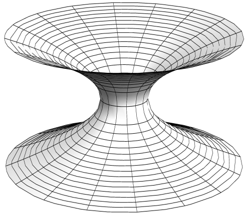
Meaning of r



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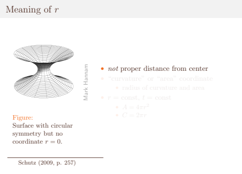
- *not* proper distance from center
- "curvature" or "area" coordinate
  - radius of curvature and area
- $r = \text{const}$, $t = \text{const}$
  - $A = 4\pi r^2$
  - $C = 2\pi r$

Figure:
Surface with circular symmetry but no coordinate $r = 0$.

Schutz (2009, p. 257)



- r is not necessary the "distance from the center"
- it is merely a coordinate
- for instance, we may have a spacetime where the center is missing
  - example: wormhole spacetime
- surface of constant (r, t) is a two-sphere of area A and circumference C

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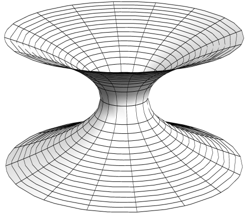

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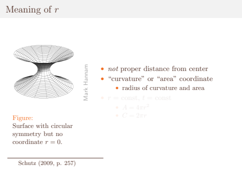

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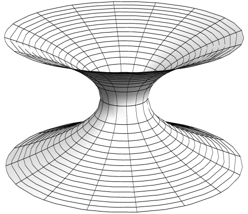

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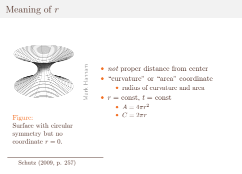

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Spherically symmetric spacetime

General metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + 2g_{0r}\,\mathrm{d}r\,\mathrm{d}t + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

g_{00}, g_{0r}, and g_{rr} functions of t and r

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g_{00}, g_{0r}, and g_{rr} functions of t and r
Schutz (2009, p. 258)

- now consider not only surface of 2-sphere, but whole spacetime
- now we have some unknown g_{00}, g_{rr}, and cross term g_{0r}
- cross terms g_{0i} for $i \in \{\theta, \phi\}$ are zero from symmetry
- need more constraints to say anything particular about them

Static spacetimes

- now I will impose the static constraint

Static spacetimes

Motivation

- leads to simple derivation of Schwarzschild metric
- unique solution to spherically symmetric, asymptotically flat Einstein vacuum field equations (Birkhoff's theorem)

Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843)

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  - it allows us to easily derive the Schwarzschild metric
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Definition

A spacetime is static if we can find a time coordinate t for which

(i) the metric independent of t

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \to -t$$

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- now I define "static"
- first condition is that the metric is independent of time
  - by itself, this condition is called "stationary"
- second condition is that metric unaffected by time reversal
- e.g. rotating stars are stationary but not static

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Time reversal

$$\mathbf{\Lambda} : (t, x, y, z) \to (-t, x, y, z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} g_{\alpha\beta} = g_{\alpha\beta}$$

Transformation

$$\Lambda^{0}{}_{\bar{0}} = x^{0}{}_{,\bar{0}} = -x^{0}{}_{,0} = -1$$
$$\Lambda^{i}{}_{\bar{i}} = x^{i}{}_{,\bar{i}} = x^{i}{}_{,i} = 1$$

Metric

$$g_{\bar{0}\bar{0}} = (\Lambda^{0}{}_{\bar{0}})^2 g_{00} = g_{00}$$
$$g_{\bar{r}\bar{r}} = (\Lambda^{r}{}_{\bar{r}})^2 g_{rr} = g_{rr}$$
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Schutz (2009, p. 258)

- now I use the static constraint to simplify the metric
- transformation
  - $(0, 0)$ term is $\mathrm{d}t/\mathrm{d}(-t)$
  - spatial terms are 1 if transformed to themselves
  - cross-terms are all zero, as coordinates independent of each other
- transformed metric
  - $(0, 0)$ term is unchanged, as -1 is squared
  - (r, r) term is unchanged, as transformation is 1
  - $(0, r)$ term is negated, but must still be equal, so it's zero
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The metric

Simplified metric

$$ds^2 = g_{00}\,dt^2 + g_{rr}\,dr^2 + r^2 d\Omega^2$$

Replacement

$$g_{00} \to -e^{2\Phi}, \quad g_{rr} \to e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

Static spherically symmetric metric

$$ds^2 = -e^{2\Phi}\,dt^2 + e^{2\Lambda}\,dr^2 + r^2 d\Omega^2$$

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \Lambda(r) = 0$$

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- we assume g_{00} to be negative, and g_{rr} to be positive
  - signature is $(-,+,+,+)$
  - holds inside stars but not black holes
- limits at infinity tell us that spacetime is *asymptotically flat*
  - $\Phi = \Lambda = 0 \implies e^{2\Phi} = e^{2\Lambda} = 1$ and $\mathbf{g} = \eta$

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Einstein Tensor

General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Einstein tensor components

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})]$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda} [\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

Schutz (2009, pp. 165, 260)

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- now we can use the metric to derive the Riemann tensor
- from that the Einstein tensor
- the derivation is involved, so we will just take them as is
- we're going to use some of these components later on
- $x' \equiv \mathrm{d}x/\mathrm{d}r$

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Static perfect fluid

Static perfect fluid

- stars are fluids – for simplicity we assume perfect
- thus we will impose additional constraints accordingly

Four-velocity

Constraints

$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

Solving for U^0

$$g_{00} U^0 U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

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- we find the only non-zero term, U^0, by relating to the dot product
- lower it with the metric, to use in next part

$$\begin{aligned} g_{00} U^0 U^0 = -1 &\implies (U^0)^2 = (-g_{00})^{-1} \\ &\implies U^0 = (-g_{00})^{-1/2} \\ &\implies U^0 = (e^{2\Phi})^{-1/2} = e^{-\Phi} \end{aligned}$$

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Stress–energy tensor

Stress-energy tensor for perfect fluid

$$T_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + p g_{\alpha\beta}$$

Components of $T_{\alpha\beta}$

Schutz (2009, p. 260)

Stress–energy tensor
Stress-energy tensor for perfect fluid
$T_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + p g_{\alpha\beta}$
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- $T_{i\alpha} = p g_{i\alpha}$ because spatial components of U are zero
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- T_{00} requires a little algebra
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Stress–energy tensor

Stress-energy tensor for perfect fluid

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Equation of state

Local thermodynamic equilibrium

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

Schutz (2009, p. 261)

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- in a static fluid we have local thermodynamic equilibrium
- pressure a function of density and specific entropy
- specific entropy assumed negligibly small

Equations of motion

Conservation of 4-momentum

$$T^{\alpha\beta}{}_{;\beta} = 0$$

- symmetries make only non-trivial solution $\alpha = r$
  TODO: prove

Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)

- first equation follows from conservation of 4-momentum
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Mass function

Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r}[r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

$m(r)$

$$m(r) \equiv \frac{1}{2} r (1 - e^{-2\Lambda}) \quad \text{or} \quad g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

Relation to energy density

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Schutz (2009, pp. 260–262)

- inspect $(0, 0)$ component of Einstein equations
- define the mass function, $m(r)$
- in Newtonian limit, $m(r)$ is mass within radius r

$$m(r) = 4\pi \int_0^r (r')^2 \rho(r') \, \mathrm{d}r'$$

- doesn't work in GR, because total energy is not localizable

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$\Phi(r)$

Einstein field equations

$$G_{rr} = 8\pi T_{rr} \implies -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi p e^{2\Lambda}$$

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$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

Schutz (2009, pp. 260–262)

- inspect (r, r) component of Einstein equations
- gives us an expression for $\Phi(r)$

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Exterior Geometry

Exterior Geometry

- until now, we've not considered whether we were inside or outside star
- properties inside different than outside (obviously)
- we're going to inspect both cases, starting with outside

Schwarzschild metric I

Condition

$$\rho = p = 0$$

Consequences

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad m(r) \equiv M$$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)} \qquad \Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

Schutz (2009, pp. 262–263)

Schwarzschild metric I
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- the external conditions just state we are in a vaccuum
  - breaks down when matter surrounds star
- $m(r)$ is constant, we call it M
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Schwarzschild metric II

First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$

Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

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- recall g_{rr} from earlier
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Far-field metric

Condition

$$r \gg M$$

Far-field Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Far-field Schwarzschild metric (Cartesian)

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

Schutz (2009, pp. 263)

- far-field metric of a star (far away)
  - mass M
  - distance R

Far-field metric

Condition

$$r \gg M$$

Far-field Schwarzschild metric

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Far-field Schwarzschild metric (Cartesian)

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

Schutz (2009, pp. 263)

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If the geometry of a given region of spacetime is:

1. spherically symmetric
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then that geometry is necessarily a subset of the Schwarzschild geometry.

(Proof given in Misner, Thorne, and Wheeler (1973, pp. 843–844))

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Interior structure

Interior structure

- now we look at the remaining, and most interesting regime
  - inside the star
- our assumptions from outside the star no longer hold

Tolman–Oppenheimer–Volkov (T–O–V) equation

Condition

$$\rho \neq 0 \quad p \neq 0$$

Recall

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

T–O–V equation

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- inside a star, we cannot assume density and pressure are zero
- revisit two earlier equations
- substitute one into the other
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System of coupled differential equations

T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

Mass function

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Equation of state

$$p = p(\rho)$$

Schutz (2009, pp. 261–262, 264)

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Schutz (2009, pp. 261–262, 264)

- T–O–V equation coupled with $\mathrm{d}m/\mathrm{d}r$ and $p(\rho)$
  - 3 equations
  - 3 unknowns (m, ρ, p)
  - $\Phi(r)$ only intermediate variable
- can integrate to find $m(r)$, $\rho(r)$, and $p(r)$

Newtonian hydrostatic equilibrium

Newtonian limit

$$p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll r$$

Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -\frac{\rho m(r)}{r^2}$$

Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3)

- in the Newtonian limit we get these constraints
- which allow us to cancel terms in the T–O–V equation
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