

Chapter 5

Preface to Curvature

5.8 Exercises

1

- (a) Repeat the argument leading to Equation 5.1, but this time assume that only a fraction $\epsilon < 1$ of the mass's kinetic energy is converted into a photon.

If only a fraction ϵ of the energy is converted into a photon, then it will start with an energy of $\epsilon(m + mgh + \mathcal{O}(v^4))$, but once it reaches the top it should have an energy of ϵm , as it loses the component due to gravitational potential energy. Thus

$$\frac{E'}{E} = \frac{\epsilon m}{\epsilon(m + mgh + \mathcal{O}(v^4))} = \frac{m}{m + mgh + \mathcal{O}(v^4)} = 1 - gh + \mathcal{O}(v^4)$$

- (b) Assume Equation 5.1 does not hold. Devise a perpetual motion device.

If we assume that the photon does not return to an energy m once it reaches the top, but instead has an energy $m' > m$, then we could create the perpetual motion device shown in Figure 5.1. A black box consumes the photon with energy m' , and splits it into a new object of mass m , and a photon of energy $m' - m$. The object repeats the action of the original falling mass, creating an infinite loop.

2 Explain why a uniform gravitational field would not be able to create tides on Earth.

Tides depend on there being a gravitational field gradient. If the curvature closer to the source of the field (e.g. the Moon) is greater than it is further away, then the closer side will move towards the source more than the further side, thus creating tides. In the absence of such a gradient, there would be no difference in curvature between the two sides, and thus they would not stretch relative to each other.

7 Calculate the components of $\Lambda^{\alpha'}_{\beta}$ and Λ^{μ}_{ν} for transformations $(x, y) \leftrightarrow (r, \theta)$.

$$\begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix} = \begin{pmatrix} \partial r / \partial x & \partial r / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} x/\sqrt{x^2 + y^2} & y/\sqrt{x^2 + y^2} \\ -y/(x^2 + y^2) & x/(x^2 + y^2) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\
&= \begin{pmatrix} \cos \theta & \sin \theta \\ -(1/r) \sin \theta & (1/r) \cos \theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\
&= \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix} \\
&= \begin{pmatrix} x/\sqrt{x^2 + y^2} & -y \\ y/\sqrt{x^2 + y^2} & x \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\Lambda^r_x &= x/\sqrt{x^2 + y^2} = \cos \theta & \Lambda^x_r &= \cos \theta = x/\sqrt{x^2 + y^2} \\
\Lambda^r_y &= y/\sqrt{x^2 + y^2} = \sin \theta & \Lambda^y_r &= \sin \theta = y/\sqrt{x^2 + y^2} \\
\Lambda^\theta_x &= -y/(x^2 + y^2) = -(1/r) \sin \theta & \Lambda^x_\theta &= -r \sin \theta = -y \\
\Lambda^\theta_y &= x/(x^2 + y^2) = (1/r) \cos \theta & \Lambda^y_\theta &= r \cos \theta = x
\end{aligned}$$

8

- (a) $f \equiv x^2 + y^2 + 2xy$, $\vec{V} \xrightarrow{(x,y)} (x^2 + 3y, y^2 + 3x)$, $\vec{W} \xrightarrow{(r,\theta)} (1, 1)$. Express $f = f(r, \theta)$, and find the components of \vec{V} and \vec{W} in a polar basis, as functions of r and θ .

$$\begin{aligned}
f &= x^2 + y^2 + 2xy = (x + y)^2 \\
&= (r \cos \theta + r \sin \theta)^2 = r^2 \sin^2 \theta + r^2 \cos^2 \theta + 2r^2 \sin \theta \cos \theta \\
&= r^2(1 + \sin(2\theta)) \\
\vec{V} &\xrightarrow{(x,y)} \begin{pmatrix} r^2 \cos^2 \theta + 3r \sin \theta \\ r^2 \sin^2 \theta + 3r \cos \theta \end{pmatrix} \\
\vec{V} &\xrightarrow{(r,\theta)} \begin{pmatrix} \cos \theta & \sin \theta \\ -(1/r) \sin \theta & (1/r) \cos \theta \end{pmatrix} \begin{pmatrix} r^2 \cos^2 \theta + 3r \sin \theta \\ r^2 \sin^2 \theta + 3r \cos \theta \end{pmatrix} \\
&\xrightarrow{(r,\theta)} \begin{pmatrix} r^2 \cos^2 \theta + 6r \sin \theta \cos \theta + r^2 \sin^3 \theta \\ -r \cos^2 \theta \sin \theta - 3 \sin^2 \theta + r \sin^2 \theta \cos \theta + 3 \cos^2 \theta \end{pmatrix} \\
&\xrightarrow{(r,\theta)} \begin{pmatrix} r^2(\sin^3 \theta + \cos^3 \theta) + 6r \sin \theta \cos \theta \\ r \sin \theta \cos \theta (\sin \theta - \cos \theta) + 3(\cos^2 \theta - \sin^2 \theta) \end{pmatrix} \\
&\xrightarrow{(r,\theta)} \begin{pmatrix} r^2(\sin^3 \theta + \cos^3 \theta) + 3r \sin(2\theta) \\ (r/2) \sin(2\theta)(\sin \theta - \cos \theta) + 3 \cos(2\theta) \end{pmatrix} \\
\vec{W} &\xrightarrow{(r,\theta)} \begin{pmatrix} \cos \theta & \sin \theta \\ -(1/r) \sin \theta & (1/r) \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&\xrightarrow{(r,\theta)} \begin{pmatrix} \cos \theta + \sin \theta \\ (1/r)(\cos \theta - \sin \theta) \end{pmatrix}
\end{aligned}$$

- (b) Express the components of \tilde{df} in (x, y) and obtain them in (r, θ) by:

(i) using direct calculation in (r, θ) :

$$\tilde{d}f \xrightarrow{(r,\theta)} (\partial f / \partial r, \partial f / \partial \theta) = (2r(1 + \sin(2\theta)), 2r^2 \cos(2\theta))$$

(ii) transforming the components in (x, y) :

$$\tilde{d}f \xrightarrow{(x,y)} (\partial f / \partial x, \partial f / \partial y) = (2(x + y), 2(x + y)) = (2r(\cos \theta + \sin \theta), 2r(\cos \theta + \sin \theta))$$

$$\begin{aligned} ((\tilde{d}f)_r, (\tilde{d}f)_\theta) &= (1, 1) \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} [2r(\cos \theta + \sin \theta)] \\ &= (2r(\cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta), 2r^2(\cos^2 \theta - \sin^2 \theta)) \\ &= (2r(1 + \sin(2\theta)), 2r^2 \cos(2\theta)) \end{aligned}$$

(c) Now find the (r, θ) components of the one-forms \tilde{V} and \tilde{W} associated with the vectors \vec{V} and \vec{W} by

(i) using the metric tensor in (r, θ) :

$$\begin{aligned} V_r &= g_{r\alpha} V^\alpha = g_{rr} V^r + g_{r\theta} V^\theta \\ &= r^2(\sin^3 \theta + \cos^3 \theta) + 3r \sin(2\theta) \\ V_\theta &= g_{\theta r} V^r + g_{\theta\theta} V^\theta = (1/2)r^3 \sin(2\theta)(\sin \theta - \cos \theta) + 3r^2 \cos(2\theta) \end{aligned}$$

$$\begin{aligned} W_r &= g_{r\alpha} W^\alpha = g_{rx} W^x + g_{ry} W^y \\ &= 1(\cos \theta + \sin \theta) + 0[(1/r)(\cos \theta - \sin \theta)] \\ &= \cos \theta + \sin \theta \\ W_\theta &= g_{\theta x} W^x + g_{\theta y} W^y = \\ &= 0(\cos \theta + \sin \theta) + r^2[r(\cos \theta - \sin \theta)] \\ &= r(\cos \theta - \sin \theta) \end{aligned}$$

(ii) using the metric tensor in (x, y) and then doing a coordinate transformation:

$$\begin{aligned} V_x &= V^x; \quad V_y = V^y \\ V_r &= \Lambda^\alpha{}_r V_\alpha = \Lambda^x{}_r V_x + \Lambda^y{}_r V_y \\ &= \cos \theta V_x + \sin \theta V_y \\ &= r^2 \cos^3 \theta + (3/2)r \sin(2\theta) + r^2 \sin^3 \theta + (3/2)r \sin(2\theta) \\ &= r^2(\cos^3 \theta + \sin^3 \theta) + 3r \sin(2\theta) \end{aligned}$$

$$\begin{aligned}
V_\theta &= \Lambda^\alpha_\theta V_\alpha = \Lambda^x_\theta V_x + \Lambda^y_\theta V_y \\
&= (-r \sin \theta) V_x + (r \cos \theta) V_y \\
&= -r^3 \cos^2 \theta \sin \theta - 3r^2 \sin^2 \theta + r^3 \sin^2 \theta \cos \theta + 3r^2 \cos^2 \theta \\
&= r^3 \sin \theta \cos \theta (\sin \theta - \cos \theta) + 3r^2 (\cos^2 \theta - \sin^2 \theta) \\
&= (1/2)r^3 \sin(2\theta)(\sin \theta - \cos \theta) + 3r^2 \cos(2\theta) \\
W_x &= W^x = W_y = W^y = 1 \\
W_r &= \Lambda^\alpha_r W_\alpha = \Lambda^x_r W_x + \Lambda^y_r W_y \\
&= \cos \theta + \sin \theta \\
W_\theta &= \Lambda^\alpha_\theta W_\alpha = \Lambda^x_\theta W_x + \Lambda^y_\theta W_y \\
&= -r \sin \theta + r \cos \theta \\
&= r(\cos \theta - \sin \theta)
\end{aligned}$$

11 Consider $V \xrightarrow{(x,y)} (x^2 + 3y, y^2 + 3x)$.

(a) Find $V_{,\beta}^\alpha$ in Cartesian coordinates.

$$V_{,x}^x = 2x; \quad V_{,y}^y = 2y; \quad V_{,y}^x = V_{,x}^y = 3.$$

(b)

$$\begin{aligned}
V^{\mu'}_{;\nu'} &= \Lambda^{\mu'}_\alpha \Lambda^{\beta}_{\nu'} V^\alpha_{,\beta} \\
V^r_{;r} &= \Lambda^r_x \Lambda^x_r V^x_{,x} + \Lambda^r_y \Lambda^y_r V^y_{,y} + \Lambda^r_x \Lambda^y_r V^x_{,y} + \Lambda^r_y \Lambda^x_r V^y_{,x} \\
&= (\cos^2 \theta)(2r \cos \theta) + (\sin^2 \theta)(2r \sin \theta) + (\sin \theta \cos \theta)(3) + (\sin \theta \cos \theta)(3) \\
&= 2r(\cos^3 \theta + \sin^3 \theta) + 3 \sin(2\theta) \\
V^\theta_{;\theta} &= \Lambda^\theta_x \Lambda^\theta_r V^x_{,x} + \Lambda^\theta_y \Lambda^\theta_r V^y_{,y} + \Lambda^\theta_x \Lambda^y_\theta V^x_{,y} + \Lambda^\theta_y \Lambda^x_\theta V^y_{,x} \\
&= (\sin^2 \theta)(2r \cos \theta) + (\cos^2 \theta)(2r \sin \theta) + (-\sin \theta \cos \theta)(3) + (-\sin \theta \cos \theta)(3) \\
&= \sin(2\theta)[r(\sin \theta + \cos \theta) - 3] \\
V^r_{;\theta} &= \Lambda^r_x \Lambda^x_\theta V^x_{,x} + \Lambda^r_y \Lambda^y_\theta V^y_{,y} + \Lambda^r_x \Lambda^y_\theta V^x_{,y} + \Lambda^r_y \Lambda^x_\theta V^y_{,x} \\
&= (-r \sin \theta \cos \theta)(2r \cos \theta) + (r \sin \theta \cos \theta)(2r \sin \theta) + (r \cos^2 \theta)(3) + (-r \sin^2 \theta) \\
&= r^2 \sin(2\theta)(\sin \theta - \cos \theta) + 3r \cos(2\theta) \\
V^\theta_{;r} &= \Lambda^\theta_x \Lambda^x_r V^x_{,x} + \Lambda^\theta_y \Lambda^y_r V^y_{,y} + \Lambda^\theta_x \Lambda^y_r V^x_{,y} + \Lambda^\theta_y \Lambda^x_r V^y_{,x} \\
&= (-1/r) \sin \theta \cos \theta (2r \cos \theta) + ((1/r) \sin \theta \cos \theta)(2r \sin \theta) + (-1/r) \sin^2 \theta (3) + ((1/r) \cos^2 \theta)(3) \\
&= \sin(2\theta)(\sin \theta - \cos \theta) + \frac{3}{r} \cos(2\theta)
\end{aligned}$$

(c) compute $V^{\mu'}_{;\nu'}$ directly in polars using the Christoffel symbols.

Recall that we have $\Gamma^{\mu}_{rr} = \Gamma^r_{r\theta} = \Gamma^{\theta}_{\theta\theta} = 0$, $\Gamma^{\theta}_{r\theta} = 1/r$, and $\Gamma^r_{\theta\theta} = -r$.

$$\begin{aligned}
 V^{\mu'}_{;\nu'} &= V^{\mu'}_{,\nu'} + V^{\alpha'} \Gamma^{\mu'}_{\alpha'\nu'} \\
 V^r_{;r} &= V^r_{,r} + V^{\alpha'} \Gamma^r_{\alpha'r} \\
 V^r_{,r} &= \partial V^r / \partial r = 2r(\sin^3 \theta + \cos^3 \theta) + 3 \sin(2\theta) \\
 V^{\alpha} \Gamma^r_{\alpha r} &= V^r \Gamma^r_{rr} + V^{\theta} \Gamma^r_{\theta r} = 0 \\
 V^r_{;r} &= V^r_{,r} = 2r(\sin^3 \theta + \cos^3 \theta) + 3 \sin(2\theta) \\
 V^{\theta}_{;\theta} &= V^{\theta}_{,\theta} + V^{\alpha'} \Gamma^{\theta}_{\alpha'\theta} \\
 V^{\theta}_{,\theta} &= \partial V^{\theta} / \partial \theta = (r/2) \sin(2\theta)(\sin \theta + \cos \theta) + r \cos(2\theta)(\sin \theta - \cos \theta) - 6 \sin(2\theta) \\
 V^{\alpha'} \Gamma^{\theta}_{\alpha'\theta} &= V^r \Gamma^{\theta}_{r\theta} + V^{\theta} \Gamma^{\theta}_{\theta\theta} \\
 &= [r^2(\sin^3 \theta + \cos^3 \theta) + 3r \sin(2\theta)](1/r) \\
 &= r(\sin^3 \theta + \cos^3 \theta) + 3 \sin(2\theta) \\
 V^{\theta}_{;\theta} &= \sin(2\theta)[r(\sin \theta + \cos \theta) - 3] \\
 V^r_{;\theta} &= V^r_{,\theta} + V^r \Gamma^r_{r\theta} + V^{\theta} \Gamma^r_{\theta\theta} \\
 &= V^r_{,\theta} + V^{\theta} \Gamma^r_{\theta\theta} = \partial V^r / \partial \theta - rV^{\theta} \\
 &= 6r \cos(2\theta) + (3/2)r^2 \sin(2\theta)(\sin \theta - \cos \theta) - ((1/2)r^2 \sin(2\theta)(\sin \theta - \cos \theta) + 3r \cos(2\theta)) \\
 &= r^2 \sin(2\theta)(\sin \theta - \cos \theta) + 3r \cos(2\theta) \\
 V^{\theta}_{;r} &= V^{\theta}_{,r} + V^r \Gamma^{\theta}_{rr} + V^{\theta} \Gamma^{\theta}_{\theta r} = V^{\theta}_{,r} + \frac{1}{r} V^{\theta} \\
 &= (1/2) \sin(2\theta)(\sin \theta - \cos \theta) + (1/2) \sin(2\theta)(\sin \theta - \cos \theta) + (3/r) \cos(2\theta) \\
 &= \sin(2\theta)(\sin \theta - \cos \theta) + (3/r) \cos(2\theta)
 \end{aligned}$$

(d) Calculate the divergence using the results from part (a)

$$V^{\alpha}_{,\alpha} = V^x_{,x} + V^y_{,y} = 2(x + y) = 2r(\sin \theta + \cos \theta)$$

(e) Calculate the divergence using the results from either part (b) or (c).

$$\begin{aligned}
 V^{\mu'}_{;\mu'} &= V^r_{;r} + V^{\theta}_{;\theta} \\
 &= 2r(\sin^3 \theta + \cos^3 \theta) + 3 \sin(2\theta) + \sin(2\theta)[r(\sin \theta + \cos \theta) - 3] \\
 &= 2r(\sin \theta + \cos \theta)
 \end{aligned}$$

(f) Compute $V^{\mu'}_{;\mu'}$ using Equation 5.56.

$$V^{\mu'}_{;\mu'} = \frac{1}{r} \frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial \theta}(V^{\theta}) = 2r(\sin \theta + \cos \theta)$$

12

$$\tilde{p}_{(x,y)} \rightarrow (x^2 + 3y, y^2 + 3x).$$

(a) Find the components $p_{\alpha,\beta}$ in Cartesian coordinates.Since $p_{\alpha,\beta} = \partial p_\alpha / \partial x^\beta$, it's simply $p_{x,x} = 2x$, $p_{y,y} = 2y$, and $p_{x,y} = p_{y,x} = 3$.(b) Find the components $p_{\mu';\nu'}$ in polar coordinates by using the transformation $\Lambda^\alpha_{\mu'} \Lambda^\beta_{\nu'} p_{\alpha,\beta}$.

$$\begin{aligned} p_{r;r} &= (\Lambda^x_r)^2 p_{x,x} + (\Lambda^y_r)^2 p_{y,y} + 2\Lambda^x_r \Lambda^y_r p_{x,y} \\ &= (\cos^2 \theta)(2r \cos \theta) + (\sin^2 \theta)(2r \sin \theta) + 2(\sin \theta \cos \theta)(3) \\ &= 2r(\sin^3 \theta + \cos^3 \theta) + 3 \sin(2\theta) \\ p_{\theta;\theta} &= (\Lambda^x_\theta)^2 p_{x,x} + (\Lambda^y_\theta)^2 p_{y,y} + 2\Lambda^x_\theta \Lambda^y_\theta p_{x,y} \\ &= (-r \sin \theta)^2 (2r \cos \theta) + (r \cos \theta)^2 (2r \sin \theta) + 2(3(-r \sin \theta)(r \cos \theta)) \\ &= r^2 \sin(2\theta)(r(\sin \theta + \cos \theta) - 3) \\ p_{r;\theta} &= \Lambda^x_r \Lambda^x_\theta p_{x,x} + \Lambda^y_r \Lambda^y_\theta p_{y,y} + \Lambda^x_r \Lambda^y_\theta p_{x,y} + \Lambda^y_r \Lambda^x_\theta p_{y,x} \\ &= (-r \sin \theta \cos \theta)(2r \cos \theta) + (r \sin \theta \cos \theta)(2r \sin \theta) + 3(r \cos^2 \theta - r \sin^2 \theta) \\ &= r^2 \sin(2\theta)(\sin \theta - \cos \theta) + 3r \cos(2\theta), \end{aligned}$$

and by the symmetry of $p_{\alpha,\beta}$ in Cartesian coordinates, $p_{\theta;r} = p_{r;\theta}$.(c) Now find $p_{\mu';\nu'}$ using the Christoffel symbols.

$$\begin{aligned} p_{r;r} &= p_{r,r} - p_r \Gamma^r_{rr} - p_\theta \Gamma^\theta_{rr} = p_{r,r} = \partial p_r / \partial r \\ &= \partial / \partial r \left[r^2(\cos^3 \theta + \sin^3 \theta) + 3r \sin(2\theta) \right] = 2r(\sin^3 \theta + \cos^3 \theta) + 3 \sin(2\theta) \\ p_{\theta;\theta} &= p_{\theta,\theta} - p_r \Gamma^r_{\theta\theta} - p_\theta \Gamma^\theta_{\theta\theta} = p_{\theta,\theta} + r p_r = \partial p_\theta / \partial \theta \\ &= \partial / \partial \theta \left[(1/2)r^3 \sin(2\theta)(\sin \theta - \cos \theta) + 3r^2 \cos(2\theta) \right] + r \left[r^2(\cos^3 \theta + \sin^3 \theta) + 3r \sin(2\theta) \right] \\ &= r^2 \sin(2\theta)[r(\sin \theta + \cos \theta) - 3] \\ p_{r;\theta} &= p_{r,\theta} - p_r \Gamma^r_{r\theta} - p_\theta \Gamma^\theta_{r\theta} = \partial p_r / \partial \theta - (1/r)p_\theta \\ &= r^2 \sin(2\theta)(\sin \theta - \cos \theta) + 3r \cos(2\theta) \\ p_{\theta;r} &= p_{\theta,r} - p_r \Gamma^r_{\theta r} - p_\theta \Gamma^\theta_{\theta r} = \partial p_\theta / \partial r - (1/r)p_\theta \\ &= r^2 \sin(2\theta)(\sin \theta - \cos \theta) + 3r \cos(2\theta) \end{aligned}$$

13 Show in polars that $g_{\mu'\nu'} V^{\alpha'}_{;\nu'} = p_{\mu';\nu'}$.

$$g_{r\alpha'} V^{\alpha'}_{;r} = g_{rr} V^r_{;r} + g_{r\theta} V^\theta_{;r}$$

$$\begin{aligned}
&= 1V_{;r}^r = p_{r;r} \\
g_{\theta\alpha'} V_{;\theta}^{\alpha'} &= g_{\theta r} V_{;\theta}^r + g_{\theta\theta} V_{;\theta}^\theta \\
&= r^2 V_{;\theta}^\theta = p_{\theta;\theta} \\
g_{r\alpha'} V_{;\theta}^{\alpha'} &= g_{rr} V_{;\theta}^r + g_{r\theta} V_{;\theta}^\theta \\
&= 1V_{;\theta}^r = p_{\theta;r} \\
g_{\theta\alpha'} V_{;r}^{\alpha'} &= g_{\theta r} V_{;r}^r + g_{\theta\theta} V_{;r}^\theta \\
&= r^2 V_{;r}^\theta = p_{\theta;r}
\end{aligned}$$

14 Compute $\nabla_\beta A^{\mu\nu}$ for the tensor A with components:

$$\begin{aligned}
A^{rr} &= r^2, & A^{r\theta} &= r \sin \theta, \\
A^{\theta\theta} &= \tan \theta, & A^{\theta r} &= r \cos \theta
\end{aligned}$$

$$\begin{aligned}
A^{rr}_{,r} &= 2r & A^{rr}_{,\theta} &= 0 \\
A^{\theta\theta}_{,r} &= 0 & A^{\theta\theta}_{,\theta} &= \sec^2 \theta \\
A^{r\theta}_{,r} &= \sin \theta & A^{r\theta}_{,\theta} &= r \cos \theta \\
A^{\theta r}_{,r} &= \cos \theta & A^{\theta r}_{,\theta} &= -r \sin \theta
\end{aligned}$$

$$\nabla_\beta A^{\mu\nu} = A^{\mu\nu}_{,\beta} + A^{\alpha\nu} \Gamma^\mu_{\alpha\beta} + A^{\mu\alpha} \Gamma^\nu_{\alpha\beta}$$

$$\begin{aligned}
\nabla_r A^{rr} &= A^{rr}_{,r} + A^{\alpha r} \Gamma^r_{\alpha r} + A^{r\alpha} \Gamma^r_{\alpha r} \\
&= A^{rr}_{,r} + A^{rr} \Gamma^r_{rr} + A^{\theta r} \Gamma^r_{\theta r} + A^{rr} \Gamma^r_{rr} + A^{r\theta} \Gamma^r_{\theta r} \\
&= A^{rr}_{,r} = 2r
\end{aligned}$$

$$\begin{aligned}
\nabla_\theta A^{rr} &= A^{rr}_{,\theta} + A^{\alpha r} \Gamma^r_{\alpha\theta} + A^{r\alpha} \Gamma^r_{\alpha\theta} \\
&= A^{rr}_{,\theta} + A^{rr} \Gamma^r_{r\theta} + A^{\theta r} \Gamma^r_{\theta\theta} + A^{rr} \Gamma^r_{r\theta} + A^{r\theta} \Gamma^r_{\theta\theta} \\
&= (A^{\theta r} + A^{r\theta}) \Gamma^r_{\theta\theta} = -r^2 (\sin \theta + \cos \theta)
\end{aligned}$$

$$\begin{aligned}
\nabla_r A^{\theta\theta} &= A^{\theta\theta}_{,r} + A^{\alpha\theta} \Gamma^\theta_{\alpha r} + A^{\theta\alpha} \Gamma^\theta_{\alpha r} \\
&= A^{\theta\theta}_{,r} + A^{r\theta} \Gamma^\theta_{rr} + A^{\theta\theta} \Gamma^\theta_{\theta r} + A^{\theta r} \Gamma^\theta_{rr} + A^{\theta\theta} \Gamma^\theta_{\theta r} \\
&= 2A^{\theta\theta} \Gamma^\theta_{\theta r} = (2/r) \tan \theta
\end{aligned}$$

$$\begin{aligned}
\nabla_\theta A^{\theta\theta} &= A^{\theta\theta}_{,\theta} + A^{\alpha\theta} \Gamma^\theta_{\alpha\theta} + A^{\theta\alpha} \Gamma^\theta_{\alpha\theta} \\
&= A^{\theta\theta}_{,\theta} + A^{r\theta} \Gamma^\theta_{r\theta} + A^{\theta\theta} \Gamma^\theta_{\theta\theta} + A^{\theta r} \Gamma^\theta_{r\theta} + A^{\theta\theta} \Gamma^\theta_{\theta\theta} \\
&= A^{\theta\theta}_{,\theta} + (A^{r\theta} + A^{\theta r}) \Gamma^\theta_{r\theta} = \sin \theta + \cos \theta + \sec^2 \theta
\end{aligned}$$

$$\nabla_r A^{r\theta} = A^{r\theta}_{,r} + A^{\alpha\theta} \Gamma^r_{\alpha r} + A^{r\alpha} \Gamma^r_{\alpha r}$$

$$\begin{aligned}
&= A^{r\theta}_{,r} + A^{r\theta}\Gamma_{rr}^r + A^{\theta\theta}\Gamma_{\theta r}^r + A^{rr}\Gamma_{rr}^\theta + A^{r\theta}\Gamma_{\theta r}^\theta \\
&= A^{r\theta}_{,r} + A^{r\theta}\Gamma_{\theta r}^\theta = 2\sin\theta \\
\nabla_\theta A^{r\theta} &= A^{r\theta}_{,\theta} + A^{\alpha\theta}\Gamma_{\alpha\theta}^r + A^{r\alpha}\Gamma_{\alpha\theta}^\theta \\
&= A^{r\theta}_{,\theta} + A^{r\theta}\Gamma_{r\theta}^r + A^{\theta\theta}\Gamma_{\theta\theta}^r + A^{rr}\Gamma_{r\theta}^\theta + A^{r\theta}\Gamma_{\theta\theta}^\theta \\
&= A^{r\theta}_{,\theta} + A^{\theta\theta}\Gamma_{\theta\theta}^r + A^{rr}\Gamma_{r\theta}^\theta = r(1 + \cos\theta - \tan\theta) \\
\nabla_r A^{\theta r} &= A^{\theta r}_{,r} + A^{\alpha r}\Gamma_{\alpha r}^\theta + A^{\theta\alpha}\Gamma_{\alpha r}^r \\
&= A^{\theta r}_{,r} + A^{rr}\Gamma_{rr}^\theta + A^{\theta r}\Gamma_{\theta r}^\theta + A^{\theta r}\Gamma_{rr}^r + A^{\theta\theta}\Gamma_{\theta r}^r \\
&= A^{\theta r}_{,r} + A^{\theta r}\Gamma_{\theta r}^\theta = 2\cos\theta \\
\nabla_\theta A^{\theta r} &= A^{\theta r}_{,\theta} + A^{\alpha r}\Gamma_{\alpha\theta}^r + A^{\theta\alpha}\Gamma_{\alpha\theta}^r \\
&= A^{\theta r}_{,\theta} + A^{rr}\Gamma_{r\theta}^r + A^{\theta r}\Gamma_{\theta\theta}^r + A^{\theta r}\Gamma_{r\theta}^r + A^{\theta\theta}\Gamma_{\theta\theta}^r \\
&= A^{\theta r}_{,\theta} + A^{rr}\Gamma_{r\theta}^r + A^{\theta\theta}\Gamma_{\theta\theta}^r = -r\sin\theta
\end{aligned}$$

15 Find the components of $V_{;\beta;\gamma}^\alpha$ for the vector $V^r = 1$, $V^\theta = 0$.

We start by finding the components of $V_{;\beta}^\alpha$.

$$V_{;\beta}^\alpha = V_{,\beta}^\alpha + V^\mu \Gamma_{\mu\beta}^\alpha.$$

By noting that $V_{,\beta}^\alpha = V^\theta = \Gamma_{rr}^r = \Gamma_{r\theta}^r = 0$, we can simplify this to

$$V_{;\beta}^\alpha = V^r \Gamma_{r\beta}^\alpha,$$

which means

$$V_{;r}^r = V_{;\theta}^\theta = V_{;r}^\theta = 0; \quad V_{;\theta}^\theta = \frac{1}{r}.$$

Now we can say

$$V_{;\beta;\mu}^\alpha = \nabla_\mu V_{;\beta}^\alpha = V_{;\beta,\mu}^\alpha + V_{;\beta}^\gamma \Gamma_{\gamma\mu}^\alpha - V_{;\gamma}^\alpha \Gamma_{\beta\mu}^\gamma.$$

Note that $V_{;\theta}^\theta$ is a function only of r , and so $V_{;\theta,r}^\theta = -1/r^2$, and all other partial derivatives are zero.

We can also see by inspecting the components, that $V_{;\mu;\nu}^\theta = V_{;\mu}^\theta \Gamma_{\nu\theta}^r$, as all other components go to zero.

Likewise, we can see that $V_{;r;\mu}^\theta = -V_{;\theta}^\theta \Gamma_{r\mu}^r$. It then becomes easy to find all the individual components.

I summarize their values in Table 5.1.

16 Repeat the steps leading from Equation 5.74 to 5.75.

Recalling that $g_{\alpha\mu;\beta} = 0$, we can rewrite Equation 5.72 as

$$g_{\alpha\beta,\mu} = \Gamma^\nu_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu_{\beta\mu} g_{\alpha\nu}.$$

Now if we switch the β and μ indices, and then switch the α and β indices, we get two more equations,

$$\begin{aligned} g_{\alpha\mu,\beta} &= \Gamma^\nu{}_{\alpha\beta}g_{\nu\mu} + \Gamma^\nu{}_{\mu\beta}g_{\alpha\nu}, \\ g_{\beta\mu,\alpha} &= \Gamma^\nu{}_{\beta\alpha}g_{\nu\mu} + \Gamma^\nu{}_{\mu\alpha}g_{\beta\nu}. \end{aligned}$$

Now we add the first two equations and subtract the third, getting

$$\begin{aligned} g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \Gamma^\nu{}_{\alpha\mu}g_{\nu\beta} + \Gamma^\nu{}_{\beta\mu}g_{\alpha\nu} + \Gamma^\nu{}_{\alpha\beta}g_{\nu\mu} + \Gamma^\nu{}_{\mu\beta}g_{\alpha\nu} - \Gamma^\nu{}_{\beta\alpha}g_{\nu\mu} - \Gamma^\nu{}_{\mu\alpha}g_{\beta\nu} \\ &= \textcolor{red}{\Gamma^\nu{}_{\alpha\mu}g_{\beta\nu}} + \textcolor{green}{\Gamma^\nu{}_{\beta\mu}g_{\alpha\nu}} + \textcolor{blue}{\Gamma^\nu{}_{\alpha\beta}g_{\nu\mu}} + \textcolor{green}{\Gamma^\nu{}_{\beta\mu}g_{\alpha\nu}} - \textcolor{red}{\Gamma^\nu{}_{\alpha\beta}g_{\nu\mu}} - \textcolor{blue}{\Gamma^\nu{}_{\alpha\mu}g_{\beta\nu}} \\ &= 2\Gamma^\nu{}_{\beta\mu}g_{\alpha\nu}. \end{aligned}$$

Recalling that $g^{\alpha\gamma}g_{\alpha\nu} = g^\gamma{}_\nu = \delta^\gamma{}_\nu$, we divide both sides by 2 and multiply by $g^{\alpha\gamma}$, arriving at Equation 5.75:

$$\begin{aligned} \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) &= \frac{2}{2}g^{\alpha\gamma}g_{\alpha\nu}\Gamma^\nu{}_{\beta\mu} \\ &= \Gamma^\nu{}_{\beta\mu} \end{aligned}$$

17 Show how $V^\beta{}_{,\alpha}$ and $V^\mu\Gamma^\beta{}_{\nu\alpha}$ transform under change of coordinates. Neither follows a tensor transformation law, but their sum does.

$$\begin{aligned} V^{\alpha'}{}_{,\beta'} &= \frac{\partial V^{\alpha'}}{\partial x^{\beta'}} = \Lambda^\beta{}_{\beta'} \frac{\partial}{\partial x^\beta} [\Lambda^{\alpha'}{}_\alpha V^\alpha] \\ &= \Gamma^\beta{}_{\beta'} \left[V^\alpha \frac{\partial}{\partial x^\beta} \Lambda^{\alpha'}{}_\alpha + \Lambda^{\alpha'}{}_\alpha \frac{\partial}{\partial x^\beta} V^\alpha \right] \\ &= \Lambda^\beta{}_{\beta'} V^\alpha \Lambda^{\alpha'}{}_{\alpha,\beta} + \Lambda^\beta{}_{\beta'} \Lambda^{\alpha'}{}_\alpha V^\alpha{}_{,\beta} \\ &\neq \Lambda^\beta{}_{\beta'} \Lambda^{\alpha'}{}_\alpha V^\alpha{}_{,\beta} \\ \frac{\partial \vec{e}_{\alpha'}}{\partial x^{\beta'}} &= \Lambda^\beta{}_{\beta'} \frac{\partial}{\partial x^\beta} [\Lambda^\alpha{}_{\alpha'} \vec{e}_\alpha] \\ &= \Lambda^\beta{}_{\beta'} \left[\Lambda^\alpha{}_{\alpha'} \frac{\partial}{\partial x^\beta} \vec{e}_\alpha + \vec{e}_\alpha \frac{\partial}{\partial x^\beta} \Lambda^\alpha{}_{\alpha'} \right] \\ &= \Lambda^\beta{}_{\beta'} \Lambda^\alpha{}_{\alpha'} \Gamma^\mu{}_{\alpha\beta} \vec{e}_\mu + \Lambda^\beta{}_{\beta'} \Lambda^\alpha{}_{\alpha',\beta} \vec{e}_\alpha \end{aligned}$$

α	β	μ	$V^\alpha{}_{;\beta;\mu}$
θ	θ	θ	0
θ	θ	r	$-1/r^2$
θ	r	θ	$-1/r^2$
θ	r	r	0
r	θ	θ	-1
r	θ	r	0
r	r	θ	0
r	r	r	0

Table 5.1: Components of the tensor in Exercise 15.

$$\neq \Lambda^\beta_{\beta'} \Lambda^\alpha_{\alpha'} \Gamma^\mu_{\alpha\beta} \vec{e}_\mu,$$

so we have shown that $\partial \vec{e}_{\alpha'} / \partial x^{\beta'}$ is not a tensor, and since V^μ is a tensor, and the product of a tensor and a non-tensor is also not a tensor, then $V^\mu \Gamma^\beta_{\nu\alpha}$ is not a tensor.

According to Carroll, the precise transformation is

$$\Gamma'^{\mu'}_{\mu'\lambda'} = \Lambda^\mu_{\mu'} \Lambda^\lambda_{\lambda'} \Lambda'^{\nu'}_{\nu} \Gamma^\nu_{\mu\lambda} + \Lambda^\mu_{\mu'} \Lambda^\lambda_{\lambda'} \Lambda'^{\nu'}_{\mu\lambda}.$$

Now we add the two expressions, in order to show that it is a tensor equation

$$\begin{aligned} V'^{\mu'}_{,\lambda'} + V^{\mu'} \Gamma'^{\nu'}_{\mu'\lambda'} &= \Lambda^\lambda_{\lambda'} V^\nu \Lambda'^{\nu'}_{\nu,\lambda} + \Lambda^\lambda_{\lambda'} \Lambda'^{\nu'}_{\nu} V^\nu_{,\lambda} + \Lambda^\lambda_{\lambda'} \Lambda'^{\nu'}_{\nu} V^\mu \Gamma^\nu_{\mu\lambda} + \Lambda^\lambda_{\lambda'} V^\nu \Lambda'^{\nu'}_{\lambda,\mu} \\ &= \Lambda^\lambda_{\lambda'} \Lambda'^{\nu'}_{\nu} (V^\nu_{,\lambda} + V^\mu \Gamma^\nu_{\mu\lambda}) \end{aligned}$$

So it does in fact transform like a tensor equation, meaning $V^\nu_{;\lambda}$ is a tensor!

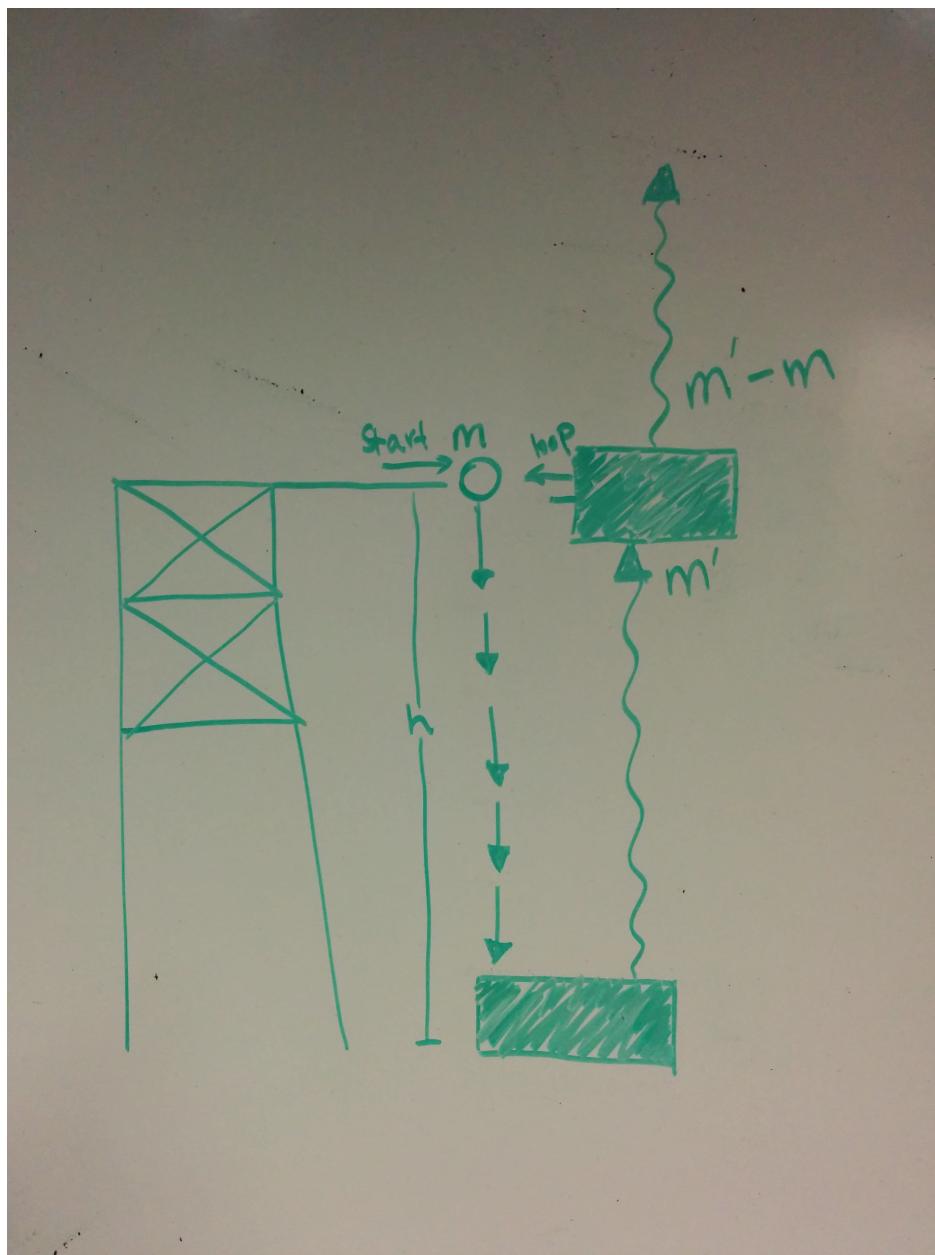


Figure 5.1: Problem 1: Perpetual motion device.