Uncertainty Principle

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Generalized Uncertainty Principle

- for any observable A, the variance can be expressed by

$$\sigma_A^2 = \left\langle (\hat{A} - \langle A \rangle)\Psi \middle| (\hat{A} - \langle A \rangle)\Psi \right\rangle$$

- we define $f := (\hat{A} - \langle A \rangle)\Psi$, and $g := (\hat{B} - \langle B \rangle)\Psi$ for any other observable B
- multiplying the variance of two observables we have

$$\sigma_A^2 \sigma_B^2 = \langle f|f \rangle \langle g|g \rangle \geq \left| \langle f|g \rangle \right|^2$$

- for any complex number z, we have

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2 = \left[\frac{1}{2i}(z - z^*) \right]^2$$

Generalized Uncertainty Principle

- now if we let $z = \langle f|g\rangle$ we have

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2$$

$$\begin{aligned}
\langle f|g\rangle &= \left\langle (\hat{A} - \langle A\rangle)\Psi \middle| (\hat{B} - \langle B\rangle)\Psi \right\rangle \\
&= \left\langle \Psi \middle| (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\Psi \right\rangle \\
&= \left\langle \Psi \middle| (\hat{A}\hat{B} - \hat{A}\langle B\rangle - \hat{B}\langle A\rangle + \langle A\rangle\langle B\rangle)\Psi \right\rangle \\
&= \left\langle \Psi \middle| \hat{A}\hat{B}\Psi \right\rangle - \langle B\rangle \left\langle \Psi \middle| \hat{A}\Psi \right\rangle - \langle A\rangle \left\langle \Psi \middle| \hat{B}\Psi \right\rangle + \langle A\rangle\langle B\rangle\langle \Psi|\Psi\rangle \\
&= \left\langle \hat{A}\hat{B} \right\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle \\
&= \left\langle \hat{A}\hat{B} \right\rangle - \langle A\rangle\langle B\rangle
\end{aligned}$$

Generalized Uncertainty Principle

- thus $\langle f|g\rangle = \langle \hat{A}\hat{B}\rangle - \langle A\rangle \langle B\rangle$ and by the same process $\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle B\rangle \langle A\rangle$
- so

$$\langle f|g\rangle - \langle g|f\rangle = \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle = \langle [\hat{A}, \hat{B}]\rangle$$

- therefore we have the general uncertainty principle:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}]\rangle \right)^2$$

Uncertainty Principle of Momentum and Position

- consider the Hermitian operators $\hat{x}$ and $\hat{p}$ whose conjugate is $[\hat{x}, \hat{p}] = i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} \langle [\hat{x}, \hat{p}] \rangle \right)^2$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{\hbar}{2} \right)^2$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Incompatible Observables

- there is an "uncertainty principle" for every pair of observables A and B such that $[A, B] \neq 0$
  - incompatible observables
- they cannot have a complete set of common eigenfunctions
- in contrast, *compatible* observables do have complete sets of simultaneous eigenfunctions, and therefore have uncertainty principles which take the form

$$\sigma_A^2 \sigma_B^2 \geq 0$$

Compatible Observables Example

- for example, we will find $\sigma_T \sigma_p$, where T and p are the kinetic energy and momentum, respectively
- $\hat{T} = \hat{p}^2/2m$

$$
\begin{aligned}
[\hat{T}, \hat{p}] &= \hat{T}\hat{p} - \hat{p}\,\hat{T} \\
&= \frac{\hat{p}^2}{2m}\hat{p} - \hat{p}\frac{\hat{p}^2}{2m} \\
&= \frac{\hat{p}^3}{2m} - \frac{\hat{p}^3}{2m} = 0
\end{aligned}
$$

$$\implies \sigma_T \sigma_p \geq 0$$

The Minimum-Uncertainty Wave Packet

- the Gaussian wave packet and the ground state of the harmonic oscillator reach the position-momentum uncertainty limit
- question: is there a most general minimum-uncertainty wave packet?
- in the derivation of the uncertainty principle, two inequalities are used; if we restrict this to equality, minimum values can be found.
- the Schwartz inequality becomes an equality when one function is a multiple of the other, $f(x) = cg(x)$, $c \in \mathbb{C}$

The Minimum-Uncertainty Wave Packet

- After the use of the Schwarz inequality, we used that $\mathrm{Re}(z) + \mathrm{Im}(z) \geq \mathrm{Im}(z)$, for some z. Making this an equality implies $\mathrm{Re}(z) = 0$, and applying this to our functions f and g we have

$$\mathrm{Re}(\langle f|g\rangle) = \mathrm{Re}(c\,\langle g|g\rangle) = 0$$

  Since $\langle g|g\rangle \neq 0$, c must only have an imaginary part, ia
- Applying this to the position-momentum uncertainty yields:

$$\left(\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x} - \langle p\rangle\right)\Psi = ia(x - \langle x\rangle)\Psi$$

- This is a differential equation with general solution

$$\Psi(x) = A\exp\Big(-a(x - \langle x\rangle)^2/2\hbar\Big)\exp\big(i\,\langle p\rangle\,x/\hbar\big)$$

- This is the form of a gaussian, thus the most general minimum uncertainty wave packet is a gaussian.

The Energy–Time Uncertainty Principle

- the position–momentum uncertainty principle is often written as

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

- it is often accompanied by the energy–time uncertainty principle

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

- don't be fooled, they may look similar but are entirely different
- position, momentum, and energy are all dynamical variables, while time is an independent variable
- now we will work towards deriving it

The Energy–Time Uncertainty Principle

- we begin by computing the time derivative of the expectation value of an observable, $Q(x, p, t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \langle \Psi | \hat{Q} \Psi \rangle$$
$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q} \Psi \right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t} \Psi \right\rangle + \left\langle \Psi \middle| \hat{Q} \frac{\partial \Psi}{\partial t} \right\rangle$$

- use the Schrödinger equation to substitute for $\frac{\partial \Psi}{\partial t}$

$$\imath\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

The Energy–Time Uncertainty Principle

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = -\frac{1}{\imath\hbar} \langle \hat{H}\Psi | \hat{Q}\Psi \rangle = +\frac{1}{\imath\hbar} \left\langle \Psi \middle| \hat{Q}\hat{H}\Psi \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

- $\hat{H}$ is hermitian, so $\langle \hat{H}\Psi | \hat{Q}\Psi \rangle = \langle \Psi | \hat{H}\hat{Q}\Psi \rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{\imath}{\hbar} \left\langle [\hat{H}, \hat{Q}] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

- if $\hat{Q}$ is time-independent, the rate of change of the expectation value is determined by the commutator of $\hat{Q}$ with $\hat{H}$
- $[\hat{H}, \hat{Q}] = 0 \implies \langle Q \rangle$ is constant

The Energy–Time Uncertainty Principle

- using $A = H$ and $B = Q$ in the generalized uncertainty principle, and assuming Q is time-independent, we see that

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2\imath} \left\langle [\hat{H}, \hat{Q}] \right\rangle \right)^2 = \left(\frac{1}{2\imath} \frac{\hbar}{\imath} \frac{\mathrm{d} \left\langle Q \right\rangle}{\mathrm{d}t} \right)^2 = \left(\frac{\hbar}{2} \right)^2 \left(\frac{\mathrm{d} \left\langle Q \right\rangle}{\mathrm{d}t} \right)^2$$

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{\mathrm{d} \left\langle Q \right\rangle}{\mathrm{d}t} \right|$$

$$\Delta E := \sigma_H, \text{ and } \Delta t := \frac{\sigma_Q}{\left| \mathrm{d}\langle Q \rangle \,/\, \mathrm{d}t \right|}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

The Energy–Time Uncertainty Principle

- Δt represents the amount of time it takes the expectation value of Q to change by one standard deviation

$$\sigma_Q = \left| \frac{\mathrm{d} \langle Q \rangle}{\mathrm{d}t} \right| \Delta t$$

- Δt depends on the observable, Q, being observed
- if ΔE is small, the rate of change for *all* observables must be small
- if *any* observable changes rapidly, the "uncertainty" in the energy must be large