$$\rho(\pi^*, a) = \mathbb{E}_z[\mathbb{1}(a \neq z)]$$

$$= \sum_x \pi^*_x \mathbb{1}(a \neq z) = 1 - \pi^*_a$$

$$\mathbb{E}_{z \sim \pi}\left[\mathbb{E}_{y \sim \underline{Pr(y|z)}} [\ell(\delta(y), z)] \right]$$

$$= \mathbb{E}_{y \sim p(y)} \left[\mathbb{E}_{z \sim \pi^*(z|y)} [\ell(\delta(y), z)] \right]$$

$$= \arg\min_{\delta} \quad '' \qquad ''$$

$$= \delta_\pi(y) = \arg\min_a \mathbb{E}_z [\ell(a, z)]$$

"Bayes rule" for π

$q \sim Beta(\alpha, \beta)$

$y \sim Binom(n, q)$

$$P(q \mid -) \propto_q \frac{q^{\alpha-1}(1-q)^{\beta-1}}{B(\alpha,\beta)} \binom{n}{y} q^{y}(1-q)^{n-y}$$

$$\propto_q q^{\alpha+y-1}(1-q)^{\beta+n-y-1}$$

$$\propto_q Beta(q; \alpha+y, \beta+n-y)$$

$$\rho(\pi^*, a) = \mathbb{E}_{q\sim\pi^*}\left[\mathbb{E}_{z\sim\pi^*}\left[\ell_q(z,a)\right]\right]$$

$$= \mathbb{E}_q\left[\pi^*_a(1-q_a)\ell_{FTP} + (1-\pi_a)\ell_{FP}\right]$$

$$= \pi^*_a(1-\mathbb{E}_q[q_a])\ell_{FTP} + (1-\pi_c)\ell_{FP}$$

posterior mean

$b_1 \ldots b_n$ Bernoulli trials

$$P(b_1 = 1) = \int q \, \mathrm{Beta}(q; \alpha, \beta)$$

$$= \frac{\alpha}{\alpha+\beta}$$

$$= \int q \, \frac{q^{\alpha-1}(1-q)^{\beta-1}}{B(\alpha,\beta)}$$

$$= \frac{1}{B(\alpha,\beta)} \int q^{\alpha}(1-q)^{\beta-1}$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$P(b_1) = \frac{B(\alpha+b_1, \beta+(1-b_1))}{B(\alpha,\beta)}$$

Posterior pred.

$$P(b_2 \mid b_1) = \int q^{b_2}(1-q)^{1-b_2} P(q \mid b_1)$$

$$= \int q^{b_2}(1-q)^{1-b_2} \mathrm{Beta}(q; \alpha+b_1, \beta+(1-b_1))$$

$$= \frac{B(\alpha + b_1 + b_2, \beta + (1-b_1) + (1-b_2))}{B(\alpha + b_1, \beta + (1-b_1))}$$

$$P(b_i \mid b_{<i}) = \frac{B\left(\alpha + \sum_{j=1}^{i} b_j, \beta + i - \sum_{j=1}^{i} b_j\right)}{B\left(\alpha + \sum_{j=1}^{i-1} b_j, \beta + i - 1 - \sum_{j=1}^{i-1} b_j\right)}$$

$$P(b_1 \cdots b_n) = \prod_{i=1}^{n} \frac{B\left(\alpha + \sum_{j=1}^{i} b_j, \beta + i - \sum_{j=1}^{i} b_j\right)}{B\left(\alpha + \sum_{j=1}^{i-1} b_j, \beta + i - 1 - \sum_{j=1}^{i-1} b_j\right)}$$

$$= \frac{\cancel{B(\alpha + b_1, \beta + 1 - b_1)}}{B(\alpha, \beta)} \times \frac{B\cancel{(\alpha + b_1 + b_2, \beta + 2 - (b_1 + b_2))}}{\cancel{B(\alpha + b_1, \beta + 1 - b_1)}}$$

$$\times \cdots \frac{B\left(\alpha + \sum_{i=1}^{n} b_i, \beta + n - \sum_{i=1}^{n} b_i\right)}{B\cancel{\left(\alpha + \sum_{i=1}^{n-1} b_i, \beta + n - 1 - \sum_{i=1}^{n} b_i\right)}}$$

telescoping

$$= \frac{B\left(\alpha + \sum_{i=1}^{n} b_i, \beta + n - \sum_{i=1}^{n} b_i\right)}{B(\alpha, \beta)}$$

$$P\left(\sum_{i=1}^{n} b_i = y\right) = \sum_{b_1=0}^{1} \cdots \sum_{b_n=0}^{1} P(b_1 \cdots b_n)\, \mathbb{1}\left(\sum_i b_i = y\right)$$

De Finetti: $P(b_1 \cdots b_n) = P(b_{\sigma(1)} \cdots b_{\sigma(n)})$

(obvious from looking at the PMF, but why?)

↳ we should expect sufficient stats invariant to order!

$$= \frac{B(\alpha+y, \beta+n-y)}{B(\alpha, \beta)} \#\left(\vec{b} : \sum_{i=1}^{n} b_i = y\right)$$

"number of binary vectors of length n whose sum equals y"

$$= \binom{n}{y} = \frac{n!}{y!(n-y)!}$$

$$\text{BetaBinom}(y; n, \alpha, \beta) = \frac{B(\alpha+y, \beta+n-y)}{B(\alpha, \beta)} \binom{n}{y}$$