STAT 301: Homework set 7

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Problem 1

Subproblem 1

$$\int f(\frac{p}{q})q \geq f(\int \frac{p}{q}q) = f(1) = 0$$

Subproblem 2

Hellinger-squared $f(x) = (\sqrt{x} - 1)^2$
Total variation $f(x) = \frac{1}{2}|x - 1|$
Kullback-Leilber $f(x) = x \log(x)$
Reverse Kullback-Leilber $f(x) = -\log(x)$
χ^2-divergence $f(x) = (x - 1)^2$

Subproblem 3

$$\begin{aligned}
D(N(\theta, I_p)||N(\mu, I_p)) &= \int \log(p) - \log(q) dQ \\
&= \frac{1}{2}\int \left[-(x-\theta)^\intercal(x-\theta) + (x-\mu)^\intercal(x-\mu)\right] dQ \\
&= \frac{1}{2}\int \left[-\mathrm{tr}((x-\theta)^\intercal(x-\theta)) + \mathrm{tr}((x-\mu)^\intercal(x-\mu))\right] dQ \\
&= \frac{1}{2}\mathbb{E}_Q \left[-\mathrm{tr}((x-\theta)^\intercal(x-\theta)) + \mathrm{tr}((x-\mu)^\intercal(x-\mu))\right] \\
&= \frac{1}{2}\left[-\mathrm{tr}(\mathbb{E}_Q(x-\theta)^\intercal(x-\theta)) + \mathrm{tr}(\mathbb{E}_Q(x-\mu)^\intercal(x-\mu))\right] \\
&= -\frac{1}{2}\left[(\mu-\theta)^\intercal(\mu-\theta) + p\right]
\end{aligned}$$

Problem 2

Subproblem 1

$$\frac{1}{2}\int(\sqrt{p} - \sqrt{q})^2 = \frac{1}{2}\int p - 2\sqrt{pq} + q = \frac{1}{2} - \int \sqrt{pq} + \frac{1}{2} = 1 - \int \sqrt{pq}$$

Subproblem 2

$$-2\log \int p^{.5}q^{.5} = -2\log(1 - H^2(P,Q)) \geq H^2(P,Q)$$

since $\log(x) \leq x - 1$ for $x \geq 0$.

Subproblem 3

$$\int \sqrt{p,q} \geq \int \sqrt{\min(p,q)^2} = \int \min(p,q)$$
$$\implies 1 - \int \sqrt{pq} \leq 1 - \int \min(p,q) = \mathrm{TV}(P,Q)$$

Subproblem 4

$$\begin{aligned}\mathrm{TV}(P,Q) = \frac{1}{2}\int |p-q| = \frac{1}{2}\int |\sqrt{p}-\sqrt{q}|\,|\sqrt{p}+\sqrt{q}| &\leq \frac{1}{2}\sqrt{\int(\sqrt{p}-\sqrt{q})^2\int(\sqrt{p}+\sqrt{q})^2} \\ &= \frac{1}{2}H(P,Q)\sqrt{2+(1-H^2(P,Q))} \\ &\leq \sqrt{2}H(P,Q)\end{aligned}$$

Subproblem 5

$$\begin{aligned}H^2(P^n,Q^n) = \frac{1}{2}\int(\sqrt{p^n}-\sqrt{q^n})^2 = \frac{1}{2}\int p^n - 2\sqrt{p^n q^n} + q^n &= 1 - \int \sqrt{p^n q^n} \\ &= 1 - (1 - H^2(P,Q))^n\end{aligned}$$

because we can factor the densities.

Problem 3

Subproblem 1

$$D_{1/2}(P,Q) = -2\log\int p^{.5}q^{.5} = -2\log\int (q/p)^{.5}dP \leq \int p\log(p/q) = D(P||Q)$$

Subproblem 2

$$D(P||Q) = \int p\log(p/q) \leq \log\int p^2q^{-1} = D_2(P,Q)$$

Subproblem 3

$$D_2(P,Q) = \log\int p^2q^{-1} = \log(1+\chi^2(P,Q)) \leq \chi^2(P,Q)$$

Problem 4

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}\log p_\theta(X_i)-\int p_{\theta^\star}\log p_\theta\right| \le \sum_{k=1}^{K}\left|\frac{1}{n}\sum_{i=1}^{n}\log p_{\theta_k}(X_i)-\int p_{\theta^\star}\log p_{\theta_k}\right| \to_{p^n_{\theta^\star}} 0$$

because we can apply LLN for each point in Θ

Problem 5

Problem 6

Subproblem 1

$$\begin{aligned}
M_n(\theta^\star+t) &= M(\theta^\star+t)+\frac{1}{\sqrt{n}}\nu_n m(x,\theta^\star+t)\\
&= M(\theta^\star)-\frac{1}{2}J_{\theta^\star}t^2+o(t^2)+\frac{1}{\sqrt{n}}\nu_n\left[m(x,\theta^\star)+t\Delta_{\theta^\star}(x)+|t|\,r(x,t)\right]\\
&= M_n(\theta^\star)+\frac{t}{\sqrt{n}}\nu_n\Delta_{\theta^\star}(x)-\frac{t^2}{2}J_{\theta^\star}+\frac{|t|}{\sqrt{n}}\nu_n r(x,t)+o(t^2)
\end{aligned}$$

Subproblem 2

Let $t=\hat\theta-\theta^\star$

$$\begin{aligned}
M_n(\theta^\star+t) &\ge M_n(\theta^\star)\\
M_n(\theta^\star)+\frac{t}{\sqrt{n}}\nu_n\Delta_{\theta^\star}(x)-\frac{t^2}{2}J_{\theta^\star}+\frac{|t|}{\sqrt{n}}\nu_n r(x,t)+o(t^2) &\ge M_n(\theta^\star)
\end{aligned}$$

$$\begin{aligned}
\frac{t^2}{2}J_{\theta^\star}-o(t^2) &\le \frac{t}{\sqrt{n}}\nu_n\Delta_{\theta^\star}(x)+\frac{|t|}{\sqrt{n}}\nu_n r(x,t)\\
&\le O_p(\frac{|t|}{\sqrt{n}})+\frac{|t|}{\sqrt{n}}o_p(1+\sqrt{n}\,|t|)
\end{aligned}$$

$$\begin{aligned}
\frac{t^2}{2}J_{\theta^\star}-o_p(t^2) &= O_p(\frac{|t|}{\sqrt{n}})\\
(\frac{1}{2}J_{\theta^\star}-o_p(1))t^2 &= O_p(\frac{|t|}{\sqrt{n}})\\
(\frac{1}{2}J_{\theta^\star}-o_p(1))\,|t| &= O_p(\frac{1}{\sqrt{n}})\\
\left|\hat\theta-\theta^\star\right| &= O_p(\frac{1}{\sqrt{n}})
\end{aligned}$$

Subproblem 3

Let $t = \hat{\theta} - \theta^\star = O_p(\frac{1}{\sqrt{n}})$

$$\begin{aligned} M_n(\theta^\star + t) &= M_n(\theta^\star) + \frac{t}{\sqrt{n}} \nu_n \Delta_{\theta^\star}(x) - \frac{t^2}{2} J_{\theta^\star} + o_p(\frac{1}{n}) \\ &= M_n(\theta^\star) - \frac{1}{2} J_{\theta^\star}(t - \frac{1}{\sqrt{n}} \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}})^2 + \frac{1}{2n} \frac{(\nu_n \Delta_{\theta^\star}(x))^2}{J_{\theta^\star}} + o_p(\frac{1}{n}) \end{aligned}$$

$$\begin{aligned} M_n(\theta^\star + t) &\geq M_n(\theta^\star + \frac{1}{\sqrt{n}} \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}}) \\ M_n(\theta^\star) - \frac{1}{2} J_{\theta^\star}(t - \frac{1}{\sqrt{n}} \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}})^2 + \frac{1}{2n} \frac{(\nu_n \Delta_{\theta^\star}(x))^2}{J_{\theta^\star}} + o_p(\frac{1}{n}) &\geq M_n(\theta^\star) + \frac{1}{2n} \frac{(\nu_n \Delta_{\theta^\star}(x))^2}{J_{\theta^\star}} + o_p(\frac{1}{n}) \\ \frac{1}{2} J_{\theta^\star}(t - \frac{1}{\sqrt{n}} \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}})^2 &= o_p(\frac{1}{n}) \\ \left| t - \frac{1}{\sqrt{n}} \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}} \right| &= o_p(\frac{1}{\sqrt{n}}) \\ \left| \sqrt{n} t - \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}} \right| &= o_p(1) \\ \sqrt{n}(\hat{\theta} - \theta^\star) &= \frac{\nu_n \Delta_{\theta^\star}(x)}{J_{\theta^\star}} + o_p(1) \\ &\rightsquigarrow N(0, \frac{E_{\theta^\star} \Delta_{\theta^\star}(x)^2}{J_{\theta^\star}^2}) \end{aligned}$$

Subproblem 4

$$\begin{aligned} \frac{E_{\theta^\star} \Delta_{\theta^\star}(x)^2}{J_{\theta^\star}^2} I_{\theta^\star} &= E_{\theta^\star} \frac{\Delta_{\theta^\star}(x)^2}{\frac{\partial^2}{\partial \theta^2} m(X, \theta)\Big|_{\theta=\theta^\star}^2} E_{\theta^\star} S_{\theta^\star}^2 \\ &\geq (\int \frac{\Delta_{\theta^\star}(x)}{\frac{\partial^2}{\partial \theta^2} m(X, \theta)\Big|_{\theta=\theta^\star}} S_{\theta^\star} dP)^2 \\ &= (\int \frac{\Delta_{\theta^\star}(x)}{\frac{\partial^2}{\partial \theta^2} m(X, \theta)\Big|_{\theta=\theta^\star}} \frac{\partial}{\partial \theta} \log p_\theta dP)^2 \\ &= (\int \frac{\Delta_{\theta^\star}(x)}{\frac{\partial^2}{\partial \theta^2} m(X, \theta)\Big|_{\theta=\theta^\star}} \frac{\partial}{\partial \theta} p_\theta)^2 \\ &= 1 \end{aligned}$$

Problem 7

$$\begin{aligned} F_{X_{(n)}}(x) &= \prod_{i=1}^{n} F_X(x) = F_X(x)^n = (\frac{x}{\theta})^n \\ f_{X_{(n)}}(x) &= \frac{d}{dx} F_{X_{(n)}}(x) = \frac{n}{\theta} (\frac{x}{\theta})^{n-1} \\ M &:= n(X_{(n)} - \theta) \\ f_M(m) &= \frac{(\frac{m}{n} + \theta)^{n-1}}{\theta^n} = \frac{1}{\theta} (\frac{m}{n\theta} + 1)^{n-1} \\ \lim_{n \to \infty} f_M(m) &= \frac{1}{\theta} e^{\frac{m}{\theta}} \end{aligned}$$

which is exponential distribution as $m < 0$.