

# Topic Models and Coordinate Ascent Variational Inference

STATS 305C: Applied Statistics

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April 25, 2022

# Mixed Membership Models

Mixed membership models are designed for *grouped data*.

Each “data point” is itself a collection of observations. For example,

- ▶ in text analysis, a document is a collection of observed words.
- ▶ in social science, a survey is a collection of observed answers.
- ▶ in genetic sequencing, a genome is a collection of observed genes.

Mixed membership models look for patterns like the components of a mixture model, but allowing each data point to involve multiple components.

# Notation for a mixed membership model

- ▶ Let  $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,D})$  denote  $n$ -th **data point**. It contains a **collection of observations**. To simplify notation, assume all data points are length  $D$ .
- ▶ Each data point reflects a **combination of  $K$  mixture components** with parameters  $\{\boldsymbol{\theta}_k\}_{k=1}^K$ . These are shared by all documents.
- ▶ Each data point has its own **mixture proportions**,  $\pi_n \in \Delta_K$ .
- ▶ Each **observation is assigned to one of the components** by  $z_{n,d} \in \{1, \dots, K\}$ .

# Topic models

The most common mixed-membership model is the **topic model**, a generative model for documents.

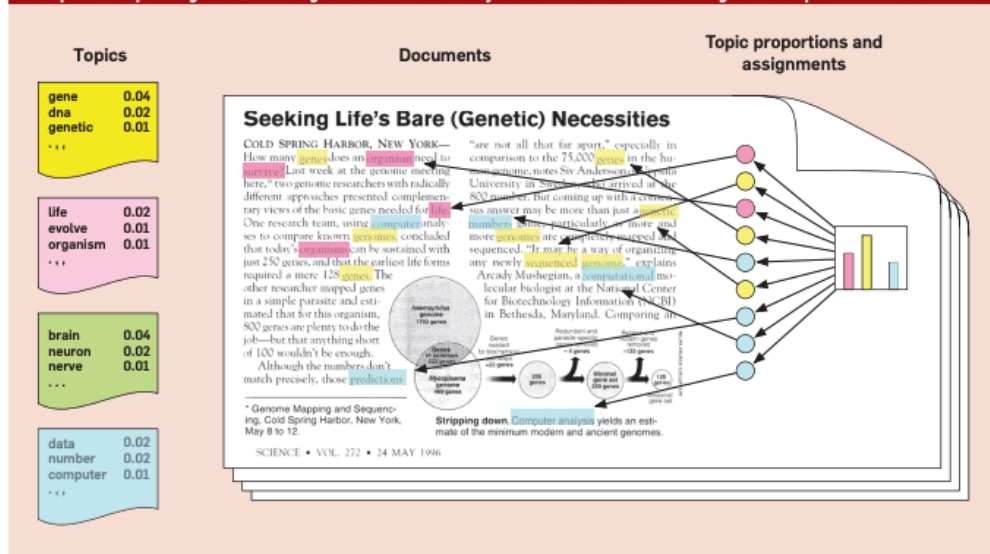
Topic models are so common, they have their own nomenclature:

<i>data set</i>	<i>corpus</i>	collection of documents
<i>data point</i>	<i>document</i>	collection of words
<i>observation</i>	<i>word</i>	one element of a document
<i>mixture component</i>	<i>topic</i>	distribution over words
<i>mixture proportions</i>	<i>topic proportions</i>	distribution over topics
<i>mixture assignment</i>	<i>topic assignment</i>	which topic produced a word

*Table:* Rosetta stone for translating between mixed membership model and topic model notation.

# Topic modeling intuition

**Figure 1. The intuitions behind latent Dirichlet allocation. We assume that some number of “topics,” which are distributions over words, exist for the whole collection (far left). Each document is assumed to be generated as follows. First choose a distribution over the topics (the histogram at right); then, for each word, choose a topic assignment (the colored coins) and choose the word from the corresponding topic. The topics and topic assignments in this figure are illustrative—they are not fit from real data. See Figure 2 for topics fit from data.**



From Blei [2012].

# Topic modeling of New York Times articles



**Figure 6.1:** Posterior topics from the *The New York Times*.

# Dynamic topic modeling of Science articles

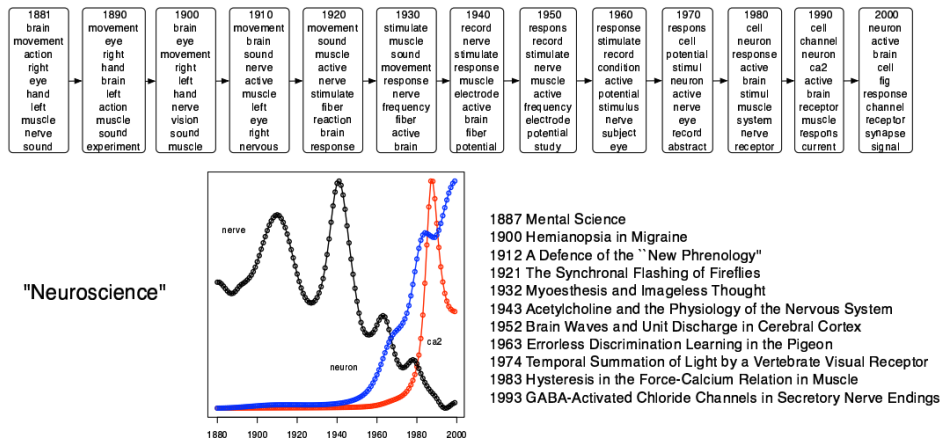
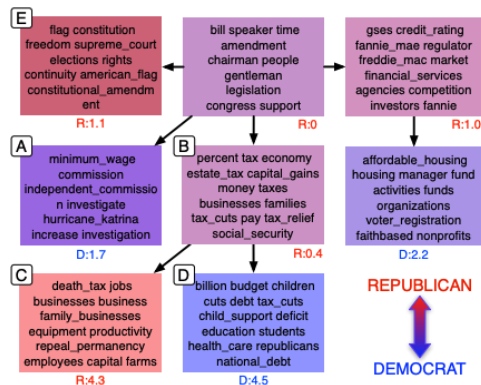


Figure 4. Examples from the posterior analysis of a 20-topic dynamic model estimated from the *Science* corpus. For two topics, we illustrate: (a) the top ten words from the inferred posterior distribution at ten year lags (b) the posterior estimate of the frequency as a function of year of several words from the same two topics (c) example articles throughout the collection which exhibit these topics. Note that the plots are scaled to give an idea of the shape of the trajectory of the words' posterior probability (i.e., comparisons across words are not meaningful).

From Blei and Lafferty [2006].

# Topic modeling of congressional voting records



**Figure 7.2:** Topics discovered from Congressional floor debates using a downstream model to capture speaker’s ideology. Many first-level topics are bipartisan (purple), while lower level topics are associated with specific ideologies (Democrats blue, Republicans red). For example, the “tax” topic (B) is bipartisan, but its Democratic-leaning child (D) focuses on social goals supported by taxes (“children”, “education”, “health care”), while its Republican-leaning child (C) focuses on business implications (“death tax”, “jobs”, “businesses”). The number below each topic denotes the magnitude of a learned regression parameter associated with that topic. Colors and the numbers beneath each topic show the regression parameter  $\eta$  associated with the topic. From Nguyen et al. [2013].



# The generative process for a mixed membership model

The generative model process is:

- ▶ for each mixture component  $k = 1, \dots, K$ , sample its parameter  $\theta_k \sim p(\theta_k \mid \phi)$
- ▶ for each data point  $n = 1, \dots, N$ :
  - ▶ sample mixture proportions  $\pi_n \sim \text{Dir}(\pi_n \mid \alpha)$
  - ▶ for each observation  $d = 1, \dots, D$ :
    - ▶ sample mixture assignment  $z_{n,d} \in \{1, \dots, K\}$  from a categorical distribution  $z_{n,d} \sim \text{Cat}(\pi_n)$
    - ▶ sample observation  $x_{n,d} \sim p(x \mid \theta_{z_{n,d}})$

The mixed membership model allows sharing at the dataset level (all data points share the same components) while allowing variability at the data point level (each data point has its own mixture proportions).

# The generative process for a topic model

Slide 9 in the language of topic modeling:

- ▶ for each **topic**  $k = 1, \dots, K$ , sample its parameter  $\theta_k \sim p(\theta_k \mid \phi)$
- ▶ for each **document**  $n = 1, \dots, N$ :
  - ▶ sample **topic proportions**  $\pi_n \sim \text{Dir}(\pi_n \mid \alpha)$
  - ▶ for each **word**  $d = 1, \dots, D$ :
    - ▶ sample **topic assignment**  $z_{n,d} \in \{1, \dots, K\}$  from a categorical distribution  $z_{n,d} \sim \text{Cat}(\pi_n)$
    - ▶ sample **word**  $x_{n,d} \in \{1, \dots, V\}$  from a categorical distribution,  $x_{n,d} \sim \text{Cat}(\theta_{z_{n,d}})$

The topic model captures sharing at the corpus level (all documents share the same topics) while allowing variability at the data point level (each document weights topics differently).

# The joint distribution

The joint probability for a general mixed membership model is,

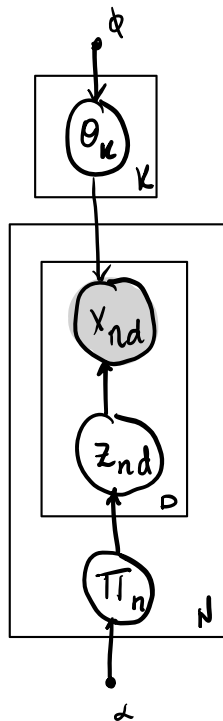
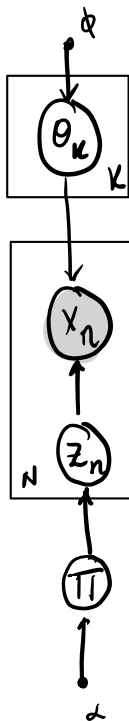
$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \mathbf{z}_n, \mathbf{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) = \prod_{k=1}^K p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left[ p(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^D p(z_{n,d} \mid \boldsymbol{\pi}_n) p(x_{n,d} \mid \boldsymbol{\theta}_{z_{n,d}}) \right] \quad (1)$$

As in mixture models, we can write this equivalently as

$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \mathbf{z}_n, \mathbf{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) = \prod_{k=1}^K p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left[ p(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^D \prod_{k=1}^K [\pi_{n,k} p(x_{n,d} \mid \boldsymbol{\theta}_k)]^{\mathbb{I}[z_{n,d}=k]} \right] \quad (2)$$

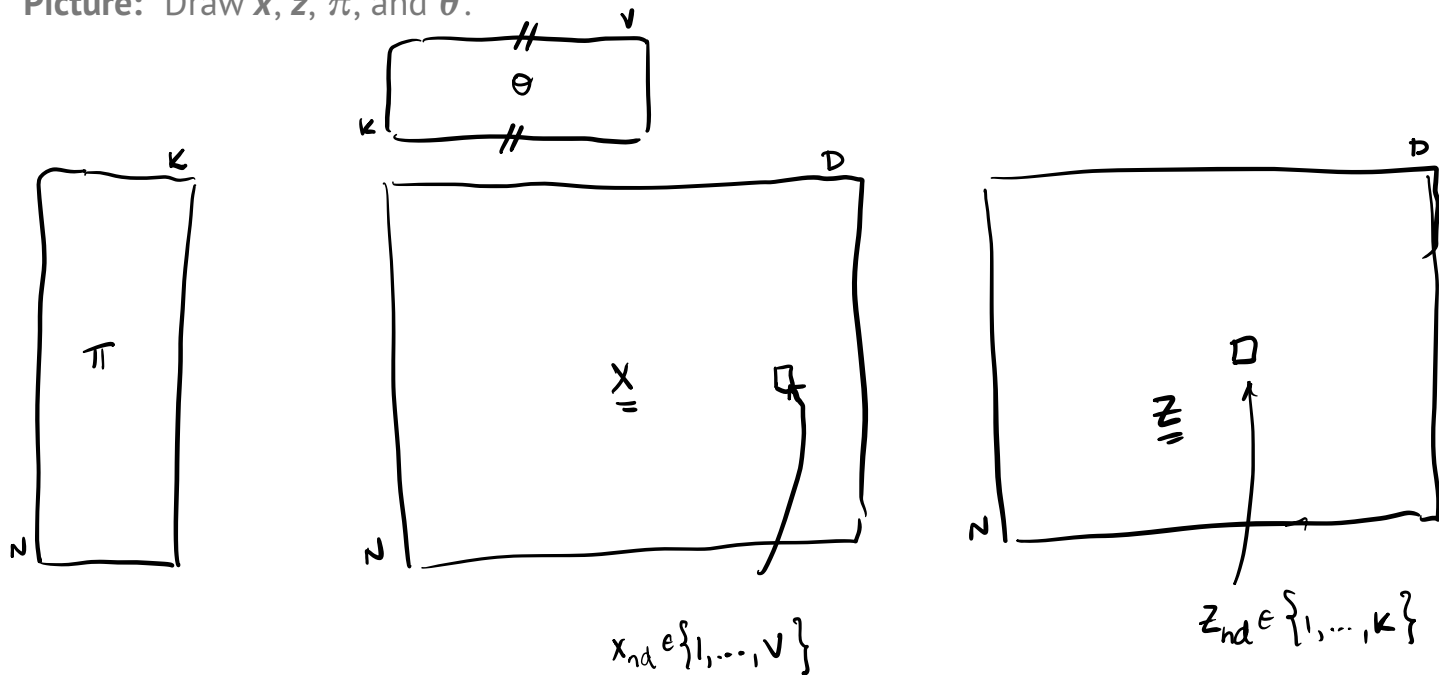
# Graphical Model

**Exercise:** Draw the graphical models for a mixture model and a mixed membership model.



# Topic model data types

Picture: Draw  $x$ ,  $z$ ,  $\pi$ , and  $\theta$ .



# Latent Dirichlet allocation

Latent Dirichlet Allocation (LDA) [Blei et al., 2003] is the most widely used topic model.

It assumes conjugate Dirichlet-Categorical model for the topics  $\theta_k \in \Delta_V$  and words  $x_{n,d} \in \{1, \dots, V\}$ ,

$$\theta_k \stackrel{\text{iid}}{\sim} \text{Dir}(\phi), \quad \Delta_V \quad (3)$$

$$\pi_n \stackrel{\text{iid}}{\sim} \text{Dir}(\alpha), \quad \Delta_K \quad (4)$$

$$z_{n,d} \stackrel{\text{iid}}{\sim} \text{Cat}(\pi_n) \quad \{1, \dots, K\} \quad (5)$$

$$x_{n,d} \stackrel{\text{iid}}{\sim} \text{Cat}(\theta_{z_{n,d}}) \quad \{1, \dots, V\} \quad (6)$$

# Latent Dirichlet allocation II

Plugging in these assumptions, the joint probability is,

$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \mathbf{z}_n, \mathbf{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) \quad (7)$$

$$= \prod_{k=1}^K \text{Dir}(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left[ \text{Dir}(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^{D_n} \pi_{n,z_{n,d}} \theta_{z_{n,d}, x_{n,d}} \right] \quad (8)$$

$$= \prod_{k=1}^K \text{Dir}(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left( \text{Dir}(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^{D_n} \left[ \prod_{k=1}^K \pi_{n,k}^{\mathbb{I}[z_{n,d}=k]} \right] \left[ \prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{\mathbb{I}[x_{n,d}=v] \mathbb{I}[z_{n,d}=k]} \right] \right) \\ \propto \left[ \prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{\phi_v-1} \right] \prod_{n=1}^N \left( \left[ \prod_{k=1}^K \pi_{n,k}^{\alpha_k + N_{n,\cdot,k}-1} \right] \left[ \prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{N_{n,v,k}} \right] \right) \quad (9)$$

where

- ▶  $N_{n,v,k} = \sum_{d=1}^D \mathbb{I}[x_{n,d} = v] \mathbb{I}[z_{n,d} = k]$  is the number of instances of word  $v$  in document  $n$  assigned to topic  $k$ , and,
- ▶  $N_{n,\cdot,k} = \sum_{v=1}^V N_{n,v,k} = \sum_{d=1}^D \mathbb{I}[z_{n,d} = k]$  is the number of words in document  $n$  assigned to topic  $k$ .

# Gibbs sampling for LDA

As usual, sample each variable from its conditional distribution, holding the rest fixed.

- **Topic assignments:** The assignments are conditionally independent given the topic parameters and proportions,

$$p(z_{n,d} = k \mid x_{n,d} = v, \{\theta_k\}_{k=1}^K, \pi_n) \propto \frac{\pi_{n,k} \theta_{k,v}}{\sum_j \pi_{n,j} \theta_{j,v}} \quad (10)$$

- **Topic proportions:** Let  $N_{n,k} = \sum_{d=1}^D \mathbb{I}[z_{n,d} = k]$  denote the number of words in document  $n$  assigned to topic  $k$ . Then,

$$p(\pi_n \mid \alpha, \mathbf{z}_n) \propto \text{Dir}(\pi_n \mid \alpha) \prod_{k=1}^K \pi_{n,k}^{N_{n,k}} = \text{Dir}([\alpha_1 + N_{n,\cdot,1}, \dots, \alpha_K + N_{n,\cdot,K}]). \quad (11)$$

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$$p(\theta_k \mid \{\mathbf{z}_n, \mathbf{x}_n\}, \phi) \propto \text{Dir}(\theta_k \mid \phi) \prod_{n=1}^N \prod_{v=1}^V \theta_{k,v}^{N_{n,v,k}} = \text{Dir}([\phi_1 + \sum_{n=1}^N N_{n,\cdot,k}, \dots, \phi_V + \sum_{n=1}^N N_{n,V,k}]).$$



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*in doc. n*

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## Why does LDA produce sharp topics?

Consider the log posterior as a function of  $\theta_k$  and  $\pi_n$ . From eq. 8, it is,

$$\sum_{k=1}^K \log p(\theta_k | \phi) + \sum_{n=1}^N \log p(\pi_n | \alpha) + \sum_{n=1}^N \sum_{d=1}^D \left( \log \pi_{n,z_{n,d}} + \log \theta_{z_{n,d},x_{n,d}} \right) \quad (13)$$

The double sum over  $n$  and  $d$  dominates this expression.

It encourages topic proportions and topic probabilities to both be large, but recall that both  $\pi_n$  and  $\theta_k$  are constrained to the simplex.

For  $\log \pi_{n,z_{n,d}}$  to be large, the posterior should assign all words to as few topics as possible.

For  $\log \theta_{z_{n,d},x_{n,d}}$  to be large, the topics should put high probability on as few words as possible.

These goals are at odds. The LDA posterior balances these goals to find topics with sharply co-occurring words.

# Topic Models and Coordinate Ascent Variational Inference

- ▶ Model: Topic Models
- ▶ **Algorithm: Coordinate Ascent Variational Inference (CAVI)**

# Taking stock

We've covered a number of posterior inference algorithms thus far:

- ▶ **Exact inference:** for simple models (e.g. conjugate exponential family models) where the posterior is available in closed form.
- ▶ **Gibbs sampling:** an MCMC algorithm that iteratively samples conditional distributions for one variable at a time. This works well for conditionally conjugate models with weak correlations.
- ▶ **Metropolis-Hastings:** a very general MCMC algorithm to sample the posterior, and the building block for many other MCMC techniques.
- ▶ **Hamiltonian Monte Carlo:** an MCMC algorithm to draw samples from the posterior by leveraging gradients of the log joint probability. This works well for more general posteriors over continuous variables.

# Variational inference

MCMC methods are asymptotically unbiased (though for finite samples there is a transient bias that shrinks as  $O(S^{-1})$ ). The real issue is variance: it only shrinks as  $O(S^{-1/2})$ .

**Motivation:** With finite computation, can we get better posterior estimates by trading asymptotic bias for smaller variance?

**Idea:** approximate the posterior by with a simple, parametric form (though not strictly a Gaussian on the mode!). Optimize to find the approximation that is as “close” as possible to the posterior.

# Notation

This notation could be a bit confusing. Let,

- ▶  $\boldsymbol{\vartheta} \in \mathbb{R}^J$  denote **all of latent variables and parameters** we wish to infer.

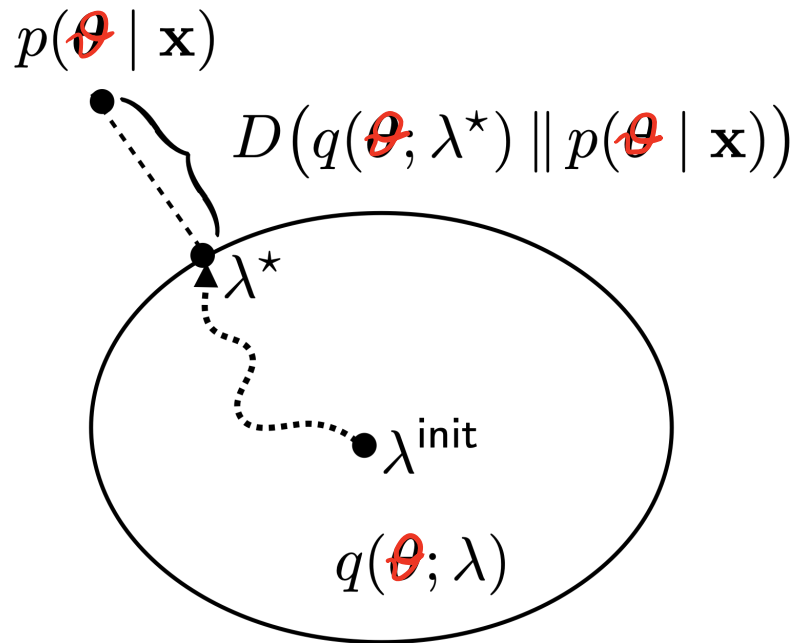
In LDA,  $\boldsymbol{\vartheta} = \{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \mathbf{z}_n\}_{n=1}^N$ .

In contrast to last week, here we will obtain a **full posterior over parameters and latent variables**.

Likewise, let

- ▶  $p(\boldsymbol{\vartheta} \mid \mathbf{x})$  denote the true posterior distribution we want to approximate.
- ▶  $q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})$  denote a parametric *variational approximation* to the posterior where...
- ▶  $\boldsymbol{\lambda}$  denotes the *variational parameters* that we will optimize.
- ▶  $D(q \parallel p)$  denote a *divergence measure* that takes in two distributions  $q$  and  $p$  and returns a measure of how similar they are.

## A view of variational inference





# Key questions

- ▶ *What parametric family should we use?*
  - ▶ This lecture: the **mean-field family**.
- ▶ *How should we measure closeness?*
  - ▶ This lecture: the **Kullback-Leibler (KL)** divergence.
- ▶ *How do we find the closest distribution in that family?*
  - ▶ This lecture: **coordinate ascent**.

These choices are what Blei et al. [2017] call **coordinate ascent variational inference** (CAVI).

# The mean-field family

The *mean-field family* gets its name from statistical mechanics. It treats each latent variable and parameter as independent with its own variational parameter,

$$q(\boldsymbol{\vartheta}; \boldsymbol{\lambda}) = \prod_{j=1}^J q(\vartheta_j; \lambda_j). \quad (14)$$

For example, in LDA the mean field approximation treats each topic, topic proportion, and topic assignment as independent,

$$q(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \mathbf{z}_n\}_{n=1}^N; \boldsymbol{\lambda}) = \prod_{k=1}^K q(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\boldsymbol{\theta})}) \prod_{n=1}^N q(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\boldsymbol{\pi})}) \prod_{n=1}^N \prod_{d=1}^D q(z_{n,d}; \boldsymbol{\lambda}_{n,d}^{(z)}) \quad (15)$$

**Question:** Is this a good approximation to the posterior?

# The Kullback-Leibler (KL) divergence

The KL divergence is a measure of closeness between two distributions. It is defined as,

$$D_{\text{KL}}(q(\boldsymbol{\vartheta}; \boldsymbol{\lambda}) \parallel p(\boldsymbol{\vartheta} \mid \boldsymbol{x})) = \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} \left[ \log \frac{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})}{p(\boldsymbol{\vartheta} \mid \boldsymbol{x})} \right] \quad (16)$$

$$= \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})] - \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log p(\boldsymbol{\vartheta} \mid \boldsymbol{x})] \quad (17)$$

It has some nice properties:

- ▶ It is non-negative.
- ▶ It is zero iff  $q(\boldsymbol{\vartheta}; \boldsymbol{\lambda}) \equiv p(\boldsymbol{\vartheta} \mid \boldsymbol{x})$ .
- ▶ It is defined in terms of expectations wrt  $q$ .

But it's also a bit weird...

- ▶ It's asymmetric ( $D_{\text{KL}}(q \parallel p) \neq D_{\text{KL}}(p \parallel q)$ ).

## The evidence lower bound (ELBO) from another angle

More concerning, the KL divergence involves the posterior  $p(\boldsymbol{\vartheta} \mid \mathbf{x})$ , which we cannot compute!

But notice that...

$$D_{\text{KL}}(q(\boldsymbol{\vartheta}; \boldsymbol{\lambda}) \parallel p(\boldsymbol{\vartheta} \mid \mathbf{x})) = \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})] - \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log p(\boldsymbol{\vartheta} \mid \mathbf{x})] \quad (18)$$

$$= \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})] - \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log p(\boldsymbol{\vartheta}, \mathbf{x})] + \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log p(\mathbf{x})] \quad (19)$$

$$= \underbrace{\mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})] - \mathbb{E}_{q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})} [\log p(\boldsymbol{\vartheta}, \mathbf{x})]}_{\text{negative ELBO, } -\mathcal{L}(\boldsymbol{\lambda})} + \underbrace{\log p(\mathbf{x})}_{\text{evidence}} \quad (20)$$

$\mathcal{L}[q, \theta]$

The first term involves the log joint, which we can compute, and the last term is independent of the variational parameters!

Rearranging, we see that  $\mathcal{L}(\boldsymbol{\lambda})$  is a lower bound on the marginal likelihood, aka the evidence,

$$\mathcal{L}(\boldsymbol{\lambda}) = \log p(\mathbf{x}) - D_{\text{KL}}(q(\boldsymbol{\vartheta}; \boldsymbol{\lambda}) \parallel p(\boldsymbol{\vartheta} \mid \mathbf{x})) \leq \log p(\mathbf{x}). \quad (21)$$

That's why we call it the **evidence lower bound (ELBO)**.

**Viewer discretion advised...**

<https://www.youtube.com/watch?v=jugUBL4rEIM>

# Optimizing the ELBO with coordinate ascent

We want to find the variational parameters  $\lambda$  that minimize the KL divergence or, equivalently, maximize the ELBO.

For the mean-field family, we can typically do this via **coordinate ascent**.

Consider optimizing the parameters for one factor  $q(\vartheta_j; \lambda_j)$ . As a function of  $\lambda_j$ , the ELBO is,

$$\mathcal{L}(\lambda) = \mathbb{E}_{q(\vartheta; \lambda)} [\log p(\vartheta, \mathbf{x})] - \dots \quad \leftarrow -\mathbb{E}_{q(\vartheta_{-j}; \lambda_{-j})} [\log q(\vartheta_j; \lambda_j)] \quad (22)$$

$$\stackrel{\text{b/c of M.F. assumption}}{\rightarrow} \mathcal{L}(\lambda) = \mathbb{E}_{q(\vartheta_j; \lambda_j)} \left[ \mathbb{E}_{q(\vartheta_{-j}; \lambda_{-j})} [\log p(\vartheta, \mathbf{x})] \right] - \mathbb{E}_{q(\vartheta_j; \lambda_j)} [\log q(\vartheta_j; \lambda_j)] + c \quad (23)$$

$$= \mathbb{E}_{q(\vartheta_j; \lambda_j)} \left[ \mathbb{E}_{q(\vartheta_{-j}; \lambda_{-j})} [\log p(\vartheta_j | \vartheta_{-j}, \mathbf{x})] \right] - \mathbb{E}_{q(\vartheta_j; \lambda_j)} [\log q(\vartheta_j; \lambda_j)] + c' \quad (24)$$

where

$$\tilde{p}(\vartheta_j) \propto \exp \left\{ \mathbb{E}_{q(\vartheta_{-j}; \lambda_{-j})} [\log p(\vartheta_j | \vartheta_{-j}, \mathbf{x})] \right\} \quad (25)$$

The ELBO is maximized wrt  $\lambda_j$  when this KL is minimized; i.e. when  $q(\vartheta_j; \lambda_j) = \tilde{p}(\vartheta_j)$ , the exponentiated expected log conditional probability, holding all other factors fixed.

## Lap 5: Mixed Membership Models and Variational Inference

- ▶ Model: Mixed Membership Models
- ▶ Algorithm: Coordinate Ascent Variational Inference (CAVI)
  - ▶ **CAVI for LDA**

# Coordinate Ascent Variational Inference for LDA

Let's derive the CAVI updates for LDA.

Assume a mean field family, and assume each factor is of the same exponential family form as the corresponding prior:

$$q(z_{n,d}; \boldsymbol{\lambda}_{n,d}^{(z)}) = \text{Cat}(z_{n,d}; \boldsymbol{\lambda}_{n,d}^{(z)}) \quad (26)$$

$$q(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) = \text{Dir}(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) \quad (27)$$

$$q(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}) = \text{Dir}(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}). \quad (28)$$

so  $\boldsymbol{\lambda}_{n,d}^{(z)} \in \Delta_K$ ,  $\boldsymbol{\lambda}_n^{(\pi)} \in \mathbb{R}_{+}^K$ , and  $\boldsymbol{\lambda}_k^{(\theta)} \in \mathbb{R}_{+}^V$  are the variational parameters.

(It turns out, for conjugate exponential family models, the optimal variational factors are of the same form as the prior anyway!)



# CAVI updates for the topic assignments

Recall that the optimal CAVI updates are of the form in Eq. 25.

We already derived the conditional distributions for Gibbs sampling (Slide 16).

For the topic assignments, the CAVI update is,

$$\log q(z_{n,d} = k; \lambda_{n,d}^{(z)}) = \mathbb{E}_{q(\pi_n)q(\theta_k)} [\log \pi_{n,k} + \log p(x_{n,d} | \theta_k)] + c \quad (29)$$

$$= \mathbb{E}_{q(\pi_n)} [\log \pi_{n,k}] + \mathbb{E}_{q(\theta_k)} [\log \theta_{k,x_{n,d}}] + c \quad (30)$$

$$= \log \text{Cat}(z_{n,d} = k; \lambda_{n,d}^{(z)}) \quad (31)$$

$$\Rightarrow \log \lambda_{n,d,k}^{(z)} = \mathbb{E}_{q(\pi_n)} [\log \pi_{n,k}] + \mathbb{E}_{q(\theta_k)} [\log \theta_{k,x_{n,d}}] + c \quad (32)$$

Since  $\lambda_{n,d}^{(z)}$  must sum to one,

$$\lambda_{n,d,k}^{(z)} = \frac{\exp \left\{ \mathbb{E}_{q(\pi_n)} [\log \pi_{n,k}] + \mathbb{E}_{q(\theta_k)} [\log \theta_{k,x_{n,d}}] \right\}}{\sum_{j=1}^K \exp \left\{ \mathbb{E}_{q(\pi_n)} [\log \pi_{n,j}] + \mathbb{E}_{q(\theta_j)} [\log \theta_{j,x_{n,d}}] \right\}} \quad (33)$$

## Expectations under Dirichlet distributions

$$\text{Dir}(\pi|\alpha) = \frac{\Gamma(\sum_k \alpha_k)}{\prod_k \Gamma(\alpha_k)} \prod_k \pi_k^{\alpha_k - 1}$$

$$= \exp\left\{\sum_k (\alpha_k - 1) \log \pi_k - A(\alpha)\right\}$$

$$A(\alpha) = \sum_k \log \Gamma(\alpha_k) - \log \Gamma(\sum_k \alpha_k)$$

The variational factors for  $\pi_n$  and  $\theta_k$  are Dirichlet distributions.

The necessary expectations have closed form expressions:

$$\mathbb{E}_{\text{Dir}(\pi; \alpha)}[\log \pi_k] = \psi(\alpha_k) - \psi\left(\sum_{j=1}^K \alpha_j\right) \quad (34)$$

where  $\psi(\cdot)$  is the *digamma function*, the logarithmic derivative of the gamma function.

$$\psi(\alpha) = \frac{d}{d\alpha} \log \Gamma(\alpha)$$

# CAVI updates for the topic proportions

Referring back to Slide 16, we see the CAVI update is,

$$\log q(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) = \mathbb{E}_{q(\mathbf{z}_n)} \left[ \log \text{Dir}(\boldsymbol{\pi}_n; \boldsymbol{\alpha}) + \sum_{k=1}^K N_{n,k} \log \pi_{n,k} \right] + c \quad (35)$$

$$= \sum_{k=1}^K (\alpha_k - 1 + \mathbb{E}_{q(\mathbf{z}_n)}[N_{n,k}]) \log \pi_{n,k} \quad (36)$$

$$= \log \text{Dir}(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) \quad (37)$$

$$\Rightarrow \boldsymbol{\lambda}_n^{(\pi)} = \left[ \alpha_1 + \mathbb{E}_{q(\mathbf{z}_n)}[N_{n,1}], \dots, \alpha_K + \mathbb{E}_{q(\mathbf{z}_n)}[N_{n,K}] \right], \quad (38)$$

where

$$\mathbb{E}_{q(\mathbf{z}_n)}[N_{n,k}] = \sum_{d=1}^D \mathbb{E}_{q(\mathbf{z}_{n,d})}[\mathbb{I}[z_{n,d} = k]] = \sum_{d=1}^D \lambda_{n,d,k}^{(z)}. \quad (39)$$

# CAVI updates for the topic parameters

The topic parameter updates are similar

$$\log q(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}) = \mathbb{E}_{q(\mathbf{z})} \left[ \log \text{Dir}(\boldsymbol{\theta}_k; \boldsymbol{\phi}) + \sum_{v=1}^V N_{k,v} \log \theta_{k,v} \right] + c \quad (40)$$

$$= \sum_{v=1}^V (\phi_v - 1 + \mathbb{E}_{q(\mathbf{z})}[N_{k,v}]) \log \theta_{k,v} \quad (41)$$

$$= \log \text{Dir}(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}) \quad (42)$$

$$\Rightarrow \boldsymbol{\lambda}_k^{(\theta)} = [\phi_1 + \mathbb{E}_{q(\mathbf{z})}[N_{k,1}], \dots, \phi_V + \mathbb{E}_{q(\mathbf{z})}[N_{k,V}]], \quad (43)$$

where

$$\mathbb{E}_{q(\mathbf{z})}[N_{k,v}] = \sum_{n=1}^N \sum_{d=1}^D \mathbb{E}_{q(z_{n,d})} [\mathbb{I}[z_{n,d} = k]] \mathbb{I}[x_{n,d} = v] = \sum_{n=1}^N \sum_{d=1}^D \lambda_{n,d,k}^{(z)} \mathbb{I}[x_{n,d} = v]. \quad (44)$$

# Calculating the ELBO

Dropping hyperparameters and variational parameters, the ELBO is,

$$\mathcal{L}(\lambda) = \mathbb{E}_q[\log p(\{\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\pi}_n\}_{n=1}^N, \{\boldsymbol{\theta}_k\}_{k=1}^K)] - \underbrace{\mathbb{E}_q[\log q(\{\mathbf{z}_n, \boldsymbol{\pi}_n\}_{n=1}^N, \{\boldsymbol{\theta}_k\}_{k=1}^K)]}_{\text{entropy of } q} \quad (45)$$

Thanks to the factorization of the joint distribution and the variational posterior, this simplifies,

$$\begin{aligned} \mathcal{L}(\lambda) = & \sum_{n=1}^N \mathbb{E}_q[\log p(\boldsymbol{\pi}_n)] + \sum_{k=1}^K \mathbb{E}_q[\log p(\boldsymbol{\theta}_k)] \quad \leftarrow \text{cross entropy} \\ & + \sum_{n=1}^N \sum_{d=1}^D \mathbb{E}_q[\log p(z_{n,d} | \boldsymbol{\pi}_n)] + \mathbb{E}_q[\log p(x_{n,d} | z_{n,d}, \boldsymbol{\theta})] \\ & - \sum_{k=1}^K \mathbb{E}_q[\log q(\boldsymbol{\theta}_k)] - \sum_{n=1}^N \mathbb{E}_q[\log q(\boldsymbol{\pi}_n)] - \sum_{n=1}^N \sum_{d=1}^D \mathbb{E}_q[\log q(z_{n,d})] \quad (46) \end{aligned}$$

The first two terms are Dirichlet cross entropies and the last three terms are entropies. Tensorflow probability has already implemented these functions for you, so this is easier to calculate than you might think!

## Calculating the ELBO II

The middle terms are,

$$\mathbb{E}_q[\log p(z_{n,d} \mid \boldsymbol{\pi}_n)] = \sum_{k=1}^K \mathbb{E}_{q(z_{n,d})}[\mathbb{I}[z_{n,d} = k]] \cdot \mathbb{E}_{q(\boldsymbol{\pi}_n)}[\log \pi_{n,k}] \quad (47)$$

$$= \sum_{k=1}^K \lambda_{n,d,k}^{(z)} \cdot \mathbb{E}_{q(\boldsymbol{\pi}_n)}[\log \pi_{n,k}] \quad (48)$$

and

$$\mathbb{E}_q[\log p(x_{n,d} \mid z_{n,d}, \boldsymbol{\theta})] = \sum_{k=1}^K \mathbb{E}_{q(z_{n,d})}[\mathbb{I}[z_{n,d} = k]] \cdot \mathbb{E}_{q(\boldsymbol{\theta}_k)}[\log \theta_{k,x_{n,d}}] \quad (49)$$

$$= \sum_{k=1}^K \lambda_{n,d,k}^{(z)} \cdot \mathbb{E}_{q(\boldsymbol{\theta}_k)}[\log \theta_{k,x_{n,d}}] \quad (50)$$

Again, these involve only simple expectations of Dirichlet distributions.