NBA Starting Lineup Optimization Via Convex Optimization

Eren Erdogan

Convex Optimization Final Project

1. <u>Introduction</u>

The idea of advanced analytics in sports is still a relatively new topic in the huge world of profession sports teams. Its rise to fame was in Major League Baseball during the Oakland Athletics 2002 season where they started to implement advanced analytics to help build a roster of players. That same year the Athletics went on to win the American League Championship even though they had the 3rd lowest payroll in baseball. They ended up losing in the World Series but the idea of using advanced analytics to construct a roster had been born. The story of the 2002 Oakland Athletics was made famous via the 2003 book Money Ball and the 2011 movie with the same title.

In recent years other professional sports leagues, not just Major League Baseball, began to also develop advanced analytics divisions with in their team offices. A notable league that followed this trend and began to implement analytics was the National Basketball Association (NBA).

The NBA is the highest-level basketball league in the world with players from all over the world. In the 2014-2015 season the league generated 5.18 Billion dollars of revenue that was split amongst its 30 teams, 15 in the Eastern Conference and 15 in the Western Conference. 29 of which are in the United States and 1 in Canada. The 2015 NBA Finals had an average viewer rating of 19.9 Million viewers per game.

It's safe to say that it's a huge league that would definitely benefit from the use of advanced analytics, considering the plethora of statistics that are taken during a typical NBA game.

A standard NBA box score from a single game looks like this below:

Warriors

STARTERS	MIN	FG	3PT	FT	OREB	DREB	REB	AST	STL	BLK	TO	PF	+/-	PTS
D. Green PF	33	5-12	1-3	1-3	2	8	10	4	2	4	1	5	+28	12
H. Barnes SF	32	3-12	0-2	1-2	4	3	7	2	1	0	0	0	+6	7
A. Bogut C	16	1-1	0-0	0-0	1	6	7	2	1	1	0	5	+15	2
S. Curry PG	20	8-13	5-7	3-3	1	6	7	2	3	0	4	0	+28	24
K. Thompson SG	33	4-14	2-6	6-6	2	3	5	4	1	0	2	2	+22	16
BENCH	**MIN**	**FG**	**3PT**	**FT**	**OREB**	**DREB**	**REB**	**AST**	**STL**	**BLK**	**TO**	**PF**	**+/-**	**PTS**
A. Varejao PF	7	0-0	0-0	0-0	1	0	1	0	0	0	0	0	-9	0
B. Rush SF	3	0-2	0-2	0-0	0	0	0	0	0	0	0	0	+1	0
A. Iguodala SF	27	1-6	0-1	0-2	0	1	1	7	0	0	2	3	+23	2
M. Speights C	15	5-11	0-2	2-2	1	4	5	1	0	0	2	2	+15	12
F. Ezeli C	11	4-5	0-0	0-2	3	3	6	1	0	1	3	2	+1	8
S. Livingston PG	25	3-8	0-0	3-4	0	3	3	3	1	1	1	2	+1	9
I. Clark SG	4	1-2	1-1	0-0	0	0	0	0	1	0	0	1	-5	3
L. Barbosa SG	16	4-5	1-1	0-0	0	1	1	0	0	0	0	0	+4	9
TEAM		**39-91**	**10-25**	**16-24**	**15**	**38**	**53**	**26**	**10**	**7**	**15**	**22**		**104**
		42.9%	**40.0%**	**66.7%**										

Entire season stats per player are also a common statistic to keep track of as well as per game average stats that are determined from season totals divided by total number of games played.

Below is an example of season statistical averages for Stephen Curry, of the Golden State Warriors.

REGULAR SEASON AVERAGES

SEASON	TEAM	GP	GS	MIN	FGM-A	FG%	3PM-A	3P%	FTM-A	FT%	OR	DR	REB	AST	BLK	STL	PF	TO	PTS
'09-'10	GS	80	77	36.2	6.6-14.3	.462	2.1-4.8	.437	2.2-2.5	.885	0.6	3.9	4.5	5.9	0.2	1.9	3.2	3.1	17.5
'10-'11	GS	74	74	33.6	6.8-14.2	.480	2.0-4.6	.442	2.9-3.1	.934	0.7	3.2	3.9	5.8	0.3	1.5	3.1	3.1	18.6
'11-'12	GS	26	23	28.2	5.6-11.4	.490	2.1-4.7	.455	1.5-1.8	.809	0.6	2.8	3.4	5.3	0.3	1.5	2.4	2.5	14.7
'12-'13	GS	78	78	38.2	8.0-17.8	.451	3.5-7.7	.453	3.4-3.7	.900	0.8	3.3	4.0	6.9	0.2	1.6	2.5	3.1	22.9
'13-'14	GS	78	78	36.5	8.4-17.7	.471	3.3-7.9	.424	3.9-4.5	.885	0.6	3.7	4.3	8.5	0.2	1.6	2.5	3.8	24.0
'14-'15	GS	80	80	32.7	8.2-16.8	.487	3.6-8.1	.443	3.9-4.2	.914	0.7	3.6	4.3	7.7	0.2	2.0	2.0	3.1	23.8
'15-'16	GS	79	79	34.2	10.2-20.2	.504	5.1-11.2	.454	4.6-5.1	.908	0.9	4.6	5.4	6.7	0.2	2.1	2.0	3.3	30.1
Career		**495**	**489**	**34.9**	**7.9-16.6**	**.477**	**3.2-7.3**	**.444**	**3.4-3.7**	**.902**	**0.7**	**3.7**	**4.3**	**6.9**	**0.2**	**1.8**	**2.5**	**3.2**	**22.4**

With so much available data, it's a no brainer that teams would want to use analytics to help them build a successful team.

2. <u>Description Of Problem In Words</u>

Now lets dive into a real world scenario where a team would want to try and implement analytics to help them build a successful team. So we are going to assume that we are working in a NBA teams analytics department and the general manager (GM) of the team comes to us with a problem he'd like for us to solve. He says to us in words,

"I want to know what the best possible starting 5-man NBA line up is. I only want experienced players, who've spent some time in the league and have been consistent. I don't want an expensive line up either. "

At first this problem may seem overwhelming however, as we begin to dig deeper into it we'll see that this is actually a convex optimization problem!

Lets begin to break down each part of what was requested and try and formulate a mathematical representation of what our GM wants us to do.

3. How to Determine "Best" Players and "Best" Team

"I want to know the best possible starting 5-man NBA line up is."

The biggest problem we face in our formulation is what determines how good a player is? Is it their points? Shooting percentage? This seems like a daunting task to figure out and pinpoint however, there is a statistic that was developed in the past 10 years that will help us greatly.

It is called Player Efficiency Rating (PER) and was developed by John Hollinger (who now works for the Memphis Grizzles in their analytics department) when he was working at ESPN. It is a complicated formula but it's essentially a rating of a player's per-minute productivity.

John Hollinger describes it himself in an ESPN article he wrote in 2011 as,

"To generate PER, I created formulas -- outlined in tortuous detail in my book "Pro Basketball Forecast" -- that return a value for each of a player's accomplishments. That includes positive accomplishments such as field goals, free throws, 3-pointers, assists, rebounds, blocks and steals, and negative ones such as missed shots, turnovers and personal fouls. Two important things to remember about PER are that it's per-minute and is pace-adjusted."

Considering the fact that he has a whole book written about the topic we will not go into detail into about how exactly it is formulated.

But is PER a better way to determine how good a player is as opposed to the old school eye test? Well let's dive into this topic.

Every year before the season starts ESPN releases their list of the top 400 NBA players in the league that they develop with their entire team of NBA Analyst. While often times controversial, it is likely the best list out that ranks players based off perceived ability.

For the sake of our task we will assume that this list represents the true rankings of NBA players for the 2015-2016 NBA seasons. Also we will assume that the best line-up would come out of the top 100 players in the league. Therefore we will only analyze the top 100 eligible players.

What determines if a player is eligible? Well our GM said to us,

"I only want experienced players, who've spent some time in the league and have been consistent."

Therefore, we'll make the assumption that a player is only eligible if they have been in the league the past 3 NBA seasons, so the 2013-2014, 2014-2015, 2015-2016 seasons.

Of those players that are eligible we want the top 100 from the ESPN 2015 NBA ranking. Below is a table of the eligible players that we will use.

NBA Rank	Player Name	NBA Rank	Player Name
1	Lebron James	26	Paul George
2	Anthony Davis	27	Carmelo Anthony
3	Kevin Durant	28	Serge Ibaka
4	Stephen Curry	29	Paul Millsap
5	James Harden	30	Andre Drummond
6	Chris Paul	31	Zach Randolph
7	Russell Westbrook	32	Gordon Hayward
8	Kawhi Leonard	33	Joakim Noah
9	Blake Griffin	34	Kyle Lowry
10	Marc Gasol	35	Pau Gasol
11	John Wall	36	Bradley Beal
12	DeMarcus Cousins	37	Eric Bledsoe
13	LaMarcus Aldridge	38	Goran Dragic
14	Dwight Howard	39	Jeff Teague
15	Damian Lillard	40	Giannis Antetokounmpo
16	Klay Thompson	41	Derrick Rose
17	Jimmy Butler	42	Rudy Gobert
18	Kyrie Irving	43	Dirk Nowitzki
19	Draymond Green	44	Khris Middleton
20	Mike Conley	45	Dwyane Wade
21	Kevin Love	46	Greg Monroe
22	Chris Bosh	47	Derrick Favors
23	Tim Duncan	48	Al Jefferson
24	Al Horford	49	Brook Lopez
25	DeAndre Jordan	50	DeMar Derozen

NBA Rank	Player Name	NBA Rank	Player Name
51	Kyle Korver	76	Kenneth Faried
52	Tony Parker	77	Robin Lopez
53	Nikola Vucevic	78	Taj Gibson
54	Andre Iguodala	79	Monta Ellis
55	Jonas Valanciunas	80	Tyreke Evans
56	Victor Oladipo	81	Jrue Holiday
57	Tristan Thompson	82	Tobias Harris
58	Danny Green	83	Brandon Knight
59	DeMarre Carroll	84	Luol Deng
60	Chandler Parsons	85	Kobe Bryant
61	Tyson Chandler	86	Trevor Ariza
62	Ricky Rubio	87	Ryan Anderson
63	Isaiah Thomas	88	Nene Hilario
64	Andrew Bogut	89	Roy Hibbert
65	Ty Lawson	90	Eric Gordon
66	Danilo Gallinari	91	Tony Allen
67	Rudy Gay	92	George Hill
68	Nicolas Batum	93	Joe Johnson
69	J.J. Redick	94	Michael Carter-Williams
70	Marcin Gortat	95	Donatas Motiejunas
71	Michael Kidd-Gilchrist	96	Thaddeus Young
72	Timofey Mozgov	97	Markieff Morris
73	Kemba Walker	98	Paul Pierce
74	Harrison Barnes	99	Wesley Matthews
75	Reggie Jackson	100	Lou Williams

Now we have our set of players who we'll use for our problem. Note, players who were in the top 100 of rankings in ESPN's rankings but have not been in the league were not used for this list since our GM asked for *"experienced"* players.

I've gone ahead and collected and organized all the data from these players from the last 3 seasons in excel sheets and then extracted that data into MATLAB for further analysis. I will go over this later on in this report as to what data was extracted. All the data was taken from Basketball-Refrence.com.

However is it fair to assume the PER will depict this perceived ranking accurately?

Well we can check this and see.

Let,

$$r_i = i; for\ i = 1 \cdots 100$$

where the variable r_i is the rank of player i. So for example Lebron James would be,

$$r_1 = 1$$

and it would go down the list for each player.

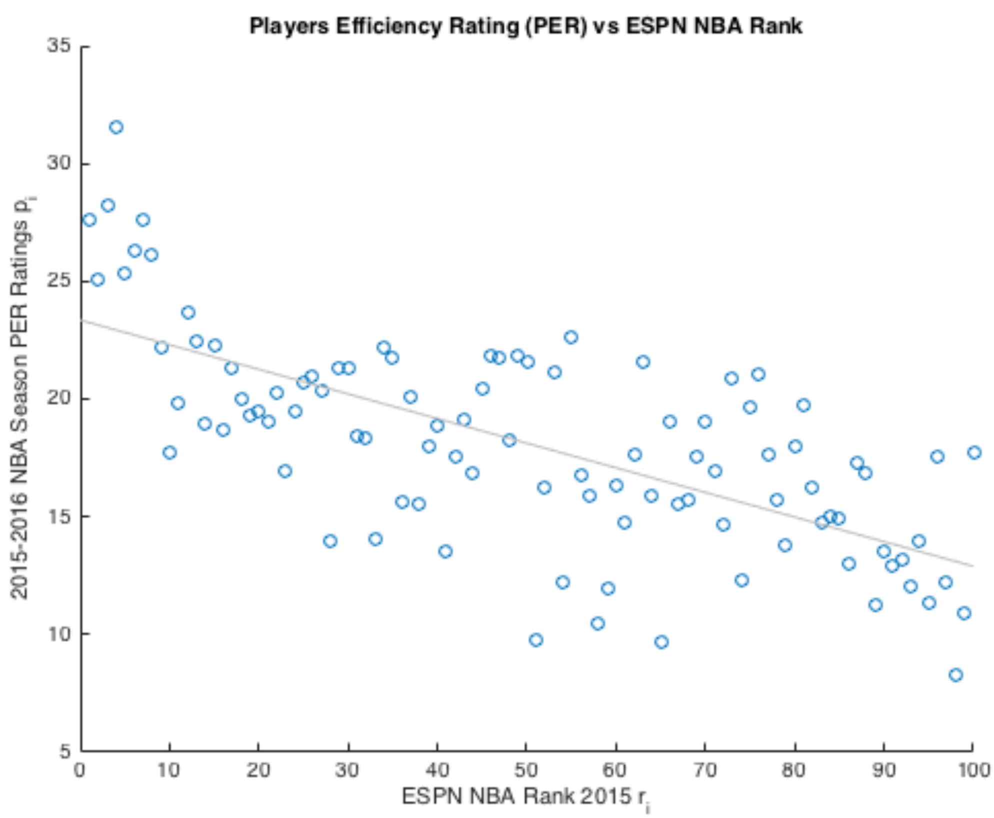
Next we'll let

$$p_i = PER\ Rating\ for\ player\ i\ for\ the\ 2015 - 2016\ season; for\ i = 1 \cdots 100$$

So for example p_1 is Lebron James PER rating for the 2015-2016 season.

$$p_1 = 27.64$$

Now we'll plot the graph of $r_i\ vs\ p_i$, in MATLAB.



We see from the best fit, least squares, line that in general the relationship between ESPN's perceived rank, and each players own PER rating is more or less linearly related. Therefore it confirms our assumption that we can use PER to represent how good an individual player is.

However our task calls for the best *"starting 5-man NBA line up".*

Is it a fair assumption that how good a team is can be determined by just totaling the individual player PER ratings up for the starting line up? Lets look into this. From this point on also, anytime the word "team" is used, it will be referring to the teams starting 5-man line-up.

We'll let,

$$tp_i = \sum_{j=1}^{5} pl_{ji}$$

where,

$$j = A\ starting\ player\ on\ team\ i;\ \ for\ j = 1,\cdots,5$$

$$i = NBA\ team\ in\ order\ of\ final\ 2015 - 2016\ NBA\ standings;\ where\ i = 1,\cdots,30$$

$$pl_{ji} = PER\ rating\ for\ player\ j\ on\ team\ i$$

$$tp_i = Team\ total\ PER\ rating\ for\ team\ i$$

This may seem confusing so lets look at a particular example. The best team in 2015-2016 NBA regular season was the Golden State Warriors.

Since they were the best team in the NBA based off their Win Percentage,

$$i = 1$$

Next we want to know their starting line up.

It is,

Golden State Warriors; *i=1*				
PG	SG	SF	PF	C
j = 1	*j = 2*	*j = 3*	*j = 4*	*j = 5*
Stephen Curry	Klay Thompson	Harrison Barnes	Draymond Green	Andrew Bogut

Now since we know i and j, we'll add the PER ratings for each player which is pl_{ji}.

	Golden State Warriors; *i=1*				
	PG	SG	SF	PF	C
	j = 1	*j = 2*	*j = 3*	*j = 4*	*j = 5*
	Stephen Curry	Klay Thompson	Harrison Barnes	Draymond Green	Andrew Bogut
PER Rating	31.56	18.67	12.32	19.31	15.92

Now, we can calculate tp_i for the Golden State Warriors

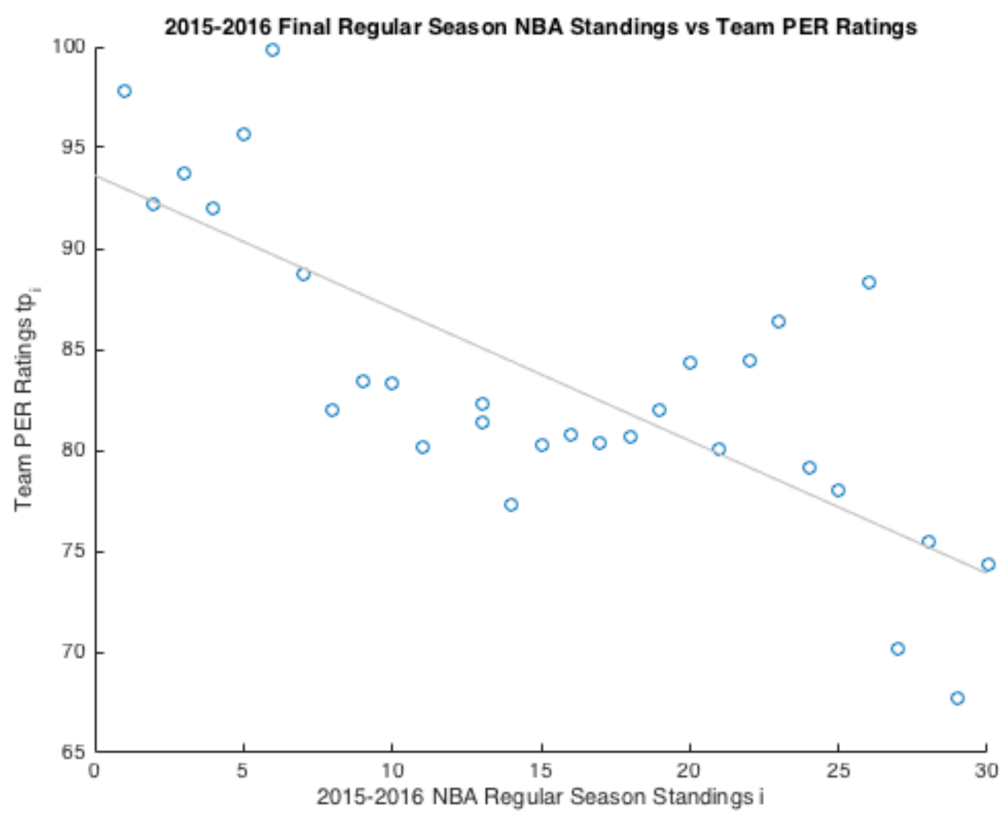
$$tp_1 = \sum_{j=1}^{5} pl_{j1} = 97.78$$

Now that we know what tp_i is we want to use it to see if it is an accurate representation of how good a team is. To do this we'll plot,

$$i \ vs \ tp_i$$

to see how tp_i is related to a teams final standing at the end of the season, which we said earlier is just i.

The plot of $tp_i \ vs \ i$ for all NBA teams can be seen below. It was generated in MATLAB, as is all plots that will come up in this report.



We can see that the best fit, least squares line has a more or less equal linear relation, which is great. So in general we can say that the better a team is (their end of the season standings), the higher Team PER rating the will have.

We'll take a closer look at this and compare the data side by side.

Rank	Team Name	Win %
1	Golden State Warriors	89.0%
2	San Antonio Spurs	81.7%
3	Cleveland Cavaliers	69.5%
4	Toronto Raptors	68.3%
5	Oklahoma City Thunder	67.1%
6	Los Angeles Clippers	64.6%
7	Miami Heat	58.5%
8	Atlanta Hawks	58.5%
9	Boston Celtics	58.5%
10	Charlotte Hornets	58.5%
11	Indiana Pacers	54.9%
12	Portland Trail Blazers	53.7%
13	Detroit Pistons	53.7%
14	Dallas Mavericks	51.2%
15	Memphis Grizzles	51.2%
16	Chicago Bulls	51.2%
17	Houston Rockets	50.0%
18	Washington Wizards	50.0%
19	Utah Jazz	48.8%
20	Orlando Magic	42.7%
21	Sacramento Kings	40.2%
22	Denver Nuggets	40.2%
23	Milwaukee Bucks	40.2%
24	New York Knicks	39.0%
25	New Orleans Pelicans	36.6%
26	Minnesota Timberwolves	35.4%
27	Phoenix Suns	28.0%
28	Brooklyn Nets	25.6%
29	Los Angeles Lakers	20.7%
30	Philadelphia 76ers	12.2%

Rank	Team Name	Team PER
1	Los Angeles Clippers	99.81
2	Golden State Warriors	97.78
3	Oklahoma City Thunder	95.65
4	Cleveland Cavaliers	93.74
5	San Antonio Spurs	92.23
6	Toronto Raptors	92.02
7	Miami Heat	88.71
8	Minnesota Timberwolves	88.35
9	Milwaukee Bucks	86.39
10	Denver Nuggets	84.42
11	Orlando Magic	84.31
12	Boston Celtics	83.45
13	Charlotte Hornets	83.27
14	Detroit Pistons	82.30
15	Utah Jazz	82.04
16	Atlanta Hawks	81.90
17	Portland Trail Blazers	81.46
18	Chicago Bulls	80.77
19	Washington Wizards	80.68
20	Houston Rockets	80.36
21	Memphis Grizzles	80.23
22	Indiana Pacers	80.16
23	Sacramento Kings	80.08
24	New York Knicks	79.15
25	New Orleans Pelicans	78.01
26	Dallas Mavericks	77.25
27	Brooklyn Nets	75.46
28	Philadelphia 76ers	74.31
29	Phoenix Suns	70.18
30	Los Angeles Lakers	67.66

Note: All the starting lineups and Team PER ratings can be found in excel sheet '2016_Team_Stats'.

From the tables on the previous page we can see in general, aside from a few exceptions, that a Team's PER ratings is an accurate way to determine how good of a team they are. We can see that the top 7 teams in both tables are the same and the bottom 4 are the same. The middles are similar but with some exceptions.

We'll do one final analysis in this topic since it is essentially the basis of our entire problem. We'll break it down to an analysis of each conference, Eastern and Western.

For the Eastern Conference we have:

PER Rank	Standings Rank	Team Name	Win %
1	1	Cleveland Cavaliers	69.5%
2	2	Toronto Raptors	68.3%
3	3	Miami Heat	58.5%
9	4	Atlanta Hawks	58.5%
6	5	Boston Celetics	58.5%
7	6	Charlotte Hornets	58.5%
12	7	Indiana Pacers	54.9%
8	8	Detroit Pistons	53.7%
10	9	Chicago Bulls	51.2%
11	10	Washington Wizards	50.0%
5	11	Orlando Magic	42.7%
4	12	Milwaukee Bucks	40.2%
13	13	New York Knicks	39.0%
14	14	Brooklyn Nets	25.6%
15	15	Philadelphia 76ers	12.2%

Standings Rank	PER Rank	Team Name	Team PER
1	1	Cleveland Cavaliers	93.74
2	2	Toronto Raptors	92.02
3	3	Miami Heat	88.71
12	4	Milwaukee Bucks	86.39
11	5	Orlando Magic	84.31
5	6	Boston Celetics	83.45
6	7	Charlotte Hornets	83.27
8	8	Detroit Pistons	82.30
4	9	Atlanta Hawks	81.90
9	10	Chicago Bulls	80.77
10	11	Washington Wizards	80.68
7	12	Indiana Pacers	80.16
13	13	New York Knicks	79.15
14	14	Brooklyn Nets	75.46
15	15	Philadelphia 76ers	74.31

For the Western Conference we have:

PER Rank	Standings Rank	Team Name	Win %
2	1	Golden State Warriors	89.0%
4	2	San Antonio Spurs	81.7%
3	3	Oklahoma City Thunder	67.1%
1	4	Los Angeles Clippers	64.6%
8	5	Portland Trail Blazers	53.7%
13	6	Dallas Mavericks	51.2%
10	7	Memphis Grizzles	51.2%
9	8	Houston Rockets	50.0%
7	9	Utah Jazz	48.8%
11	10	Sacramento Kings	40.2%
6	11	Denver Nuggets	40.2%
12	12	New Orleans Pelicans	36.6%
5	13	Minnesota Timberwolves	35.4%
14	14	Phoenix Suns	28.0%
15	15	Los Angeles Lakers	20.7%

Standings Rank	PER Rank	Team Name	Team PER
4	1	Los Angeles Clippers	99.81
1	2	Golden State Warriors	97.78
3	3	Oklahoma City Thunder	95.65
2	4	San Antonio Spurs	92.23
13	5	Minnesota Timberwolves	88.35
11	6	Denver Nuggets	84.42
9	7	Utah Jazz	82.04
5	8	Portland Trail Blazers	81.46
8	9	Houston Rockets	80.36
7	10	Memphis Grizzles	80.23
10	11	Sacramento Kings	80.08
12	12	New Orleans Pelicans	78.01
6	13	Dallas Mavericks	77.25
14	14	Phoenix Suns	70.18
15	15	Los Angeles Lakers	67.66

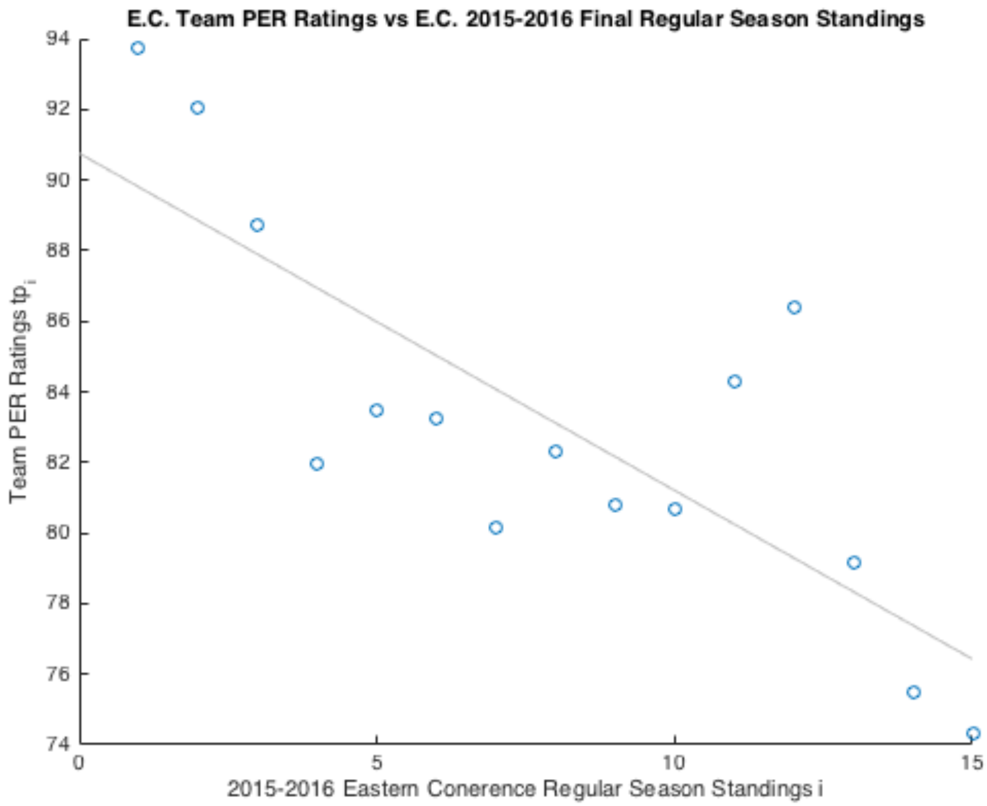
For each conference, the top 8 teams in the standings make the NBA playoffs. The Team PER rating rankings for the Eastern Conference has 6 out of the 8 teams correct.

For the Western Conference the Team PER ratings rankings have 5 out of the 8 teams correct. So we can see that it did a good but not perfect job in analyzing the best teams in each conference.

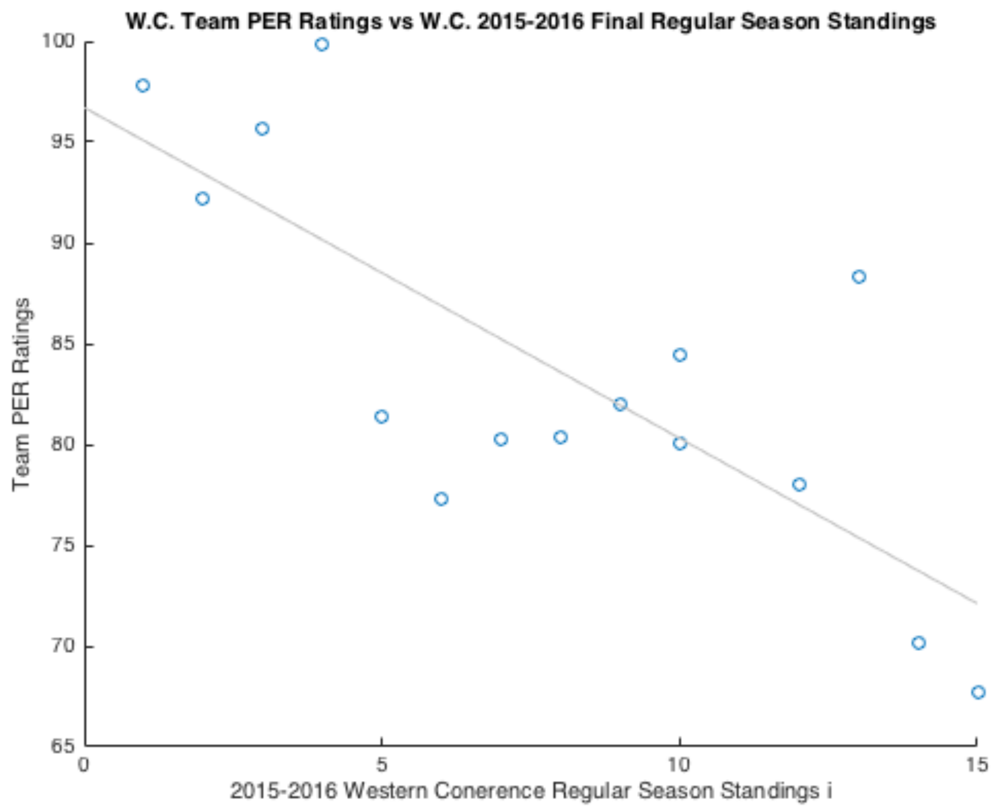
However the most important conclusion that we can make from this was that the top 7 teams for each ranking were the same in both graphs. The best teams were identical, and our GM told us to make the *"best"* team.

We'll conclude this analysis with $i\ vs\ tp_i$ graphs for each conference.

For the Eastern Conference:



For the Western Conference:



We see that even if we break it down for each conference, there is still a equal linear relationship between team PER rating and regular season standings.

So therefore, from all of this analysis, we can be confident with our assumptions that Team PER ratings is an fairly accurate way to judge how good a team is and individual PER ratings is fairly accurate to judge how good a individual player is.

4. Convex Optimization Problem Formulation

In order to formulate this as a convex optimization problem, we need to define some variables.

Let,

$$x_i = If\ player\ i\ is\ chosen\ for\ the\ line\ up; for\ i = 1, \cdots, 100$$

Therefore we'll say if player i is chosen then $x_i = 1$ and if player i is not chosen then $x_i = 0$. So, we are setting the variable $x_i$to only binary values.

This will be our optimization variable that we will use for our convex optimization problem.

Let,

$$c_i = PER\ rating\ for\ player\ i; for\ i = 1, \cdots, 100$$

So we can say from our knowledge of the last section we want to,

$$Maximize \qquad c^T x$$

Where c is a vector of all PER ratings for the eligible players. We see that this is our objective function of our convex optimization problem.

We will want to introduce constraints for this to get exactly what we were asked to get.

Lets look at the quote "*5-man NBA line up"* from our problem. So since we only want 5 players we can say that our first constraint is:

$$Subject\ To \qquad 1^T x = 5$$

Next, we'll analyze the quote "*I don't want an expensive line up either."*

Let,

$$a_i = 2015 - 2016\ NBA\ Season\ Salary\ for\ player\ i; for\ i = 1, \cdots, 100$$

Therefore we can say,

$$a^T x = Total\ team\ salary$$

Now we want to restrict it so it's not *"expensive".*

Let,

$$b = Average\ Team\ Starting\ Salary\ For\ All\ 30\ NBA\ Teams$$

So then we can say that we want to have,

$$a^T x \leq b$$

which is saying that we want our optimal total team salary to be less than or equal to the average team salary for the entire league.

Later on we will look into how loosening and tightening this constraint b will give us different results.

So far our optimization problem is,

$$\begin{array}{ll} Maximize & c^T x \\ \\ Subject\ to & 1^T x = 5 \\ & a^T x \leq b \end{array}$$

Our next quote we will look at is,

"I only want experienced players, who've spent some time in the league and have been consistent".

Earlier we said that we restricted experienced players to players who've been in the last 3 seasons. Now we want to find a way to address how a player is consistent.

To do so we'll let,

$$\begin{array}{cccc} & \mathbf{2015-2016} & \mathbf{2014-2015} & \mathbf{2013-2014} \\ F_i = & \left[\begin{array}{c} Minutes\ Per\ Game \\ Field\ Goals\ Per\ Game \\ Field\ Goal\ Percentage \\ 3-Pointers\ Per\ Game \\ 3-Point\ Percentage \\ Rebounds\ Per\ Game \\ Assist\ Per\ Game \\ Steals\ Per\ Game \\ Blocks\ Per\ Game \\ Points\ Per\ Game \end{array}\right. & \begin{array}{c} Minutes\ Per\ Game \\ Field\ Goals\ Per\ Game \\ Field\ Goal\ Percentage \\ 3-Pointers\ Per\ Game \\ 3-Point\ Percentage \\ Rebounds\ Per\ Game \\ Assist\ Per\ Game \\ Steals\ Per\ Game \\ Blocks\ Per\ Game \\ Points\ Per\ Game \end{array} & \left.\begin{array}{c} Minutes\ Per\ Game \\ Field\ Goals\ Per\ Game \\ Field\ Goal\ Percentage \\ 3-Pointers\ Per\ Game \\ 3-Point\ Percentage \\ Rebounds\ Per\ Game \\ Assist\ Per\ Game \\ Steals\ Per\ Game \\ Blocks\ Per\ Game \\ Points\ Per\ Game \end{array}\right] \end{array}$$

$$for\ player\ i; \qquad i = 1, \cdots, 100$$

So F_i is a 10 by 3 matrix since it has 10 rows and 3 columns. Columns 1, 2, 3 represent the 2015-2016, 2014-2015, 2013-2014 NBA Seasons respectively. Rows 1-10 represent Minutes Per Game (MIN), Field Goals Per Game (FG), Field Goal Percentage (FG%), 3-Pointers Per Game (3-PT), 3-Point Percentage (3-PT%), Rebounds Per Game (REB), Assists Per Game (AST), Steals Per Game (STL), Blocks Per Game (BLK), and Points Per Game (PTS) respectively.

This may be a little much to comprehend all at once so lets take a look at an example below for Stephen Curry

Stephen Curry Per Game Averages			
	2015-2016	2014-2015	2013-2014
MIN	34.2	32.7	36.5
FG	10.2	8.2	8.4
FG%	0.504	0.487	0.471
3-PT	5.1	3.6	3.3
3-PT%	0.454	0.443	0.424
REB	5.4	4.3	4.3
AST	6.7	7.7	8.5
STL	2.1	2	1.6
BLK	0.2	0.2	0.2
PTS	30.1	23.8	24

So we can see that the matrix F_i will give us exactly what we want for each player's stats from the past 3 seasons. Therefore we can say that our team's total stats will be the following,

$$x_1F_1 + x_2F_2 + \cdots + x_{100}F_{100}$$

It is extremely important to keep in mind that even though we are adding up 100 different matrices, only 5 of the 100 x_i's will be 1. The other 95 of them will be all zero and therefore reduce the corresponding F_i matrix to a matrix full of zeros. Therefore the final combination of this will result in the 5-man total team statistic for our optimized team.

Now that we have our total teams stats we'll want to restrict them to make sure our team has players who have been *"consistent."*

To do so first we'll let,

$$h = \begin{array}{c} 2015 - 2016\ League\ Starting\ Lineup\ Team\ Average \\ \begin{bmatrix} MIN \\ FG \\ FG\% \\ 3-PT \\ 3-PT\% \\ REB \\ AST \\ STL \\ BLK \\ PTS \end{bmatrix} \end{array}$$

So h is a 10 by 1 vector that takes all of the teams in the leagues 5-man starting line up team stats, adds them all up piecewise, then divides each component by 30 to get the average stats per team.

Now that we have h, we'll let

$$K = [h \quad h \quad h]$$

Then we can see that K is a 10 by 3 matrix, with the 2015-2016 league average stats repeated 3 times. Therefore we now can use K, as a constraint for our optimization problem. Finally though before we can add another constraint to our problem, for simplicity sake we'll let

$$G = (.75) * K$$

We are loosening our constraint by ¾ to allow some more feasible combinations of starting line ups. Our GM only asked for "*consistent players*", so we'll take it upon ourselves to make the assumption that a team of consistent players in general would have total team stats above at least ¾ the league average for the most recent NBA season.

Therefore our final constraint for this problem is,

$$x_1F_1 + x_2F_2 + \cdots + x_{100}F_{100} \succcurlyeq G$$

Where we are using the piece was inequality intentionally since we want each total team stat to be above the corresponding average in the G matrix.

So now for our convex optimization problem we have,

$$\begin{aligned} &Maximize && c^T x \\ &Subject\ to && 1^T x = 5 \\ & && a^T x \leq b \\ & && x_1F_1 + x_2F_2 + \cdots + x_{100}F_{100} \succcurlyeq G \end{aligned}$$

There is one last final constraint that we are going to add to our problem. Let's look again at the following quote from our problem in words, "*starting 5-man NBA line up*".

A standard starting line up in the NBA consists of a Point Guard (PG), Shooting Guard (SG), Small Forward (SF), Power Forward (PF), and a Center (C). Here's an example below, which is just the table from before for the Golden State Warriors starting line up.

2015-2016 Golden State Warriors Starting Line Up				
PG	SG	SF	PF	C
j = 1	*j = 2*	*j = 3*	*j = 4*	*j = 5*
Stephen Curry	Klay Thompson	Harrison Barnes	Draymond Green	Andrew Bogut

Even though this is the standard starting line up, position wise it is not always strictly followed. Sometimes teams may like to play a smaller line up, depending on their own personal strengths and weaknesses. A common one that the Golden State Warriors use often can be seen below,

2015-2016 Golden State Warriors Alternate Line Up				
PG	SG	SF	SF	PF
j = 1	*j = 2*	*j = 3*	*j = 4*	*j = 5*
Stephen Curry	Klay Thompson	Andre Iguodala	Harrison Barnes	Draymond Green

So now comes an issue with our convex optimization problem that we constructed just a few lines ago. There is no restriction on positions for our starting line up. So our problem will like a line up of 5 Centers, just as well as it would like a line up of 5 Points Guards. This is completely unrealistic and even if it does result in an optimal line up it wouldn't be an accurate depiction of a possible team. In order to have our results be realistic we'll have to get creative.

For each player we'll assign to them their natural position. So for example, Lebron James natural position is a SF and Stephen Curry's is a PG. Even though certain players can play different positions, we will restrict their position to their natural position.

Next we'll assign weights to each player's natural position.

Let,

$$PG = 1; \quad SG = 2; \quad SF = 3; \quad PF = 4; \quad C = 5;$$

So we are assigning a numerical weight to each position. Therefore a standard starting line would be equal to,

$$PG + SG + SF + PF + C = L$$

$$1 + 2 + 3 + 4 + 5 = 15$$

So the line up score, L, is equal to the position score of the sum of each player in the line up.

The Warriors smaller alternative line up would be equal to,

$$PG + SG + SF + SF + PF = L$$

$$1 + 2 + 3 + 3 + 4 = 13$$

Therefore we see that there will be some leeway for realistic line-ups. L doesn't have to equal exactly 15 for the line up to be realistic. Also note that just like we can have a smaller realistic line up, we can also have a bigger realistic line up. Such as one with,

$$PG + SF + PF + PF + C = L$$

$$1 + 3 + 4 + 4 + 5 = 17$$

So from this information we'll make an assumption and say that realistic line-ups are only for

$$13 \leq L \leq 17$$

Lastly we want to be able to express this as a constraint so we'll let,

$$p_i = Player\ i's\ natural\ position\ score; for\ i = 1, \cdots, 100$$

So for example Lebron James is p_1, and therefore $p_1 = 3$.

Then we can say that our last constraint is,

$$13 \leq p^T x \leq 17$$

We are limiting our 5-man line up to only our assumed realistic values. Note that only 5 values of vector x will have values of 1 and the rest will be 0.

Therefore in conclusion we can express our finalized convex optimization problem as such,

$$\begin{aligned} &\textit{Maximize} \qquad c^T x \\ &\textit{Subject to} \qquad 1^T x = 5 \\ &\qquad\qquad a^T x \leq b \\ &x_1 F_1 + x_2 F_2 + \cdots + x_{100} F_{100} \succcurlyeq G \\ &\qquad 13 \leq p^T x \leq 17 \end{aligned}$$

Where,

x = Our optimization variable and only contains binary values. The index values of the 1's in vector x will tell us what player is chosen.

1^T= Just a 100 by 1 vector of all 1's, used to add up individual values of our optimization value x.

c = PER ratings for each player expressed as a vector where,

$$c_i = 2015 - 2016\ NBA\ Season\ PER\ for\ player\ i; \quad i = 1, \cdots, 100$$

a = Player salary for the 2015-2016 NBA season expressed as a vector where,

$$a_i = Salary\ for\ player\ i\ for\ the\ 2015 - 2016\ NBA\ Season; \quad i = 1, \cdots, 100$$

b = A scalar value for the average salary per starting line up per team in the league.

F_i = A 10 by 3 matrix of player i's stats for the past 3 NBA regular seasons.

G = A 10 by 3 matrix, with the 2015-2016 league average stats repeated 3 times then multiplied by ¾

p = Position weights for each player expressed as a vector where,

$$p_i = Position\ score\ for\ player\ i; \quad i = 1, \cdots, 100$$

Note: All these above formulas and the problem in words are completely original and thought up by myself.

5. Is Our Convex Optimization Problem Actually Convex?

It's easy to say a problem is convex but to prove it is a much more daunting task. We'll see that a standard form convex optimization problem must be formulated as such and satisfy the following conditions.

$$\begin{aligned} &\mathit{Minimize} && f_0(x) \\ &\mathit{Subject\ to} && f_i(x) \leq 0, \quad i = 1,\cdots,m \\ & && h_i(x) = 0, \quad i = 1,\cdots,p \end{aligned}$$

Where,

$$f_0(x),\ f_1(x),\cdots,f_m(x)\ are\ all\ atleast\ convex\ and\ can\ be\ affine\ functions.$$

So all inequality constraints, regardless of the position of the inequality must be convex or affine.

Also,

$$h_i(x), \quad i = 1,\cdots,p;\, are\ all\ affine\ functions$$

Ok so now lets check our problem that was just formulated. We will rearrange our problem to make our analysis of it easier. So our rearranged optimization problem is,

$$\mathit{Minimize} \qquad -c^T x$$

$$\begin{gathered} \mathit{Subject\ to} \qquad 1^T x - 5 = 0 \\ a^T x - b \leq 0 \\ -(\, x_1F_1 + x_2F_2 + \cdots + x_{100}F_{100} - G) \preccurlyeq 0 \\ p^T x - 17 \leq 0 \\ -p^T x + 13 \leq 0 \end{gathered}$$

We just rearranged the problem with simple algebra and expanded the double-sided inequality to two different ones. Also we changed maximize to minimize and added a minus sign to our objective function. This has no effect on our problem since maximizing a positive function is the same as minimizing a negative one.

Now we'll break down our objective function, inequality constraints and equality constraints to check for convexity.

First let,

$$f_0(x) = -c^T x$$

This is just a coefficient vector times a variable and therefore is an affine function.

Next we'll define each constraint.

$$h_1(x) = 1^T x - 5$$

$$f_1(x) = a^T x - b$$

$$f_2(x) = -(x_1 F_1 + x_2 F_2 + \cdots + x_{100} F_{100} - G)$$

$$f_3(x) = p^T x - 17$$

$$f_4(x) = -p^T x + 13$$

All of these above constraint function are convex and affine. Therefore since our objective function is convex and affine, and our constraints are as well then our problem is indeed a convex optimization problem.

In fact our particular problem is a certain type of convex optimization problem called a Semidefinite Program (SDP). The reason why our problem is indeed a SDP is because of $f_2(x)$. $f_2(x)$ is a Linear Matrix Inequality (LMI), so therefore our problem is indeed a SDP.

The standard form SDP can be seen below,

$$\begin{array}{ll} \textit{Minimize} & c^T x \\ \textit{Subject to} & Ax = b \\ & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preccurlyeq 0 \end{array}$$

Which is the same thing as our problem however with less constraints and some different signs. These differences though don't affect the convexity of the problem.

Therefore we have adequately proved that our problem is indeed a convex optimization problem and specifically a SDP.

6. Implementation

All data for this problem was taken from http://www.basketball-reference.com/ except for the PER ratings which we taken from http://www.ESPN.com.

All the data was organized myself in excel sheets. I then used MATLAB to extract all the data into it then set up the convex optimization problem.

NOTE: Standard free CVX solvers are not able to solve this problem since they cannot handle binary variables. I used a professional solver, MOSEK, which I got for free from an academic professional license. **Please use MOSEK or Gurobi otherwise the MATLAB file WILL NOT WORK!** For information on these solvers, and how to install them please see the following webpage below.

http://cvxr.com/news/2012/08/midcp/

MATLAB Code gives thorough explanation via comments into what it is doing. Please follow those.

NOTE: Please follow instructions file, entitled 'READ_ME_FIRST_INSTRUCTIONS.pdf' while opening and setting up the MATLAB Code.

7. Duality Analysis

To get a further understanding of our problem we'll find it's Dual Problem equivalent that will let us know of a lower bound on our optimal value.

Our original problem is,

$$\begin{array}{ll} minimize & -c^T x \\ subject\ to & 1^T x - 5 = 0 \\ & a^T x - b \leq 0 \\ & -(x_1 F_1 + x_2 F_2 + \cdots + x_{100} F_{100} - G) \preccurlyeq 0 \\ & p^T x - 17 \leq 0 \\ & -p^T x + 13 \leq 0 \end{array}$$

Now we'll find the Lagrangian for it,

$$L(x, \lambda, \nu, Z) = -c^T x + \lambda(a^T x - b) + \lambda(p^T x - 17) - \lambda(p^T x - 13) + \nu(1^T x) \ldots$$
$$\ldots + trace(-Z(x_1 F_1 + x_2 F_2 + \cdots + x_{100} F_{100} - G))$$

Next we'll rearrange our equation to our liking,

$$L(x, \lambda, \nu, z) = \left(-c^T + \lambda a + \lambda p^T - \lambda p^T + v1^T - tr(ZF_i)\right)x + tr(ZG) - \lambda(b + 4)$$

We have our Lagrangian. Now we'll find our Lagrange dual function

$$g(\lambda, \nu, Z) = \ inf_x(L(x, \lambda, \nu))$$

$$g(\lambda, \nu, Z) = tr(ZG) - \lambda(b + 4); for\ -c^T + \lambda a + \lambda p^T - \lambda p^T + v1^T - tr(ZF_i) = 0$$

Otherwise,

$$g(\lambda, \nu, Z) = \ -\infty$$

Since we have our Lagrange dual function we can now set up our Dual Problem,

It is,

$$\begin{aligned} &maximize \quad tr(ZG) - \lambda(b + 4) \\ &subject\ to \\ &\qquad -c^T + \lambda a + \lambda p^T - \lambda p^T + v1^T - tr(ZF_i) = 0 \\ &\qquad \lambda \geq 0;\ \nu \geq 0; Z \succcurlyeq 0 \end{aligned}$$

The answer to this problem, d^*, will be a lower bound on our optimal value. In fact since our original problem is convex we can say that there's a very high chance that, $d^* = p^*$, with p^* being the answer to our original convex optimization also known the primal problem. This is called strong duality and usually holds for all convex problems.

8. Results

Now for the moment of truth and the answer to our problem. The optimal line up that CVX generated when we restricted the salary to the league average, which is $44,762,305.76, can be seen below.

Salary Restricted to b=$44,762,230, (league average)			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
PG	Damian Lillard	$4,236,286.00	22.25
SG	Demar Derozen	$9,500,000.00	21.58
PF	Anthony Davis	$7,070,730.00	25.10
C	Pau Gasol	$7,448,760.00	21.76
Total		$39,626,562.00	122.25

As we can see, this team PER rating is actually 22.45 points higher than the highest NBA team PER rating, the Los Angeles Clippers, while maintaining a total team salary that is less than the league average by more than 5 millions dollars.

While it may not be possible or realistic for a NBA team to make the trades to actually assemble this line up, it is really interesting to see how powerful CVX is. It was able to generate a line up with such a high team PER rating that would be considered cheap.

9. Further Results and Analysis

While we've already answered our original question, I think it'd be interesting to look at some different results that we get from merely just loosening or tightening our salary cap restriction on our team. This is called perturbation and sensitivity analysis in Convex Optimization.

Below we'll see the tables for optimal line-ups for a few different salary restrictions

Salary Restricted to b=$30,000,000			
Natural Position	Name	Salary	PER
PG	Damian Lillard	$4,236,286.00	22.25
PG	Isaiah Thomas	$6,912,869.00	21.54
SF	Harrison Barnes	$3,874,498.00	12.32
PF	Anthony Davis	$7,070,730.00	25.10
C	Pau Gasol	$7,448,760.00	21.76
Total		$29,543,143.00	102.97

Salary Restricted to b=$40,000,000			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
PG	Damian Lillard	$4,236,286.00	22.25
SG	Demar Derozen	$9,500,000.00	21.58
PF	Anthony Davis	$7,070,730.00	25.10
C	Pau Gasol	$7,448,760.00	21.76
Total		$39,626,562.00	122.25

Salary Restricted to b=$50,000,000			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
PG	Damian Lillard	$4,236,286.00	22.25
SF	Kawhi Leonard	$16,407,500.00	26.11
PF	Anthony Davis	$7,070,730.00	25.10
C	Pau Gasol	$7,448,760.00	21.76
Total		$46,534,062.00	126.78

Salary Restricted to b=$60,000,000			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
PG	Russell Westbrook	$16,744,218.00	27.64
SF	Kawhi Leonard	$16,407,500.00	26.11
PF	Anthony Davis	$7,070,730.00	25.10
C	Pau Gasol	$7,448,760.00	21.76
Total		$59,041,994.00	132.17

Salary Restricted to b=$70,000,000			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
PG	Damian Lillard	$4,236,286.00	22.25
SF	Kevin Durant	$20,158,622.00	28.25
SF	Lebron James	$22,970,500.00	27.64
PF	Anthony Davis	$7,070,730.00	25.10
Total		$65,806,924.00	134.80

Salary Restricted to b=$80,000,000			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
SF	Kawhi Leonard	$16,407,500.00	26.11
SF	Kevin Durant	$20,158,622.00	28.25
SF	Lebron James	$22,970,500.00	27.64
PF	Anthony Davis	$7,070,730.00	25.10
Total		$77,978,138.00	138.66

Salary Restricted to b=$90,000,000			
Natural Position	Name	Salary	PER
PG	Stephen Curry	$11,370,786.00	31.56
PG	Chris Paul	$21,468,695.00	26.31
SF	Kevin Durant	$20,158,622.00	28.25
SF	Lebron James	$22,970,500.00	27.64
PF	Anthony Davis	$7,070,730.00	25.10
Total		$83,039,333.00	138.86

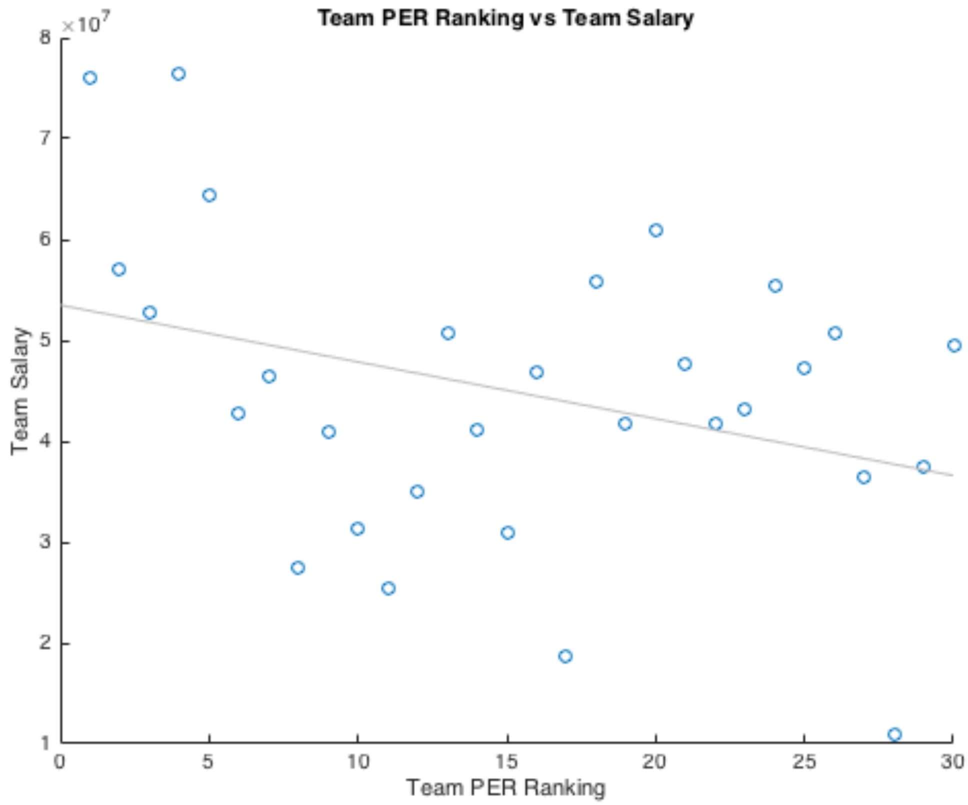
After the salary restriction hits 90 million, the optimal value and line up does not change. It's also interesting to note that from each step from 40 million dollars and up the increase in total team PER doesn't go up any more than 4 points while the salary restriction goes up 10 million each.

Also the cheapest team that was generated, with a 30 millions dollar restriction, has a team PER rating that is higher than that of the highest real team in the NBA, the Los Angeles Clippers. Lets do some further analysis and take a quick look at how team salary relates to team PER rating.

Please see the table on the on the following page,

Rank	Team Name	Team PER	Starting Line Up Salary
1	Los Angeles Clippers	99.81	$75,943,421.00
2	Golden State Warriors	97.78	$57,044,184.00
3	Oklahoma City Thunder	95.65	$52,742,680.00
4	Cleveland Cavaliers	93.74	$76,477,870.00
5	San Antonio Spurs	92.23	$64,437,500.00
6	Toronto Raptors	92.02	$42,660,482.00
7	Miami Heat	88.71	$46,327,441.00
8	Minnesota Timberwolves	88.35	$27,450,080.00
9	Milwaukee Bucks	86.39	$40,905,440.00
10	Denver Nuggets	84.42	$31,225,675.00
11	Orlando Magic	84.31	$25,409,125.00
12	Boston Celetics	83.45	$34,912,466.00
13	Charlotte Hornets	83.27	$50,744,083.00
14	Detroit Pistons	82.30	$41,163,851.00
15	Utah Jazz	82.04	$30,867,223.00
16	Atlanta Hawks	81.90	$46,746,479.00
17	Portland Trail Blazers	81.46	$18,752,847.00
18	Chicago Bulls	80.77	$55,801,823.00
19	Washington Wizards	80.68	$41,676,975.00
20	Houston Rockets	80.36	$61,008,832.00
21	Memphis Grizzles	80.23	$47,648,020.00
22	Indiana Pacers	80.16	$41,777,866.00
23	Sacramento Kings	80.08	$43,186,825.00
24	New York Knicks	79.15	$55,419,532.00
25	New Orleans Pelicans	78.01	$47,180,268.00
26	Dallas Mavericks	77.25	$50,794,834.00
27	Brooklyn Nets	75.46	$36,485,990.00
28	Philadelphia 76ers	74.31	$10,986,141.00
29	Phoenix Suns	70.18	$37,521,917.00
30	Los Angeles Lakers	67.66	$49,580,303.00

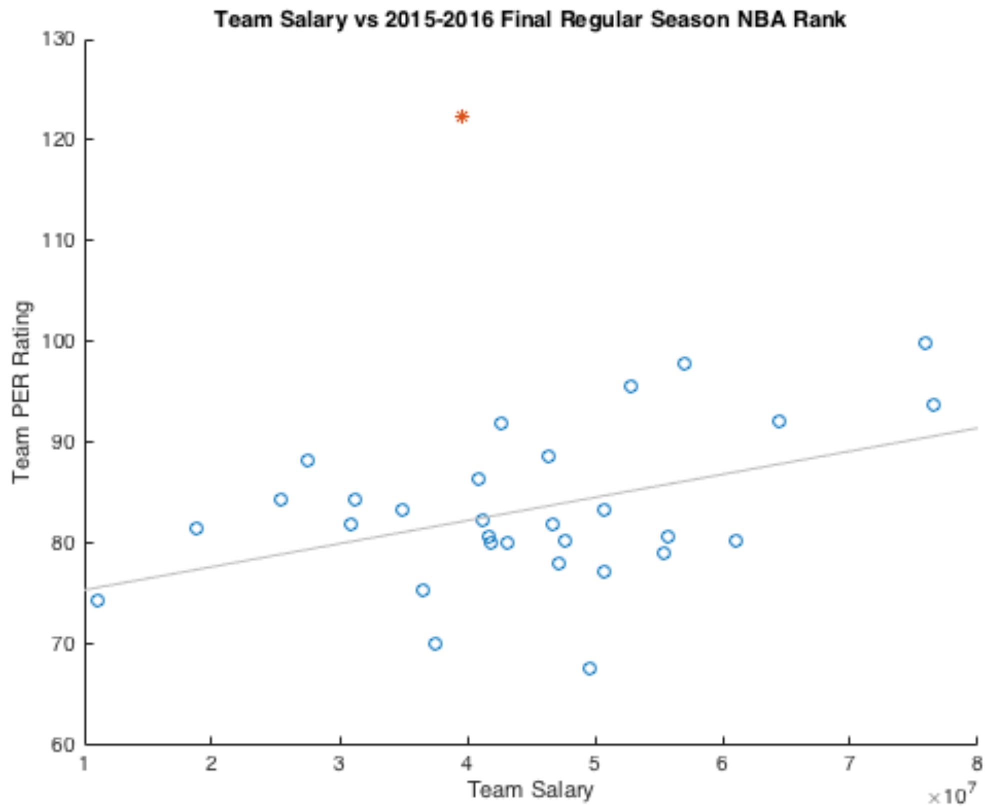
We'll graph this and try and see how the best-fit line looks with team rank, based off Team PER and not standings, on the x-axis and starting line up salary on the y-axis.



As we can see the data is pretty scattered but in general, the teams with the highest team PER rating are the ones spending the most money.

Our optimal value, for b = the average team salary, that we generated earlier with the league average would rank 1 and would have a salary $36,316,859 less than that of the Los Angeles Clippers, who have the highest Team PER.

The last analysis we'll do is to take a look at how our point compares verse all the teams in the league. To do so we'll plot team salary on the x-axis of each team, and their Team PER on the y-axis. We'll also plot our optimal point on top of this and compare these results.



We can see the red dot that is high up with an around average team salary. That point is our optimal point that we generated. It's shocking to see just how efficient our point is as it compares to the real teams in the league.

One interesting thing to note before we conclude this report is that, in every single optimal line up with differing salary restrictions there is 1 player who consistently showed up, Anthony Davis. Anthony Davis is a young player in the league, so he's still on his rookie contract, hence why he was cheap compared to other players with similar PER ratings and therefore an absolute must have. Next year though, he is scheduled to make $21 million, as opposed to his current $7 million salary, so if this analysis to be done again for next season, we would get a much different line-up.

10. Conclusion

We can see convex optimization was able to generate a line up that was far superior to any other teams line up at relatively cheap cost. While it's unrealistic to think that a team could actually trade and acquire all these players and have them as their line up, it's still really interesting to see how powerful convex optimization is in our example.

One thing that was not mentioned earlier is that NBA stats do not accurately evaluate a player's total defensive ability. While there are stats likes Blocks and Steals that are indeed defensive statistics, they don't always give a big picture evaluation of a player's defensive ability.

For example, a player who has a high steals average could mean that he likes to gamble a lot and go for many steals. When he gets the steal great, but when he doesn't then he has left his defender who is now open and has the ball. So even though steals is a great stat to have, it doesn't necessarily mean that the person is a great defender just because he has a high steals average. The same ideology could be used to describe blocks.

If the NBA was able to introduce new publically available statistics that accurately depicted a players defense, then they could be implemented into the convex optimization problem and help generate an even more accurate answer in terms of best possible team.

Also one thing that no stat will ever be able to measure is how team chemistry is and if the players on a team can get along with each other. Obviously if a team were being assembled we'd want a team of people that got along with each other. A team with bad team chemistry can have that effect the team's output on the court.

So, in conclusion we can say that our line up was the best possible line up based off the stats that were available but it ignored some real world issues that could not be taken into account.

11. Bibliography

References/Image Sources:

"Topic: National Basketball Association (NBA)." *Www.statista.com*. N.p., n.d. Web. <http://www.statista.com/topics/967/national-basketball-association/>.

"Warriors vs Rockets Box Score." *ESPN*. ESPN, n.d. Web. <http://espn.go.com/nba/boxscore?gameId=400874342>.

"Stephen Curry Statistics." *ESPN*. ESPN, n.d. Web. <http://espn.go.com/nba/player/stats/_/id/3975/stephen-curry>.

Hollinger, John. "What Is PER?" *ESPN*. ESPN, 11 Aug. 2011. Web. <http://espn.go.com/nba/columns/story?id=2850240&columnist=hollinger_john>.

"#NBArank." *ESPN*. ESPN, n.d. Web. <http://espn.go.com/nba/story/_/id/13963434/nbarank-1-5>.

Boyd, Stephen, and Lieven Vandenberghe. "Convex Optimization Lecture Slides." *Convex Optimization* (n.d.): n. pag. Web.

"2013-14 NBA Stats: Per Game | Basketball-Reference.com." *Basketball-Reference.com*. N.p., n.d. Web. <http://www.basketballreference.com/leagues/NBA_2014_per_game.html>.

"2014-15 NBA Stats: Per Game | Basketball-Reference.com." *Basketball-Reference.com*. N.p., n.d. Web. <http://www.basketballreference.com/leagues/NBA_2015_per_game.html>.

"2015-16 NBA Stats: Per Game | Basketball-Reference.com." *Basketball-Reference.com*. N.p., n.d. Web. <http://www.basketballreference.com/leagues/NBA_2016_per_game.html>.

Published papers that helped give me the idea for this project:

Omidiran, Dapo. "A New Look at Adjusted Plus/Minus for Basketball Analysis." (n.d.): n. pag. MIT Sloan Sports Conference, 4 Mar. 2011. Web. <http://www.sloansportsconference.com/wp-content/uploads/2011/08/Minus-for-Basketball-Analysis.pdf>.

Omidiran, Dapo. "Penalized Regression Models for the NBA." (n.d.): n. pag. 15 Jan. 2013. Web. <http://arxiv.org/pdf/1301.3523.pdf>.