Normal matrix

From Wikipedia, the free encyclopedia

In mathematics, a complex square matrix A is **normal** if

$$A^*A = AA^*$$

where A^* is the conjugate transpose of A. That is, a matrix is normal if it commutes with its conjugate transpose.

A real square matrix A satisfies $A^* = A^{\mathrm{T}}$, and is therefore normal if $A^{\mathrm{T}}A = AA^{\mathrm{T}}$.

A matrix is normal if and only if it is unitarily similar to a diagonal matrix, and therefore any matrix A satisfying the equation $A^*A = AA^*$ is diagonalizable.

The concept of normal matrices can be extended to normal operators on infinite dimensional Hilbert spaces and to normal elements in C*-algebras. As in the matrix case, normality means commutativity is preserved, to the extent possible, in the noncommutative setting. This makes normal operators, and normal elements of C*-algebras, more amenable to analysis.

Contents

Special cases

Among complex matrices, all unitary, Hermitian, and skew-Hermitian matrices are normal. Likewise, among real matrices, all orthogonal, symmetric, and skew-symmetric matrices are normal. However, it is *not* the case that all normal matrices are either unitary or (skew-)Hermitian. For example,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is neither unitary, Hermitian, nor skew-Hermitian, yet it is normal because

$$AA^* = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = A^*A.$$

Consequences

Proposition. A normal triangular matrix is diagonal.

Let A be a normal upper triangular matrix. Since $(A^*A)_{ii} = (AA^*)_{ii}$, one has $\langle e_i, A^*Ae_i\rangle = \langle e_i, AA^*e_i\rangle$ i.e. the first row must have the same norm as the first column:

$$\|Ae_1\|^2 = \|A^*e_1\|^2.$$

The first entry of row 1 and column 1 are the same, and the rest of column 1 is zero. This implies the first row must be zero for entries 2 through n. Continuing this argument for row–column pairs 2 through n shows A is diagonal.

The concept of normality is important because normal matrices are precisely those to which the spectral theorem applies:

Proposition. A matrix A is normal if and only if there exists a diagonal matrix Λ and a unitary matrix U such that $A = U\Lambda U^*$.

The diagonal entries of Λ are the eigenvalues of A, and the columns of U are the eigenvectors of A. The matching eigenvalues in Λ come in the same order as the eigenvectors are ordered as columns of U.

Another way of stating the spectral theorem is to say that normal matrices are precisely those matrices that can be represented by a diagonal matrix with respect to a properly chosen orthonormal basis of $\mathbf{C}^n$. Phrased differently: a matrix is normal if and only if its eigenspaces span $\mathbf{C}^n$ and are pairwise orthogonal with respect to the standard inner product of $\mathbf{C}^n$.

The spectral theorem for normal matrices is a special case of the more general Schur decomposition which holds for all square matrices. Let A be a square matrix. Then by Schur decomposition it is unitary similar to an upper-triangular matrix, say, B. If A is normal, so is B. But then B must be diagonal, for, as noted above, a normal upper-triangular matrix is diagonal.

The spectral theorem permits the classification of normal matrices in terms of their spectra, for example:

Proposition. A normal matrix is unitary if and only if its spectrum is contained in the unit circle of the complex plane.

Proposition. A normal matrix is self-adjoint if and only if its spectrum is contained in $\mathbf{R}$. In other words: A normal matrix is Hermitian if and only if all its eigenvalues are real.

In general, the sum or product of two normal matrices need not be normal. However, the following holds:

Proposition. If A and B are normal with $AB = BA$, then both AB and $A + B$ are also normal. Furthermore there exists a unitary matrix U such that UAU^* and UBU^* are diagonal matrices. In other words A and B are simultaneously diagonalizable.

In this special case, the columns of U^* are eigenvectors of both A and B and form an orthonormal basis in $\mathbf{C}^n$. This follows by combining the theorems that, over an algebraically closed field, commuting matrices are simultaneously triangularizable and a normal matrix is diagonalizable – the added result is that these can both be done simultaneously.

Equivalent definitions

It is possible to give a fairly long list of equivalent definitions of a normal matrix. Let A be a $n \times n$ complex matrix. Then the following are equivalent:

1. A is normal.
2. A is diagonalizable by a unitary matrix.
3. The entire space is spanned by some orthonormal set of eigenvectors of A.
4. $\|Ax\| = \|A^*x\|$ for every x.
5. The Frobenius norm of A can be computed by the eigenvalues of A: $\operatorname{tr}(A^*A) = \sum_j |\lambda_j|^2$.
6. The Hermitian part $\frac{1}{2}(A + A^*)$ and skew-Hermitian part $\frac{1}{2}(A - A^*)$ of A commute.
7. A^* is a polynomial (of degree $\leq n-1$) in A.[1]
8. $A^* = AU$ for some unitary matrix U.[2]
9. U and P commute, where we have the polar decomposition $A = UP$ with a unitary matrix U and some positive semidefinite matrix P.
10. A commutes with some normal matrix N with distinct eigenvalues.
11. $\sigma_i = |\lambda_i|$ for all $1 \leq i \leq n$ where A has singular values $\sigma_1 \geq \ldots \geq \sigma_n$ and eigenvalues $|\lambda_1| \geq \ldots \geq |\lambda_n|$.[3]
12. The operator norm of a normal matrix A equals the numerical and spectral radii of A. (This fact generalizes to normal operators.) Explicitly, this means:

$$\sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} |\langle Ax, x\rangle| = \max\{|\lambda| : \lambda \in \sigma(A)\}$$

Some but not all of the above generalize to normal operators on infinite-dimensional Hilbert spaces. For example, a bounded operator satisfying (9) is only quasinormal.

Analogy

It is occasionally useful (but sometimes misleading) to think of the relationships of different kinds of normal matrices as analogous to the relationships between different kinds of complex numbers:

- Invertible matrices are analogous to non-zero complex numbers
- The conjugate transpose is analogous to the complex conjugate
- Unitary matrices are analogous to complex numbers on the unit circle
- Hermitian matrices are analogous to real numbers
- Hermitian positive definite matrices are analogous to positive real numbers
- Skew Hermitian matrices are analogous to purely imaginary numbers

As a special case, the complex numbers may be embedded in the normal 2×2 real matrices by the mapping

$$a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

which preserves addition and multiplication. It is easy to check that this embedding respects all of the above analogies.

Notes

1. Proof: When A is normal, use Lagrange's interpolation formula to construct a polynomial P such that $\overline{\lambda_j} = P(\lambda_j)$, where λ_j are the eigenvalues of A.
2. Horn, pp. 109

3. Horn, Roger A.; Johnson, Charles R. (1991). *Topics in Matrix Analysis*. Cambridge University Press. p. 157. ISBN 978-0-521-30587-7.

References

- Horn, Roger A.; Johnson, Charles R. (1985), *Matrix Analysis*, Cambridge University Press, ISBN 978-0-521-38632-6.