The Effects of Risk Aversion on Optimization

Demystifying Risk Aversion Parameters in the Barra Optimizer

February, 2010

Scott Liu, PhD, CFA, Vice President
Rong Xu, PhD, CFA

Abstract

The concept of risk aversion plays an important role in modern portfolio theory. In this Research Insights paper, we discuss the influences of risk aversion on various aspects of portfolio optimization—the portfolio holdings, return, risk, utility, Sharpe ratio, efficient frontier, and the minimum-risk portfolio. We also explain the source of the default values of the risk aversion parameters in the Barra Optimizer, and propose a number of practical approaches for determining these parameters.

Our analysis reveals that, in the absence of complicated constraints or objective terms, the return, risk, and utility of the optimal mean-variance portfolio all decrease as the risk aversion increases. The number of names in a long-only optimal portfolio increases as the risk aversion increases. The Sharpe ratio, on the other hand, first increases and then decreases as the risk aversion increases. In other words, there is a portfolio with a maximum value of the Sharpe ratio when we maximize the risk-adjusted return. However, that portfolio might not be what the portfolio manager would choose.

Our main message is that the risk aversion parameters in the Barra Optimizer provide users with the flexibility to control or adjust the risk levels of their optimal portfolios. Risk aversion parameters provide valuable tools for portfolio managers to explore and customize their portfolio optimization results and investment processes.

TABLE OF CONTENTS

1. INTRODUCTION

The concept of risk aversion plays an important role in modern portfolio theory. Many practitioners, however, often find it difficult to determine the proper values for risk aversion when managing their portfolios. This paper attempts to bridge the gap between theory and practice. It aims to provide some practical illustrations for portfolio managers, especially for users of the Barra Optimizer.

We begin by reviewing some theoretical aspects of risk aversion, and show where the default values of the risk aversion parameters in the Barra Optimizer come from. Next, we summarize the influences of risk aversion on various aspects of portfolio optimization—the portfolio holdings, return, risk, utility, Sharpe ratio, efficient frontier, and the minimum-risk portfolio,[1] and present empirical results to demonstrate these influences. Then, we propose a number of practical approaches for determining the risk aversion parameters, and discuss their pros and cons. Concluding remarks are also provided.

2. WHAT DOES RISK AVERSION REPRESENT?

In his 1952 seminal work [1], Harry Markowitz laid out the foundation for modern portfolio theory and the mean-variance portfolio selection/optimization process.

He reasoned that for a fixed set of asset means, variances, and covariances, investors would want to select a portfolio that lies on an efficient frontier—that is, a portfolio with minimum variance for a given expected return or with maximum expected return for a given variance. Markowitz stopped short of pointing out that the particular portfolio on the efficient frontier an investor chooses depends on his risk aversion, which measures his reluctance to take on additional risk for any additional expected return.

Nowadays, risk aversion parameters play an important and explicit role in portfolio management. Quantitative portfolio managers often use a mean-variance objective when constructing and rebalancing their portfolios. In the simplest form, such an objective U takes on the form

$$U = \mu - \lambda \sigma^2 \tag{1}$$

where μ is the expected portfolio return, σ^2 is the portfolio variance (σ is the standard deviation), and λ ($\lambda \geq 0$) is a risk aversion parameter that specifies the relative importance of the two terms—return and variance (representing risk)—in the objective. The larger the value of λ, the more risk is to be penalized. Conversely, the smaller the value of λ, the less important the risk term will be in the objective. In the extreme case when $\lambda = 0$, the risk term is ignored and the return term completely dominates. In essence, the risk aversion parameter λ represents a trade-off between the expected return and risk.

A typical "plain vanilla" mean-variance portfolio optimization problem has the following form:[2]

(P1)
$$\underset{\mathbf{h}}{\text{Maximize}}: \quad \mathbf{r}^{\mathrm{T}}\mathbf{h} - \lambda\, \mathbf{h}^{\mathrm{T}} \boldsymbol{\Sigma}\, \mathbf{h} \tag{2}$$

$$\text{Subject to}: \quad \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \tag{3}$$

[1] Detailed analyses are provided in Appendix B.

[2] Throughout this paper, vectors and matrices are written in bold fonts, whereas scalar variables are written in regular fonts. Our discussion focuses mainly on the cases without a benchmark. Appendix B.8 briefly discusses the results of the cases involving a benchmark.

where $\mathbf{r}$ is the vector of asset excess returns, Σ is the asset-by-asset covariance matrix, $\mathbf{h}$ is the vector of portfolio holdings or weights, and $\mathbf{e}$ is a vector of 1's. Note that objectives (1) and (2) are equivalent, since

$$\mu = \mathbf{r}^{\mathrm{T}}\mathbf{h} \quad \text{and} \tag{4}$$

$$\sigma^2 = \mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} \tag{5}$$

Constraint (3) is the normal holding constraint, which requires that the sum of all asset weights be 1. When $\lambda > 0$, problem (P1) is a quadratic programming problem. When $\lambda = 0$, the (quadratic) variance term in the objective (2) vanishes, and problem (P1) reduces to a linear programming problem. [3]

How can we obtain a reasonable value for λ? Grinold and Kahn [2] have provided the following "scientific" formula:

$$\lambda = \frac{\mu_B}{2\sigma_B{}^2} \tag{6}$$

where μ_B is the expected excess return of a benchmark and σ_B is the benchmark risk.[4] Assume $\mu_B = 6$ percent and $\sigma_B = 20$ percent, then $\lambda = 0.0075$. This is the source of the default values for the risk aversion parameters in the Barra Optimizer. These values—6% return and 20% risk—represent the S&P 500's long-term average annual excess return and risk.

A "plain vanilla" objective function in the Barra Optimizer is

$$\underset{\mathbf{h}}{\text{Maximize}}: \quad \mathbf{r}^{\mathrm{T}}\mathbf{h} - \mathbf{h}^{\mathrm{T}}(\lambda_F \mathbf{XFX}^{\mathrm{T}} + \lambda_D \mathbf{D})\,\mathbf{h} \tag{7}$$

where $\mathbf{X}$ is a matrix representing asset exposures to the risk factors, $\mathbf{F}$ is the covariance matrix of the risk factors, $\mathbf{D}$ is a diagonal or block diagonal covariance matrix of the asset-specific returns, λ_F is a risk aversion parameter for the common factor risk, and λ_D is a risk aversion parameter for the asset-specific risk. Comparing (7) with (2), we observe the following:

- Barra Optimizer's objective function is more flexible since (2) is a special case of (7)—when $\lambda_F = 0$ and $\mathbf{D} = \Sigma$ or when $\lambda_D = 0$, $\mathbf{X} = \mathbf{I}$ (the identity matrix), and $\mathbf{F} = \Sigma$.
- Either one of the parameters in (7)—λ_D or λ_F—can be treated as the generic risk aversion parameter λ in (2), which establishes the relative weight of the total risk compared with the total expected return in the objective.[5]
- The ratio of the two parameters—λ_D / λ_F or λ_F / λ_D—further specifies the relative weight of the two component risk terms—common factor risk vs. asset-specific risk—in total risk.[6]

---

[3] For more information about optimization in general, and portfolio optimization in particular, please see [5], [6] and [7].
[4] See Appendix A for a derivation of this formula.
[5] This is because the total risk term in (7) can be written in the following three alternative ways:

$$\begin{aligned} & \mathbf{h}^{\mathrm{T}}(\lambda_F \mathbf{XFX}^{\mathrm{T}} + \lambda_D \mathbf{D})\mathbf{h} \\ &= \lambda_F \mathbf{h}^{\mathrm{T}}(\mathbf{XFX}^{\mathrm{T}} + (\lambda_D/\lambda_F)\mathbf{D})\mathbf{h} \\ &= \lambda_D \mathbf{h}^{\mathrm{T}}((\lambda_F/\lambda_D)\mathbf{XFX}^{\mathrm{T}} + \mathbf{D})\mathbf{h} \end{aligned}$$

[6] The λ_D/λ_F ratio is also known as the AS-CF Risk Aversion Ratio in the Barra Aegis System or the Selection Risk Multiplier in BarraOne.

- When $\lambda_D = \lambda_F$, the two parameters effectively reduce to one—the generic risk aversion parameter λ.
- The default values in the Barra Optimizer—$\lambda_D = \lambda_F = 0.0075$—are the risk aversion parameters that would make a benchmark portfolio with 6% excess return and 20% risk optimal in a standard optimization setting uncomplicated by additional constraints.[7]

3. HOW DOES RISK AVERSION INFLUENCE OPTIMIZATION?

In this section, we discuss the effects of risk aversion on various aspects of the portfolio optimization under the following simplified assumptions:

- The portfolio optimization case is a "plain vanilla" case in the form of problem (P1)[8].
- The objective function is in the form of (2),[9] which involves return and risk only. Transaction costs and penalties are not considered.
- The holding constraint is the only constraint present.

To avoid an overwhelming amount of algebraic manipulations, you can view the detailed analyses and derivations in Appendix B. We summarize the results in the next section.

3.1. Theoretical Points

Based on our analyses in Appendix B, we reach the following conclusions:

- The return, risk, and utility of the optimal mean-variance portfolio all decrease as the risk aversion increases.
- The number of names in a long-only optimal portfolio increases as the risk aversion increases.
- In the absence of a benchmark, the Sharpe-ratio of the optimal mean-variance portfolio is a "hill"-shaped non-linear function of λ with a single peak point. As λ increases from 0 to ∞, the optimal mean-variance portfolio moves along the return-variance efficient frontier from the maximum-return portfolio, through the maximum-Sharpe-ratio portfolio, to the minimum-risk portfolio.
- In the presence of a benchmark, as λ increases from 0 to ∞, the optimal mean-variance portfolio moves along the active-return-to-active-variance efficient frontier from the maximum-active-return portfolio, through a linear combination of the maximum-Sharpe-ratio and the minimum-risk portfolios, to the origin. In general, the maximum-Sharpe-ratio portfolio and the minimum-risk portfolio per se are not on this efficient frontier. Furthermore, every portfolio on this frontier has the same information ratio independent of λ.

3.2. Empirical Results

To help demonstrate the effects of the risk aversion parameters on the mean-variance optimization, we present some empirical results. The covariance matrices $\mathbf{F}$ and $\mathbf{D}$, as well as the factor exposure matrix $\mathbf{X}$, used in these simulation tests all come from the Barra US Equity Model, Long Horizon (USE3L) or Barra Europe Equity Model, Long Horizon (EUE2L) risk models. The investment universe used is the estimation universe of either the USE3L or EUE2L model.

[7] This statement results from the derivation in Appendix A.
[8] Or, it could be in the form of problem (P5) as shown in Section B.8.
[9] It could also be in the form of (56) as shown in Section B.8.

3.2.1. Forecasting

Table 1 and Table 2 show the forecasting results as of the analysis date—February 27, 2009. The values for the asset excess return vector $\mathbf{r}$ used in these cases are randomly generated to be between $[-5\%, 5\%]$.

Table 1. Forecast Results as of February 27, 2009—Varying the Risk Aversion Parameter λ

Case ID	Risk Model	Asset Bounds	λ	Return (%)	Risk (%)	SP Risk (%)	CF Risk (%)	Sharpe Ratio	Utility	#of Longs	# of Shorts
1	USE3L	[0, 1]	0.0001	4.94	30.99	15.29	26.96	0.159	0.05	4	0
2			0.001	4.80	25.63	11.03	23.13	0.187	0.04	8	0
3			0.003	4.54	22.47	8.43	20.83	0.202	0.03	12	0
4			0.0075	3.99	19.81	6.92	18.56	0.202	0.01	17	0
5			0.01	2.52	14.85	5.19	13.92	0.170	0.00	17	0
6			1	-1.85	0.15	0.05	0.14	-12.44	-0.02	17	0
7			100	-1.89	0.0015	0.0005	0.0014	-1273.0	-0.02	17	0
8		[-1, 1]	0.0001	1231.5	926.2	873.4	308.2	1.330	11.46	253	248
9			0.001	990.1	599.7	584.4	134.2	1.651	6.31	249	252
10			0.003	526.9	292.9	287.4	56.37	1.799	2.70	252	249
11			0.0075	217.5	120.9	118.9	22.30	1.798	1.08	252	249
12			0.01	990.1	599.7	584.4	134.2	1.651	6.31	249	252
13			1	-0.25	0.91	0.89	0.17	-0.271	-0.01	249	247
14			100	-1.88	0.0091	0.0089	0.0017	-206.7	-0.02	166	160
15	EUE2L	[0, 1]	0.0001	4.92	42.08	28.50	30.95	0.117	0.05	1	0
16			0.001	4.55	35.82	22.14	28.16	0.127	0.03	3	0
17			0.003	3.65	27.24	13.24	23.80	0.134	0.01	5	0
18			0.0075	2.69	23.37	10.21	21.02	0.115	-0.01	9	0
19			0.01	2.28	22.33	9.35	20.28	0.102	-0.03	9	0
20			1	-0.46	19.86	7.44	18.41	-0.023	-3.95	14	0
21			100	-0.52	19.86	7.45	18.41	-0.026	-394.4	14	0
22		[-1, 1]	0.0001	115.3	228.6	212.9	83.37	0.504	1.10	25	25
23			0.001	104.1	172.4	162.7	56.93	0.604	0.74	25	25
24			0.003	68.86	103.71	99.28	30.00	0.664	0.37	25	25
25			0.0075	32.41	49.59	46.50	17.21	0.654	0.14	26	24
26			0.01	24.41	39.13	35.83	15.73	0.624	0.09	26	24
27			1	-0.33	16.77	9.41	13.88	-0.020	-2.81	32	18
28			100	-0.58	16.76	9.39	13.88	-0.035	-281.0	32	18

Table 1 confirms that as λ increases, the portfolio's expected return, total risk, asset specific risk, common factor risk, and utility all decrease. The Sharpe-ratio, on the other hand, first increases and then decreases. For long-only cases, the number of names in the portfolio increases with λ. At the same risk aversion level, the expected return, total risk, asset specific risk, common factor risk, and utility all increase dramatically when shorting is allowed. In general, a higher risk aversion value and/or more restricted asset bounds will lead to an optimal mean-variance portfolio with lower risk but also lower expected return. The larger the value of λ, the closer the mean-variance portfolio will be to the minimum-risk portfolio.

Table 2 further reveals that if we vary λ_F while keeping λ_D at the default value, the common factor risk of the resulting optimal mean-variance portfolio decreases as λ_F increases, but the asset specific risk and total risk may or may not decrease. Similarly, increasing the value of λ_D while keeping λ_F at the default value will result in a portfolio with smaller asset specific risk, but not necessarily smaller common factor risk or total risk.

Furthermore, the ratio of λ_D/λ_F can also be used to control the number of names in the optimal long-only mean-variance portfolio. When this ratio increases, the number of names will increase, leading to a more diversified portfolio. Conversely, when it decreases, the number of names will also decrease, leading to a more concentrated portfolio.

Table 2. Forecast Results as of February 27, 2009—Varying the Risk Aversion Parameters λ_D and λ_F

Case ID	Risk Model	Asset Bounds	λ_D	λ_F	λ_D/λ_F	Return (%)	Risk (%)	SP Risk (%)	CF Risk (%)	Sharpe Ratio	Utility	# of Longs	# of Shorts
1			0.0075	0.0075	1	3.99	19.81	6.92	18.56	0.202	0.011	17	0
2				0.001	7.5	4.61	24.65	6.30	23.83	0.187	0.037	29	0
3			0.0075	0.01	0.75	2.67	15.37	5.69	14.27	0.174	0.004	16	0
4	USE3L			1	0.0075	-1.84	0.20	0.12	0.16	-9.105	-0.019	7	0
5			0.001		0.13	4.18	21.54	11.04	18.49	0.194	0.015	8	0
6			0.01	0.0075	1.33	3.77	19.09	6.36	18.00	0.20	0.009	19	0
7			1		133.33	-0.70	5.86	0.58	5.83	-0.1	-0.013	202	0
8		[0, 1]	0.0075	0.0075	1	2.69	23.37	10.21	21.02	0.115	-0.014	9	0
9				0.001	7.5	3.5	28.08	10.3	26.11	0.125	0.020	10	0
10			0.0075	0.01	0.75	2.4	22.68	10.2	20.27	0.11	-0.025	9	0
11	EUE2L			1	0.0075	-0.8	25.86	19.4	17.10	-0.029	-2.960	4	0
12			0.001		0.13	3.0	25.16	14.0	20.88	0.121	-0.004	4	0
13			0.01	0.0075	1.33	2.52	22.96	9.24	21.02	0.110	-0.016	10	0
14			1		133.33	-0.23	25.91	3.58	25.66	0.0	-0.180	50	0

3.2.2. Back Testing

In the back tests done here, we start with a cash portfolio and rebalance it monthly according to updated alphas, covariance matrices and factor exposures. The monthly asset alphas are generated based on the previous month's factor return of "Size". Shorting is not allowed.

Table 3 summarizes two sets of 10-year back test results. We see again that as λ increases, the mean monthly expected return and risk both decrease. Not surprisingly, the annualized realized return or risk also decreases with λ. The average number of names, on the other hand, increases with λ.

Table 3. Back Test Results—Varying the Risk Aversion Parameter λ

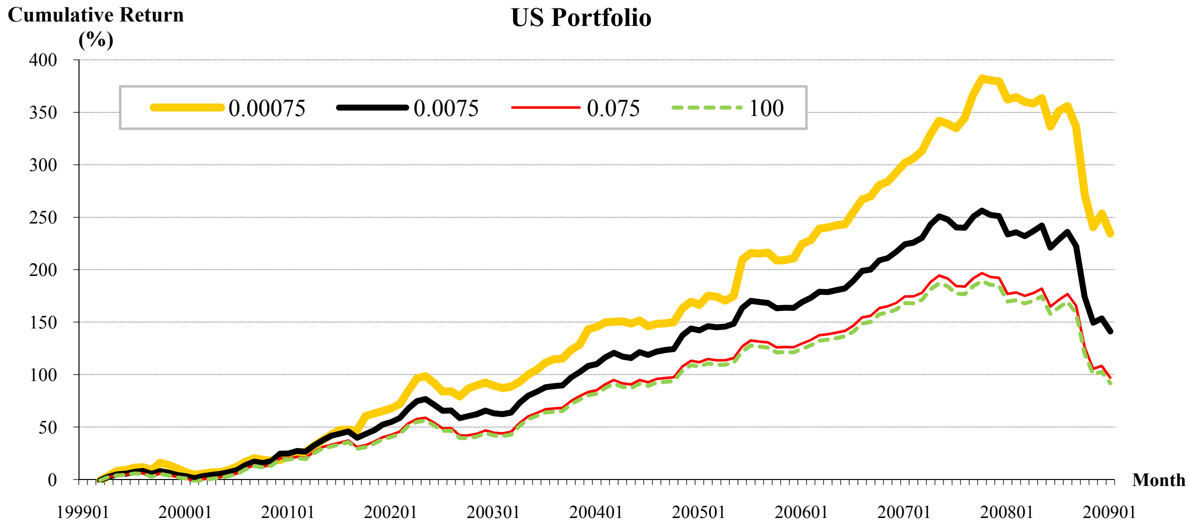
	US Portfolio				EU Portfolio			
Time Horizon	2/26/1999-2/27/2009				12/31//1999-2/27/2009			
λ	0.00075	0.0075	0.075	100	0.00075	0.0075	0.075	100
Mean Monthly:								
Expected Return (%)	0.21	0.10	0.060	0.055	0.18	0.06	0.029	0.025
Annualized Risk(%)	11.41	8.65	8.49	8.49	9.71	6.36	6.17	6.17
Annualized CF Risk(%)	10.38	8.13	8.00	8.00	8.65	5.79	5.63	5.63
Annualized SP Risk(%)	4.64	2.92	2.83	2.83	4.28	2.61	2.53	2.52
Number of Names in Portfolio	64	104	107	107	79	145	149	149
(Annualized) Realized :								
Return(%):	11.95	8.25	6.09	5.80	9.70	7.19	5.14	4.86
Risk (%):	11.62	10.38	10.22	10.20	10.00	6.46	6.60	6.65
Sharpe Ratio	1.03	0.80	0.60	0.57	0.97	1.11	0.78	0.73

Figure 1 plots the cumulative realized returns for both the US and EU portfolios. It shows that the cumulative realized returns also decrease with λ. However, the rate of decrease slows significantly as λ gets larger, indicating that the portfolios are approaching or overlapping with their minimum-risk portfolios.

4. HOW TO SET RISK AVERSION PARAMETERS IN PRACTICE?

As discussed before, the risk aversion parameters signify trade-offs between the portfolio's risk or component risk and the expected return. They represent the relative importance of the risk terms they are associated with. Because of these properties, they offer portfolio managers a way to customize the portfolio optimization process according to their own situations and needs.

For users of the Barra Optimizer, setting $\lambda_D = \lambda_F = 1$ means the risk and return terms in the objective (7) will have equal weight. Using the default values, $\lambda_D = \lambda_F = 0.0075$, on the other hand, means the risk terms are much less important than the return term in the optimization process. Setting large values for the risk aversion parameters will reduce the role of the asset excess return in portfolio selection and construction. On the other hand, adopting small values may lead to a portfolio where the risk is insufficiently accounted for.



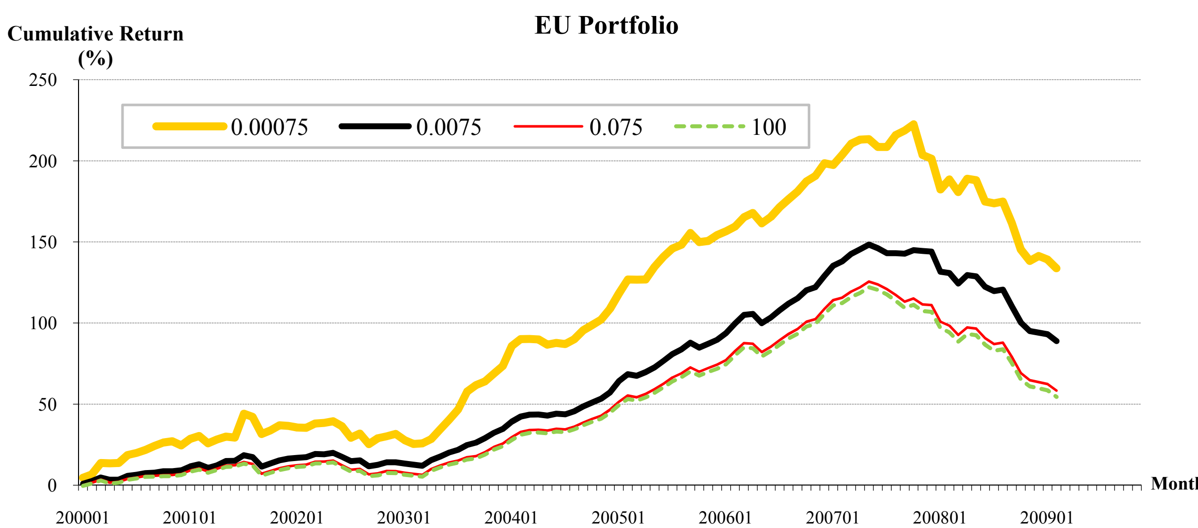


Figure 1. Cumulative Returns of Managed Portfolios Based on Different Risk Aversion Values

Choosing the values for the risk aversion parameters is about striking a proper balance between the manager's expected return and risk. It requires some basic understanding of the role risk aversion parameters play in optimization. It also depends on the perceptions and objectives of the portfolio managers. Ultimately, the parameters chosen should lead to a portfolio on the efficient frontier that reflects the manager's expectation about the portfolio return and risk.

There are several ways to set the risk aversion parameters in practice. We will briefly state what they are, and discuss their pros and cons next.

4.1. The Rule-of-Thumb Approach

Experienced portfolio managers may determine a value for λ based on their rules-of-thumb regarding the cost of risk, which measures an equivalent loss in the expected return.[10]

For instance, consider a portfolio with a historical annualized risk (standard deviation) of 20 percent. If the portfolio manager thinks that such a level of risk would translate into about 3% haircut on the expected return, then he may set $\lambda = 0.0075$ since $0.0075 \cdot 20 \cdot 20 = 3$. If he thinks the penalty for risk would be about 5% instead, then he should use $\lambda = 0.0125$ because $0.0125 \cdot 20 \cdot 20 = 5$. In general, the cost of risk is given by $\lambda \cdot \sigma^2$. Thus, the empirical formula for λ would be

$$\lambda = \text{The cost of risk} \big/ \sigma^2 \tag{8}$$

Obviously, this approach is quite subjective and requires the user to have some prior knowledge about the portfolio risk as well as the cost of risk.

4.2. The Formula-Based Approach

A more "scientific" way to obtain a reasonable value for λ is to use formula (6), as mentioned in Section 2. Generally, if some baseline or benchmark portfolio's long-term average return and risk are available, we can use the following formula to estimate λ:[11]

$$\lambda = \frac{\mu}{2\sigma^2} = \frac{SR}{2\sigma} \tag{9}$$

For instance, assume we know that our benchmark's long-term excess return is 8% and standard deviation is 20%. Then, we can substitute $\mu = 8$ and $\sigma = 20$ into formula (9) to get $\lambda = 0.01$.

Table 4 illustrates the various values of λ derived by this method. It shows that for a given value of the estimated risk, a higher return implies a higher value of the risk aversion. Similarly, for a given value of the estimated return, higher risk is associated with a lower value of the risk aversion.

Table 4. Formula-Based Risk Aversion Parameters

λ (σ \ μ)	4	6	8	10	12
10	0.02	0.03	0.04	0.05	0.06
15	0.0089	0.0133	0.0178	0.0222	0.0267
20	0.005	0.0075	0.01	0.0125	0.015
25	0.0032	0.0048	0.0064	0.008	0.0096
30	0.00222	0.00333	0.00444	0.00556	0.00667
35	0.00163	0.00245	0.00327	0.00408	0.00490
40	0.00125	0.001875	0.0025	0.003125	0.00375

[10] See Grinold and Kahn [2] for more information about this approach.
[11] The Sharpe Ratio, *SR*, is defined in (42).

Note that the λ obtained this way assumes that the underlying baseline or benchmark portfolio is feasible and efficient. It is the risk aversion that will yield the baseline or benchmark portfolio as the optimal portfolio in the "plain vanilla" setting. If this is not intended, then modifications should be made.

In the Barra Optimizer, for example, the default values of the risk aversion parameters reflect the expectation that the long-term benchmark or baseline return is 6% and risk is 20% when unconstrained. If these figures or conditions significantly differ from the user's benchmark or baseline profile, then different values of the risk aversion parameters may be needed.

4.3. The Risk-Constrained Approach

For complicated portfolio optimization problems involving numerous constraints, the formula-based approach to determine λ may be inappropriate, since it is based on a few simplified assumptions and may not represent the true situations at hand. If the portfolio manager has some risk target or range in mind, then an alternative approach is to solve a risk-constrained problem first, such as the problem (P3) in Appendix B.6, to get the dual value of the risk constraint.

As shown in Appendix B.6, the dual value for the risk constraint in (P3) equals the risk aversion value in (P1). This is true even if both (P3) and (P1) involve numerous, but identical constraints. Thus, solving a risk-constrained problem first enables the portfolio manager to get an "implied risk aversion"—the risk aversion that will make a corresponding mean-variance optimization problem yield exactly the same optimal solution.

In general, risk-constrained problems are more difficult to solve computationally and often more time consuming. Mean-variance problems, on the other hand, are more efficient and better behaved. Thus, it may be beneficial for portfolio managers to solve a risk-constrained problem first, in order to get a feel for the problem at hand and to generate an estimate for λ. Then, they could switch to the mean-variance problem settings in subsequent analyses or back tests.

4.4. The Efficient-Frontier Approach

Portfolio managers who don't have a strong sense of the risk-return profiles of their portfolios could use the efficient-frontier approach to generate an estimate for λ. They could first specify a fairly wide range for either the return or the risk, and ask the Barra Optimizer to generate a number of efficient portfolios within the range, subject to any permissible constraints they would like to have. By examining the risk-return profiles of these efficient portfolios, or inspecting the extrapolated graph for the efficient frontier, they may get an idea of their return or risk target. They could then use the "implied risk aversion" associated with the efficient portfolio closest to their return or risk target in subsequent mean-variance analyses or back tests.

In general, solving an efficient-frontier problem, which is a series of mean-variance optimization problems, is much less efficient than solving a single mean-variance optimization problem. Thus, it may be beneficial for portfolio managers to use this approach for occasional analysis or as a preliminary step to obtain the "implied risk aversion" needed for the mean-variance optimization.

5. CONCLUDING REMARKS

The effects of risk aversion on portfolio optimization can be very complicated if the case involves many constraints and objective terms. For a simple "plain vanilla" case, however, closed-form analytical formulas are available to show the impact of risk aversion on various aspects of the optimization in a straightforward and intuitive way. In such a case, as this paper shows, the expected return, risk, and utility of the optimal mean-variance portfolio all decrease as the risk aversion increases. The Sharpe-ratio, on the other hand, first increases and then decreases as the risk aversion increases. In other words, there exists a portfolio with a maximum value of the Sharpe ratio when we maximize the risk-adjusted return.

The maximum-Sharpe-ratio portfolio, however, might not be what the portfolio manager would choose. Other mean-variance portfolios may be equally good, since every one of them is on the efficient frontier. Each mean-variance portfolio corresponds to a unique risk aversion value. Exactly which one to choose really depends on the portfolio manager's willingness to take on additional risk in exchange for additional expected return.

Risk aversion parameters in the Barra Optimizer are flexible calibration tools for users to gauge their portfolio optimization results according to their risk-return expectations or historical risk-return records. For long-only cases, the parameters can also be used to control the number of names in the managed portfolio.

REFERENCES

[1] Markowitz, Harry M. (1952) "Portfolio Selection." Journal of Finance, Vol. 7, No. 1, pp. 77–91.

[2] Grinold, Richard C. and Ronald N. Kahn (2000) Active Portfolio Management McGraw-Hill, pp. 96-97.

[3] Stefek, Dan, Shucheng Scott Liu and Rong Xu (2008) *Barra Optimizer 1.1: Features and Functions Guide*.

[4] Liu, Shucheng Scott and Rong Xu (2004) "Solving Portfolio Optimization Problems with Paring Constraints." *Barra white paper*.

[5] Fletcher, R. (2000) *Practical Methods of Optimization*, John Wiley & Sons, New York; 2nd edition.

[6] Gill, P.E., W. Murray and M.H. Wright (1981) *Practical Optimization,* Academic Press, London-New York.

[7] Liu, S. (2004) "Practical Convex Quadratic Programming — Barra Optimizer for Portfolio Optimization." *Barra white paper*.

APPENDIX A: DERIVATION OF A SIMPLE FORMULA

Consider the simple problem of mixing a benchmark portfolio *B* with a risk-free asset. The expected benchmark excess return is μ_B and the benchmark risk is σ_B. The excess return and risk for the risk-free asset are both zero. Let β be the benchmark weight, and $1-\beta$ be the weight for the risk-free asset. Then problem (P1) reduces to an unconstrained problem:

$$\text{(P2)} \qquad \underset{\beta}{\text{Maximize}}: \quad \mu_B \beta - \lambda \beta^2 \sigma_B{}^2 \tag{10}$$

The first-order condition for β to be optimal is

$$\mu_B - 2\lambda \sigma_B{}^2 \beta = 0 \tag{11}$$

For the benchmark portfolio *B* to be optimal, it must also be true that $\beta = 1$, or,

$$\mu_B - 2\lambda \sigma_B{}^2 = 0 \tag{12}$$

Rearranging (12) gives

$$\lambda = \frac{\mu_B}{2\sigma_B{}^2} \tag{13}$$

Thus, λ given by (13) is the risk aversion that would lead us to choose the benchmark portfolio *B* as the optimal portfolio in the above context.

APPENDIX B: THE EFFECTS OF RISK AVERSION

To facilitate the analysis in this section, it is useful first to introduce two important portfolios—portfolio $\mathbf{h}_a$ and portfolio $\mathbf{h}_c$ —defined as follows:

$$\mathbf{h}_a = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{r}}{\mathbf{e}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{r}} \tag{14}$$

and

$$\mathbf{h}_c = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{e}} \tag{15}$$

It can be shown that portfolio $\mathbf{h}_a$ is the maximum-Sharpe-ratio fully invested portfolio, whereas portfolio $\mathbf{h}_c$ is the minimum-risk fully invested portfolio.[12] Some of their interesting and valuable properties are:

[12] It is not difficult to verify that $\mathbf{h}_a$ and $\mathbf{h}_c$ are the optimal solutions of the following two problems:

$$\begin{cases} \underset{\mathbf{h}}{\text{Maximize}}: & \dfrac{\mathbf{r}^{\mathrm{T}}\mathbf{h}}{\sqrt{\mathbf{h}^{\mathrm{T}}\boldsymbol{\Sigma}\,\mathbf{h}}} \\ \text{Subject to}: & \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \end{cases} \quad \text{and} \quad \begin{cases} \underset{\mathbf{h}}{\text{Minimize}}: & \mathbf{h}^{\mathrm{T}}\boldsymbol{\Sigma}\,\mathbf{h} \\ \text{Subject to}: & \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \end{cases}$$

respectively

- For a given set of the excess return $\mathbf{r}$ and the covariance matrix Σ, as long as Σ^{-1} exists, portfolios $\mathbf{h}_a$ and $\mathbf{h}_c$ can be calculated beforehand using formulas (14) and (15).
- Portfolios $\mathbf{h}_a$ and $\mathbf{h}_c$ are independent of the risk aversion parameter λ.
- Let μ_a and σ_a be the return and risk of $\mathbf{h}_a$, respectively. Then,[13]

$$\mu_a = \frac{\mathbf{r}^{\mathrm{T}}\Sigma^{-1}\mathbf{r}}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{r}} \quad \text{and} \tag{16}$$

$$\sigma_a{}^2 = \frac{\mu_a}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{r}} \tag{17}$$

- Let μ_c and σ_c be the return and risk of $\mathbf{h}_c$, respectively. Then,[14]

$$\mu_c = \frac{\mathbf{r}^{\mathrm{T}}\Sigma^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{e}} \quad \text{and} \tag{18}$$

$$\sigma_c{}^2 = \frac{1}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{e}} \tag{19}$$

- Note that μ_a, $\sigma_a{}^2$, μ_c, and $\sigma_c{}^2$ are all scalar values independent of λ. Furthermore, re-arranging (16)-(19) gives

$$\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{r} = \mu_a / \sigma_a{}^2 \tag{20}$$

$$\mathbf{r}^{\mathrm{T}}\Sigma^{-1}\mathbf{r} = \mu_a{}^2 / \sigma_a{}^2 \tag{21}$$

$$\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{e} = 1 / \sigma_c{}^2 \quad \text{and} \tag{22}$$

$$\mathbf{r}^{\mathrm{T}}\Sigma^{-1}\mathbf{e} = \mu_c / \sigma_c{}^2 \tag{23}$$

- By definition,

$$\mathbf{h}_a{}^{\mathrm{T}}\Sigma\,\mathbf{h}_a = \sigma_a{}^2 \quad \text{and} \tag{24}$$

$$\mathbf{h}_c{}^{\mathrm{T}}\Sigma\,\mathbf{h}_c = \sigma_c{}^2 \tag{25}$$

- Moreover,[15]

$$\mathbf{h}_a{}^{\mathrm{T}}\Sigma\,\mathbf{h}_c = \mathbf{h}_c{}^{\mathrm{T}}\Sigma\,\mathbf{h}_a = \sigma_c{}^2 \tag{26}$$

With the relations (14)-(26) in mind, we now analyze a variety of aspects that risk aversion may have on portfolio optimization, based primarily on the generic problem (P1).

[13] These are easy to derive by substituting (14) into (4) and (5).
[14] These are easy to derive by substituting (15) into (4) and (5).
[15] By utilizing the definitions (14) and (15), we have

$$\mathbf{h}_a{}^{\mathrm{T}}\Sigma\,\mathbf{h}_c = \mathbf{h}_c{}^{\mathrm{T}}\Sigma\,\mathbf{h}_a = \left(\frac{\Sigma^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{e}}\right)^{\mathrm{T}} \Sigma \left(\frac{\Sigma^{-1}\mathbf{r}}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{r}}\right) = \frac{1}{\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{e}} = \sigma_c{}^2$$

B.1. On the Optimal Portfolio Holding

Let π be the dual variable of the holding constraint (3). Then, optimization theory tells us that the first order condition for the Lagrangian

$$L = \mathbf{r}^{\mathrm{T}}\mathbf{h} - \lambda\,\mathbf{h}^{\mathrm{T}}\boldsymbol{\Sigma}\,\mathbf{h} - \pi(\mathbf{e}^{\mathrm{T}}\mathbf{h} - 1) \tag{27}$$

is

$$\partial L / \partial\,\mathbf{h} = \mathbf{r} - 2\lambda\,\boldsymbol{\Sigma}\,\mathbf{h} - \pi\mathbf{e} = 0 \tag{28}$$

or

$$\boldsymbol{\Sigma}\,\mathbf{h} = \frac{\mathbf{r} - \pi\mathbf{e}}{2\lambda}, \qquad (\lambda \neq 0) \tag{29}$$

Assuming $\boldsymbol{\Sigma}^{-1}$ exists, multiplying both sides of (29) by $\boldsymbol{\Sigma}^{-1}$ gives

$$\mathbf{h} = \frac{1}{2\lambda}\boldsymbol{\Sigma}^{-1}(\mathbf{r} - \pi\mathbf{e}) \tag{30}$$

Substituting (30) into the holding constraint (3) and rearranging yields

$$\mathbf{e}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{e}\,\pi = \mathbf{e}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{r} - 2\lambda \tag{31}$$

According to (22) and (23) the scalars $\mathbf{e}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{e}$ and $\mathbf{e}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{r}$ in (31) can be rewritten in terms of μ_c and $\sigma_c^{\,2}$. After re-arranging the terms, we have

$$\pi = \mu_c - 2\sigma_c^{\,2}\lambda \tag{32}$$

By substituting (32) into (30) and using the definitions (14) and (15), we obtain[16]

$$\mathbf{h} = \frac{\mu_c}{2\lambda\,\sigma_c^{\,2}}\mathbf{h}_a + \left(1 - \frac{\mu_c}{2\lambda\,\sigma_c^{\,2}}\right)\mathbf{h}_c \tag{33}$$

or equivalently,

$$\mathbf{h} = \frac{\lambda_a}{\lambda}\mathbf{h}_a + \left(1 - \frac{\lambda_a}{\lambda}\right)\mathbf{h}_c \tag{34}$$

where

$$\lambda_a = \frac{\mu_c}{2\sigma_c^{\,2}} \tag{35}$$

Rearranging (34) also provides

$$\mathbf{h} = \mathbf{h}_c + \frac{\lambda_a}{\lambda}(\mathbf{h}_a - \mathbf{h}_c), \qquad (\lambda \neq 0) \tag{36}$$

Formula (34) reveals that the optimal solution to the mean-variance maximization problem (P1) is a linear combination of the maximum-Sharpe-ratio fully invested portfolio and the minimum-risk

[16] More specifically,

$$\mathbf{h} = \frac{1}{2\lambda}(\Sigma^{-1}\mathbf{r} - \Sigma^{-1}\boldsymbol{e}\,\pi) = \frac{1}{2\lambda}(\mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{r}\cdot\mathbf{h}_a - \mathbf{e}^{\mathrm{T}}\Sigma^{-1}\mathbf{e}\cdot\mathbf{h}_c\,\pi) = \frac{1}{2\lambda}\left(\frac{\mu_c}{\sigma_c^{\,2}}\mathbf{h}_a - \frac{1}{\sigma_c^{\,2}}\mathbf{h}_c(\mu_c - 2\sigma_c^{\,2}\lambda)\right)$$

$$= \frac{1}{2\lambda}\left(\frac{\mu_c}{\sigma_c^{\,2}}\mathbf{h}_a + \left(2\lambda - \frac{\mu_c}{\sigma_c^{\,2}}\right)\mathbf{h}_c\right) = \frac{\mu_c}{2\lambda\sigma_c^{\,2}}\mathbf{h}_a + \left(1 - \frac{\mu_c}{2\lambda\sigma_c^{\,2}}\right)\mathbf{h}_c$$

fully invested portfolio. Since μ_c and $\sigma_c^{\ 2}$ are fixed for a given set of $\mathbf{r}$ and Σ, so is λ_a. Thus, the proportions of $\mathbf{h}_a$ and $\mathbf{h}_c$ in $\mathbf{h}$ will only depend on λ. Formula (36) further indicates that as λ approaches infinity, $\mathbf{h}$ will be approaching $\mathbf{h}_c$. This is intuitive, since as investors become more risk averse, more penalty is imposed on the risk term, and hence more emphasis of the optimization will be placed on minimizing risk rather than maximizing return. In the extreme case when the penalty on risk is prohibitively high, risk minimization will dominate the optimization process, so the minimum-risk portfolio results.

At the other end of the spectrum, formula (36) is not defined when $\lambda = 0$. However, an inspection of the objective (2) directly reveals that the resulting optimal portfolio will maximize the expected return only, and thus will allocate as much weight as possible to the asset(s) with the maximum absolute alpha. In general, as λ decreases, $|\lambda_a|/\lambda$ will increase, pushing the mean-variance optimal portfolio away from the minimum-risk portfolio, and more towards the maximum-return portfolio.

When $\lambda = \lambda_a \geq 0$, the mean-variance optimal portfolio overlaps with the maximum-Sharpe-ratio portfolio. Put in another way, λ_a is the value of λ that will make the mean-variance optimal portfolio identical to the maximum-Sharpe-ratio portfolio. Furthermore, examination of (20), (23) and (35) reveals that

$$\frac{\mu_a}{\sigma_a^{\ 2}} = \frac{\mu_c}{\sigma_c^{\ 2}} = 2\lambda_a \tag{37}$$

Figure 2 illustrates the relationship between asset weights in the optimal portfolio and the risk aversion parameter λ through two examples. In both cases shown, as λ approaches zero, one of the asset weights shoots up. This must be the asset with the highest alpha. For a long-only portfolio, in order to satisfy the holding constraint, 100% is the maximum weight for any asset because the rest of the asset weights can at most decrease to zero. When shorting is allowed, however, the maximum weight theoretically can approach infinity, since the other assets can assume negative weights to help meet the holding constraint. On the other hand, as λ increases, individual asset weights converge separately to a different fixed level for each asset. These fixed levels represent the constituent weights of the corresponding minimum-risk portfolio.

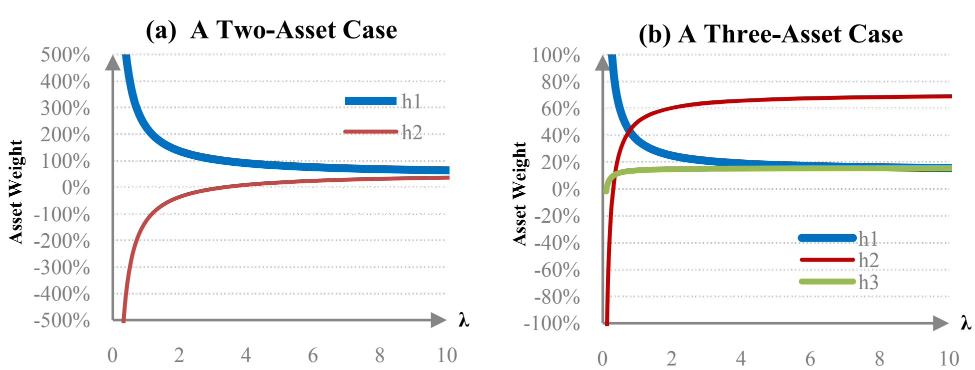


Figure 2. Optimal Asset Weights vs. Risk Aversion

An implication of the above analysis is that as λ decreases, the number of names in the optimal long-only mean-variance portfolio will also decrease. In the extreme case when $\lambda = 0$, the number of names may only be one. On the contrary, as λ increases, the number of names will generally increase until it equals to the number of names in the minimum-risk portfolio. This result is intuitive, since when investors are more risk averse, they tend to want a more diversified portfolio to reduce risk. Thus, the number of names in the optimal portfolio should increase. Conversely, when investors become more risk tolerant, they tend to choose a more concentrated portfolio with a higher expected return, overlooking any incidental higher risk.

B.2. On the Optimal Return

Multiplying both sides of (36) by $\mathbf{r}^{\mathrm{T}}$, we immediately obtain the following

$$\mu = \mu_c + \frac{\lambda_a}{\lambda}(\mu_a - \mu_c) \tag{38}$$

Formula (38) reveals that the optimal portfolio return is inversely related to the risk aversion parameter λ. The higher the value of λ, the lower the optimal μ will be. The infimum for μ is μ_c. Figure 3 depicts this relationship. We see from both the formula and the figure again that as $\lambda \to \infty$ the optimal return converges to the return of the minimum-risk portfolio, i.e., $\mu \to \mu_c$. On the other hand, when $\lambda = \lambda_a$, we have $\mu = \mu_a$.

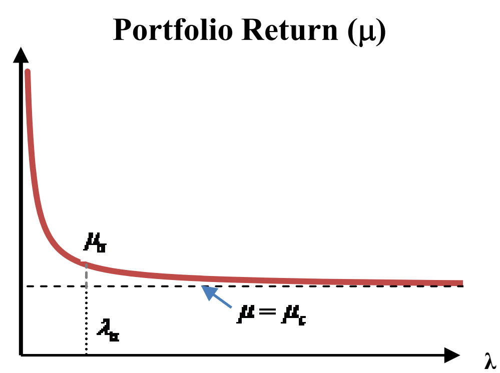


Figure 3. Optimal Portfolio Return vs. Risk Aversion

B.3. On the Optimal Risk

An analytical formula for σ^2 as a function of λ can be obtained by first substituting (36) into (5) to obtain:

$$\sigma^2 = \mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} = \left(\mathbf{h}_c + \frac{\lambda_a}{\lambda}(\mathbf{h}_a - \mathbf{h}_c)\right)^{\mathrm{T}} \Sigma \left(\mathbf{h}_c + \frac{\lambda_a}{\lambda}(\mathbf{h}_a - \mathbf{h}_c)\right) \tag{39}$$

Then, using the relations given by (24)-(26) to simplify (39) provides[17]

$$\sigma^2 = \sigma_c{}^2 + \left(\frac{\lambda_a}{\lambda}\right)^2 (\sigma_a{}^2 - \sigma_c{}^2) \tag{40}$$

Formula (40) suggests that the optimal portfolio variance is inversely related to the square of the risk aversion parameter λ. As Figure 4 shows, the effects of λ on the optimal portfolio variance are very similar to the effects on the optimal portfolio return. However, the degree of impact on variance is greater than that on the expected return. As risk aversion increases, the optimal portfolio variance decreases more rapidly than the optimal expected return. Still, the infimum for σ^2 is $\sigma_c{}^2$. As $\lambda \to \infty$, the optimal variance of the mean-variance portfolio converges to the variance of the minimum-risk portfolio, i.e., $\sigma^2 \to \sigma_c{}^2$. Not surprisingly, we also have $\sigma^2 = \sigma_a{}^2$ when $\lambda = \lambda_a$.

Portfolio Variance (σ^2)

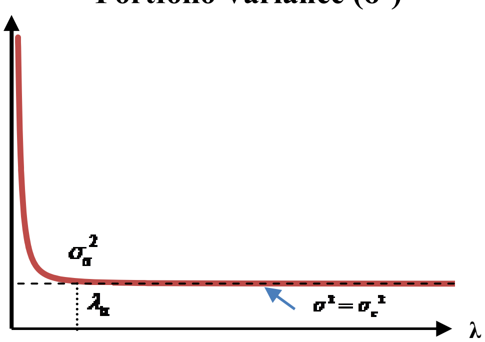


Figure 4. Optimal Portfolio Variance vs. Risk Aversion

[17] More specifically,

$$\sigma^2 = \mathbf{h}_c{}^T \Sigma \mathbf{h}_c + \frac{2\lambda_a}{\lambda}(\mathbf{h}_a - \mathbf{h}_c)^T \Sigma \mathbf{h}_c + \left(\frac{\lambda_a}{\lambda}\right)^2 (\mathbf{h}_a - \mathbf{h}_c)^T \Sigma (\mathbf{h}_a - \mathbf{h}_c)$$

$$= \sigma_c{}^2 + \frac{2\lambda_a}{\lambda}(\mathbf{h}_a{}^T \Sigma \mathbf{h}_c - \mathbf{h}_c{}^T \Sigma \mathbf{h}_c) + \left(\frac{\lambda_a}{\lambda}\right)^2 (\mathbf{h}_a{}^T \Sigma \mathbf{h}_a - 2\mathbf{h}_c{}^T \Sigma \mathbf{h}_a + \mathbf{h}_c{}^T \Sigma \mathbf{h}_c)$$

$$= \sigma_c{}^2 + \frac{2\lambda_a}{\lambda}(\sigma_c{}^2 - \sigma_c{}^2) + \left(\frac{\lambda_a}{\lambda}\right)^2 (\sigma_a{}^2 - 2\sigma_c{}^2 + \sigma_c{}^2) = \sigma_c{}^2 + \left(\frac{\lambda_a}{\lambda}\right)^2 (\sigma_a{}^2 - \sigma_c{}^2)$$

B.4. On the Optimal Utility

The utility of a portfolio essentially measures its risk-adjusted return. Given the results in Appendix B.2-B.3, the relationship between the utility of an optimal mean-variance portfolio and the risk aversion parameter λ is straightforward:[18]

$$U = \left(\mu_c - \lambda\sigma_c^2\right) + \frac{\lambda_a}{2\lambda}\left(\mu_a - \mu_c\right) \tag{41}$$

As Figure 5 shows, its curve is downward sloping. Let $U_c = \mu_c - \lambda\sigma_c^2$. Then, it is still true that the infimum for U is U_c, and as $\lambda \to \infty$, $U \to U_c$. However, U_c is not the utility of the minimum-risk portfolio[19]. Rather, it is a decreasing linear function of λ. On the other hand, by definition as well as by the relation (37), we have $U_a = \mu_a - \lambda_a\sigma_a^2 = \mu_a/2$. This shows that the utility of the maximum-Sharpe-ratio portfolio is independent of λ and happens to be one half of its portfolio return. Evaluating (41) at $\lambda = \lambda_a$, we also have $U = (\mu_c - \lambda_a\sigma_c^2) + \frac{\lambda_a}{2\lambda_a}(\mu_a - \mu_c)$ $= (\mu_c - \mu_c/2) + (\mu_a - \mu_c)/2 = \mu_a/2$. Thus, $U = U_a$ when $\lambda = \lambda_a$. In other words, the utility of the optimal mean-variance portfolio equals to the utility of the maximum-Sharpe-ratio portfolio when $\lambda = \lambda_a$.

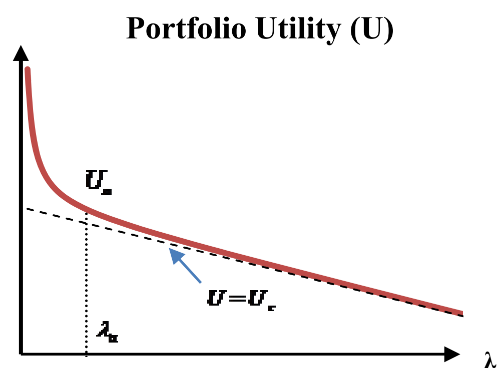


Figure 5. Optimal Portfolio Utility vs. Risk Aversion

[18] More specifically, by substituting (38) and (40) into (1), we obtain

$$U = \mu - \lambda\sigma^2 = \left(\mu_c + \frac{\lambda_a}{\lambda}(\mu_a - \mu_c)\right) - \lambda\left(\sigma_c^2 + \left(\frac{\lambda_a}{\lambda}\right)^2(\sigma_a^2 - \sigma_c^2)\right)$$

$$= \left(\mu_c - \lambda\sigma_c^2\right) + \left(\frac{\lambda_a}{\lambda}(\mu_a - \mu_c) - \frac{\lambda_a^2}{\lambda}(\sigma_a^2 - \sigma_c^2)\right) = \left(\mu_c - \lambda\sigma_c^2\right) + \frac{\lambda_a}{\lambda}\left(\mu_a - \mu_c - \lambda_a\sigma_a^2 + \lambda_a\sigma_c^2\right)$$

$$= \left(\mu_c - \lambda\sigma_c^2\right) + \frac{\lambda_a}{\lambda}\left(\mu_a - \mu_c - \mu_a/2 + \mu_c/2\right) = \left(\mu_c - \lambda\sigma_c^2\right) + \frac{\lambda_\alpha}{2\lambda}\left(\mu_a - \mu_c\right)$$

[19] The minimum risk portfolio corresponds to $\lambda \to \infty$, so its utility is approaching negative infinity.

B.5. On the Optimal Sharpe Ratio

By definition, the Sharpe ratio of a portfolio is the excess return of the portfolio divided by its standard deviation, or

$$SR = \frac{\mu}{\sigma} \tag{42}$$

An analytical formula for the Sharpe ratio of the optimal mean-variance portfolio as a function of λ is readily available by substituting (38) and (40) into (42):

$$SR = \frac{\lambda\mu_c + \lambda_a(\mu_a - \mu_c)}{\sqrt{\lambda^2\sigma_c^{\ 2} + \lambda_a^{\ 2}(\sigma_a^{\ 2} - \sigma_c^{\ 2})}} \tag{43}$$

Mathematically, formula (43) may appear daunting. Graphically, though, it is quite elegant. As Figure 6 shows, it is a "hill"-shaped curve with a fat tail on the right side, lying in the region between the two straight lines— $SR = SR_a$ and $SR = SR_c$. As $\lambda \to \infty$, we see $SR \to SR_c$ where $SR_c = \mu_c / \sigma_c$ is the Sharpe ratio of the minimum-risk portfolio. It can be shown[20] that, when $0 < \lambda < \lambda_a$, the Sharpe ratio monotonically increases with λ at a decreasing rate. When $\lambda > \lambda_a$, it monotonically decreases with λ, first at an increasing rate to a certain point and then at a decreasing rate thereafter. The Sharpe ratio of the optimal mean-variance portfolio achieves its maximum precisely at $\lambda = \lambda_a$ when it equals to the Sharpe ratio of the maximum-Sharpe-ratio portfolio, or $SR_a = \mu_a / \sigma_a$.

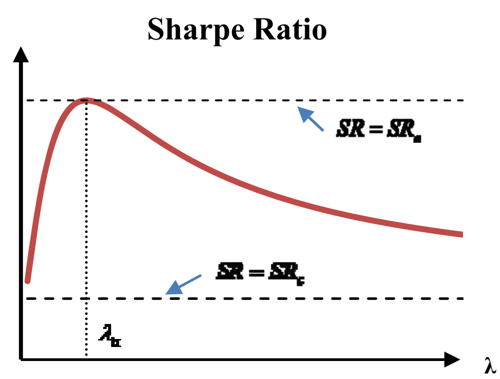


Figure 6. Sharpe Ratio of the Optimal Portfolio vs. Risk Aversion

B.6. On the Efficient Frontier

As mentioned in the beginning of Section 2, an efficient frontier is a graph of portfolios with the minimum variance for a given value of the expected return or with the maximum expected return for a given value of the portfolio variance. Mathematically, the efficient frontier is a collection of solutions to the following two optimization problems:

[20] For example, by taking first- and second-order derivatives of (43) with respect to λ.

(P3)
$$\underset{\mathbf{h}}{\text{Maximize}}: \quad \mathbf{r}^{\mathrm{T}}\mathbf{h} \tag{44}$$
$$\text{Subject to}: \quad \mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} \le \sigma_0^{\;2} \tag{45}$$
$$\mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \tag{46}$$

and

(P4)
$$\underset{\mathbf{h}}{\text{Minimize}}: \quad \mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} \tag{47}$$
$$\text{Subject to}: \quad \mathbf{r}^{\mathrm{T}}\mathbf{h} = \mu_0 \tag{48}$$
$$\mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \tag{49}$$

where μ_0 is a constant representing a given value of the expected portfolio return and $\sigma_0^{\;2}$ is a constant representing a given value of the portfolio variance.

In the next few paragraphs, we will briefly show that problems (P1), (P3) and (P4) are equivalent in the following sense—a solution to the mean-variance maximization problem (P1) is always a solution to the return-maximization problem (P3) as well as a solution to the risk-minimization problem (P4), and vice versa. Hence, a mean-variance optimal portfolio always lies on the efficient frontier. Conversely, any portfolio on the efficient frontier must be a mean-variance optimal portfolio.

Let $\pi_1 \ge 0$, π_2, π_3, and π_4 be the dual variables of the constraints (45), (46), (48) and (49), respectively. Then, the Lagrangians for (P3) and (P4) are

$$L_3 = \mathbf{r}^{\mathrm{T}}\mathbf{h} - \pi_1(\mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} - \sigma_0) - \pi_2(\mathbf{e}^{\mathrm{T}}\mathbf{h} - 1) \tag{50}$$

and

$$L_4 = \mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} - \pi_3(\mathbf{r}^{\mathrm{T}}\mathbf{h} - u_0) - \pi_4(\mathbf{e}^{\mathrm{T}}\mathbf{h} - 1) \tag{51}$$

respectively. The Lagrangian for (P1) is given by (27) already. Thus, the Karush-Kuhn-Tucker (KKT) conditions for (P1), (P3) and (P4) are

$$\begin{cases} \mathbf{r} - 2\lambda\Sigma\,\mathbf{h} - \pi\mathbf{e} = 0 \\ \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \end{cases} \tag{52}$$

$$\begin{cases} \mathbf{r} - 2\pi_1\Sigma\,\mathbf{h} - \pi_2\mathbf{e} = 0 \\ \mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} \le \sigma_0^{\;2} \\ \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \\ \pi_1(\mathbf{h}^{\mathrm{T}}\Sigma\,\mathbf{h} - \sigma_0^{\;2}) = 0 \end{cases} \tag{53}$$

and

$$\begin{cases} 2\Sigma\,\mathbf{h} - \pi_3\mathbf{r} - \pi_4\mathbf{e} = 0 \\ \mathbf{r}^{\mathrm{T}}\mathbf{h} = \mu_0 \\ \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \end{cases} \tag{54}$$

respectively.

If $(\mathbf{h}^*, \pi)$ is a solution set to the system (52), then by setting $\sigma_0{}^2 = \mathbf{h}^{*\mathrm{T}} \Sigma \mathbf{h}^*$, $\pi_1 = \lambda$ and $\pi_2 = \pi$, the set $(\mathbf{h}^*, \pi_1, \pi_2)$ will be a solution to the system (53). Similarly, setting $\mu_0 = \mathbf{r}^{\mathrm{T}} \mathbf{h}^*$, $\pi_3 = 1/\lambda$ and $\pi_4 = -\pi/\lambda$ will make the set $(\mathbf{h}^*, \pi_3, \pi_4)$ a solution to the system (54). In other words, there exists a $\sigma_0{}^2$ such that $\mathbf{h}^*$ has the maximum return among all portfolios with a variance less than or equal to $\sigma_0{}^2$, and there also exists a μ_0 such that $\mathbf{h}^*$ has the minimum risk among all portfolios with a return equal to μ_0. This $(\sigma_0{}^2, \mu_0)$ pair corresponds to a point on the efficient frontier. That is to say, the mean-variance optimal portfolio $\mathbf{h}^*$ is right on the efficient frontier.

Conversely, if $(\mathbf{h}^*, \pi_1, \pi_2)$ is a solution set to the system (53), then by setting $\lambda = \pi_1$ and $\pi = \pi_2$, the set $(\mathbf{h}^*, \pi)$ satisfies the system (52). Similarly, if $(\mathbf{h}^*, \pi_3, \pi_4)$ is a solution to the system (54), then setting $\lambda = 1/\pi_3$ and $\pi = -\pi_4/\pi_3$ will also make the set $(\mathbf{h}^*, \pi)$ a solution to the system (52). That is to say, every point on the efficient frontier is a mean-variance optimal portfolio. Furthermore, the risk aversion parameter λ in (P1) equals to the dual value π_1 in (P3), which measures the sensitivity of the return-maximizing objective (44) to a small change in the bound of the risk constraint (45). It also equals to the inverse of the dual value π_3 in (P4), which measures the sensitivity of the risk-minimizing objective (47) to a small change in the bound of the return constraint (48). In short, λ also represents a rate of change in return per unit of change in variance.

In fact, the slope of the return-variance efficient frontier is 2λ. To see this, combining (38) and (40) to obtain the following analytical form of the efficient frontier:[21]

$$\mu = \mu_c + 2\lambda(\sigma^2 - \sigma_c{}^2) \tag{55}$$

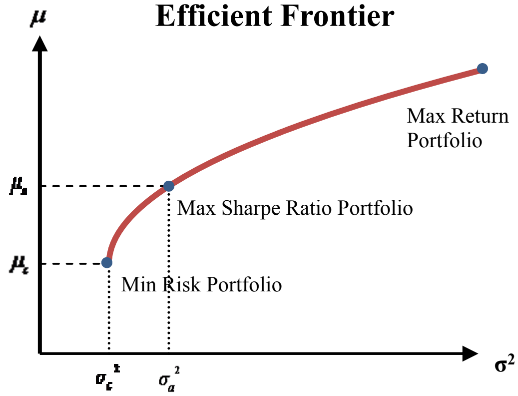


Figure 7. Return-Risk Efficient Frontier vs. Risk Aversion

[21] More specifically, based on (40), we have

$$\sigma^2 = \sigma_c{}^2 + \left(\frac{\lambda_a}{\lambda}\right)^2 (\sigma_a{}^2 - \sigma_c{}^2) = \sigma_c{}^2 + \frac{\lambda_a(\lambda_a \sigma_a{}^2 - \lambda_a \sigma_c{}^2)}{\lambda^2} = \sigma_c{}^2 + \frac{\lambda_a(\mu_a - \mu_c)}{2\lambda^2}$$

or, $\lambda_a(\mu_a - \mu_c)/\lambda = 2\lambda(\sigma^2 - \sigma_c{}^2)$, which can be plugged right into (38) to obtain (55).

Note that if λ were a constant, then μ would be a linear function of σ^2. However, as a parameter, λ varies. In fact, as the relation (40) shows, λ is inversely related to σ^2. Thus, as σ^2 increases, λ will decrease and hence the slope of the efficient frontier will also decrease. As a result, μ is a convex function of σ^2.

Figure 7 is the graphic representation of the efficient frontier. The $(\sigma_c^{\ 2}, \mu_c)$ point represents the minimum-risk portfolio, which corresponds to $\lambda \to \infty$. It has the lowest expected return as well as the lowest variance on the efficient frontier. As σ^2 increases, μ increases but at a decreasing rate, implying that the underlying λ is decreasing. The $(\sigma_a^{\ 2}, \mu_a)$ point represents the maximum-Sharpe-ratio portfolio, corresponding to $\lambda = \lambda_a$. The rightmost point on the efficient frontier is the maximum-return portfolio, which corresponds to $\lambda = 0$. It has the highest expected return as well as the highest variance. Any point with a variance lower than the variance of the minimum-risk portfolio or higher than the variance of the maximum-return portfolio would not lie on the efficient frontier.

B.7. On the Minimum-Risk Portfolio

As mentioned before, the minimum-risk fully invested portfolio $\mathbf{h}_c$ is independent of the generic risk aversion parameter λ. However, the ratio of λ_D / λ_F or λ_F / λ_D in the Barra Optimizer does have significant influence on it. When keeping everything else constant, the minimum-risk portfolio tends to exhibit smaller specific risk as λ_D gets larger, reflecting the emphasis on minimizing specific risk. Similarly, it will have smaller factor risk as λ_F gets larger, reflecting the shifted emphasis on minimizing factor risk.

In short, the generic risk aversion parameter λ determines how close the optimal mean-variance portfolio will be to the minimum-risk portfolio, whereas the ratio of λ_D / λ_F or λ_F / λ_D determines the shape and characteristics of the minimum-risk portfolio per se.

B.8. The Influence of a Benchmark

So far, our analysis has been based on the return and risk being the (excess) total return and (absolute) total risk of the portfolio, in the form of (4) and (5), respectively. If we consider the active return and active risk (or tracking error) relative to a benchmark instead, then problem (P1) would turn into

$$\text{(P5)} \qquad \underset{\mathbf{h}}{\text{Maximize}}: \quad \mathbf{r}^{\mathrm{T}}(\mathbf{h}-\mathbf{h}_B) - \lambda\,(\mathbf{h}-\mathbf{h}_B)^{\mathrm{T}}\Sigma(\mathbf{h}-\mathbf{h}_B) \qquad (56)$$

$$\text{Subject to}: \quad \mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \qquad (57)$$

where $\mathbf{h}_B$ is (the weight of) the benchmark portfolio and $\mathbf{e}^{\mathrm{T}}\mathbf{h}_B = 1$. By definition, the (excess) active return and tracking error are given by

$$\mu_{\mathrm{A}} = \mathbf{r}^{\mathrm{T}}(\mathbf{h} - \mathbf{h}_B) \tag{58}$$

and

$$\sigma_{\mathrm{TE}}^{\ 2} = (\mathbf{h} - \mathbf{h}_B)^{\mathrm{T}} \boldsymbol{\Sigma} (\mathbf{h} - \mathbf{h}_B) \tag{59}$$

All the analysis in Appendix B.1-B.7 can still be carried out. The conclusions will not be identical, but will be very similar.

In particular, the relations (36), (38), (40), (41), (43), and (55) would now become[22]

$$\mathbf{h} = \mathbf{h}_B + \frac{\lambda_a}{\lambda}(\mathbf{h}_a - \mathbf{h}_c) \tag{60}$$

$$\mu_{\mathrm{A}} = \frac{\lambda_a}{\lambda}(\mu_a - \mu_c) \tag{61}$$

$$\sigma_{\mathrm{TE}}^{\ 2} = \left(\frac{\lambda_a}{\lambda}\right)^2 (\sigma_a^{\ 2} - \sigma_c^{\ 2}) \tag{62}$$

$$U_{\mathrm{A}} = \frac{\lambda_a}{2\lambda}(\mu_a - \mu_c) \tag{63}$$

$$\frac{\mu_{\mathrm{A}}}{\sigma_{\mathrm{TE}}} = \frac{\mu_a - \mu_c}{\sqrt{\sigma_a^{\ 2} - \sigma_c^{\ 2}}} \tag{64}$$

and

$$\mu_{\mathrm{A}} = 2\lambda \sigma_{\mathrm{TE}}^{\ 2} \tag{65}$$

Note that as λ increases and approaches infinity, the mean-variance optimal portfolio now approaches the benchmark portfolio $\mathbf{h}_B$, instead of the minimum-risk portfolio $\mathbf{h}_c$. This is actually intuitive. By definition, portfolio $\mathbf{h}_c$ has the minimum total risk among all mean-variance portfolios. However, the active risk, rather than the total risk, is being penalized in the objective function (56). Portfolio $\mathbf{h}_B$, with its active risk being zero, is actually the minimum-active-risk fully invested portfolio when a benchmark is present.

Relation (64) shows that the ratio of the active return to active risk, also known as the information ratio, in the "plain vanilla" case is a constant independent of λ.

Relation (65) reveals that the active-return-to-active-variance efficient frontier is a convex curve starting from the origin, which corresponds to $\lambda \to \infty$. It will end at the maximum-active-return portfolio corresponding to $\lambda = 0$. The portfolio on this frontier corresponding to $\lambda = \lambda_a$ is $(\sigma_a^{\ 2} - \sigma_c^{\ 2}, \mu_a - \mu_c)$, which is a linear combination of the maximum-Sharpe-ratio and the minimum-risk portfolios. In general, the maximum-Sharpe-ratio portfolio $(\sigma_a^{\ 2}, \mu_a)$ and the

[22] In deriving (60), note that (29), (30), and (32) now become

$$\boldsymbol{\Sigma}(\mathbf{h} - \mathbf{h}_B) = \frac{\mathbf{r} - \pi\mathbf{e}}{2\lambda}, \quad \mathbf{h} - \mathbf{h}_B = \frac{1}{2\lambda}\boldsymbol{\Sigma}^{-1}(\mathbf{r} - \pi\mathbf{e}), \text{ and } \pi = \mu_c,$$

respectively. The last relation shows that the dual variable π, unlike in (32), is independent of λ.

minimum-risk portfolio (σ_c^2, μ_c) per se are not on this active-return-to-active-variance efficient frontier.

Furthermore, from (60) we see that the generic risk aversion parameter λ now determines how close the optimal mean-variance portfolio will be to the benchmark portfolio. The ratio of λ_D / λ_F or λ_F / λ_D in the Barra Optimizer, on the other hand, will determine the characteristics of the minimum-active-risk portfolio and how much it resembles the benchmark portfolio.

B.9. Caveats

We want to reiterate that all the analyses done in Appendix B.1-B.8 are based on the following simplified assumptions:

- The portfolio optimization case is a “plain vanilla” case in the form of problem (P1) or (P5).
- The objective function is in the form of (2) or (56) which involves return and risk only. Transaction costs and penalties are not considered.
- The holding constraint is the only constraint present.

Obviously, any violation of the above assumptions might render the conclusions drawn here incomplete or invalid. Additional constraints are more likely to obscure or even distort the patterns shown. We want to emphasize that what we have provided here is some basic understanding and a framework for evaluating the impact of risk aversion. Complicated practical cases undoubtedly will require further investigation. In most cases, closed-form formulas are not available, and simulation analysis needs to be carried out to obtain numerical instead of analytical results.

Contact Information

clientservice@mscibarra.com

Americas	
Americas	1.888.588.4567 (toll free)
Atlanta	+ 1.404.551.3212
Boston	+ 1.617.532.0920
Chicago	+ 1.312.675.0545
Montreal	+ 1.514.847.7506
Monterrey	+ 52.81.1253.4020
New York	+ 1.212.804.3901
San Francisco	+ 1.415.836.8800
Sao Paulo	+ 55.11.3706.1360
Stamford	+1.203.325.5630
Toronto	+ 1.416.628.1007

Europe, Middle East & Africa	
Amsterdam	+ 31.20.462.1382
Cape Town	+ 27.21.673.0100
Frankfurt	+ 49.69.133.859.00
Geneva	+ 41.22.817.9777
London	+ 44.20.7618.2222
Madrid	+ 34.91.700.7275
Milan	+ 39.02.5849.0415
Paris	0800.91.59.17 (toll free)
Zurich	+ 41.44.220.9300

Asia Pacific	
China North	10800.852.1032 (toll free)
China South	10800.152.1032 (toll free)
Hong Kong	+ 852.2844.9333
Seoul	+ 822.2054.8538
Singapore	800.852.3749 (toll free)
Sydney	+ 61.2.9033.9333
Tokyo	+ 81.3.5226.8222

www.mscibarra.com

Notice and Disclaimer

- This document and all of the information contained in it, including without limitation all text, data, graphs, charts (collectively, the "Information") is the property of MSCI Inc. ("MSCI"), Barra, Inc. ("Barra"), or their affiliates (including without limitation Financial Engineering Associates, Inc.) (alone or with one or more of them, "MSCI Barra"), or their direct or indirect suppliers or any third party involved in the making or compiling of the Information (collectively, the "MSCI Barra Parties"), as applicable, and is provided for informational purposes only. The Information may not be reproduced or redisseminated in whole or in part without prior written permission from MSCI or Barra, as applicable.

- The Information may not be used to verify or correct other data, to create indices, risk models or analytics, or in connection with issuing, offering, sponsoring, managing or marketing any securities, portfolios, financial products or other investment vehicles based on, linked to, tracking or otherwise derived from any MSCI or Barra product or data.

- Historical data and analysis should not be taken as an indication or guarantee of any future performance, analysis, forecast or prediction.

- None of the Information constitutes an offer to sell (or a solicitation of an offer to buy), or a promotion or recommendation of, any security, financial product or other investment vehicle or any trading strategy, and none of the MSCI Barra Parties endorses, approves or otherwise expresses any opinion regarding any issuer, securities, financial products or instruments or trading strategies. None of the Information, MSCI Barra indices, models or other products or services is intended to constitute investment advice or a recommendation to make (or refrain from making) any kind of investment decision and may not be relied on as such.

- The user of the Information assumes the entire risk of any use it may make or permit to be made of the Information.

- NONE OF THE MSCI BARRA PARTIES MAKES ANY EXPRESS OR IMPLIED WARRANTIES OR REPRESENTATIONS WITH RESPECT TO THE INFORMATION (OR THE RESULTS TO BE OBTAINED BY THE USE THEREOF), AND TO THE MAXIMUM EXTENT PERMITTED BY LAW, MSCI AND BARRA, EACH ON THEIR BEHALF AND ON THE BEHALF OF EACH MSCI BARRA PARTY, HEREBY EXPRESSLY DISCLAIMS ALL IMPLIED WARRANTIES (INCLUDING, WITHOUT LIMITATION, ANY IMPLIED WARRANTIES OF ORIGINALITY, ACCURACY, TIMELINESS, NON-INFRINGEMENT, COMPLETENESS, MERCHANTABILITY AND FITNESS FOR A PARTICULAR PURPOSE) WITH RESPECT TO ANY OF THE INFORMATION.

- Without limiting any of the foregoing and to the maximum extent permitted by law, in no event shall any of the MSCI Barra Parties have any liability regarding any of the Information for any direct, indirect, special, punitive, consequential (including lost profits) or any other damages even if notified of the possibility of such damages. The foregoing shall not exclude or limit any liability that may not by applicable law be excluded or limited, including without limitation (as applicable), any liability for death or personal injury to the extent that such injury results from the negligence or wilful default of itself, its servants, agents or sub-contractors.

- Any use of or access to products, services or information of MSCI or Barra or their subsidiaries requires a license from MSCI or Barra, or their subsidiaries, as applicable. MSCI, Barra, MSCI Barra, EAFE, Aegis, Cosmos, BarraOne, and all other MSCI and Barra product names are the trademarks, registered trademarks, or service marks of MSCI, Barra or their affiliates, in the United States and other jurisdictions. The Global Industry Classification Standard (GICS) was developed by and is the exclusive property of MSCI and Standard & Poor's. "Global Industry Classification Standard (GICS)" is a service mark of MSCI and Standard & Poor's.

About MSCI Barra

MSCI Barra is a leading provider of investment decision support tools to investment institutions worldwide. MSCI Barra products include indices and portfolio risk and performance analytics for use in managing equity, fixed income and multi-asset class portfolios.

The company's flagship products are the MSCI International Equity Indices, which include over 120,000 indices calculated daily across more than 70 countries, and the Barra risk models and portfolio analytics, which cover 59 equity and 48 fixed income markets. MSCI Barra is headquartered in New York, with research and commercial offices around the world.