Optimal holdings of active, passive and smart beta strategies

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Abstract

The growing dominance of the core and explore model – a large passive index combined with a collection of high tracking error satellite portfolios – in conjunction with the growth of factor investing has renewed interest in how to allocate among different equity strategies. We study this problem from the perspective that investment risk is not having what you need when you need it. We find that portfolios that minimize this investment risk differ substantially from portfolios generated using conventional methods.

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The past decade has witnessed a marked shift in the composition of institutional equity portfolios. Traditional style boxes and a heavy reliance on active management have made way for large allocations to a passive core combined with high tracking error, high conviction, satellite strategies – the core and explore approach.[1] More recently, the emergence of low fee, factor based strategies has further upended this trend. These changes raise the natural question: how should an asset owner allocate among different equity strategies? This basic question goes beyond the active passive debate. After all, factor (i.e. smart beta) strategies have blurred the line between these two approaches. Our main contribution is to answer this question using an allocation framework based on the idea that investment risk is not having what you need when you need it. We find that the portfolios generated by this framework differ substantially from the portfolios generated using conventional methods.

The typical domestic equity core and explore model has about two thirds in the index and one third in satellite portfolios.[2] One way to understand the appeal of core and explore is to adopt a policy portfolio approach (which we do throughout this paper) and to think in terms of expected excess return relative to the policy benchmark.[3] For instance, a typical annualized tracking error for a collection of high tracking error strategies is on the order of 400 basis points. If we apply an expected information ratio (IR) of 0.6, this translates to an average annual excess return of 240 basis points. If we then subtract 72 basis points (30% of the gross excess return) for fees, the asset owner has an expected net return on the collection of high conviction strategies of 168 basis points. This means that an equity portfolio with 65% in the index core (with fees of three basis points), and 35% in the high conviction satellites has an expected after fee excess return of 59 basis points. Rounding this value to 60 basis points, we can think of core and explore as driven by the investor's desire to beat the benchmark by 60 basis points net, per year.

For an investor who needs an average excess return of 60 basis points per year, the core and explore strategy is a relatively simple way to achieve this goal. But what about the role of other, lower orbit, satellite strategies? These lower tracking error strategies are typically quantitatively managed and include enhanced index and factor based (i.e. smart beta) portfolios. While these strategies have lower tracking errors, they also have lower excess returns. Viewed through a mean-tracking error lens – minimize expected tracking error subject to the constraint that expected excess return is equal to 60 basis points – these other strategies can only improve the portfolio's overall tracking error characteristics. This obviously improves the information ratio of the portfolio, but the expected excess return is unchanged because it is fixed by the optimization goal of 60 basis points. While an improved information ratio is desirable, it comes at a cost. This cost is not the

[1] Throughout this study, we shall interchangeably refer to this approach as core and explore or the two asset strategy.
[2] Eight public plans in the US representing nearly 300 billion USD in assets have an average asset weighted mix in their domestic equity portfolios of 64% in passive and 36% in active strategies. Frasier-Jenkins et. al (2017) document a large and growing passive allocation for the overall US market. They predict that by January 2018, 50% of equity AUM will be passively managed.
[3] See Grinold and Kahn (2000) for modern portfolio theory applied to active management.

management fees, as quantitative and factor based strategies tend to have lower fees. Rather, the increased costs are manager search and monitoring costs.[4]

An asset owner focused primarily on achieving a return target (e.g. 60 basis points above benchmark) is less likely to endure the increased searching and monitoring costs if the only improvement is a modest increase in the information ratio of the portfolio. These hidden, or soft costs are higher for active strategies than for more passive factor portfolios, explaining, at least in part, the growing attraction of factor based strategies.

This analysis however, is not the whole story. The analysis suffers from two drawbacks: (i) it assumes that tracking error is an adequate description of investment risk, and (ii) there is no role for investment horizon. Tarlie (2016) introduces an extension of the mean variance approach that addresses these two shortcomings. The main idea is to define investment risk as not having what you need when you need it. Not having what you need introduces the notion of shortfall, and when you need it introduces horizon.

We find that optimal expected shortfall portfolios differ substantially from both the 65/35 core and explore portfolio, and the portfolios generated using the conventional mean tracking error approach. While the core and explore approach ignores the lower tracking error strategies (e.g. enhanced index and smart beta), we find that for modest target excess compounding rates (e.g. ~60 basis points per year) and long portfolio horizons (e.g. greater than 3 years), these strategies play an important role in reducing investment risk[5]. Furthermore, we show how investors with modest target excess compounding rates and long portfolio horizons can have their cake and eat it too. Having their cake means that the optimal expected shortfall portfolios have lower expected shortfall than the corresponding optimal mean variance portfolios.[6] And, eating it too means that these optimal expected shortfall portfolios also have significantly higher expected surplus and significantly lower shortfall probability than their mean variance counterparts.

These shortfall, surplus and probability benefits do come at a cost. The cost is a higher tracking error. In this framing, higher tracking error is not risk. Rather, it is the *cost* necessary to minimize *investment risk* – not having what you need when you need it. For the problems we study in this paper, the optimal expected shortfall portfolios are also mean variance efficient. The framework therefore provides a mechanism to identify how much volatility, or tracking error, is necessary to minimize investment risk. For example, for a target excess compounding rate of about 60 basis points per year and a portfolio horizon of about 20 years, the tracking error that minimizes investment risk (not having what you need when you need it) is almost twice the tracking error of the corresponding mean variance solution. Investors seeking to minimize investment risk – not

[4] Beck et. al. (2016) make this point as well, but they also point out that actively managed strategies can benefit from more flexible trading schedules, thereby reducing trading costs relative to passive funds.
[5] Unless otherwise specified below, we use investment risk to mean not having what you need when you need it.
[6] For simplicity we use the term mean variance to refer to mean tracking error throughout the paper.

having what you need when you need it - therefore need to aim significantly higher from both an expected excess return and tracking error perspective than implied by the conventional approach.

Our work has a number of implications. First, asset owners should revisit their equity allocations from the perspective that investment risk is not having what you need when you need it. This means paying particular attention to excess return targets and portfolio horizon. Second, asset owners should reevaluate their stance towards tracking error. It is a cost not a risk. Furthermore, the tracking error that minimizes investment risk depends on portfolio horizon. Long horizons allow asset owners to have their cake and eat it too, but this requires aiming higher from a tracking error perspective than implied by the conventional mean tracking error approach. Third, those asset owners that have adopted the core and explore approach and that have modest excess return expectations and long portfolio horizons, should allocate away from the passive index to the lower tracking error strategies with positive (after fee and transaction costs) expected excess returns. While the soft costs of managing these strategies is clearly higher than managing a passive index, enduring these costs is warranted by the consequences of falling short of what is needed.

The problems with tracking error as a measure of portfolio risk are well known in the literature. Notable papers are Roll (1992) and Jorion (2003), among others. These papers illustrate that mean tracking error optimal portfolios are not optimal with respect to the absolute return and volatility characteristics of the portfolio. Jorion (2003), for example, explicitly advocates for plan sponsors to concentrate on total portfolio volatility as risk instead of tracking error. Papers that use tracking error as a measure of risk include Blitz and Hottinga (2001), who maximize information ratio as a means of determining optimal tracking error allocations, and Markus et. al. (1999), who optimize over various forms of linear tracking error models. In our work, we use tracking error as a means of classifying different strategies, and we focus on investment risk as not having what you need and when you need it, not on tracking error or portfolio volatility.

Our work also focuses on the role of factor (i.e. smart beta) strategies. These strategies have recently received a lot of attention. Ang (2014), Chapter 14, provides a good motivation for investors to consider factor investing, and Homescu (2015) provides a comprehensive analysis of factor investing, concentrating on the factors and their combinations. Our paper complements these two studies by providing an allocation framework for how factor portfolios fit with other equity strategies. Our allocation framework differs from that in Carson et. al. (2017), who provide a smart beta glide path within the context of a predefined stock-bond allocation. Their approach is to maximize the Sharpe ratio of the smart beta glide path with a constraint on the tracking error relative to the stock-bond glide path. Keller (2014) builds optimal portfolios using smart beta portfolios based on a variant of maximum Sharpe ratio. Finally, a good review of the actual performance of smart beta ETFs is provided by Glushkov (2016), who also offers a classification of the most popular ETFs.

1. Stylized Example

Our basic question is how to allocate between a passive index, an enhanced index, a factor (i.e. smart beta) portfolio, and a collection of high conviction strategies. Our key assumption is that the goal is to beat the policy benchmark by 60 basis points per annum. We begin, in Section 1.1, by articulating our assumptions regarding strategy characteristics, and then in Section 1.2 we apply the conventional mean variance approach to illustrate explicitly that adding these two lower tracking error strategies only modestly improves tracking error and information ratio characteristics. In the presence of hidden and soft costs, these modest improvements in tracking error may not warrant including these two lower tracking error strategies.

Motivated by the notion that investment risk is not having what you need when you need it, in Section 1.3 we introduce the notion of shortfall and examine the shortfall characteristics of the core and explore and optimal mean variance portfolios. While the optimal mean variance portfolio has more desirable shortfall characteristics than the core and explore portfolio, we show in Section 1.4 that we can do substantially better by optimizing over expected shortfall directly.

Our main result, contained in Section 1.4, is that if investment horizon is long enough, then investors can have their cake and eat it too. Having their cake means that expected shortfall is optimal (by construction), and eating it too means that the portfolios also have highly desirable surplus and probability characteristics, even though we only optimize over shortfall.

Furthermore, for the shortfall preferences we consider in this paper, optimal expected shortfall portfolios are also mean variance efficient.[7] This means that optimal expected shortfall portfolios lie on the efficient frontier – they do not represent an improved frontier. Specifying investment horizon and target compounding rate simply defines the location on the efficient frontier that minimizes investment risk (i.e. shortfall). Optimizing expected shortfall for a given horizon and target compounding rate therefore determines the appropriate amount of tracking error needed to minimize investment risk.

1.1 Stylized Strategy Characteristics

Our focus in this paper is on understanding how to allocate between four strategies: pure index, enhanced index, factor (i.e. smart beta), and high conviction. We classify the available strategies by tracking error, from low to high, as summarized in Table 1. The passive index matches the policy benchmark and has no tracking error, the enhanced index has low tracking error, a factor-based strategy has medium tracking error, and high conviction strategy has high tracking error.

The core and explore strategy combines a large allocation to a passive core with a collection of high conviction strategies. The passive core is simply a market capitalization weighted index and

[7] In general, optimal expected shortfall portfolios are mean variance efficient as long as target compounding rates are not too high and horizons not too short. See Tarlie (2016) for details.

represents the default strategic asset allocation option. The high conviction strategy represents a blend of high tracking error portfolios. These portfolios, termed high conviction because of their demonstrated willingness to bear a high tracking error, tend to be concentrated in a few securities with large deviations from underlying index weights and frequently include meaningful out of benchmark holdings. While an index has no tracking error by definition, the high conviction portfolio represents the other extreme.

Table 1. Classification scheme for investment strategies by tracking error

Tracking Error	Zero	Low	Medium	High
Investment strategy	Passive index	Enhanced Index	Factor Portfolio	High Conviction
Typical tracking error	0%	$\lesssim$1.5%	1.5% ~ 4%	$\gtrsim$4%

The enhanced index strategies are characterized by lower tracking error, and are typically well diversified with many small deviations from benchmark weights, and few, if any, out of benchmark holdings. These portfolios generate modest levels of tracking error by tilting the portfolio, at the level of single stocks, towards a variety of characteristics (a.k.a. factors), such as value, quality or size.

The success of bottom up factor tilting and the ability to replicate some active return streams has naturally led asset owners to consider owning the top down factors directly. The resulting smart beta or factor portfolios typically have low fees and high levels of transparency. This reduces some of the monitoring and searching costs associated with active strategies. The increased transparency, however, is often at the cost of greater portfolio efficiency. As a result, although smart beta and enhanced index strategies have similar modest expected excess returns, smart beta portfolios tend to have intermediate or medium levels of expected tracking errors.

We base our initial analysis on a set of stylized assumptions to describe the four strategies; in Section 2.2, we explore the empirical basis for these assumptions. Table 2 shows our stylized assumptions for the four assets. Because we take the perspective of the asset owner, we care about net returns to the investor. As a result, we assume that all active strategies charge 30% of the expected gross excess return. This means that the projected fee is 15 basis points for the enhanced index but 72 basis points for the high conviction portfolio. Furthermore, we assume that the smart beta portfolio charges 20% of the expected gross excess return, the lower percentage reflecting the simpler portfolio construction rules.

The gross and net returns in Table 2 are assumed to be net of transaction costs. Our stylized assumptions are motivated by the empirical results detailed in Section 2.2. For the enhanced index and high conviction strategies, these empirical results are net of transaction costs. For the factor strategy, we estimate transaction costs of 0.24%, in line with Chow et. al. (2017) who estimate market impact costs in the neighborhood of 0.20% for multi factor smart beta portfolios.

Table 2. Stylized asset characteristics for passive index, enhanced index, factor portfolio and high conviction portfolio. All returns are annualized geometric. All values are in percentage points, except information ratios and correlations.

	Index (Passive)	Enhanced Index	Factor Portfolio	High Conviction
Gross Excess Return	0	0.50	0.50	2.40
Fees	0.03	0.15	0.10	0.72
Net Excess Return	-0.03	0.35	0.40	1.68
Tracking Error	0	1.25	2.00	4.00
Gross Information Ratio	N/A	0.40	0.25	0.60
Net Information Ratio	N/A	0.28	0.20	0.42
			Excess Return Correlations	
Enhanced Index		1	0.25	0
Factor Portfolio			1	0
High Conviction				1

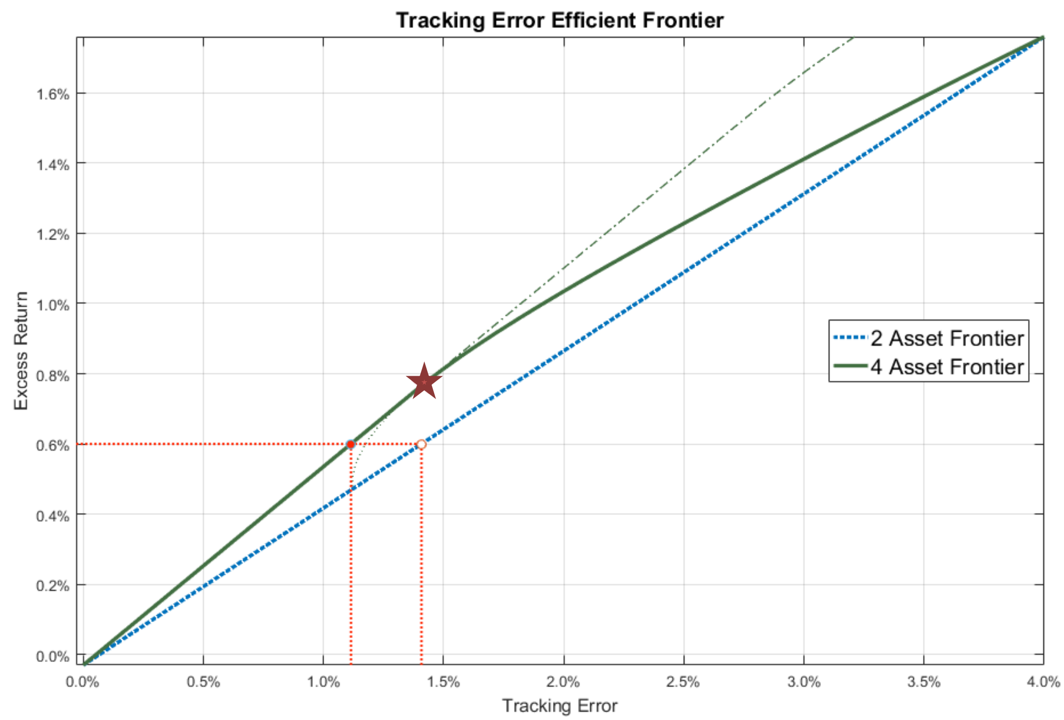
In terms of correlations, we assume that the high conviction portfolio excess return is uncorrelated with both the factor portfolio and the enhanced index. We assume a correlation of 0.25, however, between the enhanced index and the factor portfolio. We assume a small positive correlation, even though the correlation in the empirical data is zero; see Table 9 in Section 2.2. An important difference between the enhanced index strategies and the factor (smart beta) strategies is that the former use a bottom up, stock selection framework, whereas the latter use a top down, factor orientation. Although factor portfolios and enhanced index portfolios may start with similar factors, the application of top down selection of stocks in the smart beta portfolios and the bottom up selection for enhanced index portfolios, as well as different risk management approaches and design decisions, produces divergent excess return patterns, resulting in a low correlation. We nonetheless assume a small correlation to reflect the idea that the enhanced index and factor strategies are typically constructed from the same underlying characteristics such as value, quality, and size.

1.2 Mean Variance Optimization

In this section we find the optimal strategy mix based on minimizing expected tracking error subject to the constraint that expected excess return exceeds 60 basis points. We start with the simple two asset problem where assets are the passive index and the high conviction portfolio.

Because the passive index is risk free (i.e. zero tracking error) in active space, the frontier starts just below the origin, reflecting the three basis points of fees, and is linear (the dotted blue line in Figure 1). Every portfolio on the two asset frontier has the same information ratio of 0.43.

Figure 1. Tracking Error Efficient Frontier. The thick dotted line is the two asset efficient frontier based on the passive index and the high conviction portfolio. The thick solid line is the four asset efficient frontier without leverage. The traditional tracking error frontier (without the zero tracking error passive index) is composed of the thin dotted line below the maximum IR portfolio and the solid line above the maximum IR portfolio. In the presence of a zero tracking error portfolio, and without leverage the efficient frontier is composed of the straight solid line below the maximum IR portfolio and the curved solid line above the maximum IR portfolio. Also shown (dashed dotted line) is the extension of the four asset efficient frontier with leverage.



In the standard textbook approach, the investor's aversion to tracking error determines the appropriate portfolio on the efficient frontier. Alternatively, the investor can choose to constrain the optimization to target a desired tracking error or excess return. These three approaches – tracking error aversion, target tracking error, target excess return – are functionally equivalent as investor preferences boil down to a single parameter. Thus, solving for a 60 basis point excess return target produces the prototypical core and explore portfolio with 65% in the passive index, 35% in the high conviction composite, and an annualized tracking error of 141 basis points. This portfolio is illustrated by the open circle in Figure 1 and weights and statistics are shown in the first column of Table 3.

What happens when we add the enhanced index and factor portfolios? Instead of a straight line, the frontier (shown as the solid green line) now curves slightly reflecting the increased diversification available. If we solve for the same 60 basis point excess return target, the optimal four asset mean variance portfolio, illustrated by the red circle, allocates 20% to the index, 23% to the high conviction portfolio, 42% to the enhanced index, and the remaining 15% to the factor portfolio. The four asset optimal mean variance portfolio has the same expected return as the core and explore portfolio, but a lower expected tracking error of 112 basis points. This lower tracking error results in a modestly higher net information ratio of 0.54 compared to 0.43. This portfolio is illustrated by the open circle in Figure 1 and weights and statistics are shown in the second column of Table 3.

Table 3. Portfolio weights and characteristics for the two asset core and explore portfolio, the four asset mean variance portfolio, and the maximum IR portfolio. All values are in percentage points, except information ratio.

	2 Asset Core and Explore	4 Asset Mean Variance	Maximum IR
		Weights	
Index	65	20	0
Enhanced Index	0	42	52
Factor Portfolio	0	15	19
High Conviction Portfolio	35	23	29
		Portfolio characteristics	
Net Excess Return	0.60	0.60	0.78
Tracking Error	1.41	1.12	1.42
Net Information Ratio	0.43	0.54	0.54

A useful way to understand the optimal four asset portfolio is to start with the maximum information ratio (IR) portfolio. The maximum IR portfolio, denoted by the star in Figure 1, is located at the kink in the efficient frontier that separates the linear and the curved portions of the frontier. As shown in the last column of Table 3, this maximum IR portfolio has no weight in the pure index strategy, 52% in the enhanced index strategy, 19% in the factor strategy, and 29% in the high conviction strategy, implying an expected excess return of 78 basis points and tracking error of 142 basis points. All the portfolios with target excess returns below 78 basis points have some combination of the passive index and the maximum IR portfolio. Since we do not allow for leverage, if the target expected return is above 78 basis points then the optimal portfolio lies on the portion of the efficient frontier above and to the right of the maximum IR portfolio.

An interesting feature of the maximum IR portfolio is that it is dominated by the enhanced index strategy. The enhanced index strategy is 52% of the maximum IR portfolio, compared to 29% for the high conviction strategy. The enhanced index strategy dominates the high conviction strategy even though the net IR of the enhanced index strategy is 0.28, compared to 0.40 for the high conviction strategy. Why is this? The intuition is that in the limit of zero correlations, portfolio weight is proportional to mean divided by variance, not mean divided by standard deviation.[8] From Table 2, (net) excess return divided by tracking error squared is 22.4 for the enhanced index strategy, but only 10.5 for the high conviction strategy, explaining the dominance of the enhanced index strategy.

All else equal, lower tracking error, and hence higher information ratio, is desirable. However, all else is rarely equal. For example, mean variance does not account for some of the real world costs faced by institutional investors. When an investment manager adds an additional client, the costs of managing a marginal dollar are typically small. By contrast, asset owners face the opposite problem. Each additional active strategy adds to the burden on the investment staff's limited time and attention. The *soft costs* of active management include the time and effort devoted to monitoring managers, as well as the career risk associated with inevitable periods of manager underperformance. Furthermore, industry peer rankings for asset owners tend to emphasize absolute returns instead of risk adjusted returns.[9] Under these conditions, a modestly lower information ratio may well be a price many asset owners are willing to pay to avoid operational complexity.

1.3 Shortfall

In this section, we develop shortfall as an expression of the notion that investment risk is not having what you need. We then incorporate horizon to address the question of when you need it.

Relative Portfolio Value

The concept of relative portfolio value is central to our development of shortfall as a meaningful objective for an investor trying to outperform their policy benchmark. For an excess return problem, what matters is not the absolute portfolio value but rather the relative portfolio value. How do we expect the portfolio to perform relative to its benchmark?

It is useful to cast the issue in terms of simple balance sheet items by framing the benchmark as the investor's liability (L) and the portfolio as the corresponding asset (A). The standard accounting approach measures equity as the difference between assets and liabilities, i.e. $A - L$. However, for an investor who needs to compound the asset value at a rate greater than the liability, it is more natural to work with the relative portfolio value, i.e. the ratio of assets to liabilities

$$W = \frac{A}{L}. \tag{1}$$

The ratio of assets to liabilities, unlike the difference, more directly captures the effect of compounding over time. To illustrate, suppose for simplicity that the liability is expected to grow

[8] While correlations are not zero in our problem, the only nonzero correlation is between the enhanced index and the factor portfolio, and the value is 0.25.

[9] See NACUBO-Commonfund Study of Endowments 2016 http://www.nacubo.org/Research/NACUBO-Commonfund_Study_of_Endowments/Public_NCSE_Tables.html

at a constant rate l while the asset is expected to grow at a constant rate a. If both the asset and the liability start with the same values at $t = 0$, then the relative portfolio value at time t is given by

$$W = e^{(a-l)*t}\ . \tag{2}$$

For a completely passive portfolio, assets match liabilities perfectly, i.e. $a = l$, and $W = 1$. For active portfolios, relative portfolio value can take on any positive value depending on how the assets have performed relative to the benchmark. For instance, a portfolio with a 2% excess return over a single year has a relative portfolio value of $e^{0.02} \approx\ 1.02$.

In general, the asset and liability growth rates (i.e. geometric returns) a and l are random variables that depend on time and are not constant. For normally distributed excess returns, the implied target excess compounding (i.e. geometric) rate, in terms of the target excess (arithmetic) return μ^* and portfolio tracking error σ_p, is given by

$$\gamma^* = \mu^* - 0.5 * \sigma_p^2. \tag{3}$$

In our portfolio example, the reference core and explore portfolio has a target excess return of $\mu_p =$ 60 basis points per year and annualized tracking error of $\sigma_p = 141$ basis points, implying a target excess compounding rate of $\gamma^* = 59$ basis points per year.

For an investment horizon of one year, we can think of the target excess compounding rate as implying a target relative portfolio value of $W^* = e^{0.0059} \approx 1.0059$. More generally, for horizon T (measured in years) the target relative portfolio value is

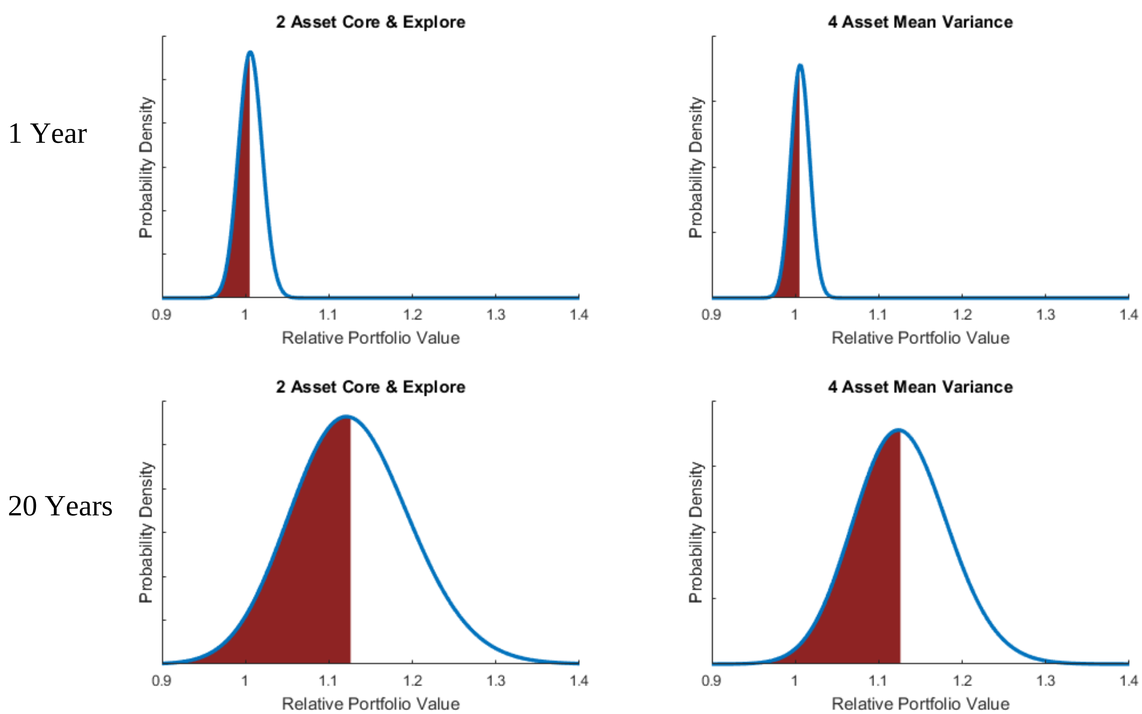
$$W^*(T) = e^{\gamma^* T}, \tag{4}$$

which in our example is $e^{0.0059 * T}$. Throughout this paper, we measure time in years.

We assume that excess arithmetic returns for the two asset and four asset optimal mean variance portfolios are independent and identically (normally) distributed – iid normal –with a mean of $0.0060 * T$ and tracking errors of $0.0141 * \sqrt{T}$ and $0.0112 * \sqrt{T}$, respectively. Assuming normally distributed excess returns means that relative portfolio value is lognormally distributed.

The graphs in Figure 2 show the distributions of relative portfolio values for the two asset core and explore portfolio and the four asset optimal mean variance portfolio for horizon $T = 1$ and $T = 20$. These probability density functions illustrate the range of potential relative portfolio values and the associated probabilities. Although we expect both the two asset core and explore and the four asset optimal mean variance portfolios to outperform (i.e. achieving relative portfolio values in excess of one) on average over a T year horizon, there are many outcomes where the relative portfolio falls short of its target of $e^{0.0059*T}$. Correspondingly, there are also many outcomes where the relative portfolio value exceeds its target. The probability density chart reflects this uncertainty by showing the range of outcomes and the associated probabilities. The shape of the probability density is governed by the expected excess return and tracking error over the entire investment horizon.

Figure 2. Probability density for the two asset core and explore and the four asset mean variance portfolios after one year and twenty years. Red shaded areas are the relative portfolio values below the desired target $W^*(T) = e^{0.0059*T}$



Expected Shortfall as Investment Risk

The investment risk for an investor trying to outperform a policy benchmark is that the relative portfolio value falls below target. Tarlie (2016) provides a framework for specifying how much the investor cares about falling short of their target relative to exceeding it. A useful special case that captures the essence of the basic proposition that investment risk is not having what you need is to measure investment risk as the expected percentage deviation of relative portfolio value below the target. As shown in Tarlie (2016) this is analogous to a linear utility function below target and a flat utility function above target.

Accordingly, we measure expected shortfall as the probability weighed sum of the percentage shortfall of the relative portfolio values below the target, i.e.

$$\Phi(T) = \int_0^{W^*(T)} \left(\frac{W^*(T) - W}{W^*(T)}\right) P(W)dW, \tag{5}$$

where $P(W)$ is the lognormal distribution. The probability of shortfall is simply the sum of the probabilities for relative portfolio values below the target. In a related manner, the expected surplus is the probability weighted sum of the percentage surplus of the relative portfolio values above the target, i.e.

$$\Pi(T) = \int_{W^*(T)}^{\infty} \left(\frac{W - W^*(T)}{W^*(T)}\right) P(W)dW, \tag{6}$$

and the probability of surplus is simply the sum of the probabilities for relative portfolio values above the target.

The shaded area in the charts in Figure 2 correspond to the relative portfolio values below the relative portfolio target of $W^*(T) = e^{0.0059*T}$ for horizons $T = 1$ and $T = 20$. Table 4 contains the expected shortfall, expected surplus, probability of shortfall, and probability of surplus statistics for the two asset and four asset portfolios for horizons $T = 1$ and $T = 20$. The probability of underperformance is the probability that the relative portfolio value is below one.

Table 4. Expected shortfall and surplus with associated probabilities for the two asset core and explore and four asset mean variance portfolios for horizons of one and 20 years. Probability of underperformance is the probability that the relative portfolio value is below one. All values are in percentage points.

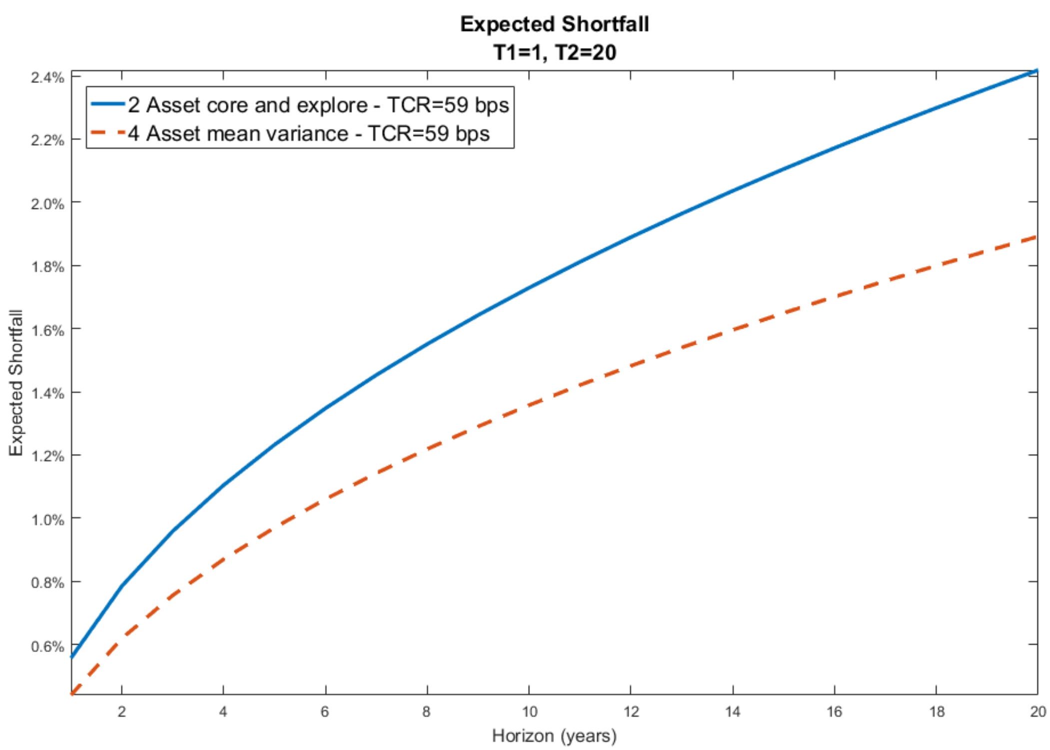
	2 Asset Core and Explore	**4 Asset Mean Variance**
	One year horizon	
Expected shortfall	0.56	0.44
Expected surplus	0.57	0.45
Probability of shortfall	50	50
Probability of surplus	50	50
Probability of underperformance	35	32
	20 year horizon	
Expected shortfall	2.42	1.93
Expected surplus	2.62	2.06
Probability of shortfall	50	50
Probability of surplus	50	50
Probability of underperformance	3	1

For the one year horizon, there is not much difference between the shortfall and surplus characteristics of the two asset and four asset portfolios. Since we assume a target compounding rate equal to the expected excess geometric return, the probability of shortfall is 50% in both cases. The shortfall of the less diversified core and explore portfolio is 12 basis points worse (56 basis points vs 44 basis points) than the more diversified four asset portfolio, and it also has a slightly higher probability of overall underperformance. But these differences are small.

What about the longer term? What is the impact of the passage of time? In Table 4 we see that at the twenty year horizon, expected shortfall is now about 49 (2.42-1.93) basis points worse for the two asset portfolio than the four asset portfolio. While the expected surplus is higher, this is simply due to the higher tracking error of the portfolio.

But what about other, intermediate horizons? Figure 3 plots the expected shortfall for the two asset (dashed line) and four asset portfolios (solid line) as a function of portfolio horizon. We see that both curves are upward sloping, and that the expected shortfall for the four asset portfolio has a lower expected shortfall at every horizon. Furthermore, the gap between expected shortfall for the two portfolios increases as horizon grows.

Figure 3: Expected shortfall at each point in time for two asset core and explore and four asset mean variance portfolios over 20 years for target relative portfolio value $\boldsymbol{W^*(T) = e^{0.0059*T}}$



These results suggest that an investor who (i) cares about falling short of the target more than exceeding the target, and (ii) has a long investment horizon, prefers the four asset portfolio relative to the two asset portfolio. The source of this preference is rooted in the basic idea that investment risk is not having what you need when you need it. This shortfall lens illustrates how the benefits of lower tracking error – a single period estimate – accrue over time.

These results also beg the question: if the investor cares about shortfall and understands the importance of horizon, why not find the portfolio that minimizes expected shortfall? We turn to this question next.

1.4 Expected Shortfall Optimization

Our goal in this section is to generate portfolios by optimizing expected shortfall. In the conventional paradigm, risk is volatility, and the investor chooses the optimal portfolio from the set of efficient ones, e.g. those that minimize volatility for a given expected return. The issue with this approach is that there simply aren't enough degrees of freedom in the formulation of the mean variance problem to handle the questions most relevant to the asset owner, such as: What do you have? What do you need? How much do you care about not achieving what you need? And, when do you need it?

The idea of downside risk (e.g. shortfall) is not new and continues to resonate with practitioners and academics alike. There is a rich literature that explores the tails of the return distribution to highlight worst case return scenarios.[10] A common measure of downside risk is expected tail loss. This expected loss is generally defined as the average expected shortfall below some probability threshold. For an asset owner who cares about having what they need, expected shortfall is more naturally defined as the average expected portfolio value below a desired target *portfolio value*. Furthermore, conventional shortfall approaches usually substitute variance for shortfall so that the investment objective is mean shortfall rather than mean variance. By contrast, the objective function that we apply naturally incorporates expected return, variance, target return, and investment horizon.

Tarlie (2016) presents an extension of the mean variance framework, firmly rooted in expected utility theory, which addresses these four questions. The essential elements of this framework are (i) a target compounding rate, (ii) an investment horizon, and (iii) asymmetric preferences with respect to shortfall and surplus.

[10] Some notable references include: Roy (1952), which emphasizes the investor's preference for return above a certain level and is closely related to Markowitz (1952); Fishburn (1977) which frames shortfall in a utility theory context; Kahnemann and Tversky (1977), which modifies conventional utility by introducing a reference level and assigning different values to gains and losses; and Bertsimas et. al. (2004) who solve a mean shortfall optimization problem.

In the following sections, we apply this framework to the case of expected shortfall. The next two subsections introduce the basic objective function, and the third presents our main portfolio results.

Single Horizon Objective Function

For horizon T and target (excess) compounding rate γ^*, the objective function for expected shortfall (see Eq. (13) in Tarlie (2016)) is

$$\Phi(x, \gamma^*, T) = N(z_1) - \left[e^{-z_1\bar{\sigma}_T + \frac{\bar{\sigma}_T^2}{2}}\right] N(z_2) \tag{7}$$

$$z_1 = \frac{\gamma^* T - \bar{\gamma}_T}{\bar{\sigma}_T}, \quad z_2 = z_1 - \bar{\sigma}_T, \tag{8}$$

where $N(\cdot)$ is the standard cumulative normal. Two notable points about the expected shortfall objective function. First, it results from a standard application of expected utility theory and the explicit evaluation of a standard integral.[11] Second, the functional form is familiar, as it resembles the formula for the price of a European option. However, the resemblance, which arises because the utility of shortfall is assumed linear in (relative) portfolio value below the target, and flat above the target just like the payoff of a put option, is only superficial.[12]

The quantities $\bar{\gamma}_T$ and $\bar{\sigma}_T^2$ are the mean and variance, respectively, of the log of relative portfolio value at horizon T. We see from Eq. (7) that the objective function Φ naturally incorporates notions of mean, variance, target return, and investment horizon.

The two quantities $\bar{\gamma}_T$ and $\bar{\sigma}_T^2$ depend on the model of asset returns and on the term structure of portfolios, i.e. the portfolios over the entire investment horizon. We assume that expected (log) excess returns $\bar{\alpha}_i$ and tracking error $\sigma_{Ri}(> 0)$ are constant so that the dynamics of log excess returns follow

$$d\ln R_i(t) = \bar{\alpha}_i dt + \sigma_{Ri} dB_i(t), \tag{9}$$

where $dB_i(t)$ are temporally uncorrelated Brownian increments. Cross sectional excess return correlations ρ_{ij} are defined by $E\left[dB_i(t)dB_j(t)\right] = \rho_{ij}dt$. If $x(t)$ is the time series vector of portfolio weights for $t \in [0, T]$, where $t = 0$ is the "here and now" and $t = T$ is the investment horizon, then we have

[11] For expected shortfall, the integral is given in Eq. (5). The objective function in Eq. (7) results from evaluating the integral in Eq. (5) assuming that wealth is lognormally distributed.

[12] The economic interpretation of the expected shortfall objective is very different from that of the option pricing formula. The option pricing formula embeds fundamental notions of no arbitrage and replicating portfolios. By contrast, the expected shortfall objective embeds relative portfolio value and the associated attitudes to shortfall and surplus.

$$\bar{\sigma}_T^2 = \int_0^T dt \sum_{ij} x_i(t) x_j(t)\, \sigma_{Ri} \sigma_{Rj} \rho_{ij} \tag{10}$$

$$\bar{\gamma}_T = \int_0^T dt \sum_i x_i(t) \left(\bar{\alpha}_i + \frac{\sigma_{Ri}^2}{2} \right) - \frac{\bar{\sigma}_T^2}{2}, \tag{11}$$

where $x_i(t)$ is the weight in asset i at time t. The portfolio weights are normalized to one so that $\sum_i x_i(t) = 1$ for $t \in [0, T]$.

Multi-horizon Objective Function

The objective function in the previous section is for a single investment horizon. In general, however, an investor who is averse to shortfall risk cares about this risk over a range of times, not only a single point in time. We follow the standard approach of time discounting so that the multi horizon objective function has the form

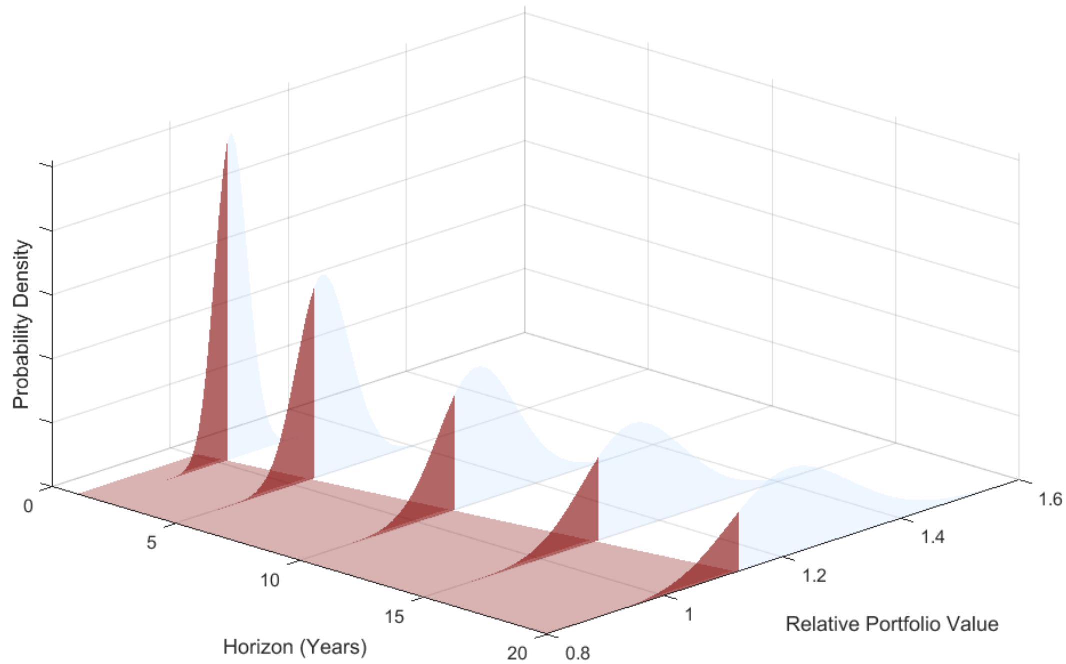
$$\overline{\Phi}(x, \gamma^*, T_1, T_2) = \int_{T_1}^{T_2} dT\ e^{-\beta T} \Phi(x, \gamma^*, T), \tag{12}$$

where β is the time preference parameter. The quantity $x(t)$ is the time series vector of portfolio weights, i.e. the term structure of portfolios, for $t \in [0, T_2]$.

In this formulation of the multi horizon objective, the investor cares about shortfall for all horizons between times T_1 and T_2. We can think of times less than T_1 as the patient phase – the investor does not care about shortfall up to time T_1 – and times between T_1 and T_2 as the target phase – the investor cares about falling short of the target. The plot in Figure 4 illustrates the basic idea for $T_1 = 1$ and $T_2 = 20$. The red shading in the time-wealth plane indicates that for these times the investor cares about falling short of the relative portfolio value, with the upper boundary of this region defined by $W^*(T) = e^{\gamma^* T}$.

The multi horizon objective function given in Eq. (12) depends on $x(t)$, the term structure of portfolios for times between $t = 0$ and $t = T_2$. Optimizing Eq. (12) generates the optimal term structure of portfolios. But for our purposes the portfolio that matters is $x(0)$, the "here and now" portfolio. The portfolios for $t > 0$ are optimal, conditional on information known at $t = 0$. Consequently, these portfolios reflect the uncertainty of the evolution of the relative portfolio value. As time passes and relative portfolio values become known, the investor reoptimizes and rebalances to reflect this new information; see Tarlie (2016) for additional discussion.

Figure 4: Relative portfolio value probability density for $T = 2, 5, 10, 15$ and 20. Dark shaded areas are the relative portfolio values below the desired target $W^*(T) = e^{0.0059*T}$ for $T_1 = 1$ and $T_2 = 20$. Probability densities are illustrated assuming a geometric portfolio excess return of 100 basis points and 192 basis points of tracking error.



Optimal Expected Shortfall Portfolios

We now have all the elements necessary to build optimal portfolios that minimize expected shortfall for different horizon preferences. For simplicity, we set the time preference parameter $\beta = 0$.

We start by considering a long portfolio horizon $T_2 = 20$ and a short patient phase of one year so that $T_1 = 1$. These horizons correspond to an asset owner with a long portfolio horizon but who also cares about deviations from target over most of the portfolio horizon.

Table 5 shows the optimal expected shortfall portfolio $x(0)$ and associated portfolio statistics for a target compounding rate of 59 basis points. The left panel shows the weights for the four asset optimal expected shortfall portfolio, the four asset optimal mean variance portfolio, and the two asset core and explore portfolio. The main difference between the four asset optimal expected shortfall portfolio and the four asset optimal mean variance portfolio is that the optimal mean variance portfolio has 20% weight in the index, whereas the optimal expected shortfall portfolio has zero weight in the index. For the optimal expected shortfall portfolio, the high conviction strategy absorbs 15 of the 20 percentage points, with the remaining five going to the factor and enhanced index strategies.

Table 5. Comparison of weights and characteristics for the two asset core and explore (C&E), four asset mean variance (MV), and four asset expected shortfall (ESF) portfolios. The target excess compounding rate is 59 basis points, and the horizons are $T_1 = 1$ and $T_2 = 20$. All values are in percentage points except for IR, which is the ratio of expected alpha to tracking error. Expected alpha and TE are in annualized units. All return statistics are annualized geometric returns.

	Weights				Statistics					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	0	44	19	37	1.00	1.92	0.52	0.85	5.53	33
4 asset MV	20	42	16	22	0.59	1.12	0.53	1.34	1.41	50
2 asset C&E	65	0	0	35	0.59	1.41	0.42	1.68	1.79	50

Legend: TE = tracking error, IR = information ratio, ESF = expected shortfall, ESP = expected surplus, PSF = probability of shortfall.

The right hand panel in Table 5 shows some statistics for these three portfolios. The four asset optimal ESF portfolio has an average expected shortfall over the entire investment horizon of 0.85%. This compares to 1.34% for the four asset optimal mean variance portfolio and 1.68% for the two asset optimal mean variance portfolio. This result is not a surprise, after all, the optimal expected shortfall portfolio has the lowest expected shortfall by optimization.

The striking feature of the results in Table 5 is that the average expected surplus (ESP) and probability of shortfall (PSF) are substantially lower for the expected shortfall optimal portfolio, even though it is only expected shortfall that is optimized. In particular, the average probability of shortfall for the four asset ESF portfolio is 33%, substantially below the 50% average probability for the optimal mean variance portfolios. Furthermore, the average expected surplus for the four asset ESF portfolio is 5.53%, substantially above the 1.41% and 1.79% values for both the two asset core and explore and the four asset optimal mean variance portfolios.

The improved shortfall, surplus, and probability statistics of the four asset optimal ESF portfolio relative to the four asset optimal mean variance portfolio comes at a cost. The cost is increased tracking error. The expected tracking error for the four asset optimal ESF portfolio is 1.92%, compared to 1.12% for the four asset optimal mean variance portfolio.

To gain more insight into these results, let's consider the case of a short portfolio horizon of one year. Table 6 shows the optimal expected shortfall portfolio $x(0)$ and associated portfolio statistics for a target compounding rate of 59 basis points, and a one year patient phase so that $T_1 = T_2 = 1$.

We see in Table 6 that reducing the portfolio horizon T_2 results in an optimal ESF portfolio with low expected excess return (0.33%) and low tracking error (0.65%). Because the portfolio has a lower expected excess return than the target compounding rate, the probability of shortfall is high at 61%. Furthermore, the expected surplus is relatively low at 0.15%. In terms of weights, the four asset expected shortfall (ESF) portfolio has a significantly larger allocation to the passive core (54% versus 20%) than the equivalent mean variance portfolio. The shift is financed mainly by a reduction in the enhanced index weight (42% versus 24%).

The results of Table 5 and Table 6 suggest that increasing portfolio horizon T_2 is analogous to decreasing risk aversion, or increasing target tracking error in a mean variance context. We explore this further in the next subsection.

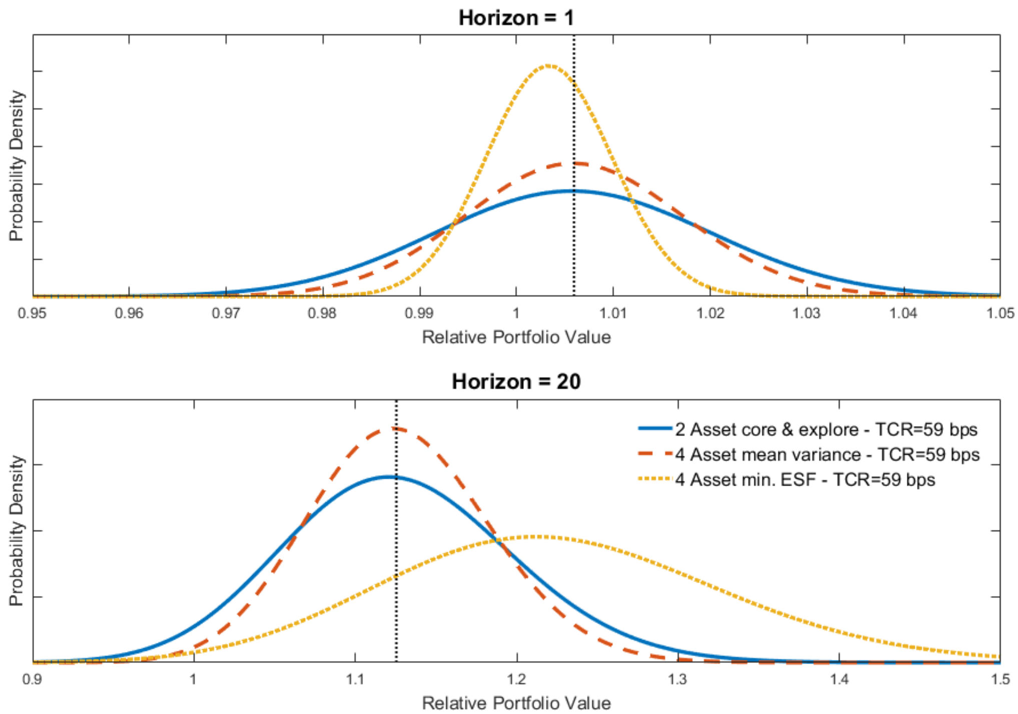
Table 6. Comparison of weights and characteristics for the two asset core and explore (C&E), four asset mean variance, and four asset ESF portfolios. The target excess compounding rate is 59 basis points, and the horizons are $T_1 = 1$ and $T_2 = 1$. All values are in percentage points except for IR, which is the ratio of expected alpha to tracking error. Expected alpha and TE are in annualized units. All return statistics are annualized geometric returns.

	Weights				Statistics					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	54	24	9	13	0.33	0.65	0.52	0.40	0.15	61
4 asset MV	20	42	16	22	0.59	1.12	0.53	0.44	0.45	50
2 asset C&E	65	0	0	35	0.59	1.41	0.42	0.56	0.57	50

Legend: TE = tracking error, IR = information ratio, ESF = expected shortfall, ESP = expected surplus, PSF = probability of shortfall.

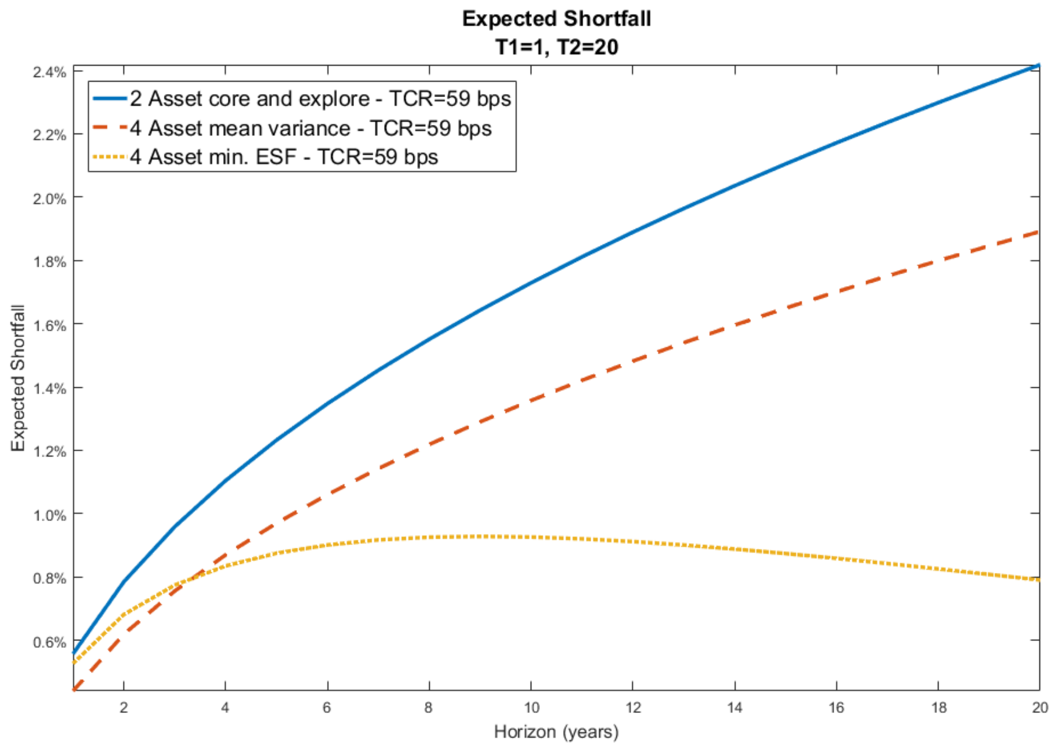
To better visualize the difference between the characteristics of the three portfolios, Figure 5 plots the distributions of relative portfolio value for $T_2 = 1$ (top panel) and $T_2 = 20$ (bottom panel). In the top panel, the optimal ESF portfolio is shifted to the left of other two portfolios and is more peaked due to a lower expected return and tracking error. Despite a higher probability of shortfall, expected shortfall is lower. In the bottom panel however, the distribution of outcomes for the optimal ESF portfolio is shifted to the right of the distributions for the other two portfolios, hence the larger surplus and lower expected probability of shortfall.

Figure 5. Probability density of relative portfolio values for the two asset core and explore (wide dashed line), four asset mean variance (narrow dashed line) and four asset ESF (solid line) at 1 year and 20 years for 59 basis point target compounding rate. Also shown (dotted vertical line) is the relative portfolio value target $W^*(T) = e^{0.0059*T}$



To complete the discussion, Figure 6 repeats the plot of expected shortfall for the two asset core and explore and four asset mean variance portfolio and includes the expected shortfall for the four asset optimal ESF portfolio (dotted line) for $T_1 = 1$ and $T_2 = 20$. The ESF curve is upward sloping at first but curves downward as horizon increases. By contrast, the core and explore and four asset mean variance curves slope upward for all horizons. Furthermore, the expected shortfall for the ESF portfolio is lower than for each of the other portfolios for every horizon over four years.

Figure 6: Expected shortfall at each point for two asset core and explore, four asset mean variance and four asset minimum expected shortfall portfolios over 20 years.



Having Your Cake and Eating It Too (if your horizon is long enough)

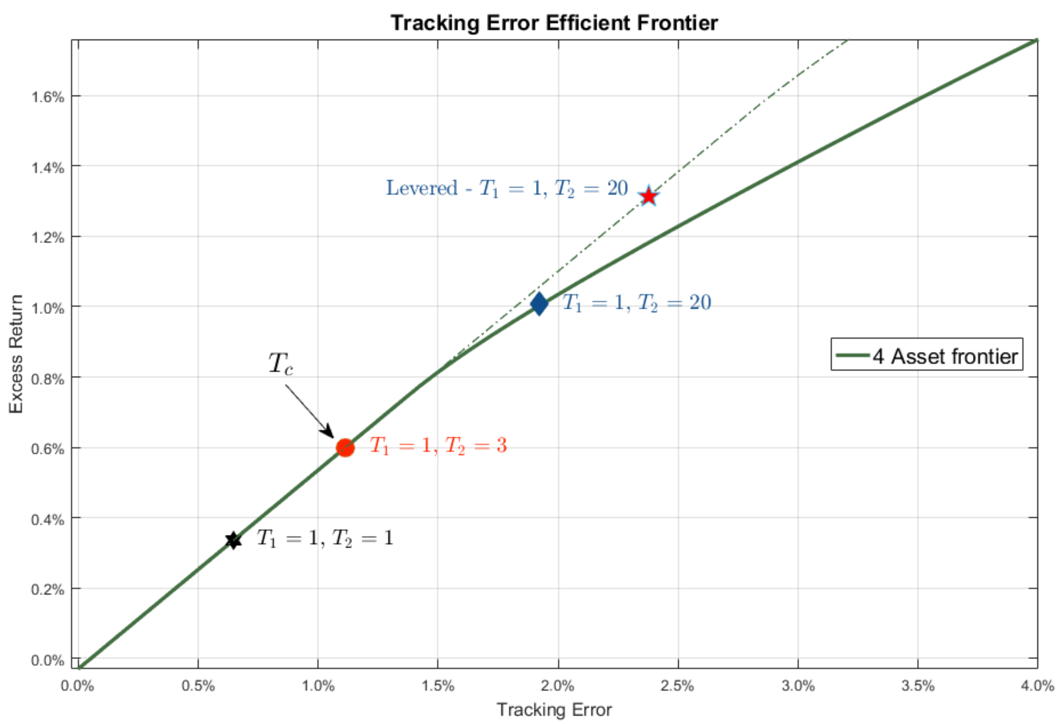
Our main result, illustrated in Table 5, is that investors with moderate target excess compounding rates (e.g. 59 basis points) and long investment horizons, can have their cake – low expected shortfall – and eat it too – high expected surplus and low probability of shortfall. Of course, these desirable attributes come at the cost of higher tracking error.

To understand this result a little more deeply, let us hold the target excess compounding rate γ^* constant at 59 basis points and fix the patient phase at $T_1 = 1$. As shown in Tarlie (2016), optimal expected shortfall portfolios are mostly mean variance efficient, and the problems we consider in this paper are mean variance efficient. This means that holding the target compounding rate constant and increasing the investment horizon is equivalent to moving along the efficient frontier from lower left to upper right, i.e. as horizon increases both the expected return and the volatility of returns increases.

For short horizons, the expected return is lower than the target compounding rate, and for long horizons the expected return is higher. This means that there is a critical horizon, T_c, for which the expected (geometric) return matches the target compounding rate. For the four asset problem with $\gamma^* = 0.59\%$ and $T_1 = 1$, we find that $T_c = 3$. Thus, if $T_2 > T_c = 3$, then investors can have their cake and eat it too, but if $T_2 < T_c = 3$ this is not the case.

This concept of a critical horizon T_c is illustrated in Figure 7. The solid circle indicates the location on the efficient frontier associated with the critical horizon $T_c = 3$ for the problem with $\gamma^* = 0.59\%$ and $T_1 = 1$. In general, T_c is an increasing function of the target excess compounding rate γ^*. This is intuitive: demanding more return requires more patience.

Figure 7: Four asset tracking error efficient frontier (solid line) and four asset efficient frontier with leverage (dashed dotted line). The solid hexagram, circle and diamond correspond to ESF portfolios with fixed target excess compounding rate γ^* (59 basis points) and different horizon preferences $T_1 = 1$, and $T_2 = 1, 3, 20$. Also shown (solid star) is the portfolio for $T_1 = 1$, and $T_2 = 20$ allowing for leverage .



Increasing the portfolio horizon T_2 from $T_c = 3$ moves the portfolio up and to the right along the efficient frontier. The portfolio for $T_2 = 20$ is shown as the solid diamond. In a similar fashion, reducing the portfolio horizon T_2 from $T_c = 3$ moves the portfolio down and to the left along the efficient frontier. The portfolio for $T_2 = 1$ is shown as the solid hexagram.

This picture illustrates a key point. For the problems we consider in this paper, optimal expected shortfall portfolios are mean variance efficient. This means that optimizing over expected shortfall for a target excess return and set of investment horizons defines the tracking error needed to minimize investment risk. As the portfolio horizon lengthens, tracking error increases. This is intuitive, investors with short horizons should not invest much in the more volatile assets and should therefore have portfolios with lower tracking error. If you need your money tomorrow, don't invest in stocks.

To emphasize that the optimal expected shortfall portfolios are mean variance efficient, we relax the no leverage constraint and solve the problem for $\gamma^* = 0.59\%$, $T_1 = 1$, and $T_2 = 20$. The star in Figure 7 indicates the location of the portfolio, which we see lies on the leverage enabled efficient frontier.

2. Sensitivity Analysis

In this section, we examine the sensitivity to investor preferences and assumptions about asset characteristics. For investor preferences, we focus on changing (i) the target excess compounding rate, and (ii) the investment horizon. For strategy characteristics (e.g. expected excess returns, tracking errors, and correlations), we present estimates based on data from the eVestment database, and use these estimates to compute optimal portfolios and compare the results to our stylized assumptions.[13] Our main results are robust to the differences in strategy characteristics.

2.1 Investor Preferences

Target Compounding Rate

In this subsection, we change the target compounding rate but keep the investment horizon parameters fixed at $T_1 = 1$ and $T_2 = 20$. Table 7 illustrates the portfolio weights and statistics for the four asset optimal expected shortfall, the four asset optimal mean variance, and the two asset core and explore portfolios for target excess compounding rates of 30 basis points (top panel) and 116 basis points (bottom panel). In both cases, we see the same general trend as in Table 5, viz. the four asset optimal expected shortfall portfolio has more desirable expected shortfall, expected surplus, and probability of shortfall characteristics than the other two portfolios, but also that these more desirable characteristics come at the cost of an increased tracking error.

For the top panel, the target excess compounding rate of 30 basis points, we see that the four asset optimal ESF portfolio has weights that are fairly balanced across all four assets. By contrast, the four asset optimal mean variance and two asset core and explore portfolios are much more concentrated in the index. This is expected, because at a target of 30 basis points, which is lower than the expected returns on all the non-index strategies, the asset owner does not need to take a

[13] Results in the eVestment database are self reported and suffer from survivorship bias. A similar analysis we conducted using the CRSP database on mutual funds finds broadly similar results.

large position in the higher yielding strategies, and weights can be shifted to the enhanced index and factor strategies and away from the core. However, when the target excess return is 116 basis points, which is considerably more aggressive, the asset owner needs to assume more tracking error and therefore allocates more to the high-conviction active and factor strategies and away from the passive core and the enhanced index portfolio.

Table 7. Comparison of weights and characteristics for the two asset core and explore (C&E), 4-asset mean variance, and 4-asset ESF portfolios. The horizons are $T_1 = 1$ and $T_2 = 20$. The top panel is for a target excess compounding rate of 30 basis points, and the bottom panel is for a target excess compounding rate of 116 basis points. All values are in percentage points except for IR, which is the ratio of expected alpha to tracking error. Expected alpha and TE are annualized. All return statistics are annualized geometric returns.

	Weights				**Statistics**					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	10	48	17	25	0.78	1.44	0.54	0.32	5.75	24
4 asset MV	57	22	9	11	0.30	0.59	0.51	0.71	0.73	50
2 asset C&E	81.5	0	0	18.5	0.30	0.74	0.40	0.90	0.92	50

	Weights				**Statistics**					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	0	10	18	72	1.59	3.51	0.45	2.48	7.6	40
4 asset MV	0	22	18	60	1.17	2.43	0.48	2.84	3.15	50
2 asset C&E	31	0	0	69	1.16	2.75	0.42	3.19	3.59	50

Legend: TE = tracking error, IR = information ratio, ESF = expected shortfall, ESP = expected surplus, PSF = probability of shortfall.

Investment Horizon

In this subsection, we focus on changing the two horizon parameters T_1 and T_2 but keeping the target excess compounding rate constant at 59 basis points. The top panel in Table 8 shows the portfolio weights and statistics for $T_1 = 1$ and $T_2 = 10$, and the bottom panel for $T_1 = 3$ and $T_2 = 20$. In terms of the expected shortfall and surplus characteristics, we see the same basic pattern as in Table 5 for $T_1 = 1$ and $T_2 = 20$, viz. that the optimal expected shortfall portfolios have a better expected shortfall statistic, by design, but also have better expected surplus and probability of shortfall characteristics. The cost, however, is a higher tracking error.

Table 8. Comparison of weights and characteristics for the two asset core and explore (C&E), four asset mean variance, and four asset ESF portfolios. The target excess compounding rate is 58 basis points The top panel is for $T_1 = 1$ and $T_2 = 10$, and the bottom panel is for $T_1 = 3$ and $T_2 = 20$. All values are in percentage points except for IR, which is the ratio of expected alpha to tracking error. Expected alpha and TE are annualized. All return statistics are annualized geometric returns.

	Weights				Statistics					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	0	50	19	31	0.87	1.61	0.54	0.82	2.44	40
4 asset MV	20	42	16	22	0.59	1.12	0.53	0.98	1.02	50
2 asset C&E	65	0	0	35	0.59	1.41	0.42	1.24	1.29	50

	Weights				Statistics					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	0	38	19	43	1.03	1.98	0.52	0.87	6.2	31
4 asset MV	20	42	16	22	0.59	1.12	0.53	1.43	1.5	50
2 asset C&E	65	0	0	35	0.59	1.41	0.42	1.8	1.91	50

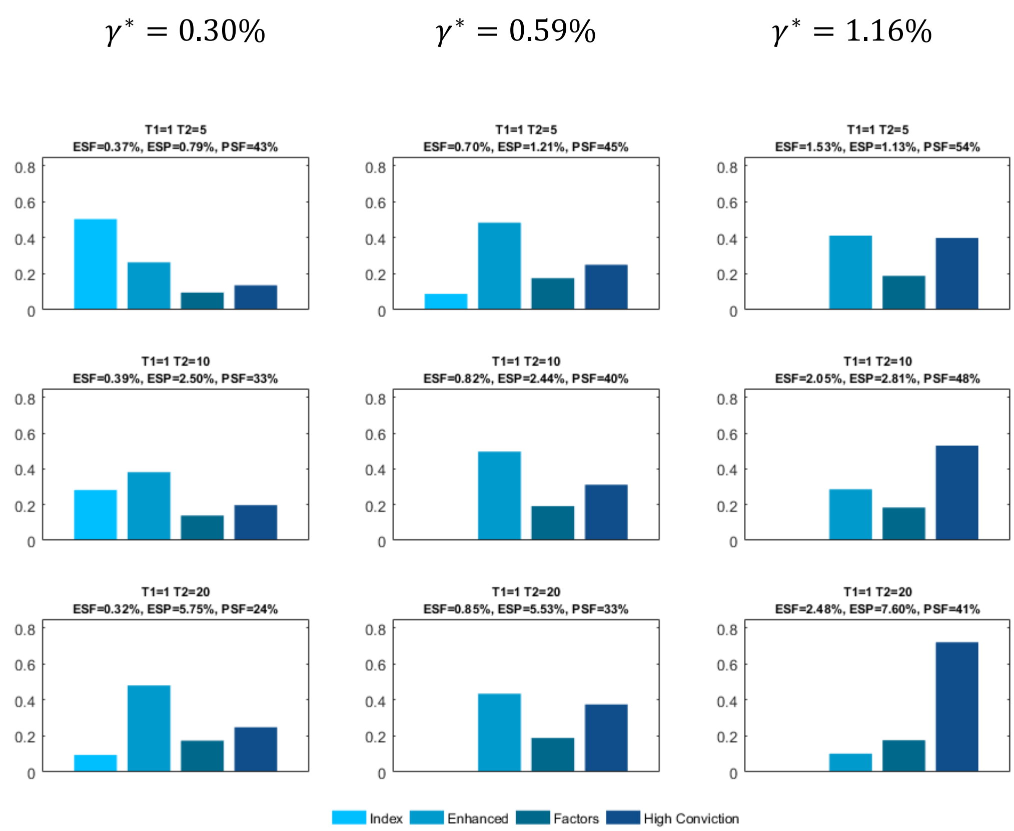
Legend: TE = tracking error, IR = information ratio, ESF = expected shortfall, ESP = expected surplus, PSF = probability of shortfall.

In terms of portfolio weights, keeping $T_1 = 1$ but lowering the portfolio horizon from $T_2 = 20$ to $T_2 = 10$ (top panel) increases the weight in the enhanced index asset by about six percentage points (from 44% to 50%), and decreases the weight in the high conviction asset by the same amount (from 37% to 31%). This is intuitive because with a shorter horizon the asset owner sensitive to shortfall must lower tracking error by reducing the holding in the highest tracking error strategy. On the other hand, increasing T_1 from one to three years (bottom panel) but keeping $T_2 = 20$ decreases the weight in the enhanced index asset by six percentage points (from 44% to 38%), and increases the weight in the high conviction asset by the same amount(from 37% to 43%). This is because the asset owner is able to absorb more tracking error given the longer horizon. Interestingly, the weight in the factor portfolio stays fixed at 19%.

Target Excess Compounding Rates and Horizon

We now focus solely on the optimal expected shortfall portfolios and their sensitivity to both target excess compounding rates and horizon. Differences in allocations shed light on the sensitivity of the horizon sensitive optimization to the various preferences. The three columns in Figure 8 show the optimal allocations under the assumption of (geometric) target excess returns of 30, 59, and 116 basis points, which are roughly equivalent to arithmetic returns of 30, 60, and 120 basis points and tracking error of 141 basis points.

Figure 8. Optimal expected shortfall portfolios for three values of the target excess compounding rate γ^*, and for three horizons. The horizons have $T_1 = 1$, and $T_2 = 5, 10, 20$. ESF is expected shortfall, ESP is expected surplus and PSF is probability of shortfall.



Moving down the first column, for a target excess compounding rate of 30 basis points, we see the sensitivity of the optimal allocations to keeping the patient phase fixed at $T_1 = 1$ and extending the portfolio horizon T_2 from 5 years in the first row, to 10 and 20 years in the subsequent rows. With a portfolio horizon of 5 years, the largest allocation is to the index, because the first column assumes a low target of 30 basis points, which is easy to achieve with modest allocations to the active assets. However, as the portfolio horizon increases, the allocation to the index decreases, virtually disappearing when $T_2 = 20$, because the longer horizon allows the asset owner to absorb more tracking error. Further, note that although the factor portfolio and the enhanced index portfolio have lower IRs than the high conviction portfolio, because of their lower tracking errors, their allocations are nonetheless substantial. It is also worthwhile to note that the expected shortfall for $T_2 = 20$ is lower than for $T_2 = 10$, a consequence of a low target excess compounding rate relative to the expected excess returns of the available assets.

Moving down the second column with the same horizons, but with a target excess compounding rate of 59 basis points, we see that there is only a small allocation to the index for $T_2 = 5$, and this allocation disappears as T_2 increases. For $T_2 = 5$, the largest allocation is to the enhanced index portfolio, but as T_2 increases, the weight in the quantitative portfolio decreases, while the weight in the high conviction portfolio increases commensurately. This is intuitive, because the longer horizon allows the asset owner to bear more tracking error, and to reduce expected shortfall a higher excess return strategy is selected.

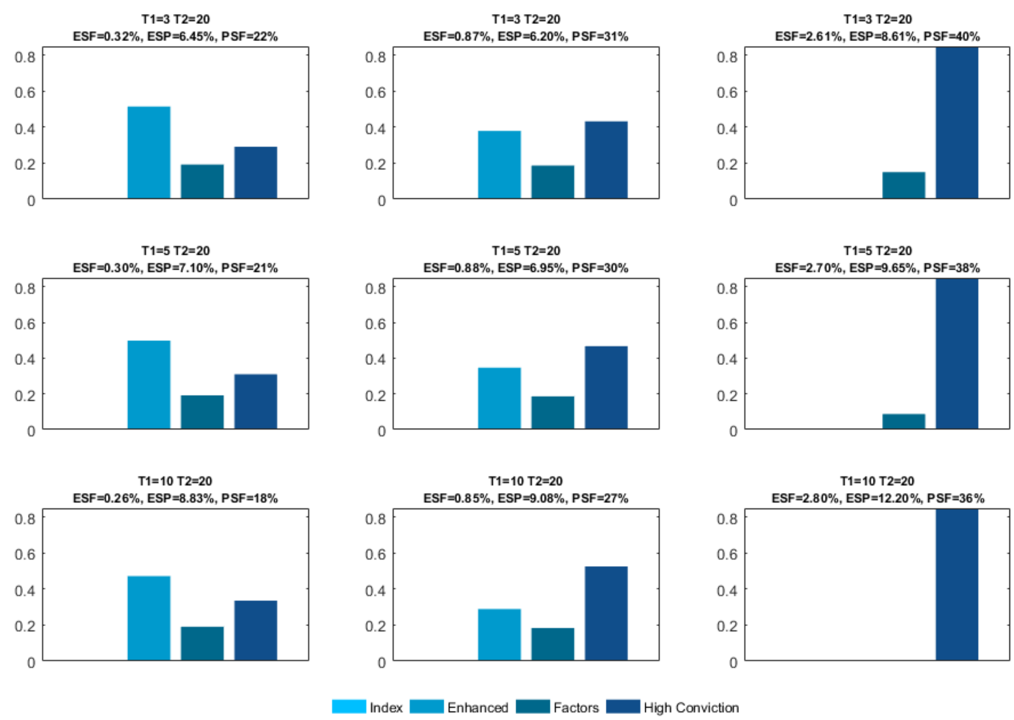
Moving down the third column with the same horizons, but now with a high target excess compounding rate of 116 basis points, we see that there is no allocation to the passive index for any horizon. The absence of the passive index is due to the high target excess compounding rate. The basic pattern as portfolio horizon T_2 increases is for the weight in the high conviction portfolio to rise and the weights for both the enhanced index and the factor portfolios to decline.

Finally, it is interesting to examine the behavior holding the horizons fixed and varying the target excess compounding rate. For a portfolio horizon of 5 years, as the target excess compounding rate increases, the weight in the index decreases and the weights in the other three portfolios increases. However, for a portfolio horizon of 10 years, the weight in the index falls, and the weight in the high conviction portfolio rises, but the weights in the enhanced index and factor portfolios rise at first but then decline. This makes intuitive sense because for a moderate target compounding rate of 59 basis points, the enhanced index strategy provides an attractive enough return with a low tracking error, which reduces the probability of shortfall. However, for a high target excess compounding rate, the enhanced index portfolio does not provide sufficient excess return. Similarly, for a portfolio horizon of 20 years, the same basic pattern repeats as for the 10 year horizon, with the only difference that the weight in the enhanced index portfolio is fairly constant for the first two values of the target excess compounding rate, then declining subsequently.

Figure 9 provides similar analysis to Figure 8, except that now $T_2 = 20$ and the patient phase $T_1 = 3, 5, 10$. First, we see that there is no allocation to the index fund, since the longer patient period allows the asset owner to bear more tracking error. Second, for target excess compounding rates of 30 basis points and 59 basis points, as T_1 increases, the weight in the enhanced index asset goes down, the weight in the high conviction portfolio goes up, and the weight in the factor portfolio stays relatively constant. For example, for $\gamma^* = 0.59\%$ and $T_1 = 3$, the weights in the enhanced index and high conviction portfolios are 38% and 43%, respectively. Increasing T_1 to 10, changes these two weights to 29% and 53%, respectively, because the asset owner is able to absorb more tracking error. Third, for a target excess compounding rates of 116 basis points, no weight is allocated to the enhanced index portfolio, and the factor portfolio receives a small allocations for $T_1 = 3$ and 5, but for $T_1 = 10$ the entire portfolio is invested in the high conviction asset.

Figure 9. Optimal expected shortfall portfolios for three values of the target excess compounding rate γ^*, and for three horizons. The horizons have $T_1 = 3, 5, 10$, and $T_2 = 20$. ESF is expected shortfall, ESP is expected surplus and PSF is probability of shortfall.

$\gamma^* = 0.30\%$ $\gamma^* = 0.59\%$ $\gamma^* = 1.16\%$



2.2 Empirical Asset Characteristics

In this subsection, we compare the results using our stylized asset characteristics with the results using asset characteristics estimated from historical data. We start by discussing the data, then move onto the portfolio results.

Data and Variables

To get a more realistic analysis of the above optimization framework, we use data about investment products from the eVestment database. We use all available fund return data starting in December 1997 and ending in December 2016. Each December, we retrieve all domestic products that are benchmarked to the Standard & Poors (S&P) 500 Index or the Russell 1000 Index, which self-identify as quantitative, which have at least 100 positions, and have assets under management (AUM) of at least $500 million.[14] We exclude all funds with low volatility in their name. We also require the products to have at least 36 months of prior returns. The selected sample for the low tracking error quantitative products has 1 product in 1997, growing to 28 in 2016, with an average of 20 per year. We use the average of this set as a proxy for the enhanced index portfolio.

In a similar manner, each December we select all products from eVestment that are *not* benchmarked to the S&P 500 or Russell 1000, which are self-designated as fundamental, which have AUM of at least $300 million, and also have at least 36 months of prior returns. In addition, we require these products to have fewer than 100 positions. The selected sample of fundamental products has 3 funds in 1997, 950 in 2016, and 538 on average. We use the average of this set as a proxy for the high conviction portfolio.

To estimate the prior tracking error of each product, we first calculate the difference between log of (1+ the return on the product) and log of (1+ the return on the benchmark) in each of the prior 36 months. The benchmark return for the enhanced index products is their chosen benchmark (S&P 500 or Russell 1000). We benchmark the high conviction products to the S&P 500. The tracking error is the standard deviation of these differences over the prior 36 months.

To estimate the future returns of a product, we use the monthly returns on each product following the December portfolio formation date, minus the same benchmark used to calculate the tracking error above. If a product stops reporting returns during the holding period, we assume that the proceeds from the product are reinvested in the benchmark. Our results are based on monthly returns for the 12 months after the product selection month (December).

Because dedicated single and multi-factor products are a recent phenomenon, we do not have a long enough history of actual product performance as we do for the enhanced index and high

[14] As Livnat et al (2017) point out using the CRSP survivorship bias free database, such products exhibit lower drawdown, lower tracking error, lower active share, and more consistent returns over time, which are characteristic of quantitative products.

conviction products. Instead, we create a proxy smart beta portfolio from three equal-weighted simple factor portfolios based on the constituents of the Russell 1000. We build a value portfolio based on using the bottom quartile of price to book. We build a size portfolio based on the bottom third of market capitalization within the 1000 largest stocks. And finally, we build a high quality portfolio that constitutes one quarter of the market capitalization of the top 1000 stocks based on a combination of high level of ROE, low volatility of ROE, and low debt to equity. The value and size factor portfolios are rebalanced monthly, and the quality portfolio is rebalanced semi-annually. We restrict ourselves to these three factors because of their long established histories in both academic and practitioner spheres. We avoid purely statistical or price based factors for the same reasons.

While the eVestment returns include transaction costs, the proxy smart beta portfolio does not. To estimate transaction costs for the proxy smart beta portfolio we use an industry standard transaction cost model that is calibrated to match the models used by broker-dealers. This model includes costs for commissions, bid-ask spread, and price impact. The price impact term dominates for portfolios with anything but very low turnover. For each stock, impact cost is the product of dollars traded relative to a 30 day trailing average of daily volume, and price volatility. Using monthly data from 1998 – 2016, we find that an equally weighted portfolio of value, quality, and size within the Russell 1000 has an average annualized geometric excess return of 1.22%. We estimate transaction costs of 0.24%, consistent with Chow et. al. (2015), resulting in an after transaction cost estimate of 0.98%. We also estimate the tracking error of this portfolio to be 2.96%.

The top panel of Table 9 reports the observed characteristics of the different available investment approaches for the asset owner, and the bottom panel shows the excess return correlations.

Table 9. Asset characteristics. All returns are annualized geometric. All values are in percentage points, except information ratios and correlations.

	Index (Passive)	Enhanced Index	Factor Portfolio	High Conviction
Gross Excess Return	0	0.44	0.98	3.21
Fees	0.03	0.13	0.20	0.96
Net Excess Return	-0.03	0.31	0.78	2.25
Tracking Error	0	1.37	2.96	5.09
Gross Information Ratio	N/A	0.32	0.33	0.63
Net Information Ratio	N/A	0.22	0.26	0.44
			Excess Return Correlations	
Enhanced Index		1	0.0	0.0
Factor Portfolio			1	0.2
High Conviction				1

The performance of the empirical proxy portfolios is broadly in line with our stylized example above. The high conviction proxy has a gross alpha of 321 basis points relative to the Russell 1000 with an associated tracking error of 509 basis points, giving a gross information ratio of 0.63. The factor portfolio has an empirical gross alpha of 98 basis points, 296 basis points of tracking error, for a gross information ratio of 0.33. Finally, the enhanced index proxy has a gross alpha of 44 basis points, 137 basis points of tracking error and a gross information ratio of 0.32.

In our stylized example, for the high conviction portfolio we assume gross alpha of 240 basis points and tracking error of 400 basis points, giving a gross information ratio of 0.6. Our choice of 240 basis points is motivated by the fact that a significant portion of the empirical 321 basis point value added and the 509 tracking error values comes from one event, the technology, media, and telecom bubble of the late 1990s, early 2000s. The value of 240 basis points comes from assuming a tracking error of 400 basis points and then applying an IR of 0.6, a value consistent with the empirical data.

In our stylized example, we assume that a collection of factor portfolios has an expected excess return of 50 basis points. The empirical value is 98 basis points, a value significantly higher than our stylized example. Our rationale for using the lower value is based on (i) a belief that the particular time period (1997 – 2016) was unusually favorable to these factors, (ii) the return series are purely hypothetical.

Another difference is in the correlations. While the quantitative excess return is uncorrelated with either of the other two portfolios, the factor and the high conviction portfolios have a positive correlation of 0.2. We confirm this with a style analysis of the excess returns; it shows that this positive correlation comes from the embedded loading of the high conviction portfolio on the size factor. In contrast, we find no correlation between the enhanced index and the factor portfolios. This confirms that despite using similar factors the application of bottom up security selection versus top down factor construction leads to very different portfolios.

Results

In Table 10, we show the portfolio weights and statistics for a target excess compounding rate of 59 basis points and horizon parameters $T_1 = 1$ and $T_2 = 20$. The main difference between the four asset ESF portfolio using empirical asset characteristics and our stylized asset characteristics is that the high conviction portfolio sheds six percentage points that are redistributed equally between the factor portfolio and the enhanced index. By contrast, the main difference in the four asset optimal mean variance portfolio is in the weight of the index. For the portfolio based on the empirical asset characteristics, the index weight is fifteen percentage points higher than for the portfolio based on our stylized asset characteristics, with six percentage points coming from both the enhanced index and factor portfolios .

Although the optimal portfolios are slightly affected by the choice of asset characteristics, the main result of Table 5, viz. the four asset optimal ESF portfolio not only has a more attractive expected shortfall, but also has a more attractive expected surplus and probability of shortfall, even though it is only optimized to reduce expected shortfall. But, as we saw in Table 5, these benefits come at the cost of higher tracking error. We therefore conclude that our main result (Table 5) is robust to our asset characteristic assumptions.

Table 10. Comparison of weights and characteristics for the two asset core and explore (C&E), four asset mean variance, and four asset ESF portfolios using the empirical asset characteristics. The target compounding rate is 59 basis points and the horizons are $T_1 = 1$ and $T_2 = 20$. All values are in percentage points except for IR, which is the ratio of expected alpha to tracking error. Expected alpha and TE are annualized.

	Weights				Statistics					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	0	47	22	31	1.21	2.24	0.54	0.70	7.92	28
4 asset MV	35	36	13	16	0.59	1.10	0.54	1.32	1.40	50
2 asset C&E	74	0	0	26	0.59	1.33	0.44	1.59	1.69	50

*Legend: TE = tracking error, IR = information ratio, ESF = expected shortfall, ESP = expected surplus, PSF = probability of shortfall***.**

3. Conclusion

We study the basic problem of allocating amongst a set of equity strategies given a policy benchmark. The fact that many institutional plans today allocate about two thirds of the portfolio to the benchmark and the remainder to a collection of higher tracking error strategies (the core and explore model), reveals an investor preference for a modest target excess return of about 0.60% per year. We argue that although lower tracking error strategies, such as quantitatively driven enhanced index and factor portfolios, can lower the tracking error and improve the information ratio, these benefits may not outweigh the increase in soft costs, such as finding and monitor costs, not to mention additional career risk.

However, the tracking error and information ratio analysis has two main shortcomings. First, investment risk is not tracking error. Second, there is no accounting for investment horizon. We address these two shortcomings by defining investment risk as not having what you need when you need it. Using this definition of investment risk, we find that asset owners with long investment horizons can benefit in terms of both shortfall and surplus by incorporating a combination of low, medium, and high tracking error strategies. This means that asset owners should be willing to bear the soft costs associated with these strategies because the consequences of falling short are far more meaningful than suggested by the modest tracking error benefits.

Finally, our results have three implications for investors. First, we show that lower tracking error strategies – enhanced index and smart beta factor portfolios – in conjunction with the high conviction strategies play an important role in minimizing investment risk – not having what you need when you need it. This means that investors aiming to minimize investment risk must consider a variety of *different* strategies when building their equity portfolio. Second, our methodology determines the necessary tracking error required to minimize investment risk. Tracking error is not investment risk. Rather, it is a cost that must be borne in order to minimize investment risk. For moderate excess target compounding rates and long investment horizons, we find that the appropriate tracking error is about twice as large as for the corresponding mean variance solution. Third, investors with a long enough horizon can have their cake and eat it too. Having their cake corresponds to owning portfolios that minimize investment risk, and eating it too corresponds to portfolios that also have larger expected surplus and lower probability of shortfall. This means that investors with modest target excess returns and long horizons that have also adopted the core and explore approach should allocate away from the passive index to the lower tracking error strategies with positive expected returns.

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