Hermitian matrix

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In mathematics, a **Hermitian matrix** (or **self-adjoint matrix**) is a complex square matrix that is equal to its own conjugate transpose—that is, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j:

$a_{ij} = \overline{a_{ji}}$ or $A = \overline{A^{\mathrm{T}}}$, in matrix form.

Hermitian matrices can be understood as the complex extension of real symmetric matrices.

If the conjugate transpose of a matrix A is denoted by A^{H}, then the Hermitian property can be written concisely as

$$A = A^{\mathrm{H}}.$$

Hermitian matrices are named after Charles Hermite, who demonstrated in 1855 that matrices of this form share a property with real symmetric matrices of always having real eigenvalues.

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In this section, the conjugate transpose of matrix A is denoted as A^{H}, the transpose of matrix A is denoted as A^{T} and conjugate of matrix A is denoted as A^*.

See the following example:

$$\begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix}$$

The diagonal elements must be real, as they must be their own complex conjugate.

Well-known families of Pauli matrices, Gell-Mann matrices and their generalizations are Hermitian. In theoretical physics such Hermitian matrices are often multiplied by imaginary coefficients,[1][2] which results in *skew-Hermitian* matrices (see below).

Here we offer another useful Hermitian matrix using an abstract example. If a square matrix A equals the multiplication of a matrix and its conjugate transpose, that is, $A = BB^{\mathrm{H}}$, then A is a Hermitian positive semi-definite matrix. Furthermore, if B is row full-rank, then A is positive definite.

Properties

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- The entries on the main diagonal (top left to bottom right) of any Hermitian matrix are necessarily real, because they have to be equal to their complex conjugate.
- Because of conjugation, for complex valued entries the off-diagonal elements cannot be symmetric (or same). Hence, a matrix that has only real entries is Hermitian if and only if it is a symmetric matrix, i.e., if it is symmetric with respect to the main diagonal. A real and symmetric matrix is simply a special case of a Hermitian matrix.
- Every Hermitian matrix is a normal matrix.
- The finite-dimensional spectral theorem says that any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries. This implies that all eigenvalues of a Hermitian matrix A with dimension n are real, and that A has n linearly independent eigenvectors. Moreover, Hermitian matrix has orthogonal eigenvectors for distinct eigenvalues. Even if there are degenerate eigenvalues, it is always possible to find an orthogonal basis of $\mathbf{C}^n$ consisting of n eigenvectors of A.
- The sum of any two Hermitian matrices is Hermitian.
- The inverse of an invertible Hermitian matrix is Hermitian as well.
- The product of two Hermitian matrices A and B is Hermitian if and only if $AB = BA$. Thus A^n is Hermitian if A is Hermitian and n is an integer.
- For an arbitrary complex valued vector v the product $v^{\mathrm{H}}Av$ is real because of $v^{\mathrm{H}}Av = (v^{\mathrm{H}}Av)^{\mathrm{H}}$. This is especially important in quantum physics where hermitian matrices are operators that measure properties of a system e.g. total spin which have to be real.
- The Hermitian complex n-by-n matrices do not form a vector space over the complex numbers, since the identity matrix I_n is Hermitian, but $i\,I_n$ is not. However the complex Hermitian matrices *do* form a vector space over the real numbers $\mathbf{R}$. In the $2n^2$-dimensional vector space of complex $n \times n$ matrices over $\mathbf{R}$, the complex Hermitian matrices form a subspace of dimension n^2. If E_{jk} denotes the n-by-n matrix with a 1 in the j,k position and zeros elsewhere, a basis can be described as follows:

  E_{jj} for $1 \leq j \leq n$ (n matrices)

  together with the set of matrices of the form

  $E_{jk} + E_{kj}$ for $1 \leq j < k \leq n$ ($\frac{n^2 - n}{2}$ matrices)

  and the matrices

  $i(E_{jk} - E_{kj})$ for $1 \leq j < k \leq n$ ($\frac{n^2 - n}{2}$ matrices)

  where i denotes the complex number $\sqrt{-1}$, known as the imaginary unit.

- If n orthonormal eigenvectors $u_1, \dots, u_n$ of a Hermitian matrix are chosen and written as the columns of the matrix U, then one eigendecomposition of A is $A = U\Lambda U^{\mathrm{H}}$ where $UU^{\mathrm{H}} = I = U^{\mathrm{H}}U$ and therefore

  $$A = \sum_j \lambda_j u_j u_j^{\mathrm{H}},$$

where λ_j are the eigenvalues on the diagonal of the diagonal matrix Λ.

Further properties

Additional facts related to Hermitian matrices include:

- The sum of a square matrix and its conjugate transpose $(C + C^{\mathrm{H}})$ is Hermitian.
- The difference of a square matrix and its conjugate transpose $(C - C^{\mathrm{H}})$ is skew-Hermitian (also called antihermitian). This implies that commutator of two Hermitian matrices is skew-Hermitian.
- An arbitrary square matrix C can be written as the sum of a Hermitian matrix A and a skew-Hermitian matrix B:

$$C = A + B \quad \text{with} \quad A = \frac{1}{2}(C + C^{\mathrm{H}}) \quad \text{and} \quad B = \frac{1}{2}(C - C^{\mathrm{H}})$$

- The determinant of a Hermitian matrix is real:

Proof: $\det(A) = \det(A^{\mathrm{T}}) \quad \Rightarrow \quad \det(A^{\mathrm{H}}) = \det(A)^*$
Therefore if $A = A^{\mathrm{H}} \quad \Rightarrow \quad \det(A) = \det(A)^*$.

(Alternatively, the determinant is the product of the matrix's eigenvalues, and as mentioned before, the eigenvalues of a Hermitian matrix are real.)

Rayleigh quotient

In mathematics, for a given complex Hermitian matrix *M* and nonzero vector *x*, the Rayleigh quotient[3] $R(M, x)$, is defined as:[4][5]

$$R(M, x) := \frac{x^{\mathrm{H}} M x}{x^{\mathrm{H}} x}.$$

For real matrices and vectors, the condition of being Hermitian reduces to that of being symmetric, and the conjugate transpose x^{H} to the usual transpose x^{T}. Note that $R(M, cx) = R(M, x)$ for any non-zero real scalar c. Recall that a Hermitian (or real symmetric) matrix has real eigenvalues. It can be shown that, for a given matrix, the Rayleigh quotient reaches its minimum value $\lambda_{\min}$ (the smallest eigenvalue of M) when x is $v_{\min}$ (the corresponding eigenvector). Similarly, $R(M, x) \leq \lambda_{\max}$ and $R(M, v_{\max}) = \lambda_{\max}$.

The Rayleigh quotient is used in the min-max theorem to get exact values of all eigenvalues. It is also used in eigenvalue algorithms to obtain an eigenvalue approximation from an eigenvector approximation. Specifically, this is the basis for Rayleigh quotient iteration.

The range of the Rayleigh quotient (for matrix that is not necessarily Hermitian) is called a numerical range (or spectrum in functional analysis). When the matrix is Hermitian, the numerical range is equal to the spectral norm. Still in functional analysis, $\lambda_{\max}$ is known as the spectral radius. In the context of C*-algebras or algebraic quantum mechanics, the function that to M associates the Rayleigh quotient $R(M, x)$ for a fixed x and M varying through the algebra would be referred to as "vector state" of the algebra.

See also

- Skew-Hermitian matrix (anti-Hermitian matrix)
- Haynsworth inertia additivity formula
- Hermitian form
- Self-adjoint operator
- Unitary matrix

References

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2. Physics 125 Course Notes (http://www.hep.caltech.edu/~fcp/physics/quantumMechanics/angularMomentum/angularMomentum.pdf) at California Institute of Technology
3. Also known as the **Rayleigh–Ritz ratio**; named after Walther Ritz and Lord Rayleigh.
4. Horn, R. A. and C. A. Johnson. 1985. *Matrix Analysis*. Cambridge University Press. pp. 176–180.
5. Parlet B. N. *The symmetric eigenvalue problem*, SIAM, Classics in Applied Mathematics,1998

External links

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- Visualizing Hermitian Matrix as An Ellipse with Dr. Geo (https://www.cyut.edu.tw/~ckhung/b/la/hermitian.en.php), by Chao-Kuei Hung from Chaoyang University, gives a more geometric explanation.
- "Hermitian Matrices" (http://www.mathpages.com/home/kmath306/kmath306.htm). *MathPages.com*.