Unitary matrix

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In mathematics, a complex square matrix U is **unitary** if its conjugate transpose U^* is also its inverse—that is, if

$$U^*U = UU^* = I,$$

where I is the identity matrix. In physics, especially in quantum mechanics, the Hermitian conjugate of a matrix is denoted by a dagger (†) and the equation above becomes

$$U^\dagger U = UU^\dagger = I.$$

The real analogue of a unitary matrix is an orthogonal matrix. Unitary matrices have significant importance in quantum mechanics because they preserve norms, and thus, probability amplitudes.

Contents

Properties

For any unitary matrix U of finite size, the following hold:

- Given two complex vectors x and y, multiplication by U preserves their inner product; that is, $\langle Ux, Uy\rangle = \langle x, y\rangle$.
- U is normal
- U is diagonalizable; that is, U is unitarily similar to a diagonal matrix, as a consequence of the spectral theorem. Thus, U has a decomposition of the form

  $$U = VDV^*$$

  where V is unitary and D is diagonal and unitary.
- $|\det(U)| = 1$.
- Its eigenspaces are orthogonal.
- U can be written as $U = e^{iH}$, where e indicates matrix exponential, i is the imaginary unit, and H is a Hermitian matrix.

For any nonnegative integer n, the set of all n-by-n unitary matrices with matrix multiplication forms a group, called the unitary group $U(n)$.

Any square matrix with unit Euclidean norm is the average of two unitary matrices.[1]

Equivalent conditions

If U is a square, complex matrix, then the following conditions are equivalent:

1. U is unitary.
2. U^* is unitary.
3. U is invertible with $U^{-1} = U^*$.
4. The columns of U form an orthonormal basis of $\mathbb{C}^n$ with respect to the usual inner product.
5. The rows of U form an orthonormal basis of $\mathbb{C}^n$ with respect to the usual inner product.
6. U is an isometry with respect to the usual norm.
7. U is a normal matrix with eigenvalues lying on the unit circle.

Elementary constructions

2 × 2 unitary matrix

The general expression of a 2 × 2 unitary matrix is:

$$U = \begin{bmatrix} a & b \\ -e^{i\varphi} b^* & e^{i\varphi} a^* \end{bmatrix}, \qquad |a|^2 + |b|^2 = 1,$$

which depends on 4 real parameters (the phase of a, the phase of b, the relative magnitude between a and b, and the angle φ). The determinant of such a matrix is:

$$\det(U) = e^{i\varphi}.$$

The sub-group of such elements in U where $\det(U) = 1$ is called the special unitary group SU(2).

The matrix U can also be written in this alternative form:

$$U = e^{i\varphi/2} \begin{bmatrix} e^{i\varphi_1} \cos\theta & e^{i\varphi_2} \sin\theta \\ -e^{-i\varphi_2} \sin\theta & e^{-i\varphi_1} \cos\theta \end{bmatrix},$$

which, by introducing $\phi_1 = \psi + \Delta$ and $\phi_2 = \psi - \Delta$, takes the following factorization:

$$U = e^{i\varphi/2} \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{bmatrix}.$$

This expression highlights the relation between 2 × 2 unitary matrices and 2 × 2 orthogonal matrices of angle θ.

Many other factorizations of a unitary matrix in basic matrices are possible.

See also

- Orthogonal matrix
- Hermitian matrix
- Symplectic matrix
- Orthogonal group O(n)
- Special Orthogonal group SO(n)
- Unitary group U(n)

- Special Unitary group SU(*n*)
- Unitary operator
- Matrix decomposition
- Quantum gate

References

1. Li, Chi-Kwong; Poon, Edward (2002). "Additive decomposition of real matrices". *Linear and Multilinear Algebra*. **50** (4): 321–326. doi:10.1080/03081080290025507 (https://doi.org/10.1080%2F03081080290025507).

External links

- Weisstein, Eric W. "Unitary Matrix" (http://mathworld.wolfram.com/UnitaryMatrix.html). *MathWorld*.
- Ivanova, O. A. (2001) [1994], "Unitary matrix" (https://www.encyclopediaofmath.org/index.php?title=U/u095540), in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4