Eigendecomposition of a matrix

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In linear algebra, **eigendecomposition** or sometimes **spectral decomposition** is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors. Only diagonalizable matrices can be factorized in this way.

Contents

Fundamental theory of matrix eigenvectors and eigenvalues

A (non-zero) vector $\mathbf{v}$ of dimension N is an **eigenvector** of a square ($N{\times}N$) matrix $\mathbf{A}$ if it satisfies the linear equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

where λ is a scalar, termed the **eigenvalue** corresponding to $\mathbf{v}$. That is, the eigenvectors are the vectors that the linear transformation $\mathbf{A}$ merely elongates or shrinks, and the amount that they elongate/shrink by is the eigenvalue. The above equation is called the **eigenvalue equation** or the **eigenvalue problem**.

This yields an equation for the eigenvalues

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

We call $p(\lambda)$ the **characteristic polynomial**, and the equation, called the **characteristic equation**, is an Nth order polynomial equation in the unknown λ. This equation will have N_λ distinct solutions, where $1 \le N_\lambda \le N$. The set of solutions, that is, the eigenvalues, is called the **spectrum** of $\mathbf{A}$.[1][2][3]

We can factor p as

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k} = 0.$$

The integer n_i is termed the **algebraic multiplicity** of eigenvalue λ_i. The algebraic multiplicities sum to N:

$$\sum_{i=1}^{N_\lambda} n_i = N.$$

For each eigenvalue, λ_i, we have a specific eigenvalue equation

$$\left(\mathbf{A} - \lambda_i \mathbf{I}\right) \mathbf{v} = 0.$$

There will be $1 \le m_i \le n_i$ linearly independent solutions to each eigenvalue equation. The m_i solutions are the eigenvectors associated with the eigenvalue λ_i. The integer m_i is termed the **geometric multiplicity** of λ_i. It is important to keep in mind that the algebraic multiplicity n_i and geometric multiplicity m_i may or may not be equal, but we always have $m_i \le n_i$. The simplest case is of course when $m_i = n_i = 1$. The total number of linearly independent eigenvectors, $N_\mathbf{v}$, can be calculated by summing the geometric multiplicities

$$\sum_{i=1}^{N_\lambda} m_i = N_\mathbf{v}.$$

The eigenvectors can be indexed by eigenvalues, *i.e.* using a double index, with $\mathbf{v}_{i,j}$ being the jth eigenvector for the ith eigenvalue. The eigenvectors can also be indexed using the simpler notation of a single index $\mathbf{v}_k$, with $k = 1, 2, ..., N_\mathbf{v}$.

Eigendecomposition of a matrix

Let $\mathbf{A}$ be a square ($N{\times}N$) matrix with N linearly independent eigenvectors, $q_i \ (i = 1, \ldots, N)$. Then $\mathbf{A}$ can be factorized as

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$$

where $\mathbf{Q}$ is the square ($N{\times}N$) matrix whose i^{th} column is the eigenvector q_i of $\mathbf{A}$ and $\mathbf{\Lambda}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, *i.e.*, $\mathbf{\Lambda}_{ii} = \lambda_i$. Note that only diagonalizable matrices can be factorized in this way. For example, the defective matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ cannot be diagonalized.

The eigenvectors $q_i \ (i = 1, \ldots, N)$ are usually normalized, but they need not be. A non-normalized set of eigenvectors, $v_i \ (i = 1, \ldots, N)$, can also be used as the columns of $\mathbf{Q}$. That can be understood by noting that the magnitude of the eigenvectors in $\mathbf{Q}$ gets canceled in the decomposition by the presence of $\mathbf{Q}^{-1}$.

The decomposition can be derived from the fundamental property of eigenvectors: $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and thus $\mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{\Lambda}$ which yields $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$.

Example

Taking a 2×2 real matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ as an example to be decomposed into a diagonal matrix through multiplication of a non-singular matrix $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \text{ for some real diagonal matrix } \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}.$$

Shifting $\mathbf{B}$ to the right hand side:

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

The above equation can be decomposed into 2 simultaneous equations:

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ax \\ cx \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} by \\ dy \end{bmatrix} \end{cases}$$

Factoring out the eigenvalues x and y:

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = y \begin{bmatrix} b \\ d \end{bmatrix} \end{cases}$$

Letting $\vec{a} = \begin{bmatrix} a \\ c \end{bmatrix}, \vec{b} = \begin{bmatrix} b \\ d \end{bmatrix}$, this gives us two vector equations:

$$\begin{cases} A\vec{a} = x\vec{a} \\ A\vec{b} = y\vec{b} \end{cases}$$

And can be represented by a single vector equation involving 2 solutions as eigenvalues:

$$\mathbf{Au} = \lambda\mathbf{u}$$

where λ represents the two eigenvalues x and y, $\mathbf{u}$ represents the vectors $\vec{a}$ and $\vec{b}$.

Shifting $\lambda\mathbf{u}$ to the left hand side and factorizing $\mathbf{u}$ out

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}$$

Since $\mathbf{B}$ is non-singular, it is essential that $\mathbf{u}$ is non-zero. Therefore,

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Thus

$$(1 - \lambda)(3 - \lambda) = 0$$

Giving us the solutions of the eigenvalues for the matrix $\mathbf{A}$ as $\lambda = 1$ or $\lambda = 3$, and the resulting diagonal matrix from the eigendecomposition of $\mathbf{A}$ is thus $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

Putting the solutions back into the above simultaneous equations

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 1 \begin{bmatrix} a \\ c \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 3 \begin{bmatrix} b \\ d \end{bmatrix} \end{cases}$$

Solving the equations, we have $a = -2c, c \in \mathbb{R}$ and $b = 0, d \in \mathbb{R}$

Thus the matrix $\mathbf{B}$ required for the eigendecomposition of $\mathbf{A}$ is $\begin{bmatrix} -2c & 0 \\ c & d \end{bmatrix}, [c, d] \in \mathbb{R}$. i.e. :

$$\begin{bmatrix} -2c & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2c & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, [c, d] \in \mathbb{R}$$

Matrix inverse via eigendecomposition

If matrix $\mathbf{A}$ can be eigendecomposed and if none of its eigenvalues are zero, then $\mathbf{A}$ is nonsingular and its inverse is given by

$$\mathbf{A}^{-1} = \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^{-1}$$

Furthermore, because $\mathbf{\Lambda}$ is a diagonal matrix, its inverse is easy to calculate:

$$\left[\Lambda^{-1}\right]_{ii} = \frac{1}{\lambda_i}$$

Practical implications[4]

When eigendecomposition is used on a matrix of measured, real data, the inverse may be less valid when all eigenvalues are used unmodified in the form above. This is because as eigenvalues become relatively small, their contribution to the inversion is large. Those near zero or at the "noise" of the measurement system will have undue influence and could hamper solutions (detection) using the inverse.

Two mitigations have been proposed: 1) truncating small/zero eigenvalues, 2) extending the lowest reliable eigenvalue to those below it.

The first mitigation method is similar to a sparse sample of the original matrix, removing components that are not considered valuable. However, if the solution or detection process is near the noise level, truncating may remove components that influence the desired solution.

The second mitigation extends the eigenvalue so that lower values have much less influence over inversion, but do still contribute, such that solutions near the noise will still be found.

The reliable eigenvalue can be found by assuming that eigenvalues of extremely similar and low value are a good representation of measurement noise (which is assumed low for most systems).

If the eigenvalues are rank-sorted by value, then the reliable eigenvalue can be found by minimization of the Laplacian of the sorted eigenvalues:[5]

$$\min \left|\nabla^2 \lambda_s\right|$$

where the eigenvalues are subscripted with an 's' to denote being sorted. The position of the minimization is the lowest reliable eigenvalue. In measurement systems, the square root of this reliable eigenvalue is the average noise over the components of the system.

Functional calculus

The eigendecomposition allows for much easier computation of power series of matrices. If $f(x)$ is given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

then we know that

$$f(\mathbf{A}) = \mathbf{Q}f(\mathbf{\Lambda})\mathbf{Q}^{-1}$$

Because $\mathbf{\Lambda}$ is a diagonal matrix, functions of $\mathbf{\Lambda}$ are very easy to calculate:

$$\left[f(\mathbf{\Lambda})\right]_{ii} = f(\lambda_i)$$

The off-diagonal elements of $f(\mathbf{\Lambda})$ are zero; that is, $f(\mathbf{\Lambda})$ is also a diagonal matrix. Therefore, calculating $f(\mathbf{A})$ reduces to just calculating the function on each of the eigenvalues.

A similar technique works more generally with the holomorphic functional calculus, using

$$\mathbf{A}^{-1} = \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^{-1}$$

from above. Once again, we find that

$$[f(\mathbf{\Lambda})]_{ii} = f(\lambda_i)$$

Examples

$$\mathbf{A}^2 = (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1})(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}) = \mathbf{Q}\mathbf{\Lambda}(\mathbf{Q}^{-1}\mathbf{Q})\mathbf{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{\Lambda}^2\mathbf{Q}^{-1}$$
$$\mathbf{A}^n = \mathbf{Q}\mathbf{\Lambda}^n\mathbf{Q}^{-1}$$

Decomposition for special matrices

Normal matrices

A complex normal matrix ($A^*A = AA^*$) has an orthogonal eigenvector basis, so a normal matrix can be decomposed as

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$$

where **U** is a unitary matrix. Further, if **A** is Hermitian ($A = A^*$), which implies that it is also complex normal, the diagonal matrix **Λ** has only real values, and if **A** is unitary, **Λ** takes all its values on the complex unit circle.

Real symmetric matrices

As a special case, for every *N×N* real symmetric matrix, the eigenvalues are real and the eigenvectors can be chosen such that they are orthogonal to each other. Thus a real symmetric matrix **A** can be decomposed as

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$$

where **Q** is an orthogonal matrix, and **Λ** is a diagonal matrix whose entries are the eigenvalues of A.

Useful facts

Useful facts regarding eigenvalues

- The product of the eigenvalues is equal to the determinant of **A**

$$\det(\mathbf{A}) = \prod_{i=1}^{N_\lambda} \lambda_i^{n_i}$$

Note that each eigenvalue is raised to the power n_i, the algebraic multiplicity.

- The sum of the eigenvalues is equal to the trace of **A**

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N_\lambda} n_i \lambda_i$$

Note that each eigenvalue is multiplied by n_i, the algebraic multiplicity.

- If the eigenvalues of **A** are λ_i, and **A** is invertible, then the eigenvalues of $\mathbf{A}^{-1}$ are simply λ_i^{-1}.
- If the eigenvalues of **A** are λ_i, then the eigenvalues of $f(\mathbf{A})$ are simply $f(\lambda_i)$, for any holomorphic function f.

Useful facts regarding eigenvectors

- If **A** is Hermitian and full-rank, the basis of eigenvectors may be chosen to be mutually orthogonal. The eigenvalues are real.
- The eigenvectors of $\mathbf{A}^{-1}$ are the same as the eigenvectors of **A**.
- Eigenvectors are defined up to a phase, i.e. if $Av = \lambda v$ then $e^{i\theta}v$ is also an eigenvector, and specifically so is $-v$.
- In the case of degenerate eigenvalues (an eigenvalue appearing more than once), the eigenvectors have an additional freedom of rotation, i.e. any linear (orthonormal) combination of eigenvectors sharing an eigenvalue (i.e. in the degenerate sub-space), are themselves eigenvectors (i.e. in the subspace).

Useful facts regarding eigendecomposition

- **A** can be eigendecomposed if and only if

$$N_{\mathbf{v}} = N$$

- If $p(\lambda)$ has no repeated roots, i.e. $N_\lambda = N$, then **A** can be eigendecomposed.
- The statement "**A** can be eigendecomposed" does *not* imply that **A** has an inverse.
- The statement "**A** has an inverse" implies that **A** can be eigendecomposed.

Useful facts regarding matrix inverse

- **A** can be inverted if and only if

$$\lambda_i \neq 0 \; \forall \, i$$

- If $\lambda_i \neq 0 \; \forall \, i$ and $N_{\mathbf{v}} = N$, the inverse is given by

$$\mathbf{A}^{-1} = \mathbf{Q} \mathbf{\Lambda}^{-1} \mathbf{Q}^{-1}$$

Numerical computations

Numerical computation of eigenvalues

Suppose that we want to compute the eigenvalues of a given matrix. If the matrix is small, we can compute them symbolically using the characteristic polynomial. However, this is often impossible for larger matrices, in which case we must use a numerical method.

In practice, eigenvalues of large matrices are not computed using the characteristic polynomial. Computing the polynomial becomes expensive in itself, and exact (symbolic) roots of a high-degree polynomial can be difficult to compute and express: the Abel–Ruffini theorem implies that the roots of high-degree (5 or above) polynomials cannot in general be expressed simply using nth roots. Therefore, general algorithms to find eigenvectors and eigenvalues are iterative.

Iterative numerical algorithms for approximating roots of polynomials exist, such as Newton's method, but in general it is impractical to compute the characteristic polynomial and then apply these methods. One reason is that small round-off errors in the coefficients of the characteristic polynomial can lead to large errors in the eigenvalues and eigenvectors: the roots are an extremely ill-conditioned function of the coefficients.[6]

A simple and accurate iterative method is the power method: a random vector v is chosen and a sequence of unit vectors is computed as

$$\frac{Av}{\|Av\|}, \; \frac{A^2 v}{\|A^2 v\|}, \; \frac{A^3 v}{\|A^3 v\|}, \ldots$$

This sequence will almost always converge to an eigenvector corresponding to the eigenvalue of greatest magnitude, provided that v has a nonzero component of this eigenvector in the eigenvector basis (and also provided that there is only one eigenvalue of greatest magnitude). This simple algorithm is useful in some practical applications; for example, Google uses it to calculate the page rank of documents in their search engine.[7] Also, the power method is the starting point for many more sophisticated algorithms. For instance, by keeping not just the last vector in the sequence, but instead looking at the span of *all* the vectors in the sequence, one can get a better (faster converging) approximation for the eigenvector, and this idea is the basis of Arnoldi iteration.[6] Alternatively, the important QR algorithm is also based on a subtle transformation of a power method.[6]

Numerical computation of eigenvectors

Once the eigenvalues are computed, the eigenvectors could be calculated by solving the equation

$$(\mathbf{A} - \lambda_i \mathbf{I}) \, \mathbf{v}_{i,j} = 0$$

using Gaussian elimination or any other method for solving matrix equations.

However, in practical large-scale eigenvalue methods, the eigenvectors are usually computed in other ways, as a byproduct of the eigenvalue computation. In power iteration, for example, the eigenvector is actually computed before the eigenvalue (which is typically computed by the Rayleigh quotient of the eigenvector).[6] In the QR algorithm for a Hermitian matrix (or any normal matrix), the orthonormal eigenvectors are obtained as a product of the Q matrices from the steps in the algorithm.[6] (For more general matrices, the QR algorithm yields the Schur decomposition first, from which the eigenvectors can be obtained by a backsubstitution procedure.[8]) For Hermitian matrices, the Divide-and-conquer eigenvalue algorithm is more efficient than the QR algorithm if both eigenvectors and eigenvalues are desired.[6]

Additional topics

Generalized eigenspaces

Recall that the *geometric* multiplicity of an eigenvalue can be described as the dimension of the associated eigenspace, the nullspace of $\lambda I - A$. The algebraic multiplicity can also be thought of as a dimension: it is the dimension of the associated **generalized eigenspace** (1st sense), which is the nullspace of the matrix $(\lambda I - A)^k$ for *any sufficiently large k*. That is, it is the space of **generalized eigenvectors** (1st sense), where a generalized eigenvector is any vector which *eventually* becomes 0 if $\lambda I - A$ is applied to it enough times successively. Any eigenvector is a generalized eigenvector, and so each eigenspace is contained in the associated generalized eigenspace. This provides an easy proof that the geometric multiplicity is always less than or equal to the algebraic multiplicity.

This usage should not be confused with the *generalized eigenvalue problem* described below.

Conjugate eigenvector

A **conjugate eigenvector** or **coneigenvector** is a vector sent after transformation to a scalar multiple of its conjugate, where the scalar is called the **conjugate eigenvalue** or **coneigenvalue** of the linear transformation. The coneigenvectors and coneigenvalues represent essentially the same information and meaning as the regular eigenvectors and eigenvalues, but arise when an alternative coordinate system is used. The corresponding equation is

$$Av = \lambda v^*.$$

For example, in coherent electromagnetic scattering theory, the linear transformation A represents the action performed by the scattering object, and the eigenvectors represent polarization states of the electromagnetic wave. In optics, the coordinate system is defined from the wave's viewpoint, known as the Forward Scattering Alignment (FSA), and gives rise to a regular eigenvalue equation, whereas in radar, the coordinate system is defined from the radar's viewpoint, known as the Back Scattering Alignment (BSA), and gives rise to a coneigenvalue equation.

Generalized eigenvalue problem

A **generalized eigenvalue problem** (2nd sense) is the problem of finding a vector $\mathbf{v}$ that obeys

$$A\mathbf{v} = \lambda B\mathbf{v}$$

where A and B are matrices. If $\mathbf{v}$ obeys this equation, with some λ, then we call $\mathbf{v}$ the **generalized eigenvector of A and B** (in the 2nd sense), and λ is called the **generalized eigenvalue of A and B** (in the 2nd sense) which corresponds to the generalized eigenvector $\mathbf{v}$. The possible values of λ must obey the following equation

$$\det(A - \lambda B) = 0.$$

In the case we can find $n \in \mathbb{N}$ linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ so that for every $i \in \{1, \ldots, n\}$, $A\mathbf{v}_i = \lambda_i B\mathbf{v}_i$, where $\lambda_i \in \mathbb{F}$ then we define the matrices $\mathbf{P}$ and $\mathbf{D}$ such that

$$P = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \equiv \begin{pmatrix} (\mathbf{v}_1)_1 & \cdots & (\mathbf{v}_n)_1 \\ \vdots & & \vdots \\ (\mathbf{v}_1)_n & \cdots & (\mathbf{v}_n)_n \end{pmatrix}$$

$$(D)_{ij} = \begin{cases} \lambda_i, & \text{if } i = j \\ 0, & \text{else} \end{cases}$$

Then the following equality holds

$$\mathbf{A} = \mathbf{BPDP^{-1}}$$

And the proof is

$$\mathbf{AP} = \mathbf{A}\begin{pmatrix} | & & | \\ \mathbf{v_1} & \cdots & \mathbf{v_n} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ A\mathbf{v_1} & \cdots & A\mathbf{v_n} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 B\mathbf{v_1} & \cdots & \lambda_n B\mathbf{v_n} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ B\mathbf{v_1} & \cdots & B\mathbf{v_n} \\ | & & | \end{pmatrix}\mathbf{D} = \mathbf{BPD}$$

And since $\mathbf{P}$ is invertible, we multiply the equation from the right by its inverse, finishing the proof.

The set of matrices of the form $A - \lambda B$, where λ is a complex number, is called a *pencil*; the term *matrix pencil* can also refer to the pair (A,B) of matrices.[9] If B is invertible, then the original problem can be written in the form

$$B^{-1}A\mathbf{v} = \lambda\mathbf{v}$$

which is a standard eigenvalue problem. However, in most situations it is preferable not to perform the inversion, but rather to solve the generalized eigenvalue problem as stated originally. This is especially important if A and B are Hermitian matrices, since in this case $B^{-1}A$ is not generally Hermitian and important properties of the solution are no longer apparent.

If A and B are Hermitian and B is a positive-definite matrix, the eigenvalues λ are real and eigenvectors v_1 and v_2 with distinct eigenvalues are B-orthogonal ($\mathbf{v_1^* B v_2 = 0}$).[10] Also, in this case it is guaranteed that there exists a basis of generalized eigenvectors (it is not a defective problem).[9] This case is sometimes called a *Hermitian definite pencil* or *definite pencil*.[9]

See also

- Matrix decomposition
- List of matrices
- Eigenvalue, eigenvector and eigenspace
- Spectral theorem
- Singular value decomposition
- Householder transformation
- Frobenius covariant
- Sylvester's formula
- Eigenvalue perturbation

Notes

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External links

- Interactive program & tutorial of Spectral Decomposition (http://people.revoledu.com/kardi/tutorial/LinearAlgebra/SpectralDecomposition.html).

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