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RISK MEASUREMENT WHEN SHARES ARE SUBJECT TO INFREQUENT TRADING

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When shares are traded infrequently, beta estimates are often severely biased This paper reviews the problems introduced by infrequent trading, and presents a method for measuring beta when share price data suffer from this problem. The method is used with monthly returns for a one-in-three random sample of all U K Stock Exchange shares from 1955 to 1974. Most of the bias in conventional beta estimates is eliminated when the proposed estimators are used in their place.

1. Introduction

1.1. *Infrequent trading*

Comprehensive databases have recently become available for stock exchanges in which many securities are traded only intermittently.[1] These databases considerably extend the scope for empirical studies of stock prices. However, infrequent trading also introduces serious biases into much of this work.

The major source of bias is the tendency for prices recorded at the end of a time period to represent the outcome of a transaction which occurred earlier in or prior to the period in question.[2] Fisher (1966) pointed out that this causes an index constructed from such share price data to be an average of the temporally ordered underlying values of the shares. Consequently, positive serial correlation is induced into returns which are calculated from

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[1] For example, the British, Canadian, Australian, French, Japanese, O.T.C. and daily American databases reported by Smithers (1977), Rorke and Love (1974a), Ball, Brown and Finn (1977), Altman, Jacquillat and Levasseur (1974), Lau, Quay and Ramsey (1974), Benston and Hagerman (1974), and Scholes and Williams (1977), respectively.

[2] Other sources of bias associated with non-trading include a widening of the bid-ask spread and an increase in measurement error e.g. due to price rounding, as non-traded stocks frequently have a low share price. See, for example, Tinic and West (1974) and Praetz (1976).

the index[3] and the estimated variance of returns on the index is biased downward.

Shares which suffer from non-trading also have their covariance with the market substantially underestimated. The downward bias in the covariance of frequently traded shares is, however, much smaller. Thus, infrequently traded securities have a beta estimate which is biased downwards, while the figure for frequently traded securities is upward biased.[4] It is the objective of this article to present a method for obtaining an unbiased estimate of the systematic risk of a share, when the share and some or all of the securities in the market are subject to infrequent trading.

1.2. The intervaling effect

The market model asserts that there is a linear relationship between the return R_t, on a security and the return, M_t, on the market,

$$R_t = \alpha + \beta M_t + \varepsilon_t. \tag{1}$$

A phenomenon which has been encountered in estimating the parameters of the market model is the intervaling effect. This is a tendency for the explanatory power of the regression equation and the mean value of beta, estimated from value weighted indexes, to rise as the differencing interval is increased. As Schwartz and Whitcomb (1977a) explain, the intervaling effect is indicative of a non-trading problem, though it can be generated by any kind of error in measuring returns. The prevalence of the intervaling effect is therefore a necessary, though not a sufficient, condition for regarding non-trading as a potentially widespread problem.

The intervaling effect has, in fact, been documented in numerous studies of different security markets.[5] For example, Smith (1978) found that for a sample of 200 stocks, the coefficient of determination increased monotonically as the differencing interval was progressively lengthened from one to twelve months, and the mean estimated beta measured against a value weighted index increased in an almost equally consistent way.[6] Similar results will be presented in section 4.1 for the London Share Price Database.

[3]For a more detailed exposition of this problem, see Fama (1965).

[4]The upward bias for frequently traded shares follows because the mean beta of all securities is by definition unity.

[5]Some of the share price databases which have exhibited the intervaling effect are the I.S.L., C.R.S.P., City University, DataStream, C.E.S.A. and Eurofinance files. See Black and Scholes (1973), Hawawini and Vora (1979), Russell (1972), Cunningham (1973), Altman, Jacquillat and Levasseur (1974), and Pogue and Solnik (1974), respectively.

[6]Smith also observed a tendency for the estimated betas of portfolios with above-average systematic risk to increase with the differencing interval, while portfolios with below-average systematic risk had beta estimates which decreased. He interprets this as supportive of the

There is therefore a general need to consider the impact of infrequent trading on market model parameter estimates.

1.3. Recent work on non-trading

Despite Fisher's (1966) early identification of the phenomenon of non-trading, the problems that it introduces were ignored for several years. Working (1960) pioneered research on the biases generated by averaging temporally ordered prices, but the theoretical and empirical work which followed was specific to the problems engendered by explicit price averaging.[7] While the prevalence of the Fisher effect was confirmed in studies of different securities markets,[8] its impact has been ignored by many authors.[9] The problems that it poses for risk measurement only received recognition as an important issue relatively recently.[10]

Three approaches have been suggested in the literature for estimating the risk of infrequently traded shares. Some researchers have introduced lagged market returns as additional independent variables in their market model regressions.[11] Other writers have calculated their returns on a trade-to-trade basis, and regressed these returns on market movements calculated over precisely the same trade-to-trade time intervals.[12] Finally, Scholes and Williams (1977) have shown that it is possible to combine these ideas and use non-synchronous plus synchronous market returns as explanatory variables for trade-to-trade returns. In addition, several authors have tried to

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Levhari–Levy (1977) hypothesis on the effect of the choice of investment horizon. Since the constituents of each portfolio differed according to the differencing interval, this interpretation is open to question.

[7]Thus, for example, Daniels (1966) and Rosenberg (1971) treated only those averaging processes which are invariant under time-reversal, while the empirical work of Meyer and Corbeau (1975) examined the biases from using a high–low average as a proxy for the closing price.

[8]For example, U.S., Canadian and U.K. market indexes have been estimated to have a monthly serial correlation of 0.19, 0.26 and 0.25 respectively. [See Fisher (1966), Rorke and Love (1974b), Dimson (1974a) and section 3.3 below].

[9]Infrequent trading is a likely explanation for the apparent nonrandomness of the prices of smaller companies' shares which was observed by Kemp and Reid (1971), Girmes and Benjamin (1975), Jennergren and Korsvold (1974), Pogue and Solnik (1974), Conrad and Jutner (1973), and others. It is also the probable reason for the mounting evidence of pervasive heteroscedasticity in market model residuals for non-U.S. stock markets [e.g., see Praetz (1969) and Belkaoui (1977)] compared to relatively little heteroscedasticity in the case of the New York Stock Exchange [e.g., see Martin and Klemkowsky (1975)].

[10]See Ball (1977) for a simplified analysis of the effects of measurement errors such as those generated by non-trading on estimates of systematic risk.

[11]See, for example, Ibbotson (1975), Dimson (1974b) and Schwert (1977). Pogue and Solnik (1974) discussed this option but rejected it.

[12]This procedure appears to have been used only by Marsh (1979), Schwert (1977), and Franks, Broyles and Hecht (1977). Others, such as Lau, Quay and Ramsey (1974), omitted all returns calculated from prices which originated more than ten days prior to the month-end.

reconstruct the market index so as to reduce the serial correlation induced by intermittent trading.[13]

While each of these approaches is valuable in particular circumstances, none is of general applicability. As will become clear in section 2.3 the first method can only be justified if the constituents of the market index do not suffer from more than a negligible amount of non-trading. The trade-to-trade method requires each share price to be labelled with a transaction date, and also needs frequent recordings of a market index which must have negligible non-trading. The Scholes–Williams method again requires transaction dates, and fails to make use of share prices which are not preceded or followed by a trade in an immediately adjacent time period.

The method that is proposed in this paper, which is a development of the lagged market returns approach, largely overcomes these drawbacks. In particular, it requires neither the market index to be continuously traded nor supplementary data, such as transactions information, to be available.

The proposed method for estimating beta is derived and described in section 2. In section 3, the data and methodology used to evaluate the method is described. Section 4 presents the results of the empirical work, and section 5 discusses some issues which arise from this study and presents some concluding remarks.

2. The AC method

2.1. The process generating changes in value

Changes in value are assumed to be generated by the market model (1), where R_t and M_t are the serially and cross-serially uncorrelated returns on the security and the market in period t,[14] β is the stationary systematic risk of the security and is equal to $\mathrm{cov}(R_t, M_t)/\mathrm{var}(M_t)$, and α is an intercept which is not time dependent.[15] The zero mean error term ε_t is uncorrelated with M_t. The estimators $\hat{\alpha}$ and $\hat{\beta}$ are usually obtained from ordinary least squares simple regression, using measured returns $\hat{R}_t$ and $\hat{M}_t$,

$$\hat{R}_t = \hat{\alpha} + \hat{\beta}\hat{M}_t + v_t. \tag{2}$$

[13]For some different approaches to this problem, see Brealey (1970), Officer (1975), and Schwartz and Whitcomb (1977b).

[14]Thus, true returns are generated in 'chronological', and not in 'operational' or trade-to-trade, time as Clark (1973), Press (1971) and Oldfield, Rogalski and Jarrow (1977) have suggested.

[15]The latter introduces a slight misspecification if the return generating process is the Sharpe (1964) – Lintner (1965) or Black (1972) capital asset pricing model. However, Brenner's (1979) empirical analyses disclosed negligible bias in estimated beta arising from an incorrect choice of eq. (1); also see Kaplan and Roll (1972) and Ball and Brown (1968). The choice of equation is, of course, important when cumulative returns are being calculated; see Brenner (1977), Bar-Yosef and Brown (1977), Ohlson (1978), and Ball (1978).

It is also assumed that at time period t, the probability of a security having been most recently traded in period $t-i$ ($i \geqq 0$) is equal to θ_i. Securities trade at least once every n periods. Similarly, the proportion of the market portfolio which was last traded in period $t-i$ is equal to ϕ_i. The distributions of θ_i and ϕ_i are stationary and identically distributed over time. It follows that

$$\theta_i \geqq \theta_{i+j}, \quad \phi_i \geqq \phi_{i+j}, \qquad j>0, \tag{3a}$$

$$\sum_{i=0}^{n} \theta_i = \sum_{i=0}^{n} \phi_i = 1. \tag{3b}$$

2.2. The process generating observed returns

Since securities are traded intermittently, an observed price $\hat{P}_t$ may represent a transaction price P_t in the same period t or a price P_{t-i} established in the last trade which occurred in period $t-i$ ($i>0$).[16] Observed prices therefore have an expected value which is a weighted average of a sequence of true prices, where the latter are the transaction prices which would arise if trading were continuous,

$$E(\hat{P}_t) = \sum_{i=0}^{n} \theta_i P_{t-i}. \tag{4}$$

It follows that the observed price changes $\Delta\hat{P}_t$ also have an expected value which is a weighted average of current and previous true price changes $\Delta\hat{P}_t$ which would be apparent if trading were continuous,

$$E(\Delta\hat{P}_t) = \sum_{i=0}^{n} \theta_i \Delta P_{t-i}. \tag{5}$$

The continuously compounded return $\hat{R}_t$ based on observed prices is similarly obtained from the first difference of logarithmic prices in (4). It can be shown [see Fowler, Rorke and Riding (1977)] that the expected value of $\hat{R}_t$ is given by

$$E(\hat{R}_t) = E(\ln \hat{P}_t - \ln \hat{P}_{t-1})$$
$$= \sum_{i=0}^{n} \theta_i R_{t-i}, \tag{6}$$

[16]In the terminology of Fowler, Rorke and Riding (1977) the exposition presented here deals with 'infrequently' traded securities rather than 'moderately' traded securities. The latter are securities which trade at least once per period though not necessarily at the close of the period. The method presented here can easily be extended to cover moderately traded securities.

where the R_{t-i} are the corresponding true, but unobservable, returns on the security.

The coefficients θ_i are shown in table 1 for shares whose probability of trading per period is $\theta_0=0.1, 0.3, \ldots, 0.9$. Since the values of $\hat{\theta}_i$ $(i=1,2,\ldots)$ decay fast $\theta_0 \geqq 0.3$ or thereabouts, we may express (6) in ex-post form, for both the security and the market, as follows:[17]

$$\hat{R}_t = \sum_{i=0}^{n} \theta_i R_{t-i} + u_{Rt}, \tag{7a}$$

$$\hat{M}_t = \sum_{i=0}^{n} \phi_i M_{t-i} + u_{Mt}, \tag{7b}$$

where u_{Rt} and u_{Mt} are zero-mean error terms uncorrelated with θ_i and R_{t-i} and with ϕ_i and M_{t-i} respectively.

Table 1

Relationship between observed and true returns: Simulated data.[a]

$$\hat{R}_t = \hat{a} + \sum_{i=0}^{4} \hat{\theta}_i R_{t-i} + u_{Rt}.$$

Probability of a trade θ_0	Regression coefficients $\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	Standard error of $\hat{\theta}_i^b$	Multiple correlation coefficient[b]
0.1	0.10	0.09	0.08	0.07	0.07	0.010	0.19
0.3	0.30	0.21	0.15	0.10	0.07	0.009	0.41
0.5	0.50	0.25	0.12	0.06	0.03	0.008	0.57
0.7	0.70	0.21	0.06	0.02	0.01	0.007	0.72
0.9	0.90	0.09	0.01	0.00	0.00	0.004	0.91

[a]This table shows that, where the probability, θ_0, of a trade occurring in a time period is at least 0.3, the regression coefficients $\hat{\theta}_i = \theta_0(1-\theta_0)^i$ decay fast. Thus, in a regression of observed returns on true returns, only a few terms are usually required in (7a) adequately to represent the relationship.

[b]Based on simulation results.

Eqs. (3a) and (7a, b) imply that the covariance of a share's observed return with the market, $\mathrm{cov}(\hat{R}_t, \hat{M}_t)$, is positively related to its trading frequency. Since the mean $\hat{\beta}$ of all shares is unity, simple regression (2) must generate upward biased estimates of the risk of frequently traded shares and downward biased estimates of the risk of infrequently traded shares.

[17]Relationship (7a) was verified by a simulation reported in an earlier version of this paper. It was found that (7a) was a close approximation for dollar price changes and percentage changes as well as for continuously compounded returns. The simulation generated true (but unobservable) prices according to the formula:

$$P_t = \exp(\ln P_{t-1} + N\sigma),$$

2.3. The aggregated coefficients (AC) method

Consider the multiple regression of observed returns on preceding, synchronous and subsequent market returns. The regression equation is

$$\hat{R}_t = \hat{\alpha} + \sum_{k=-n}^{n} \hat{\beta}_k \hat{M}_{t+k} + w_t, \tag{8}$$

and the estimated intercept and slope coefficients are given by $\hat{\alpha}$ and the $\hat{\beta}_k$. We shall examine the characteristics of the latter by substituting the true returns, R_t and M_t, for the observed returns in this equation.

The left-hand side of (8) is given by (7a). By substituting the market model (1) and utilising (3b) one obtains

$$\begin{aligned} \hat{R}_t &= \sum_{i=0}^{n} \theta_i R_{t-i} + u_{Rt} \\ &= \sum_{i=0}^{n} \theta_i(\alpha + \beta M_{t-i} + \varepsilon_{t-i}) + u_{Rt}, \\ &= \alpha + \sum_{i=0}^{n} M_{t-i}\theta_i\beta + \sum_{i=0}^{n} \theta_i\varepsilon_{t-i} + u_{Rt}. \end{aligned} \tag{9}$$

The right-hand side of (8) is obtained in a similar way from (7b) and is then re-arranged,

$$\begin{aligned} \hat{R}_t &= \hat{\alpha} + \sum_{k=-n}^{n} \hat{\beta}_k\left(\sum_{i=0}^{n} \phi_i M_{t+k-i} + u_{Mt+k}\right) + w_t \\ &= \hat{\alpha} + \sum_{k=-2n}^{n} M_{t+k} \sum_{i=0}^{n} \phi_i \hat{\beta}_{i+k} + \sum_{k=-n}^{n} \hat{\beta}_k u_{Mt+k} + w_t, \end{aligned} \tag{10}$$

where the values of $\hat{\beta}_{i+k}$ are zero for $|i+k| > n$.

Since (9) and (10) are equal, we may equate the coefficients of M_{t+k} for all values of k. Hence, the following set of equations is obtained:

$$\left.\begin{matrix} \theta_{-k}\beta \\ 0 \end{matrix}\right\} = \sum_{i=0}^{n} \phi_i \hat{\beta}_{i+k} \quad \left\{\begin{matrix} 0 \geqq k \geqq -n, \\ k > 0, k < -n. \end{matrix}\right. \tag{11}$$

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where the standard deviation of returns σ was 0.1 and N is a standard normal random variate. The observed prices occurred when a trade S_t took place,

$$\hat{P}_t = \hat{P}_{t-1} + S_t(P_t - \hat{P}_{t-1}),$$

where $S_t = 0$ (if $U > \theta_0$) or 1 (if $U < \theta_0$) and where U is a uniformly distributed random variate in the range 0–1. 10,000 periods were simulated and regression (7a) was performed.

In a sampling context, we may replace β in (11) by $\hat{\beta}$, where plim $(\hat{\beta})=\beta$. Summing eq. (11) over all values of k and substituting (3b) results in the relationship

$$\sum_{k=0}^{n} \theta_k \hat{\beta} = \sum_{k=-2n}^{n} \sum_{i=0}^{n} \phi_i \hat{\beta}_{i+k},$$

$$\hat{\beta} \sum_{k=0}^{n} \theta_k = \sum_{k=-n}^{n} \hat{\beta}_k \sum_{i=0}^{n} \phi_i,$$

$$\hat{\beta} = \sum_{k=-n}^{n} \hat{\beta}_k. \tag{12}$$

In other words, the true systematic risk in (1) can be obtained from security price data which is subject to infrequent trading. All that need be done is to run the multiple regression (8) of security returns against lagged, matching and leading market terms. A consistent estimate of beta is obtained by aggregating the slope coefficients from this regression.

2.4. *The characteristics of the non-synchronous coefficients*

When infrequently traded shares are sold, their observed price change reflects true market returns since the last transaction, as well as contemporaneous returns on the market. The observed movement in the market index gives greatest weight to recent true returns [by (3a) and (7b)]. Thus, the value of $\hat{\beta}_0$ is larger than the value of $\hat{\beta}_k$ $(k<0)$. In addition, the value of a lagged coefficient $\hat{\beta}_k$ $(k<0)$ is larger when there is a higher probability of there having been no transaction within the last k periods. The value of the lagged coefficients is therefore negatively related to trading frequency.

Similarly, when frequently traded shares are sold, their observed price change is related to contemporaneous true market returns. These returns are reflected in subsequent observed market movements, but [again, by (3a)] the contemporaneous observed market movement gives greater weight than any subsequent movement to the current true returns on the market. Hence, the value for $\hat{\beta}_0$ exceeds that of $\hat{\beta}_k$ $(k>0)$. It also follows that frequently traded shares must have observed returns which comove with subsequent observed returns on the index. They therefore have values for $\hat{\beta}_k$ $(k>0)$ which are positively related to trading frequency.

The systematic risk of the market is unity, and it is necessary for $\hat{\beta}_k$ $(k\neq 0)$ to have a value of zero for the market as a whole. This gives us the following expected values for $\hat{\beta}_k$ conditional on the share's trading frequency per period, θ_0, relative to the trading frequency of the market, ϕ_0:

$$E(\hat{\beta}_k \mid k<0) \lesseqgtr 0 \quad \text{as} \quad \theta_0 \gtrless \phi_0, \tag{13a}$$

$$E(\hat{\beta}_k \mid k=0) \gtrless \beta \quad \text{as} \quad \theta_0 \gtrless \phi_0, \tag{13b}$$

$$E(\hat{\beta}_k \mid k>0) \gtrless 0 \quad \text{as} \quad \theta_0 \gtrless \phi_0. \tag{13c}$$

Note that if the market is very frequently traded relative to a share, i.e. $\phi_0 \gg \theta_0$, then the leading coefficients will be small compared to the lagged coefficients. Consequently, when shares are being regressed on an index which is composed of large companies and/or which is value weighted, it is the lagged coefficients which are of importance.[18]

As the number of non-synchronous terms used in the multiple regression is increased, i.e., as n in (8) takes the values $0, 1, 2, \ldots$, the bias of the AC estimator (12) is reduced. On the other hand, the efficiency of the AC method also declines. This is because the lagged and leading coefficients, $\hat{\beta}_k$ $(k \neq 0)$, suffer from estimation error.

We may assess the statistical importance of the non-synchronous $\hat{\beta}_k$ by estimating the variance of the true values β_k of these coefficients in the population. Assuming independence between β_k and measurement errors, $\mathrm{VAR}(\beta_k)$ is estimated as the difference between the cross-sectional variance of the $\hat{\beta}_k$ and the mean error variance,[19]

$$\mathrm{V\hat{A}R}(\beta_k) \simeq \mathrm{VAR}(\hat{\beta}_k) - \mathrm{MEAN}(\mathrm{var}(\hat{\beta}_k)). \tag{14}$$

Thus, lagged and leading market terms contribute to producing an unbiased estimate of beta as long as there is a positive implied cross-sectional variance of the β_k. Beyond a certain number of lags (and leads), however, the value of $\mathrm{V\hat{A}R}(\beta_k)$ will be small and will tend to fluctuate around zero as $|k|$ is increased. This gives an indication of the maximum number of lags and leads which are required to avoid bias in the estimator.

Further reduction of the estimation error in the AC beta estimate can only be achieved by one of two courses of action. Either fewer lagged (leading) terms may be included, in the knowledge that a residue of non-trading bias will be re-introduced into the least (most) frequently traded shares. Alternatively, the beta estimate may be explicitly adjusted to take account of the estimation error in each of the synchronous and non-synchronous coefficients.

If the interdependencies between the $\hat{\beta}_k$ for a share are negligible, the individual coefficients may be adjusted back towards their mean values of

[18]See Dimson (1974b) for an exposition of this approach or Ibbotson (1975) for an early application of the lagged coefficient method. Fama and Schwert (1977) follow this approach in dealing with the relationship between asset returns and inflation.

[19]MEAN and VAR (in capitals) refer to cross-sectional means and variances, and not to estimates and error variances for individual securities.

zero ($k \neq 0$) or unity ($k=0$), following the method proposed by Vasicek (1973). This involves giving a weight w_k, which is equal to $\hat{\mathrm{VAR}}(\beta_k)/\{\hat{\mathrm{VAR}}(\beta_k)+\mathrm{var}(\hat{\beta}_k)\}$, to each estimated coefficient $\hat{\beta}_k$, and a weight of $1-w_k$ to the prior for the value of $\hat{\beta}_k$, namely zero or unity. Thus, the Bayesian estimator equivalent to (12) is

$$\mathrm{Adj}\,\hat{\beta} = 1 - w_0 + \sum_{k=-n}^{n} w_k \hat{\beta}_k. \tag{15}$$

Of course, if more is known about the share's trading frequency or about the company, more informed priors may be devised.[20]

2.5. Comparison with other methods

There are at least four alternative methods for risk measurement when share prices suffer from infrequent trading. These are adjusted simple regression, simple regression with overlapping observations, trade-to-trade regression and the Scholes–Williams (1977) method. Adjusted simple regression involves running the market model (2) with observations for periods $t=1,2,\ldots,T$; and adjusting the estimator $\hat{\beta}$ for non-trading bias. The extent to which the $\hat{\beta}$ must be altered depends on the trading frequency of the share relative to the market. Unfortunately, the appropriate adjustment can only be determined if the bias in $\hat{\beta}$ is known. Thus, adjusted simple regression cannot be used correctly unless another method has also been used to estimate true betas.

Simple regression (2) with overlapping observations has been used by a number of commercial beta services. This involves increasing the length of the differencing interval from one to m periods, until the $\hat{\beta}$'s are largely invariant to increases in m. Consequently, the number of observations declines. To offset this effect, overlapping m-period observations are used for $t=m,\ m+1,\ldots,T-m+1$. There is therefore a loss of $2(m-1)$ observations, a further $2(m-1)$ periods receive less weight in the regressions than the remaining periods, and because of the serial dependence in the observations significance levels are misstated.

The viable alternatives are therefore the trade-to-trade and Scholes–Williams methods, which will be discussed in turn. The trade-to-trade estimator is given by a variant of (2), using an index based on frequently traded shares. Since heteroscedasticity in the residual is usually a problem [see Marsh (1979)], $\hat{\beta}$ is estimated from the multiple regression,

[20]Better priors can be based on the mean industry beta, the extent of overseas activity, the time since receiving a listing [see Downes (1975)], the size of the company [see Dimson and Marsh (1979)] and various accounting data [see Myers (1977) and the references therein].

$$R_s(t_s-t_{s-1})^{-\frac{1}{2}}=\hat{\alpha}(t_s-t_{s-1})^{-\frac{1}{2}}+\hat{\beta}M_s(t_s-t_{s-1})^{-\frac{1}{2}}+v_s, \tag{16}$$

where returns are measured from transaction $(s-1)$ to transaction (s) throughout the interval $t_s=1,\dots,T$. When market returns are matched exactly to the times of transactions, only the index can suffer from non-trading problems, i.e., $\theta_0 \mp 1 \geqq \phi_0$. Accurately computed trade-to-trade estimates of beta therefore tend to overstate the true risk of a share [by (13b)].

The trade-to-trade method requires an index of frequently traded share prices which is recorded many times per period. If the index is sufficiently broad to suffer from negligible residual variance, the trade-to-trade estimator is relatively efficient. While the coverage or method of computation of the index may sometimes be unsuitable,[21] the main drawback of the trade-to-trade method is its data requirement. The method cannot be used when the times of recording share prices within a time interval are unknown, or when a good proxy for a continuously recorded index of transaction prices is unavailable.

The Scholes–Williams method merely requires a record of whether, and not when, a share was traded within a time period. In this approach, a return is calculated and used only if a transaction is known to have occurred in consecutive time periods. The market index is defined to be the mean of all such returns. With this definition of the index, simple regression (2) overstates the beta of shares which are traded as frequently as the market. Shares which trade infrequently or very frequently tend to have their risk underestimated. These biases differ from those described in section 2.2.

With the type of data described above, beta is estimated by another variant of (2) in which the market model is run with either a synchronous, a lagged or a leading market return,

$$\hat{R}_t=\hat{\alpha}+\hat{\beta}_k\hat{M}_{t+k}+v_t, \qquad k=-1,0,1. \tag{17}$$

The unbiased estimator is given by the sum of these slope coefficients, divided by one plus twice the autocorrelation, ρ, of the market. That is,

$$\hat{\beta}=\sum_{k=-1}^{1}\hat{\beta}_k\Big/(1+2\rho). \tag{18}$$

Because multi-period returns are discarded in the Scholes–Williams approach, there is a loss of a proportion $1-\theta_0$ of the observations available for

[21]For example, the constituents of the index may not be representative of the market as a whole; the index may be a price-weighted or geometric index; it may exclude dividend income; or it may be value weighted when an equally weighted index is desired. These are problems which can be overcome with other beta estimation techniques, in which the index is computed from returns drawn from a market-wide database.

use with the method.[22] For example, a share which trades in about one period in two will have only one Scholes–Williams observation in every four periods. The Scholes–Williams estimator is therefore inefficient compared to its trade-to-trade equivalent.

It is difficult to analyse the efficiency of the AC method, relative to the Scholes–Williams alternative, except under simplifying assumptions.[23] With the AC method, one has the option to vary the efficiency of the estimator by changing the number of independent variables. This may re-introduce some bias to the most severely non-traded securities, though the Scholes–Williams method may generate too few observations on the latter for any meaningful regressions to be performed. When shares are traded infrequently and sufficient lagged market terms have been introduced to eliminate virtually all non-trading bias, the AC estimator (12) generally remains more efficient than the Scholes–Williams estimator. However, for shares which trade in almost every period, the main source of bias is the tendency for closing prices to occur at different times within an interval. For these shares, the Scholes–Williams approach compares favourably with the AC method.

To investigate further the bias and efficiency of the three principal alternatives to simple regression, these methods were applied to a simulated database of 100 securities. 1260 trading days (i.e., 60 months) were simulated, with daily changes in value being generated by (1). Each share's trading frequency (i.e., probability of trading in a day), F, was generated as a function of its market capitalisation (in £ millions), C, as follows:

$$\ln C = \mu_C + \sigma_C N_1, \tag{19}$$

$$\ln F = \mu_F + \sigma_F(\rho N_1 + \sqrt{1-\rho^2}\, N_2), \tag{20}$$

where N_1, N_2 represent random unit normal deviates, ρ is the correlation between $\ln C$ and $\ln F$, and μ_C, σ_C and μ_F, σ_F are the parameters of the lognormal distributions of C and F respectively. Monthly observed prices were then generated,[24] and beta was estimated using (2), (16), (18) and (12).

[22]Runs of i trades in succession have probabilities of $(1-\theta_0)\theta_0^i$ $(i \geqq 0)$. Thus, the proportion of time periods which generate transactions, and hence returns, is $\sum_{i=1}^{\infty}(1-\theta_0)\theta^i = \theta_0$. Since runs of i consecutive trades produce only $i-1$ observations for the Scholes–Williams method, the proportion of time periods generating returns from this approach is $\sum_{i=2}^{\infty}(1-\theta_0)\theta_0^i = \theta_0^2$ Thus the number of Scholes–Williams returns is θ_0 times the number of trade-to-trade returns.

[23]Assume that the variance of the market index, σ_M^2, is invariant to the estimation method and that the non-synchronous coefficients in (17) and (8) have no explanatory power. Then $\operatorname{var}(\hat{\beta})$ takes the approximate values of $\sigma_v^2/\theta_0 T\sigma_M^2$ for the trade-to-trade (16), $3\sigma_v^2/\theta_0^2 T\sigma_M^2$ for the Scholes–Williams (17) and $N\sigma_v^2/\theta T\sigma_M^2$ for the aggregated coefficients methods, where N is the number of independent variables in (8). This would imply efficiencies for the three methods of 1, $\theta_0/3$ and $1/N$ respectively If the assumptions are relaxed, the trade-to-trade method is less than perfectly efficient, while the relative efficiencies of the latter two methods is higher than stated here. Moreover, adjustment for regression bias in the non-synchronous coefficients of the AC method compensates for the loss in efficiency.

[24]Using methods similar to those in footnote 17. Further details of the simulation are

Shares were ranked by their market capitalisation (1 = very large capitalisation/very frequent trading, 10 = very low capitalisation/severe non-trading) and assigned to deciles, and the beta estimate was computed for each decile as the mean of its constituents' betas.

In order to highlight the biases in beta, all shares were assigned a 'true' beta in (1) of unity. Table 2 shows the mean decile estimates of beta for the simulations, based on empirically estimated parameters for (1), (19) and (20).[25] The table confirms the results reported above, namely: (i) There is a

Table 2

Ratio of estimated beta to true beta: Simulated data.[a]

	Simple regression		Alternative estimator for beta		
Decile of market capitalisation/trading frequency	Using all observed returns $\hat{\beta}$ in (2)	Scholes–Williams variant of simple regression $\hat{\beta}_0$ in (17)	Trade-to-trade $\hat{\beta}$ in (16)	Scholes–Williams $\hat{\beta}$ in (18)	Aggregated coefficients (3 lags, 1 lead) $\hat{\beta}$ in (12)
1 (Frequent)	1.21	1.05	1.10	0.99	1.00
2	1.19	1.04	1.06	0.99	1 01
3	1.17	1.03	1.06	1.01	1.03
4	1.10	0.99	1.02	0.96	1.02
5	1.10	1.00	1.04	1.03	1 02
6	1.02	0.95	1.02	0.96	0.98
7	0.97	0.96	1 06	1.00	1.02
8	0.90	0.92	1.02	1 00	0.98
9	0.79	0.91	1.07	1.11	1.05
10 (Infrequent)	0 54	0.74	0.98	0 93	0.88
All shares	1.00	1 00	1.05	1.00	1.00
(Cross-sectional standard deviation)	(0.35)	(0.25)	(0.17)	(0.48)	(0.36)

[a]This table shows alternative estimates of beta for a database whose shares all have a true beta of unity. Note the downward bias of simple regression estimates for infrequently traded shares, and the difference in the cross-sectional dispersion of the various estimators

available from the author. Note that prices were assumed to have a zero bid–ask spread. The numbers reported in table 2 are the mean of 10 simulations.

[25]The parameters for (1) were drawn from Dimson and Brealey (1978). Those for (19) and (20) were based on the 2197 returns series on the LSPD (see section 3.1) at end-1978. The distributions of C and F were characterised by the following statistics:

Variable	*Mean*	*Standard deviation*	*Skewness*	*Kurtosis*	*Correlation*
$\ln C$	3.3	1.7	0.3	3.2	0.8
$\ln F$	−1 9	1.2	−0.6	3.2	
Unit normal	0.0	1.0	0.0	3.0	

In the simulation, $\ln C$ was truncated at the first and ninety-ninth centiles, and $\ln F$ was truncated at zero.

marked non-trading bias in simple regression betas. (ii) This bias persists even if returns are computed only from transactions occurring in successive time periods. The downward bias of $\hat{\beta}_0$ in (17) for very frequently traded shares, which has been shown to occur by Scholes and Williams (1977), is too small to observe. (iii) The trade-to-trade method, used with a value weighted index, slightly over-estimates beta but is relatively efficient. (iv) The Scholes-Williams estimator appears not to suffer from non-trading bias, but is very inefficient. (v) The AC method (with three lagged and one leading market term) suffers from a residue of non-trading bias in decile 1, but is not as inefficient as the Scholes–Williams method.

3. Data and methodology

3.1. The database

To evaluate the AC method empirically, returns were calculated from the London Share Price Database (LSPD) file. The LSPD consists of returns and ancillary data on four overlapping samples of companies listed on the London Stock Exchange between January 1955 and December 1974:[26]

(1) A one third random sample of all companies existing in, or gaining a listing after, January 1955.
(2) The 500 largest companies by market value in 1955.
(3) The 200 largest companies by market value in 1972.
(4) The 1,000 largest companies by market value in 1976.

A unique characteristic of the database is that in addition to quoted share prices, the latest monthly trading prices have been collected along with the date on which the transaction was marked to have occurred. An analysis of the age of the transaction prices recorded for the random sample is shown in table 3. It can be seen that, while a third of the sample had prices which were on average no more than a couple of days out of date, a third of the sample had prices which were at least a week out of date. Moreover, even frequently traded shares had a proportion of prices which were recorded several days, sometimes weeks, before the end of the month. Though comparable figures are unavailable for the CRSP[27] file it is obvious that, compared to its U.S. equivalent, the LSPD presents enormous non-trading problems.[28]

[26]The database has since been extended to cover all listed shares from January 1975 onwards.

[27]Center for Research in Security Prices of the University of Chicago Graduate School of Business. See Fisher and Lorie (1964, 1977).

[28]The most extreme example of non-trading in the Stock Exchange Daily Official List, the source document for LSPD prices, was on 29th May 1964. At that date, the last trade in Lamot Ltd. was marked as having occurred on 23rd October 1955.

Table 3

The ages of prices in the LSPD random sample, 1955–1974.[a]

Decile of trading frequency	Range of mean price ages (days)	Distribution of price ages (days) 0 (%)	1 (%)	2–5 (%)	6–10 (%)	11–20 (%)	21–30 (%)	31–40 (%)	41–50 (%)	Over 50 (%)	Total (%)	Number of observations ('000s)
1	0.0– 0.4	94	3	2	—	—	—	—	—	—	100	23
2	0.4– 1.0	77	10	10	2	—	—	—	—	—	100	21
3	1.0– 1.8	65	13	15	4	2	—	—	—	—	100	18
4	1.8– 2.6	54	13	20	8	3	1	—	—	—	100	19
5	2.6– 3.8	45	13	22	11	5	1	1	—	1	100	18
6	3.8– 5.4	38	12	24	13	8	3	1	—	1	100	20
7	5.4– 7.8	30	11	23	16	12	4	2	1	2	100	18
8	7.8–11.6	22	8	22	17	15	7	3	1	4	100	19
9	11.6–20.1	16	6	17	16	17	10	5	3	10	100	17
10	20.1 and over	8	3	10	11	14	11	7	4	33	100	14

[a]*Source* Smithers (1977).

3.2. The sample

Of the 1,900 companies in the random sample of the LSPD, only 421 maintained a continuous listing throughout the 1955–74 period. These price series formed the basis for the sample. From the price series, a returns file was created to CRSP format where the return on a security in month t is given by

$$\hat{R}_t = \ln(\hat{P}_t + d_t) - \ln \hat{P}_{t-1}, \tag{21}$$

where $\hat{P}_t$ is the last recorded transaction price and d_t is the dividend payment for the security in month t. Prices were fully adjusted for capital changes and returns were screened for extreme outliers.[29] All those shares which effectively have a complete price history[30] were admitted to the sample used in this study.

Finally, shares were ranked by the average age of their month-end prices, and assigned to deciles according to their trading frequency (1 = very frequent trading, 10 = severe non-trading). The range of average ages in each decile is shown in table 4 for the complete twenty year period. An analysis of successive subperiods shows that the average frequency of trading of our

Table 4

The mean price ages of continuously listed shares, 1955–1974.

Decile of trading frequency	Mean price age (days)
1 (Frequent)	0.0– 0.1
2	0.1– 0.4
3	0.4– 0.9
4	0.9– 1.5
5	1.5– 2.4
6	2.4– 3.9
7	3.9– 5.3
8	5.3– 8.8
9	8.8–12.3
10 (Infrequent)	12.3 and over

[29]Monthly continuously compounded returns outside the range ± 2 were deemed to be errors and were treated as missing observations.

[30]By 'effectively having a complete price history' we mean having at most two (and in virtually all cases at most one) missing observations. The latter is inevitable for the month of February 1956 when the Stock Exchange suffered a printing strike and kept the sketchiest of hand-written records. By replacing the missing return observation with the corresponding Financial Times index movement, the sample size was more than doubled to 314 with the introduction of minimal bias. Of these, 14 companies, the majority of whose returns were zero, were eliminated from the sample to ensure that there was sufficient data for the regressions.

sample has increased over the years. While Smithers (1977) has shown this to be a general trend, it is undoubtedly accentuated in our sample because of the survivorship bias in the method used for sample selection. Note that the relative frequency with which these shares are traded causes our sample to understate the problems caused by non-trading.

3.3. Construction of the index

The file was used both with an equally weighted (EW) market index and with the value weighted Financial Times-Actuaries (FTA) All Share Index. The FTA index is an arithmetic index of price relatives which covers some 650 of the larger U.K. companies.[31]

The EW index is an arithmetic mean of the return on all N securities in the returns file.[32] It is computed from the formula

$$\hat{M}_t = \sum_{j=1}^{N} \hat{R}_{jt}/N, \tag{22}$$

where the return $\hat{R}_{jt}$ on the jth security is given by (21).

As might have been expected, the EW index is highly correlated with the FTA index, the correlation coefficient being 0.91 (or 0.95 if quarterly or half-yearly returns are used). However, the greater weight accorded to infrequently traded shares in the EW index, relative to the FTA index, is clear from table 5. The table shows a much lower first order serial correlation coefficient for the FTA than for the EW index, 0.07 as against 0.25.

[31]Unlike the returns file, the FTA excludes mining and plantation companies, and includes relatively few investment trusts (closed end funds) and companies operating largely overseas. To cover the period prior to the date on which the FTA was introduced, 10 April 1962, the FTA was linked back to 1955 using the Financial Times Ordinary (FTO) Share Index, an equally weighted geometric index of 30 large U.K. industrial companies. It was found by Franks, Broyles and Hecht (1977) that use of the FTO in place of the FTA (during the period in which they overlap) had no significant effect on measures of risk, while Theobald and Whitman (1978) confirmed that the two indices are very highly correlated.

[32]Hodges and Schaefer (1974) have pointed out the arithmetic mean of the log relatives cannot be interpreted as a potentially attainable portfolio return, even if funds are continuously reallocated between securities. For this reason, other runs were undertaken using an arithmetic EW index, by changing (21) to

$$\hat{R}_t = (\hat{P}_{t+} + d_t)/\hat{P}_{t-1} - 1,$$

and leaving (22) as above. The results were scarcely any different and are not reported here. The present index has the advantage of better meeting OLS distributional assumptions. It also more closely approximates true wealth relatives as the alternative index grossly outperforms the buy-and-hold policy, in part because of the Fisher effect and in part because of the survivorship bias of the sample. The returns of the mean log relative have an $\hat{\alpha}$ if regressed on FTA returns using (2), of -0.0009.

Table 5

Serial correlation of the market indexes, June 1955–July 1974.

$\hat{M}_t = \hat{\alpha} + \hat{\rho}_k \hat{M}_{t+k} + u_t, \qquad k = 1, 2, \ldots, 5.$

Market Index	Serial correlation coefficients[a]				
	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\rho}_3$	$\hat{\rho}_4$	$\hat{\rho}_5$
Financial Times-Actuaries (FTA)	0.07	0.03	0.13	0.14	0.04
Equally Weighted (EW)	0.25	0.09	0.22	0.16	0.01
Standard error[b]	0.07	0.07	0.07	0.07	0.07

[a]Each coefficient is the average of the slopes in the regressions of the observed market return, $\hat{M}_t$, on the lagged and leading returns, $\hat{M}_{t-k}$ and $\hat{M}_{t+k}$.

[b]Under the null hypothesis of zero serial correlation, the standard error is the reciprocial of the square root of the number of observations.

4. Empirical results

4.1. The bias of conventional beta estimates

An examination of the betas obtained from simple regression using (2) reveals the biases generated by infrequent trading. When monthly returns were regressed on the FTA index, the mean estimated beta was 0.73, and the FTA index only explained 21 percent of the variation in returns. The low value for beta is not because our sample is a relatively low risk group of shares, for if this were the case the choice of differencing interval would be largely immaterial. In fact, the estimate of beta rises to 0.87 when half-yearly returns are used. The mean $\bar{R}^2$ also rises from 21 percent to 36 percent. The existence of a strong intervaling effect is evidence of the seriousness of the measurement errors in calculating the true returns of these shares.

In table 6, beta estimates and $\bar{R}^2$ computed from the EW index are shown for each decile. Since the EW index constituents correspond exactly to the sample which is being studied, the mean beta is always unity. Note, however, that the intervaling effect persists in relation to the $\bar{R}^2$, which rises from 23 percent with monthly data to 38 percent when observations are semiannual.

The positive relationship between trading frequency and estimated beta and $\bar{R}^2$ is very noticeable. Very frequently traded shares have betas of around 1.2 and an $\bar{R}^2$ of about 35 percent when monthly data is used. For very infrequently traded shares, these figures drop to around 0.6 and 8 percent respectively. If we assume that each decile has roughly average true systematic risk, then deciles 1–2 and 9–10 would appear to produce estimates that are on average 20 percent too high and 40 percent too low respectively. The movements of the beta estimates towards unity, as the differencing

Table 6

Simple regression estimates of beta for January 1955–December 1974.[a]

Decile of trading frequency	Differencing interval: 1 month $\hat{\beta}$	1 month $\bar{R}^2$(%)	3 months $\hat{\beta}$	3 months $\bar{R}^2$(%)	6 months $\hat{\beta}$	6 months $\bar{R}^2$(%)
	1.15	36	1.03	45	0.99	48
2	1.23	34	1.11	43	1.12	48
3	1.16	30	1.07	40	1.09	45
4	1.19	27	1.13	38	1.09	43
5	1.12	25	1.17	39	1.15	45
6	1.03	22	1.01	31	0.96	34
7	1.00	23	1.01	34	1.03	42
8	0.95	15	1.01	26	0.99	30
9	0.67	11	0.81	21	0.85	29
10 (Infrequent)	0.50	6	0.65	14	0.72	20
All shares	1.00	23	1.00	33	1.00	38

[a]This table shows that, as the differencing interval is increased, the beta estimated using (2) rises for infrequently traded shares and falls for shares which are frequently traded relative to the market index

interval is extended, also indicates that much of the variation in betas is due to non-trading bias.

4.2. *The Aggregated Coefficients method*

The results from using the proposed AC estimator as a replacement for the simple regression beta estimate are presented in this section. In order to avoid any risk of data-mining, all regressions used five lags and five leads, and there was no experimentation to determine the number of non-synchronous coefficients to be used. (This should not be interpreted as a recommendation to use as many market terms in other empirical studies.)

When betas are estimated using the AC method (12) with multiple regression (8) there is a much more even distribution of estimated betas across deciles of trading frequency. As table 7 shows, the range of betas is reduced from 0.47 through 1.20, in the case of the simple regression estimate using (2), to 0.85 through 1.13, in the case of the five lags and leads AC estimate. Since the deciles were constructed without any attempt at ranking by beta, the narrower distribution accords with our priors. In addition the $\bar{R}^2$ increases from a mean of 22.1 percent to 23.8 percent, an improvement over the fit of the simple regression model (2).[33]

[33]This improvement shows that the non-synchronous market terms have some explanatory power The rise in $\bar{R}^2$ is less than that which accompanies increases in the length of the differencing interval. This is because the correlation coefficient remains attenuated through the use of a noisier data series. See Kendall and Stuart (1974, ex 26 17).

Table 7

Aggregated coefficients estimates of beta for June 1955–July 1974.[a]

Decile of trading frequency	Beta estimates		Estimated $\bar{R}^2$ (%)	
	Simple regression (2)	Aggregated coefficients (12)	Simple regression (2)	Aggregated coefficients (12)
1 (Frequent)	1.16	0.93	34.6	36.6
2	1.20	1.04	32.0	33.5
3	1.16	1.04	28.9	30.6
4	1.17	1.07	26.1	27.1
5	1.14	1.13	24.4	25.1
6	1.04	0.99	21.7	22.9
7	1.01	1.02	22.8	24.5
8	0.95	1.03	14.9	15.6
9	0.68	0.85	10.6	13.4
10 (Infrequent)	0.47	0.91	5.3	8.4
All shares	1.00	1.00	22.1	23.8

[a]This table compares the mean decile betas and coefficients of determination obtained using simple regression (2) with those obtained using the AC method (12).

As terms are added to the regression, the decile 9 and 10 beta estimates continue to rise while the other estimates fall very marginally.[34] However, they do not exceed 0.90, and an explanation of why this is the case is offered in table 8, which compares the constituents of deciles 1 and 10. Apart from their small size decile 10 companies are quite frequently engaged in business overseas. For example a quarter of these firms are rubber or tea plantation businesses. By contrast, only one of the decile 1 firms (a mining company) is

Table 8

Comparison of deciles 1 and 10.[a]

(a) Market value in 1972 (£m)	Percentage of companies	
	Decile 1	Decile 10
0–1	—	60
1–10	—	33
10–100	33	7
100–1,000	57	—
1,000–10,000	10	—
All sizes	.00	100

(b) Industrial classification in 1972	Percentage of companies	
	Decile 1	Decile 10
Commercial and industrial	63	50
Oil and gas	13	—
Insurance and banking	13	
Rubber and tea plantations	—	23
Others (all < 10% of total)	11	27
All classifications	100	100

[a]This table discloses the small size and overseas orientation of decile 10 (severely non-traded) companies, as compared to decile 1 (very frequently traded) companies.

principally based outside the U.K., though several decile 1 companies are multinationals. It is the presence of the latter which leads to a lower estimated beta for decile 1 companies than for other relatively frequently traded firms, in line with the assertions of Agmon and Lessard (1977).

These results suggest that the introduction of lagged and leading terms into the market model can markedly improve beta estimates. We may assess the importance of these non-synchronous terms by estimating the variance of true values of the coefficients in the population. The variance of the true coefficients β_k ($k=-5, -4, \ldots, 5$) may then be expressed as a percentage of

[34]This asymmetry reflects the asymmetry of the distribution of trading frequencies

the cross-sectional variance of the estimated coefficients $\hat{\beta}_k$. If the estimated coefficients $\hat{\beta}_k$ are measured without error, then all of the cross-sectional variance would be attributable to true difference between coefficients. If the estimated coefficients represent no underlying relationship and are purely noise, then none of the cross-sectional variance would be attributable to true difference between coefficients.

Table 9 shows the percentages of the cross-sectional variance of estimated coefficients which were found to be attributable to true differences between

Table 9

Percentage of cross-sectional variance of estimated coefficients which is attributable to true differences between coefficients.[a]

Lag or lead k	Percentage attributable to true differences $\widehat{\mathrm{VAR}}(\beta_k)/\mathrm{VAR}(\hat{\beta}_k)$
−5	9
−4	21
−3	33
−2	23
−1	61
0	85
1	21
2	10
3	11
4	5
5	28

[a]This table expresses the estimated cross-sectional variance of the true coefficients (14) as a proportion of the cross-sectional variance of the estimated coefficients. In the case of one of the leading and several of the lagged coefficients, a quarter or more of the cross-sectional variance is attributable to differences in the true coefficients.

coefficients. 85 percent of the variance of the synchronous estimated coefficients $\hat{\beta}_0$ is attributable to true differences between securities in their level of systematic risk. Corresponding figures for the CRSP data were estimated by Blume (1975) to be 92, 92, 89, 82 and 75 percent over the seven-year periods ending 1933, 1940, 1947, 1954 and 1961.

Of all the lagged and leading coefficients, only the one-period lag coefficient $\hat{\beta}_{-1}$ was comparable with the synchronous coefficient $\hat{\beta}_0$, 61 percent of its cross-sectional variance being attributable to true difference between the one-period lag coefficients. However, true difference between coefficients with a lag of two, three or four periods or with a lead of one period still explained around a quarter of their respective cross-sectional variances.[35] Contrary to the American studies of Ibbotson (1975), Schwert (1977) and Schwartz and Whitcomb (1977b), this study indicates that with monthly U.K. data a leading and several lagged market terms are needed in (8) if risk measures are to take account of the effects of infrequent trading.

In table 10, the Bayesian-adjusted coefficients are presented for the five lags/five leads multiple regression (8) over the period 1955–74. The expected signs of the coefficients were specified in (13), and the table shows results which are consistent with those predictions. Frequently traded shares have lagged coefficients which have negative values while infrequently traded shares have positive lagged coefficients. The synchronous coefficients are all positive, and are overestimates of beta for frequently traded shares and underestimates for infrequently traded shares. The leading coefficients are positive for frequently traded shares, while for infrequently traded shares the predicted negative values are too small (in absolute terms) to be shown on the table.

It appears that the lagged coefficients are much more important in magnitude than the leading coefficients. This is an outcome of the skewed distribution of trading frequencies in the sample. For some databases, severe non-trading is the norm for most securities, and the relative importance of lagged versus leading coefficients could be different.

The final column of table 10 presents the estimates of beta using (15). It is interesting to note the similarity between these estimates and the corresponding figure if simple regression is used with a differencing interval chosen to minimise the effects of non-trading, e.g., the half-yearly interval used in table 6.[36] The shares which suffer from the most severe non-trading, decile 10, now have a mean beta estimate of 0.67.[37] However, most deciles have beta estimates which are close to the figures obtained with the AC method unadjusted for regression bias and presented in table 7.

Because the non-synchronous coefficients in table 10 have been adjusted towards zero there is a discernable tendency for these beta estimates to suffer from non-trading bias, and this is confirmed in replications of the table for shorter estimation periods. In practical applications of the AC method, the choice of prior is thus important. Fortunately, the LSPD provides users with the average price age for each share, and priors can therefore be obtained for

[35]The relatively high percentage of cross-sectional variance explained by the five-period leading coefficient is probably an anomaly caused by the ex-post serial correlation of U.K. stock market returns during 1973–4. The percentage explained by other lagged and leading coefficients is however, very similar in the 1971–4 period as in the four 4-year periods prior to 1971.

[36]The correlation between the estimates in table 10 and the half-yearly results in table 6 is 0.97.

[37]For the purpose of comparison, the decile 10 mean beta was also calculated using the trade-to-trade method (16) The value was 0.51 as compared to the mean for all deciles of 0.75. This is very similar to the above figure of 0.67 relative to a global mean of 1.00.

Table 10

Lagged, matching and leading Bayesian adjusted coefficients for 1955–1974.[a]

Decile of Trading frequency	Lag or lead −5	−4	−3	−2	−1	0	1	2	3	4	5	Bayesian adjusted AC estimator
1	—[b]	—	−0.04	—	−0.22	1.20	0.02	—	−0.01	—	0.04	0.98
2	—	—	−0.02	−0.01	−0.01	1.25	—	0.01	—	—	0.02	1.10
3	—	—		0.01	−0.12	1.19	—	—	—	—	0.02	1.05
4	—	−0.01	−0.01	−0.01	−0.04	1.19	0.01	−0.01	—	—	—	1.12
5	0.01	−0.02	−0.01	—	−0.03	1.15	0.01	—	−0.01	—	—	1.09
6	—	−0.01	0.01	−0.02	0.02	1.03	−0.03	—	—	—	−0.01	1.01
7	—	−0.01	—	−0.03	0.09	0.95	0.01	−0.01	0.01	—	0.01	1.01
8	—	—	0.02	—	0.08	0.92	—	—	—	—	—	1.01
9	—	0.01	0.01	−0.01	0.17	0.67	—	—	—	—	−0.01	0.85
10	—	0.01	0.05	0.02	0.14	0.45	—	—	—	—	—	0.67
All shares	—	—	—	—	—	1.00	—	—	—	—	—	1.00

[a]This table gives the mean Bayesian adjusted slope coefficients, for each decile, from the regression (8) of security returns on lagged, matching and leading market returns. The coefficients have been aggregated to produce the estimator (15) for beta.
[b]Coefficients which are equal to 0.00 to two places of decimals are omitted from this table.

the share's degree of trading frequency. Alternatively, the priors might be estimated statistically from the relationship between mean price age and estimated lagged and leading coefficients.

5. Discussion

5.1. Selecting an estimator for beta

With the advent of various methods for measuring risk when shares are subject to infrequent trading, empirical evaluation of alternative techniques becomes an important subject for research. However, a number of possible pitfalls face the investigator in this area, because infrequent trading not only biases estimates of beta – it also introduces problems into most research which is designed to evaluate the quality of alternative beta estimates. In this section, we discuss some of these issues in order that suitable criteria can be devised to evaluate methods of estimating beta. Our starting point must be the classic tests of stability of beta in the U.S., the studies undertaken by Blume (1971) and by Sharpe and Cooper (1972).

Blume (1971) estimated the beta of portfolios which were constructed from shares ranked by their estimated beta in a previous, non-overlapping period. He found a perfect rank correlation between the portfolio betas and their values in the previous period, and interpreted this as evidence for moderate stability in beta. He concluded that betas regress only part of the way back to their mean value, even when the beta estimate is adjusted for regression bias [see Blume (1975)]. Sharpe and Cooper (1972) assigned shares to risk deciles on the basis of their estimated beta. They calculated the probability of a share falling in the same decile in two successive, non-overlapping periods, or within one or two deciles of its classification in the earlier period. Sharpe and Cooper asserted that a high probability of falling in similar risk classes in successive periods indicates at least some stability in beta. There is general agreement that although betas are not stationary they are to some extent predictable [see Schaefer, Brealey, Hodges and Thomas (1975), Brenner and Smidt (1977) and the references cited in the latter].

Unfortunately, infrequent trading will bias beta estimates so as to cause the estimates to appear stable. Infrequently traded securities will have low beta estimates, while frequently traded securities will have high estimates. Provided that frequency of trading is serially correlated, the beta estimates will be relatively stable, regressing somewhat to the mean. Even if there is no stability whatsoever in the underlying beta, the methods of Blume and of Sharpe and Cooper will create the illusion of stability provided that trading is sufficiently infrequent!

Because of these problems, it is not possible to compare the quality of alternative beta estimates by examining their stability. When trading is infrequent, unbiased estimates of beta can be inferior to conventional estimators as predictors of the beta estimate in a subsequent period.

An alternative approach to the problem of evaluating beta estimates was devised by Fisher and Kamin (1975). They examined the mean squared prediction error which results when security returns are conditionally forecast over the month following the period used to estimate beta. Their conditional forecast of return was based on the market return during the month, and the beta which had been estimated in the preceding period.

When trading is infrequent, this research design also suffers from the problems which afflict tests of the stability of beta. To illustrate this, compare the conditional forecasts made for two shares which have a true beta of unity, one share being as frequently traded as the market index and the other being infrequently traded. The conventional, simple regression (2) estimates of beta would therefore have expected values of unity and less than unity, respectively. Accordingly, the conditional forecast of return for the first share would be the market return, and for the infrequently traded share would be in between the market return for that period and the mean market return.[38]

What returns would actually be calculated from prices which record the last trade of the month? The first share would appear to move in line with the market, but the second share would provide observed returns which reflect both matching and previous market returns. In the long run, the infrequently traded share would provide observed returns that are in between the matching market return and the mean return on the market. This is in line with the forecast based on the conventional estimate of beta.

On the other hand, an unbiased estimate of the shares' beta would have led to the prediction of the same observed return for both shares. The unbiased estimate of beta could therefore produce conditional forecasts of *observed* returns with a higher forecasting error than the biased estimates of beta.[39]

One way around this difficulty might be to compare conditional forecasts of return over longer periods of time, but this reduces the number of observations and accentuates the impact of outlying returns. An alternative solution might be to compare conditional forecasts between precise days on which trades are known to have occurred. Unfortunately, the consequential need for a daily market index makes it impossible to use an index constructed from the share prices held on the database. This too could easily show in a spuriously favourable light an unsatisfactory method of estimating beta.

[38]The conditional returns have been derived under the assumption that the expected value of α in (1) is zero.

[39]The bias noted here has the opposite effect of the non-trading bias noted by Fisher and Kamin. They argued that the bias in beta resulting from non-trading would cause an increase in the mean squared prediction error. Their view ignores non-trading during the period in which returns are measured.

We conclude that, when non-trading is a problem, it is unlikely that empirical evidence will enable us to determine the ideal estimation method. Our choice of method must be based on the extent to which an estimator is theoretically attractive and meets our subjective priors in empirical work. On this basis, the AC method is an attractive technique for estimating beta when the times of transactions are not known.

5.2. Conclusion

The article commenced with a review of the biases in estimates of beta which arise when some securities are traded infrequently. Evidence was presented on the pervasiveness of the non-trading problem, and a number of recent approaches for dealing with the problem were reviewed.

In section 2, the most important assumption was the serial independence of the population distribution of changes in value of a security. It is, of course, acknowledged that dependencies may arise as an ex-post phenomenon in a given sample. But dependencies in observed returns based on transaction prices have a population distribution which is serially independent. A model of the process generating observed returns was therefore proposed. This was used to develop the aggregated coefficients (AC) method for estimating beta. It was shown that an unbiased estimator for beta is the sum of the slope coefficients in a regression of security returns on lagged, matching and leading market returns. The characteristics of these slope coefficients were described, and the AC method was compared with the principal alternative techniques for dealing with infrequent trading.

In section 3, a one-in-three random sample of all shares listed in the U.K. throughout 1955–74, taken from the London Share Price Database, was described; and an equally weighted index of the LSPD returns was calculated. The first part of section 4 showed that large biases arise when the beta of random sample shares is calculated in the conventional way, using either the Financial Times Actuaries Index or an equally weighted index.

The remainder of section 4 examined the characteristics of betas estimated using the AC method. The AC method appears to eliminate most of the bias in beta attributable to non-trading. Whereas conventionally estimated betas for companies ranked by trading frequency ranged from a mean of 1.20 down to a mean of 0.47 for the decile of least frequently traded shares, the range of AC estimates of beta was 1.13 (for decile 5) to 0.85 (for decile 9). It was found that at least one leading term and four lagged terms would be required if the multiple regression is to include coefficients which explain a quarter or more of the cross-sectional variance of coefficient estimates.

The last section discussed the problems of testing the predictive value of a beta estimate. When trading is infrequent, there are major obstacles to selecting an estimator on the basis of empirical tests. It therefore becomes necessary for us to select a method which accords with priors derived from other empirical and theoretical work.

In conclusion, this paper has presented a method for estimating beta when share prices suffer from non-trading. Its distinguishing feature is that it requires no supplementary data, such as transactions information, to be available. The results of using the proposed method on U.K. data seem promising. It is hoped that this paper will therefore make easier the study of companies whose shares are thinly traded. While these companies are admittedly small, almost every empirical study of an interesting question seems to include an amazing number of them.

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