Direct sum of modules

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In abstract algebra, the **direct sum** is a construction which combines several modules into a new, larger module. The direct sum of modules is the smallest module which contains the given modules as submodules with no "unnecessary" constraints, making it an example of a coproduct. Contrast with the direct product, which is the dual notion.

The most familiar examples of this construction occur when considering vector spaces (modules over a field) and abelian groups (modules over the ring **Z** of integers). The construction may also be extended to cover Banach spaces and Hilbert spaces.

Contents

Construction for vector spaces and abelian groups

We give the construction first in these two cases, under the assumption that we have only two objects. Then we generalise to an arbitrary family of arbitrary modules. The key elements of the general construction are more clearly identified by considering these two cases in depth.

Construction for two vector spaces

Suppose V and W are vector spaces over the field K. The cartesian product $V \times W$ can be given the structure of a vector space over K (Halmos 1974, §18) by defining the operations componentwise:

- $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$
- $\alpha (v, w) = (\alpha v, \alpha w)$

for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\alpha \in K$.

The resulting vector space is called the *direct sum* of V and W and is usually denoted by a plus symbol inside a circle:

$$V \oplus W$$

It is customary to write the elements of an ordered sum not as ordered pairs (v, w), but as a sum $v + w$.

The subspace $V \times \{0\}$ of $V \oplus W$ is isomorphic to V and is often identified with V; similarly for $\{0\} \times W$ and W. (See *internal direct sum* below.) With this identification, every element of $V \oplus W$ can be written in one and only one way as the sum of an element of V and an element of W. The dimension of $V \oplus W$ is equal to the sum of the dimensions of V and W. One elementary use is the reconstruction of a finite vector space from any subspace W and its orthogonal complement:

$$\mathbb{R}^n = W \oplus W^\perp$$

This construction readily generalises to any finite number of vector spaces.

Construction for two abelian groups

For abelian groups G and H which are written additively, the direct product of G and H is also called a direct sum (Mac Lane & Birkhoff 1999, §V.6). Thus the cartesian product $G \times H$ is equipped with the structure of an abelian group by defining the operations componentwise:

- $(g_1, h_1) + (g_2, h_2) = (g_1 + g_2, h_1 + h_2)$

for g_1, g_2 in G, and h_1, h_2 in H.

Integral multiples are similarly defined componentwise by

- $n(g, h) = (ng, nh)$

for g in G, h in H, and n an integer. This parallels the extension of the scalar product of vector spaces to the direct sum above.

The resulting abelian group is called the *direct sum* of G and H and is usually denoted by a plus symbol inside a circle:

$$G \oplus H$$

It is customary to write the elements of an ordered sum not as ordered pairs (g, h), but as a sum $g + h$.

The subgroup $G \times \{0\}$ of $G \oplus H$ is isomorphic to G and is often identified with G; similarly for $\{0\} \times H$ and H. (See *internal direct sum* below.) With this identification, it is true that every element of $G \oplus H$ can be written in one and only one way as the sum of an element of G and an element of H. The rank of $G \oplus H$ is equal to the sum of the ranks of G and H.

This construction readily generalises to any finite number of abelian groups.

Construction for an arbitrary family of modules

One should notice a clear similarity between the definitions of the direct sum of two vector spaces and of two abelian groups. In fact, each is a special case of the construction of the direct sum of two modules. Additionally, by modifying the definition one can accommodate the direct sum of an infinite family of modules. The precise definition is as follows (Bourbaki 1989, §II.1.6).

Let R be a ring, and $\{M_i : i \in I\}$ a family of left R-modules indexed by the set I. The *direct sum* of $\{M_i\}$ is then defined to be the set of all sequences (α_i) where $\alpha_i \in M_i$ and $\alpha_i = 0$ for cofinitely many indices i. (The direct product is analogous but the indices do not need to cofinitely vanish.)

It can also be defined as functions α from I to the disjoint union of the modules M_i such that $\alpha(i) \in M_i$ for all $i \in I$ and $\alpha(i) = 0$ for cofinitely many indices i. These functions can equivalently be regarded as finitely supported sections of the fiber bundle over the index set I, with the fiber over $i \in I$ being M_i.

This set inherits the module structure via component-wise addition and scalar multiplication. Explicitly, two such sequences (or functions) α and β can be added by writing $(\alpha + \beta)_i = \alpha_i + \beta_i$ for all i (note that this is again zero for all but finitely many indices), and such a function can be multiplied with an element r from R by defining $r(\alpha)_i = (r\alpha)_i$ for all i. In this way, the direct sum becomes a left R-module, and it is denoted

$$\bigoplus_{i \in I} M_i.$$

It is customary to write the sequence (α_i) as a sum $\Sigma\alpha_i$. Sometimes a primed summation $\Sigma'\alpha_i$ is used to indicate that cofinitely many of the terms are zero.

Properties

- The direct sum is a submodule of the direct product of the modules M_i (Bourbaki 1989, §II.1.7). The direct product is the set of all functions α from I to the disjoint union of the modules M_i with $\alpha(i) \in M_i$, but not necessarily vanishing for all but finitely many i. If the index set I is finite, then the direct sum and the direct product are equal.
- Each of the modules M_i may be identified with the submodule of the direct sum consisting of those functions which vanish on all indices different from i. With these identifications, every element x of the direct sum can be written in one and only one way as a sum of finitely many elements from the modules M_i.
- If the M_i are actually vector spaces, then the dimension of the direct sum is equal to the sum of the dimensions of the M_i. The same is true for the rank of abelian groups and the length of modules.
- Every vector space over the field K is isomorphic to a direct sum of sufficiently many copies of K, so in a sense only these direct sums have to be considered. This is not true for modules over arbitrary rings.
- The tensor product distributes over direct sums in the following sense: if N is some right R-module, then the direct sum of the tensor products of N with M_i (which are abelian groups) is naturally isomorphic to the tensor product of N with the direct sum of the M_i.
- Direct sums are also commutative and associative (up to isomorphism), meaning that it doesn't matter in which order one forms the direct sum.
- The group of R-linear homomorphisms from the direct sum to some left R-module L is naturally isomorphic to the direct product of the sets of R-linear homomorphisms from M_i to L:

$$\mathrm{Hom}_R\left(\bigoplus_{i \in I} M_i, L\right) \cong \prod_{i \in I} \mathrm{Hom}_R(M_i, L).$$

  Indeed, there is clearly a homomorphism τ from the left hand side to the right hand side, where $\tau(\theta)$(i) is the R-linear homomorphism sending $x \in M_i$ to $\theta(x)$ (using the natural inclusion of M_i into the direct sum). The inverse of the homomorphism τ is defined by
  $$\tau^{-1}(\beta)(\alpha) = \sum_{i \in I} \beta(i)(\alpha(i))$$

for any α in the direct sum of the modules M_i. The key point is that the definition of τ^{-1} makes sense because $\alpha(i)$ is zero for all but finitely many i, and so the sum is finite.
In particular, the dual vector space of a direct sum of vector spaces is isomorphic to the direct product of the duals of those spaces.

- The *finite* direct sum of modules is a biproduct: If

  $$p_k : A_1 \oplus \cdots \oplus A_n \to A_k$$

  are the canonical projection mappings and

  $$i_k : A_k \mapsto A_1 \oplus \cdots \oplus A_n$$

  are the inclusion mappings, then

  $$i_1 \circ p_1 + \cdots + i_n \circ p_n$$

  equals the identity morphism of $A_1 \oplus \cdots \oplus A_n$, and

  $$p_k \circ i_l$$

  is the identity morphism of A_k in the case $l=k$, and is the zero map otherwise.

Internal direct sum

Suppose M is some R-module, and M_i is a submodule of M for every i in I. If every x in M can be written in one and only one way as a sum of finitely many elements of the M_i, then we say that M is the **internal direct sum** of the submodules M_i (Halmos 1974, §18). In this case, M is naturally isomorphic to the (external) direct sum of the M_i as defined above (Adamson 1972, p.61).

A submodule N of M is a **direct summand** of M if there exists some other submodule N' of M such that M is the *internal* direct sum of N and N'. In this case, N and N' are **complementary subspaces**.

Universal property

In the language of category theory, the direct sum is a coproduct and hence a colimit in the category of left R-modules, which means that it is characterized by the following universal property. For every i in I, consider the *natural embedding*

$$j_i : M_i \to \bigoplus_{k \in I} M_k$$

which sends the elements of M_i to those functions which are zero for all arguments but i. If $f_i : M_i \to M$ are arbitrary R-linear maps for every i, then there exists precisely one R-linear map

$$f : \bigoplus_{i \in I} M_i \to M$$

such that $f \circ j_i = f_i$ for all i.

Dually, the direct product is the product.

Grothendieck group

The direct sum gives a collection of objects the structure of a commutative monoid, in that the addition of objects is defined, but not subtraction. In fact, subtraction can be defined, and every commutative monoid can be extended to an abelian group. This extension is known as the Grothendieck group. The extension is done by defining equivalence classes of pairs of objects, which allows certain pairs to be treated as inverses. The construction, detailed in the article on the Grothendieck group, is "universal", in that it has the universal property of being unique, and homomorphic to any other embedding of an abelian monoid in an abelian group.

Direct sum of modules with additional structure

If the modules we are considering carry some additional structure (e.g. a norm or an inner product), then the direct sum of the modules can often be made to carry this additional structure, as well. In this case, we obtain the coproduct in the appropriate category of all objects carrying the additional structure. Two prominent examples occur for Banach spaces and Hilbert spaces.

In some classical texts, the notion of direct sum of algebras over a field is also introduced. This construction, however, does not provide a coproduct in the category of algebras, but a direct product (*see note below* and the remark on direct sums of rings).

Direct sum of algebras

A direct sum of algebras *X* and *Y* is the direct sum as vector spaces, with product

$$(x_1 + y_1)(x_2 + y_2) = (x_1 x_2 + y_1 y_2).$$

Consider these classical examples:

$\mathbf{R} \oplus \mathbf{R}$ is ring isomorphic to split-complex numbers, also used in interval analysis.
$\mathbf{C} \oplus \mathbf{C}$ is the algebra of tessarines introduced by James Cockle in 1848.
$\mathbf{H} \oplus \mathbf{H}$, called the split-biquaternions, was introduced by William Kingdon Clifford in 1873.

Joseph Wedderburn exploited the concept of a direct sum of algebras in his classification of hypercomplex numbers. See his *Lectures on Matrices* (1934), page 151. Wedderburn makes clear the distinction between a direct sum and a direct product of algebras: For the direct sum the field of scalars acts jointly on both parts: $\lambda(x \oplus y) = \lambda x \oplus \lambda y$ while for the direct product a scalar factor may be collected alternately with the parts, but not both: $\lambda(x, y) = (\lambda x, y) = (x, \lambda y)$. Ian R. Porteous uses the three direct sums above, denoting them $^2R,\ ^2C,\ ^2H$, as rings of scalars in his analysis of *Clifford Algebras and the Classical Groups* (1995).

The construction described above, as well as Wedderburn's use of the terms *direct sum* and *direct product* follow a different convention from the one in category theory. In categorical terms, Wedderburn's *direct sum* is a categorical product, whilst Wedderburn's *direct product* is a coproduct (or categorical sum), which (for commutative algebras) actually corresponds to the tensor product of algebras.

Composition algebras

A composition algebra (*A*, *, n) is an algebra over a field *A*, an involution * and a "norm" n(*x*) = *x x**. Any field *K* gives rise to a series of composition algebras beginning with *K*, and the trivial involution, so that n(*x*) = x^2. The inductive step in the series involves forming the direct sum A $\oplus$ A and using the new involution $(x, y)^* = x^* - y$.

Leonard Dickson developed this construction doubling quaternions for Cayley numbers, and the doubling method involving the direct sum A ⊕ A is called the Cayley–Dickson construction. In the instance beginning with $K = \mathbb{R}$, the series generates complex numbers, quaternions, octonions, and sedenions. Beginning with $K = \mathbb{C}$ and the norm $n(z) = z^2$, the series continues with bicomplex numbers, biquaternions, and bioctonions.

Max Zorn realized that the classical Cayley–Dickson construction missed constructing some composition algebras that arise as real subalgebras in the $(\mathbb{C}, z^2)$ series, in particular the split-octonions. A modified Cayley–Dickson construction, still based on use of the direct sum A ⊕ A of a base algebra A, has since been used to exhibit the series $\mathbb{R}$, split-complex numbers, split-quaternions, and split-octonions.

Direct sum of Banach spaces

The direct sum of two Banach spaces X and Y is the direct sum of X and Y considered as vector spaces, with the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ for all x in X and y in Y.

Generally, if X_i is a collection of Banach spaces, where i traverses the index set I, then the direct sum $\bigoplus_{i\in I} X_i$ is a module consisting of all functions x defined over I such that $x(i) \in X_i$ for all $i \in I$ and

$$\sum_{i\in I} \|x(i)\|_{X_i} < \infty.$$

The norm is given by the sum above. The direct sum with this norm is again a Banach space.

For example, if we take the index set $I = \mathbf{N}$ and $X_i = \mathbf{R}$, then the direct sum $\bigoplus_{i\in\mathbf{N}} X_i$ is the space l_1, which consists of all the sequences (a_i) of reals with finite norm $\|a\| = \sum_i |a_i|$.

A closed subspace A of a Banach space X is **complemented** if there is another closed subspace B of X such that X is equal to the internal direct sum $A \oplus B$. Note that not every closed subspace is complemented, e.g. c_0 is not complemented in ℓ^∞.

Direct sum of modules with bilinear forms

Let $\{(M_i,b_i) : i \in I\}$ be a family indexed by I of modules equipped with bilinear forms. The **orthogonal direct sum** is the module direct sum with bilinear form B defined by[1]

$$B\left((x_i),(y_i)\right) = \sum_{i\in I} b_i\left(x_i, y_i\right)$$

in which the summation makes sense even for infinite index sets I because only finitely many of the terms are non-zero.

Direct sum of Hilbert spaces

If finitely many Hilbert spaces $H_1,\ldots,H_n$ are given, one can construct their orthogonal direct sum as above (since they are vector spaces), defining the inner product as:

$$\langle (x_1,\ldots,x_n),(y_1,\ldots,y_n)\rangle = \langle x_1,y_1\rangle + \ldots + \langle x_n,y_n\rangle.$$

The resulting direct sum is a Hilbert space which contains the given Hilbert spaces as mutually orthogonal subspaces.

If infinitely many Hilbert spaces H_i for i in I are given, we can carry out the same construction; notice that when defining the inner product, only finitely many summands will be non-zero. However, the result will only be an inner product space and it will not necessarily be complete. We then define the direct sum of the Hilbert spaces H_i to be the completion of this inner product space.

Alternatively and equivalently, one can define the direct sum of the Hilbert spaces H_i as the space of all functions α with domain I, such that α(i) is an element of H_i for every i in I and:

$$\sum_i \left\|\alpha_{(i)}\right\|^2 < \infty.$$

The inner product of two such function α and β is then defined as:

$$\langle \alpha, \beta \rangle = \sum_i \langle \alpha_i, \beta_i \rangle.$$

This space is complete and we get a Hilbert space.

For example, if we take the index set I = **N** and X_i = **R**, then the direct sum $\bigoplus_{i\in\mathbf{N}} X_i$ is the space l_2, which consists of all the sequences (a_i) of reals with finite norm $\|a\| = \sqrt{\sum_i \|a_i\|^2}$. Comparing this with the example for Banach spaces, we see that the Banach space direct sum and the Hilbert space direct sum are not necessarily the same. But if there are only finitely many summands, then the Banach space direct sum is isomorphic to the Hilbert space direct sum, although the norm will be different.

Every Hilbert space is isomorphic to a direct sum of sufficiently many copies of the base field (either **R** or **C**). This is equivalent to the assertion that every Hilbert space has an orthonormal basis. More generally, every closed subspace of a Hilbert space is complemented: it admits an orthogonal complement. Conversely, the Lindenstrauss–Tzafriri theorem asserts that if every closed subspace of a Banach space is complemented, then the Banach space is isomorphic (topologically) to a Hilbert space.

See also

- Biproduct
- Indecomposable module
- Jordan–Hölder theorem
- Krull–Schmidt theorem
- Split exact sequence

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