Stochastic Portfolio Optimization with Log Utility

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Abstract

A portfolio optimization problem on an infinite time horizon is considered. Risky asset price obeys a logarithmic Brownian motion, and the interest rate varies according to an ergodic Markov diffusion process. Moreover, the interest rate fluctuation is correlated with the risky asset price fluctuation. The goal is to choose optimal investment and consumption policies to maximize the infinite horizon expected discounted log utility of consumption. A dynamic programming principle is used to derive the dynamic programming equation (DPE). The explicit solutions for optimal consumption and investment control policies are obtained. In addition, for a special case, an explicit formula for the value function is given.

Keywords: Portfolio optimization, dynamic programming equations, subsolution and supersolutions.

1 Introduction

In the classical Merton portfolio optimization problem, there are two investment options, a riskless asset with constant interest rate and a risky asset whose price fluctuates from time to time. An investor dynamically allocates wealth between the risky and the riskless asset and chooses a consumption rate, with the goal of maximizing total expected discounted utility of consumption. For hyperbolic average risk aversion (HARA) type utility function, we can get a simple explicit solution. See for example Fleming and Soner [FlSo].

However, unlike the constant interest rate in the classical Merton model, the interest rate is not always fixed in our real life. For example, even for the money in the bank, the interest rate may fluctuate from time to time. In addition, the interest rate fluctuation can be strongly correlated with the price fluctuation of the risk asset. In the US market, what we can see is that the US Federal Reserve will usually adjust the prime interest rate according to the performance of the US stock markets. For example, the Federal Reserve increased the interest rate several times in late 90s when the NASDAQ market was increasing crazily. Later, due to the bad performance of the stock market after the high-tech stock bubble blew up, the Federal Reserve lowered the interest rate many times. The recent news says that the Federal Reserve has increased the interest rate because of the recovery of the stock market as well as the US economy. All above suggest that the interest rate can fluctuate from time to time, and it can be correlated with the risky asset price change.

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Therefore, in the present paper we allow the interest rate for the riskless asset to change with time. More particular, we assume that the "riskless" interest rate r_t is an ergodic Markov diffusion process. A typical example is the Vasicek model (see (5.1)), in which r_t fluctuates around a certain value at most of the time. Moreover, we will assume that the r_t process to describe the interest rate is correlated to the risky asset price process. See equation (2.4).

There have been many research works related to the classical Merton's portfolio optimization problem. Bielecki and Pliska [BiPl], Fleming and Sheu [FlSh1, FlSh2] have considered the cases in which there is no consumptions and the goal is to maximize the long term growth rate of the utility based on the wealth. There are also some models involving stochastic volatility instead of constant volatility, such as Avellaneda and Zhu [AZ], Fleming and Hernandez-Hernandez [FlHH], Zariphopoulou [Z], Fouque, Papanicolaou and Sircar [F-P-S], etc. In [F-P, P1, P2], some investment-consumption models when the interest rate is fluctuating and correlated with the performance of the stock market are considered. Although the model considered in this paper is similar to those considered in [P1, P2], the method used here is different. In addition, the utility function we used is also different.

Besides of maximizing the total expected consumption-based utilty on an infinite time horizon, portfolio management problems on finite time horizon have also been consider by many authors. The predictability of stock return is described by geometric Brownian motion and its generalization by assuming random or time-dependent draft and diffusion coefficients for stock price process. Kim and Omberg [KO] derives an exact solution by assuming that the investor cannot only consume over a finite time horizon and the utility is based on the terminal wealth only. Compbell and Vicerra [CV1] gives an approximate analytical solution for an infinitely-lived investor. For utility based on consumption, Wachter [W] derived an exact, closed form solution for portfolio weights over an finite time horizon. Recently, Detemple and Rindisbacher [DR] discussed the optimal portfolio selection problem with stochastic interest rate and investment constraints. The problem they considered is also defined on a finite time horizon and the utility function is the power utility function based on the terminal wealth. One thing needs to be mentioned is that for optimization problems on finite time horizon, martingale method and convex duality can be used (see [CH, CK, KLS, Pl]). However, for the infinite time horizon problem considered in this paper, it is convenient to use dynamic programming method, which can also be used for finite time horizon problems.

The stochastic control problem we considered here has state variables x_t, r_t, where x_t is the total wealth. The controls are the fraction u_t of wealth in the risky asset and $c_t = \frac{C_t}{x_t}$ where C_t is the consumption rate. The state dynamics are the stochastic differential equations (2.8), (2.2). The goal is to choose optimal control policies to maximize the objective function defined by (2.12). Since now r_t is also a state variable, the dynamic programming equation (DPE) for the value function is a second order nonlinear partial differential equations (see (2.14)).

In this paper, we consider the log utility case where $U(cx) = \log cx$, corresponding to the HARA utility function when $\gamma = 0$. Similar models with non-log HARA utility have been considered in [F-P, P1, P2], where the DPE for the value function turned out to be a nonlinear second order partial differential equation. Due to its nonlinearity, the equation is usually not able to be solved. In [F-P], a sub-super solution method is introduced to solve this kind of issues. For the log utility case, although we are able to obtain a linear modified DPE (2.16), we can still use the sub-super solution method conveniently. This will be done in Section 4. The explicit formula for optimal control policy is obtained. As mentioned, we consider the case when r_t is correlated to the change of the risky asset. From the results (see (4.14)), we can see that the correlation coefficient does not appear in the solutions under the log utility situation. However, this is no longer true if the utility function is not logarithmic. See [F-P, P1, P2].

The paper is organized as the follows. In Section 2, the problem will be formulated as a stochastic portfolio optimization problem. Some useful properties about the interest rate process is given in Section 3. In Section 4, dynamic programming principle and ODE/PDE method are used to get the classical solution of the DPE (2.14), and then verify that the solution is actually equal to the value function defined by (2.13). The explicit solutions for optimal control policies are also given. For the Vasicek Model, besides the explicit solution for optimal control policies, we can also obtain an explicit formula for the value function, which will be given in Section 5. The explicit solution will be very useful to test numerical methods that used to solve the DPEs. In Section 6, some possible extensions of the model are discussed.

2 Problem Formulation

We follow the set up for the classic Merton's model. Consider an investor who can allocate his or her wealth to a riskless asset (e.g. bond) and a risky asset (e.g. stock). We assume that the unit price P_t for the risky asset follows a geometric Brownian motion:

$$dP_t = P_t[bdt + \sigma_1 dw_{1,t}], \tag{2.1}$$

where b, σ_1 are positive constants and $w_{1,t}$ is a standard 1-dimensional Brownian motion. Unlike the constant interest rate in the classical Merton's model, we assume that the instantaneous interest rate r_t will change from time to time according to an Ito process:

$$dr_t = f(r_t)dt + \sigma_2 d\tilde{w}_t, \tag{2.2}$$

$$r_0 = r, \tag{2.3}$$

where σ_2 is a positive constants and $\tilde{w}_t$ is another standard Brownian motion which is correlated with $w_{1,t}$. In addition, $f(r)$ is a smooth function of r and it satisfies certain conditions (see (2.7)). More particularly, we can assume that $w_t = (w_{1,t}, w_{2,t})'$ is a standard 2-dimensional Brownian motion and $\tilde{w}_t$ is given by

$$d\tilde{w}_t = \rho dw_{1,t} + \sqrt{1-\rho^2}dw_{2,t}, \tag{2.4}$$

where $|\rho| \leq 1$ is a constant which is referred as the correlation coefficient. Since $w_{1,t}$ and $w_{2,t}$ are independent, we have

$$E[dw_{1,t} \cdot d\tilde{w}_t] = \rho dt.$$

When $|\rho| \neq 1$, the market will be incomplete.

The stock market risk is given by a stochastic process defined by

$$\theta_t = \frac{b - r_t}{\sigma_1}. \tag{2.5}$$

It is easy to verify that θ_t satisfies

$$d\theta_t = -\frac{1}{\sigma_1}f(b - \sigma_1\theta_t)dt - \frac{\sigma_2}{\sigma_1}d\tilde{w}_t. \tag{2.6}$$

If the market is incomplete, i.e., $|\rho| \neq 1$, there will be un-traded interest risk. In other words, the pure interest rate risk will be completely un-hedgeable. In this case, to make the market complete, we need to have a tradable financial instrument, such as a forward contract, a fix-income derivative or a mutual fund designed specifically to hedge the interest rate risk. This is beyond what we consider in this paper. However, as we can see later, for the model we considered here with log utility function, the value of ρ does not make any difference to our results. One reason is that for log utility function, the relative risk-averse coefficient is 1, which corresponding to a myopia investor. In this case, the incompleteness ceases to matter. Similar results are obtained for the finite time horizon case in [DR]. When the investor is not myopia, i.e., when the relative risk-averse coefficient is not 1, we will see that ρ does make a difference (see [F-P, P2]). In other words, logarithmic utility function separates the behavior of individual investors that are more or less risk-averse than a myopia investor.

A common used model for interest rate, the Vasicek model, is to take $f(r) = -c(r - r_1)$ in (2.2), where $c > 0$ and r_1 are constants. Here we will consider a generalized Vasicek model. More particularly, we only assume that $f(r)$ satisfy the following conditions:

$$f(r) \in \mathbf{C}^2(\mathbf{R}); \quad |f_{rr}(r)| \le K(1 + |r|^\alpha); \quad c_2 \le f_r(r) \le c_1; \tag{2.7}$$

where $K > 0, \alpha > 0, c_1$ and c_2 are constants. Later in Section 5, we will deal with the Vasicek model as a special case.

Remark 2.1 *There is one thing worth to mention about the conditions of $f(r)$. Unlike the Vasicek model, we do not require that f_r is negative. Actually, c_1 can be a positive number as large as $\frac{\beta}{2}$. See (4.7). In other words, even if the interest rate process is not mean-reverting, we can still get the same results.*

We assume $x_t > 0$ is the investor's total wealth at time t. In addition, assume that u_t is the fraction of the investor's wealth invested on the risky asset and C_t is the consumption rate at time t. For technical reasons, we use $c_t \equiv C_t/x_t$ instead of C_t as a control variable. Given above assumptions, we can get that the stochastic differential equations for x_t are:

$$dx_t = x_t[r_t + (b - r_t)u_t - c_t]dt + \sigma_1 u_t x_t dw_{1,t}, \tag{2.8}$$

$$x_0 = x, \tag{2.9}$$

where $x > 0$ is the initial wealth.

The stat variables are x_t, r_t which solve $(2.8-2.9)$ and $(2.2-2.3)$, respectively. Our control variables are u_t and c_t. Here we do not impose any constraints for u_t. A negative u_t stands for the dis-investment or short-sell. We also assume that there is no transaction costs and the stock (risky asset) can be traded any time.

We require that the control $(u_t, c_t) \in \mathbf{R}^2$ for any $t \ge 0$, and it is $\mathcal{F}_t$-progressively measurable for some w_t-adapted increasing family of σ-algebras $(\mathcal{F}_t, t \ge 0)$. See Fleming and Soner [FlSo] Chapter 4 for details. In some cases, (u_t, c_t) may be obtained from a local Lipschitz continuous control policy $(\underline{u}, \underline{c})$

$$u_t = \underline{u}(t, x_t, r_t), \quad c_t = \underline{c}(t, x_t, r_t).$$

We assume that $c_t \ge 0$ and there is an upper bound Λ for c_t, where $\Lambda > 0$ is large enough to guarantee the feasibility of the optimal consumption control. In addition, we assume there is no constraints for the value of u_t. We also require that

$$E \int_0^T u_t^2 dt < \infty, \quad \forall T > 0. \tag{2.10}$$

In this paper, our admissible control space Π is defined by

$$\Pi = \left\{ u_t, c_t : \lim_{T \to \infty} e^{-\beta T} E \int_0^T u_t^2 dt = 0; \quad 0 \le c_t \le \Lambda < \infty \right\}, \tag{2.11}$$

where $\Lambda > 0$ is a constant.

Log utility is considered in this paper. The goal is to maximize the objective function

$$J(x, r, u., c.) \equiv E_{x,r} \int_0^\infty e^{-\beta t} \log(c_t x_t) dt. \tag{2.12}$$

Thus, the value function is

$$V(x, r) = \max_{u.,c.} E_{x,r} \int_0^\infty e^{-\beta t} \log(c_t x_t) dt. \tag{2.13}$$

Using the dynamic programming principle, we can get the DPE for $V(x,r)$

$$\begin{aligned}\beta V &= \max_u \left[(b-r)uxV_x + \frac{1}{2}\sigma_1^2u^2x^2V_{xx} + \rho\sigma_1\sigma_2uxV_{xr}\right] + rxV_x \\ &\quad + f(r)V_r + \frac{1}{2}\sigma_2^2V_{rr} + \max_{c\geq 0}\left[-cxV_x + \log(cx)\right]. \qquad (2.14)\end{aligned}$$

Look for the solution of the form:

$$V(x,r) = A\log x + W(r).$$

By substitution, we have

$$A = 1/\beta.$$

Thus,

$$V(x,r) = \frac{1}{\beta}\log x + W(r). \qquad (2.15)$$

By virtue of (2.14) $-$ $(valuef)$, we can derive the equation for $W(r)$ which is

$$\begin{aligned}\beta W &= \frac{1}{\beta}\max_u\left[(b-r)u - \frac{1}{2}\sigma_1^2u^2\right] + \frac{r}{\beta} + f(r)W_r + \frac{1}{2}\sigma_2^2W_{rr} \\ &\quad + \max_{c\geq 0}\left[-\frac{c}{\beta} + \log c\right]. \qquad (2.16)\end{aligned}$$

The potential optimal control policy is

$$u^*(x,r) = \frac{b-r}{\sigma_1^2}, \quad c^*(x,r) = \beta. \qquad (2.17)$$

Given above equalities, we can rewrite the ODE for $W(r)$ as

$$\frac{1}{2}\sigma_2^2W_{rr} + f(r)W_r - \beta W + \bar{Q}(r) = 0, \qquad (2.18)$$

where

$$\bar{Q}(r) = \frac{1}{\beta}\left[\frac{(b-r)^2}{2\sigma_1^2} + r\right] + \log\beta - 1, \qquad (2.19)$$

which is quadratic and lower-bounded. Actually,

$$\bar{Q}(r) \geq \frac{1}{\beta}\left[b - \frac{1}{2}\sigma_1^2\right] + \log\beta - 1. \qquad (2.20)$$

The next step is to derive a classical solution $W(r)$ of the equation (2.18) and verify that the function given by (2.15) is actually our value function defined by (2.13). In addition, the control policy given by (2.17) will be verified to be our optimal control policy. This will be done in Section 4. Before we turn to that part, let us state some properties of the solution r_t of equations (2.2) $-$ (2.3) in the next section.

3 Properties of Interest Rate Process

The following lemmas will be used in the sections followed:

Lemma 3.1 *Assume that $f(r)$ satisfies (2.7). Then (2.2) $-$ (2.3) possesses a unique strong solution r_t. In addition, for any $\epsilon > 0$ and any integer $m > 0$, we have*

$$\begin{aligned}E|r_t|^{2m} &\leq \Lambda_1, & if\ c_1 \leq 0; \qquad (3.1)\\ E|r_t|^{2m} &\leq \Lambda_2 e^{2m(c_1+\epsilon)t}, & if\ c_1 > 0; \qquad (3.2)\end{aligned}$$

where Λ, Λ_1 and Λ_2 are positive constants which are independent of t.

Proof. In this proof, unless specified, Λ_1, Λ_2 stand for general positive constants which are independent of t unless specified. First, under the condition (2.7), it is not hard to show that $(2.2) - (2.3)$ has a unique, strong solution r_t. In addition, we have

$$Er_t^{2m} < \Lambda_1 e^{\Lambda_2 t}, \quad \forall t \in [0, T], \tag{3.3}$$

where Λ_1, Λ_2 are constants which only depend on m, c_1, c_2 and T. For details, please refer to Friedman [Fr] Theorem 1.1 and Theorem 2.3 in Chapter 5.

On the other hand, via Ito's rule, we have

$$\begin{aligned} dr_t^{2m} &= 2mr_t^{2m-1}dr_t + m(2m-1)\sigma_2^2 r_t^{2m-2}dt \\ &= [2mr_t^{2m-1}f(r_t) + m(2m-1)\sigma_2^2 r_t^{2m-2}]dt + 2m\sigma_2 r_t^{2m-1}d\tilde{w}_t. \end{aligned}$$

Thus, we can get

$$r_t^{2m} = r^{2m} + \int_0^t [2mr_s^{2m-1}f(r_s) + m(2m-1)\sigma_2^2 r_s^{2m-2}]ds + m_t$$

where

$$m_t = \int_0^t 2m\sigma_2 r_s^{2m-1} d\tilde{w}_s.$$

By virtue of (3.3), it is not hard to verify that m_t is a martingale. Then we can get

$$Er_t^{2m} = r^{2m} + \int_0^t E[2mr_s^{2m-1}f(r_s) + m(2m-1)\sigma_2^2 r_s^{2m-2}]ds.$$

Define

$$\mu_m(t) \equiv Er_t^{2m}.$$

Then $\mu_m(t) \in \mathbf{C}^1([0, \infty))$. Using (2.7), we can get

$$r_s^{2m-1}f(r_s) \le c_1 r_s^{2m} + f(0)r_s^{2m-1}.$$

Therefore, using the Cauchy-Schwarz Inequality, if $c_1 < 0$, we have

$$\begin{aligned} \frac{d\mu_m(t)}{dt} &= E[2mr_t^{2m-1}f(r_t) + m(2m-1)\sigma_2^2 r_t^{2m-2}] \\ &\le E[2mc_1 r_t^{2m} + 2mf(0)r_t^{2m-1} + m(2m-1)\sigma_2^2 r_t^{2m-2}] \\ &\le E[mc_1 r_t^{2m} + \Lambda_2] \\ &= mc_1\mu_m(t) + \Lambda_2. \end{aligned}$$

Then it is not hard to get that

$$\frac{de^{-mc_1 t}\mu_m(t)}{dt} \le e^{-mc_1 t}\Lambda_2.$$

Integrate it and we can get

$$\begin{aligned} \mu_m(t) &\le e^{mc_1 t} r^{2m} + \frac{\Lambda_2}{mc_1}[e^{mc_1 t} - 1] \\ &\le r^{2m} \\ &\le \Lambda_1. \end{aligned}$$

By the definition of $\mu_m(t)$, (3.1) holds.

On the other hand, if $c_1 > 0$, following the same procedure, we have

$$\begin{aligned}
\frac{d\mu_m(t)}{dt} &= E[2mr_t^{2m-1}f(r_t) + m(2m-1)\sigma_2^2 r_t^{2m-2}] \\
&\leq E[2mc_1 r_t^{2m} + 2mf(0)r_t^{2m-1} + m(2m-1)\sigma_2^2 r_t^{2m-2}] \\
&\leq E[2m(c_1+\epsilon)r_t^{2m} + \Lambda_2] \\
&= 2m(c_1+\epsilon)\mu_m(t) + \Lambda_2.
\end{aligned}$$

Then it is not hard to get that

$$\frac{de^{-2m(c_1+\epsilon)t}\mu_m(t)}{dt} \leq e^{-2m(c_1+\epsilon)t}\Lambda_2.$$

Integrate it and we can get

$$\begin{aligned}
\mu_m(t) &\leq e^{2m(c_1+\epsilon)t}r^{2m} + \frac{\Lambda_2}{2m(c_1+\epsilon)}[e^{2m(c_1+\epsilon)t} - 1] \\
&\leq \Lambda_2 e^{2m(c_1+\epsilon)t}.
\end{aligned}$$

That is, (3.2) holds. **Q.E.D.**

From the above results, it is easy to get the following lemma:

Lemma 3.2 *Suppose that* (2.7) *holds and* $\{r_t\}_{t\geq 0}$ *is a solution of* (2.2) − (2.3). *For any* $\beta > 0$ *and any integer* $m > 0$, *if*

$$\beta - 2mc_1 > 0, \tag{3.4}$$

then we have

$$\lim_{T\to\infty} e^{-\beta T} E r_T^{2m} = 0, \tag{3.5}$$

$$\lim_{T\to\infty} e^{-\beta T} E \int_0^T r_t^{2m} dt = 0. \tag{3.6}$$

4 Obtain the Value Function

In this section, we will use the subsolution-supersolution method to get a classical solution $\tilde{W}(r)$ of (2.16). Then we will verify that

$$\tilde{V}(x,r) \equiv \frac{1}{\beta}\log x + \tilde{W}(r)$$

is equal to the value function $V(x,r)$ defined by (2.13).

Define

$$\begin{aligned}
LW &= \frac{1}{2}\sigma_2^2 W_{rr} + f(r)W_r, \qquad (4.1) \\
h(r,W) &= \bar{Q}(r) - \beta W \qquad (4.2)
\end{aligned}$$

where $\bar{Q}$ is given by (2.19). Obviously, L is uniformly elliptic. Given above notations, the equation of $W(r)$ (2.18) can be written as

$$-LW = h(r,W). \tag{4.3}$$

Now let us define subsolutions and supersolutions.

Definition 4.1 *A function $\underline{W}(r)$ is said to be a subsolution of* (4.3) *if*

$$-L\underline{W} \le h(r, \underline{W}), \quad \forall r \in \mathbf{R}. \tag{4.4}$$

On the other hand, a function $\bar{W}(r)$ is a supersolution of (4.3) *if*

$$-L\bar{W} \ge h(r, \bar{W}), \quad \forall r \in \mathbf{R}. \tag{4.5}$$

In addition, if $\underline{W}, \bar{W}$ also satisfy

$$\underline{W}(r) \le \bar{W}(r), \quad \forall r \in \mathbf{R},$$

then, we call $\underline{W}$ and $\bar{W}$ an ordered pair of subsolution/supersolution.

If there are an ordered pair of subsolution/supersolution of (4.3), under certain conditions, we can prove that (4.3) has a classical solution. The result will later be given in Theorem 4.1. But first we need to find an ordered pair of subsolution/supersolution.

Lemma 4.1 *Define*

$$K_1 \equiv \frac{1}{\beta^2}\left[b - \frac{\sigma_1^2}{2} + \beta \log \beta - \beta\right]. \tag{4.6}$$

Then, any constant $K_2 < K_1$ is a subsolution of (4.3).

Proof. Since K_1 is a constant, we have

$$-LK_1 = 0.$$

On the other hand, since $\bar{Q}(r)$ is quadratic, by the definition of h and K_1, it is not hard to verify that

$$h(r, K_1) \ge 0.$$

Thus, we can get

$$-LK_1 \le h(r, K_1).$$

By definition, K_1 is a subsolution of (4.3). Actually, it is not easy to verify that for any $K_2 < K_1$,

$$h(r, K_2) > 0.$$

Therefore, K_2 is also a subsolution of (4.3). **Q.E.D.**

In the next lemma, we will show that (4.3) possesses a supersolution which is a quadratic function of r. In addition, the supersolution we find here will lie above the subsolution we obtained above.

Lemma 4.2 *Suppose*

$$\beta - 2c_1 > 0. \tag{4.7}$$

Then there exist constants $a_1 > 0, a_2 > K_1$ such that

$$\bar{W}(r) \equiv a_1 r^2 + a_2 \tag{4.8}$$

is a supersolution of (4.3).

Proof. For $\bar{W}(r)$ defined by (4.8), by virtue of (2.7) and the mean-value theorem, we can get

$$\begin{aligned} -L\bar{W} &= -a_1\sigma_2^2 - 2a_1 f(r) r \\ &= -a_1\sigma_2^2 - 2a_1 r\left[f_r(\xi) r + f(0)\right] \\ &\ge -a_1\sigma_2^2 - 2c_1 a_1 r^2 - 2a_1 f(0) r, \end{aligned}$$

where $\xi \in [0, r]$. On the other hand, we have

$$\begin{aligned} h(r, \bar{W}) &= \bar{Q}(r) - \beta \bar{W}(r) \\ &= \left[\frac{1}{2\beta\sigma_1^2} - \beta a_1\right] r^2 + \frac{1}{\beta}\left[1 - \frac{b}{\sigma_1^2}\right] r + \log\beta - 1 - \beta a_2. \end{aligned}$$

Therefore, first we can take $a_1 > 0$ such that

$$-2c_1 a_1 + \beta a_1 - \frac{1}{2\beta\sigma_1^2} > 0, \tag{4.9}$$

which is doable if $\beta - 2c_1 > 0$. Then we can take $a_2 > K_1$ large enough such that

$$-L\bar{W} \geq h(r, \bar{W}).$$

That is, $\bar{W}(r)$ is a supersolution of (4.3). **Q.E.D.**

Noting that $a_2 > K_1$, we can get that $\bar{W}(r) > K_1$ for all $r \in \mathbf{R}$. Therefore, $\langle K_1, \bar{W}(r)\rangle$ is an ordered pair of subsolution and supersolution. Now we can establish the existence results:

Theorem 4.1 *The equation* (4.3) *has a classical solution* $\tilde{W}(r)$ *such that*

$$K_1 \leq \tilde{W}(r) \leq \bar{W}(r), \tag{4.10}$$

where K_1 *and* $\bar{W}(r)$ *are defined by* (4.6) *and* (4.8), *respectively.*

Proof. According to Lemma 4.1 and 4.2, K_1 and $\bar{W}(r)$ are ordered subsolution and supersolution of (4.3). Now by virtue of Pao [Pa] Theorem 7.5.2(page 322), we can get the result. **Q.E.D.**

The following lemma is needed for the Verification Theorem which will be given later.

Lemma 4.3 *If* $\tilde{W}(r)$ *is a classical solution of* (4.3) *and it satisfies* (4.10), *then we have*

$$E\left[\tilde{W}_r^2(r_t)\right] < \Lambda(T), \quad \forall t \in [0, T], \tag{4.11}$$

where $\Lambda(T)$ *is a constant which does not depend on* t.

Proof. Since $\tilde{W}(r)$ is a classical solution of (4.3), we must have

$$\frac{\sigma_2^2}{2}\tilde{W}_{rr} + f(r)\tilde{W}_r - \beta\tilde{W} + \bar{Q}(r) = 0.$$

For any fixed $R > 0$, integrate it over $[0, R]$, then we can get

$$\tilde{W}_r(R) = \tilde{W}_r(0) + \frac{2}{\sigma_2^2}\left[\beta\int_0^R \tilde{W}(r)dr - \int_0^R \bar{Q}(r)dr - \int_0^R f(r)\tilde{W}_r(r)dr\right].$$

Integrating by parts, we have

$$\int_0^R f(r)\tilde{W}_r(r)dr = f(R)\tilde{W}(R) - f(0)\tilde{W}(0) - \int_0^R f_r(r)\tilde{W}(r)dr.$$

Using the above equality and by virtue of (2.7), (4.10) and the Cauchy-Schwarz Inequality, we can get

$$\left|\int_0^R f(r)\tilde{W}_r(r)dr\right| \leq p_1(R),$$

where p_1 is a polynomial of R with an order m such that

$$m > 4.$$

In the following part, we use p_1 to stand for a general polynomial of R of order m. Similarly, by (4.10), (2.19) and the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \left| \int_0^R \tilde{W}(r)dr \right| &< p_1(R), \\ \left| \int_0^R \bar{Q}(r)dr \right| &< p_1(R). \end{aligned}$$

Then, using the Jensen's Inequality, we can get

$$\begin{aligned} \tilde{W}_r^2(R) &\leq 2\tilde{W}_r(0)^2 + \frac{8}{\sigma_2^4}\left[3\beta^2 \left| \int_0^R \tilde{W}(r)dr \right|^2 \right. \\ &\quad \left. + 3\left| \int_0^R \bar{Q}(r)dr \right|^2 + 3\left| \int_0^R f(r)\tilde{W}_r(r)dr \right|^2 \right] \\ &\leq p_2(R), \end{aligned}$$

where p_2 is a polynomial of R with an order of $2m$. Given this, we can easily prove (4.11) by virtue of Lemma 3.2. **Q.E.D.**

Next we need a verification theorem.

Theorem 4.2 (Verification Theorem) *Suppose $\beta - 2c_1 > 0$ and $\tilde{W}(r)$ is a classical solution of* (4.3) *such that* (4.10) *holds. Define*

$$\tilde{V}(x, r) \equiv \frac{1}{\beta} \log x + \tilde{W}(r). \tag{4.12}$$

Then we have
(a) For every admissible control process $(u_t, c_t) \in \Pi$,

$$\tilde{V}(x, r) \geq E_{x,r} \int_0^\infty e^{-\beta t} \log(c_t x_t)dt. \tag{4.13}$$

(b)If

$$u^*(x_t, r_t) = \frac{b - r_t}{\sigma_1^2}, \quad c^*(x_t, r_t) = \beta, \tag{4.14}$$

then $(u^*, c^*) \in \Pi$ and we can get

$$\tilde{V}(x, r) = E_{x,r} \int_0^\infty e^{-\beta t} \log(c_t^* x_t^*)dt. \tag{4.15}$$

That is, $\tilde{V}(x, r) \equiv V(x, r)$, where $V(x, r)$ is defined by (2.13).

Proof. For any control $(u_t, c_t) \in \Pi$, by Ito's rule, we can get

$$d[e^{-\beta t}\tilde{V}(x_t, r_t)] = e^{-\beta t} d\tilde{V}(x_t, r_t) - \beta e^{-\beta t}\tilde{V}(x_t, r_t)dt. \tag{4.16}$$

Noting (2.14), we have

$$\begin{aligned}
d\tilde{V}(x_t,r_t) &= \tilde{V}_x dx_t + \tilde{V}_r dr_t + \frac{1}{2}\tilde{V}_{xx}(dx_t)^2 + \frac{1}{2}\tilde{V}_{rr}(dr_t)^2 + \tilde{V}_{xr}dx_t\ dr_t \\
&= \Big[(b-r_t)u_t x_t \tilde{V}_x + \frac{1}{2}\sigma_1^2 u_t^2 x_t^2 \tilde{V}_{xx} + \rho\sigma_1\sigma_2 u_t x_t \tilde{V}_{xr} + r_t x_t \tilde{V}_x \\
&\quad + f(r_t)\tilde{V}_r + \frac{1}{2}\sigma_2^2 \tilde{V}_{rr} - c_t x_t \tilde{V}_x\Big] dt + \sigma_1 u_t x_t \tilde{V}_x dw_{1,t} + \sigma_2 \tilde{V}_r d\tilde{w}_t \\
&\le \beta\tilde{V} - \log(c_t x_t) + \sigma_1 u_t x_t \tilde{V}_x dw_{1,t} + \sigma_2 \tilde{V}_r d\tilde{w}_t.
\end{aligned} \tag{4.17}$$

For $\tilde{V}(x,r)$ of the form (4.12) and for $(u_t,c_t)\in\Pi$, by virtue of (2.10), (4.11), it is easy to show that

$$\int_0^T \sigma_1 e^{-\beta t}\tilde{V}_x u_t x_t dw_{1,t} \tag{4.18}$$

and

$$\int_0^T \sigma_2 e^{-\beta t}\tilde{V}_r d\tilde{w}_t \tag{4.19}$$

are martingales. Therefore, by virtue of (4.17), we have

$$E\int_0^T e^{-\beta t} d\tilde{V}(x_t,r_t) - E\int_0^T \beta e^{-\beta t}\tilde{V}(x_t,r_t)dt \le -E\int_0^T e^{-\beta t}\log(c_t x_t)dt.$$

Combined with (4.16), it implies

$$Ee^{-\beta T}\tilde{V}(x_T,r_T) - \tilde{V}(x,r) \le -E\int_0^T e^{-\beta t}\log(c_t x_t)dt. \tag{4.20}$$

Thus, we get

$$\tilde{V}(x,r) \ge E\int_0^T e^{-\beta t}\log(c_t x_t)dt + Ee^{-\beta T}\tilde{V}(x_T,r_T). \tag{4.21}$$

Next, we will show that

$$\limsup_{T\to\infty} e^{-\beta T} E\tilde{V}(x_T,r_T) \ge 0. \tag{4.22}$$

By (4.10), we can get

$$\limsup_{T\to\infty} e^{-\beta T} E\tilde{W}(r_T) \ge 0. \tag{4.23}$$

Now it is sufficient to show

$$\limsup_{T\to\infty} e^{-\beta T} E[\log x_T] \ge 0. \tag{4.24}$$

According to (2.8) and Ito's rule, we have

$$\begin{aligned}
d\log x_t &= \frac{dx_t}{x_t} - \frac{1}{2x_t^2}\sigma_1^2 u_t^2 x_t^2 dt \\
&= \left[-\frac{1}{2}\sigma_1^2 u_t^2 + (b-r_t)u_t + r_t - c_t\right] dt + \sigma_1 u_t dw_{1,t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[\log x_T] &= E\left[\int_0^T d\log x_t\right] + \log x \\
&= E\int_0^T \left[r_t + (b-r_t)u_t - \frac{1}{2}\sigma_1^2 u_t^2\right] dt - E\int_0^T c_t dt + \log x.
\end{aligned} \tag{4.25}$$

Since $\beta - 2c_1 > 0$, by Lemma 3.2, we can get

$$\lim_{T\to\infty} e^{-\beta T} E r_T^2 = 0, \tag{4.26}$$

$$\lim_{T\to\infty} e^{-\beta T} E \int_0^T r_t^2 dt = 0. \tag{4.27}$$

Then, for $(u_t, c_t) \in \Pi$, we have

$$\begin{aligned}
\limsup_{T\to\infty} e^{-\beta T} E \int_0^T r_t dt &\geq \liminf_{T\to\infty} e^{-\beta T} \left(-E \int_0^T |r_t| dt \right) \\
&\geq -\limsup_{T\to\infty} \frac{1}{2} e^{-\beta T} \left(T + E \int_0^T r_t^2 dt \right) \\
&= 0; \\
\limsup_{T\to\infty} e^{-\beta T} E \int_0^T (b - r_t) u_t dt &\geq \limsup_{T\to\infty} e^{-\beta T} \left(E \int_0^T b u_t dt - E \int_0^T r_t u_t dt \right) \\
&\geq -\lim_{T\to\infty} \frac{1}{2} e^{-\beta T} \left(b^2 T + E \int_0^T u_t^2 dt \right) \\
&\quad - \lim_{T\to\infty} \frac{1}{2} e^{-\beta T} E \left[\int_0^T r_t^2 dt + \int_0^T u_t^2 dt \right] \\
&= 0; \\
\limsup_{T\to\infty} e^{-\beta T} E \int_0^T \left(-\frac{1}{2} \sigma_1^2 u_t^2 \right) dt &\geq -\lim_{T\to\infty} \frac{1}{2} e^{-\beta T} \sigma_1^2 E \int_0^T u_t^2 dt \\
&= 0; \\
\limsup_{T\to\infty} e^{-\beta T} \left(-E \int_0^T c_t dt \right) &\geq -\lim_{T\to\infty} e^{-\beta T} \Lambda T \\
&= 0.
\end{aligned}$$

Noting (4.25), we can get

$$\limsup_{T\to\infty} e^{-\beta T} E \log x_T \geq 0. \tag{4.28}$$

Thus, we have proved that

$$\limsup_{T\to\infty} e^{-\beta T} \tilde{V}(x_T, r_T) \geq 0, \quad \forall (u_t, c_t) \in \Pi, \tag{4.29}$$

which implies (a), combined with (4.21).

For control policy

$$u^*(x_t, r_t) = \frac{b - r_t^*}{\sigma_1^2}, \quad c^*(x_t, r_t) = \beta, \tag{4.30}$$

it is easy to check that $(u^*, c^*) \in \Pi$ and

$$u^* \in \arg\max_u \left[(b - r) u x \tilde{V}_x + \frac{1}{2} \sigma_1^2 u^2 x^2 \tilde{V}_{xx} + \rho \sigma_1 \sigma_2 u \tilde{V}_{xr} \right], \tag{4.31}$$

$$c^* \in \arg\max_{c\geq 0} \left[-c x \tilde{V}_x + \log(cx) \right]. \tag{4.32}$$

Similar to (4.21), now we can get

$$\tilde{V}(x, r) = E \int_0^T e^{-\beta t} \log(c_t^* x_t^*) dt + E e^{-\beta T} \tilde{V}(x_T^*, r_T^*). \tag{4.33}$$

We need to show that

$$\liminf_{T\to\infty} e^{-\beta T} E\tilde{V}(x_T^*, r_T^*) \le 0. \tag{4.34}$$

Since

$$\tilde{V}(x_T^*, r_T^*) = \frac{1}{\beta}\log x_T^* + \tilde{W}(r_T^*), \tag{4.35}$$

it is sufficient to show that

$$\liminf_{T\to\infty} e^{-\beta T} E\log x_T^* \le 0, \tag{4.36}$$

and

$$\liminf_{T\to\infty} e^{-\beta T} E\tilde{W}(r_T^*) \le 0. \tag{4.37}$$

Similarly, by (4.25), we have

$$\begin{aligned} E\log x_T^* &= E\int_0^T d\log x_t^* + \log x \\ &= E\int_0^T \left[r_t^* + (b-r_t^*)u_t^* - \frac{1}{2}\sigma_1^2 u_t^{*2}\right]dt - E\int_0^T c_t^* dt + \log x \\ &= \frac{1}{2\sigma_1^2}E\int_0^T \left[(r_t^*)^2 + 2(\sigma_1^2 - b)r_t^* + b^2\right]dt - \beta T + \log x \\ &\le \frac{1}{2\sigma_1^2}E\int_0^T \left[2(r_t^*)^2 + (\sigma_1^2 - b)^2 + b^2\right]dt - \beta T + \log x. \end{aligned} \tag{4.38}$$

By virtue of Lemma 3.2, it is easy to know that (4.36) holds.

On the other hand, by virtue of (4.10) and Lemma 3.2, it is not hard to show that (4.37) holds. Therefore, we have proved that

$$\tilde{V}(x,r) = E_{x,r}\int_0^\infty e^{-\beta t}\log(c_t^* x_t^*)dt, \tag{4.39}$$

which is (b). **Q.E.D.**

Remark 4.2 *From above theorem, we can see that ρ plays no role in the final results. However, it is not the case for non-log HARA utility functions. Please refer to [F-P, P2] for details.*

5 The Vasicek Model: Explicit Solution

In this section, as an example, we consider the Vasicek Model:

$$dr_t = -c_1(r_t - \bar{r})dt + \sigma_2 d\tilde{w}_t, \tag{5.1}$$

where $c_1 > 0$ and $\bar{r} > 0$ are constants. That is, in (2.2), we take

$$f(r) = -c_1(r - \bar{r}). \tag{5.2}$$

Let $\theta_t = \frac{b-r_t}{\sigma_1}$. As we mentioned earlier, the process θ_t determines the price of risk in the economy, or the reward, in term of expected return, of taking on a unit of risk. It is easy to verify that θ_t satisfies an OU process:

$$d\theta_t = -c_1\left(\theta_t - \frac{b-\bar{r}}{\sigma_1}\right)dt - \frac{\sigma_2}{\sigma_1}d\tilde{w}_t. \tag{5.3}$$

In fact, we can get an explicit solution in this case.

Now the DPE for the value function $V(x,r)$ is

$$\begin{aligned}\beta V &= \max_u \left[(b-r)uxV_x + \frac{1}{2}\sigma_1^2 u^2 x^2 V_{xx} + \rho\sigma_1\sigma_2 uxV_{xr}\right] + rxV_x \\ &\quad - c_1(r-\bar{r})V_r + \frac{1}{2}\sigma_2^2 V_{rr} + \max_{c\geq 0}\left[-cxV_x + \log(cx)\right]. \end{aligned} \tag{5.4}$$

The optimal control policy (will be verified later) is

$$u^*(x_t, r_t) = \frac{b-r_t}{\sigma_1^2}, \quad c^*(x_t, r_t) = \beta. \tag{5.5}$$

As we can see here, the optimal investment control u_t is proportional to the market risk process Z_t given by (5.3)–the higher the market price of the risk, the more should be invested onto the risky asset.

Given the potential optimal control policies u^*, c^* given by (5.5), we can look for the solution of the form:

$$V(x,r) = A\log x + W(r). \tag{5.6}$$

By substitution, we have

$$A = 1/\beta.$$

In addition, the ODE for $W(r)$ is

$$\frac{1}{2}\sigma_2^2 W_{rr} - c_1(r-\bar{r})W_r - \beta W + \bar{Q}(r) = 0, \tag{5.7}$$

where

$$\bar{Q}(r) = \frac{1}{\beta}\left[\frac{(b-r)^2}{2\sigma_1^2} + r\right] + \log\beta - 1, \tag{5.8}$$

which is quadratic. Thus, we can consider a solution of the form

$$W(r) = a_1 r^2 + a_2 r + a_3. \tag{5.9}$$

Then, by setting the coefficients of $1, r, r^2$ equal to 0, we can get the equations for a_1, a_2 and a_3 as the following

$$-(\beta+2c_1)a_1 + \frac{1}{2\beta\sigma_1^2} = 0, \tag{5.10}$$

$$2c_1\bar{r}a_1 - (\beta+c_1)a_2 + \frac{1}{\beta}\left[1 - \frac{b}{\sigma_1^2}\right] = 0, \tag{5.11}$$

$$\sigma_2^2 a_1 + c_1\bar{r}a_2 - \beta a_3 + \frac{b^2}{2\beta\sigma_1^2} + \log\beta - 1 = 0. \tag{5.12}$$

Therefore, we can get

$$a_1 = \frac{1}{2\beta\sigma_1^2(\beta+2c_1)}, \tag{5.13}$$

$$a_2 = \frac{1}{\beta(\beta+c_1)}\left[\frac{c_1\bar{r} + (\beta+2c_1)(\sigma_1^2 - b)}{\sigma_1^2(\beta+2c_1)}\right], \tag{5.14}$$

$$\begin{aligned} a_3 &= \frac{1}{\beta}\left[\frac{\sigma_2^2}{2\sigma_1^2\beta(\beta+2c_1)} + \frac{c_1\bar{r}}{\beta(\beta+c_1)}\left[\frac{c_1\bar{r} + (\beta+2c_1)(\sigma_1^2-b)}{\sigma_1^2(\beta+2c_1)}\right]\right. \\ &\quad \left. + \frac{b^2}{2\sigma_1^2\beta} + \log\beta - 1\right]. \end{aligned} \tag{5.15}$$

Then,

$$V(x,r) = \frac{1}{\beta}\log x + a_1 r^2 + a_2 r + a_3. \tag{5.16}$$

By virtue of Theorem 4.2 in Section 4, it is easy to know that $u^*(x,r)$ and $c^*(x,r)$ are optimal control policies and $V(x,r)$ is the value function.

6 Extension and Future Works

In this paper, we considered the generalized Merton's problem with log utility on an infinite time horizon. The market of only one risky asset and one riskless asset is considered. In the log utility case, it turns out that whether the market is complete does not make any difference here. However, it is no longer true when the utility function is non-log utility function.

The interest rate model we consider here is given by a stochastic process driven a single factor $\tilde{w}_t$. Actually, we can also consider the situation when the interest rate model is driven by many Markovian stochastic factors:

$$dr_t = f(r_t)dt + \sum_{i=1}^{N} \tilde{\sigma}_i d\tilde{w}_{it}. \tag{6.1}$$

For log utility case as we consider here, the results will be the same and there will be no essential technical difficulties.

Another interest rate model we may consider is the CIR model given by

$$dr_t = -c(r_t - \bar{r})dt + \sigma_2\sqrt{r_t}d\tilde{w}_t. \tag{6.2}$$

or even a generalized CIR model:

$$dr_t = f(r_t)dt + g(\sqrt{r_t})d\tilde{w}_t. \tag{6.3}$$

In this model, the modified dynamic programming equation for $W(r)$ (see (2.16)) will no longer be uniformly elliptic(the coefficient of W_{rr} will be a function of r), and it can be degenerated. The expected results can not be obtained easily from the sub-super solution method. More works need to be done to obtain the similar results for CIR model.

In addition, we can also consider the market with multiple risky assets. In other words, we may consider a similar problem while P_t in (2.1) is multi-dimensional. Although similar results are expected for this case, some technical difficulties need be overcome. For example, we will no longer have a simple explicit formula or the lower bound for $\bar{Q}$ similar to $(2.19)-(2.20)$ and that will cause some trouble to get the explicit subsolution and supersolution for our modified dynamic programming equation (4.3).

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