Optimization Problems with Stochastic Dominance Constraints

Darinka Dentcheva

Stevens Institute of Technology, Hoboken, New Jersey, USA

Research supported by NSF awards DMS-0604060 and CMII-0965702

Borovec, Bulgaria, September 12-16, 2011

Motivation

Risk-Averse Optimization Models

Choose a decision $z \in Z$, which results in a random outcome $G(z) \in \mathscr{L}_p(\Omega, \mathscr{F}, P)$ with "good" characteristics; special attention to low probability-high impact events.

- Utility models apply a nonlinear transformation to the realizations of $G(z)$ (expected utility) or to the probability of events (rank dependent utility/distortion). Expected utility models optimize $\mathbb{E}[u(G(z))]$
- Probabilistic / chance constraints impose prescribed probability on some events: $P[G(z) \geq \eta]$
- Mean–risk models optimize a composite objective of the expected performance and a scalar measure of undesirable realizations $\mathbb{E}[G(z)] - \varrho[G(z)]$ (risk/ deviation measures)
- Stochastic ordering constraints compare random outcomes using stochastic orders and random benchmarks

Outline

Stochastic Orders

Integral Univariate Stochastic Orders

For $X, Y \in \mathscr{L}_1(\Omega, \mathscr{F}, P)$

$$X \succeq_{\mathfrak{F}} Y \Leftrightarrow \int_{\Omega} u(X(\omega))\, P(d\omega) \geq \int_{\Omega} u(Y(\omega))\, P(d\omega) \quad \forall\, u(\cdot) \in \mathfrak{F}$$

Collection of functions $\mathfrak{F}$ is the generator of the order.

Stochastic Dominance Generators

$\mathfrak{F}_1 = \{$nondecreasing functions $u : \mathbb{R} \to \mathbb{R}\}$ generates the usual stochastic order or first order stochastic dominance ($X \succeq_{(1)} Y$)

Introduced in statistics in 1947 by Mann and Whitney

Blackwell (1953), Lehmann (1955)

$\mathfrak{F}_2 = \{$nondecreasing concave $u : \mathbb{R} \to \mathbb{R}\}$ generates the second order stochastic dominance relation ($X \succeq_{(2)} Y$)

Risk-consistency facilitates applications in economics pioneered by Quirk and Saposnik (1962), Fishburn (1964), Hadar and Russell (1969)

Stochastic Dominance Relation via Distribution Functions

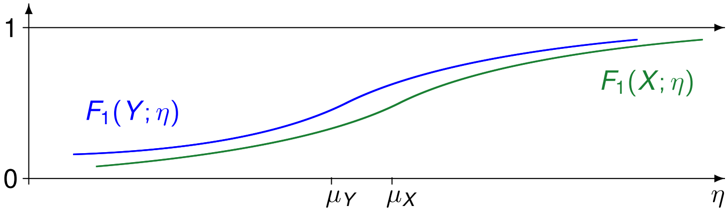
Distribution Functions

$$
\begin{aligned}
F_1(X;\eta) &= \int_{-\infty}^{\eta} P_X(dt) = P\{X \le \eta\} \ \text{ for all } \eta \in \mathbb{R} \\
F_k(X;\eta) &= \int_{-\infty}^{\eta} F_{k-1}(X;t)\,dt \ \text{ for all } \eta \in \mathbb{R}, \quad k = 2, 3, \ldots
\end{aligned}
$$

*k*th order Stochastic Dominance

$$
X \succeq_{(k)} Y \quad \Leftrightarrow \quad F_k(X,\eta) \le F_k(Y,\eta) \quad \text{for all } \eta \in \mathbb{R}
$$

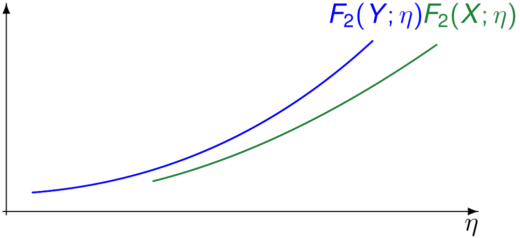
First Order Stochastic Dominance
Usual Stochastic Order



$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_1(X;\eta) \le F_1(Y;\eta) \quad \text{for all } \eta \in \mathbb{R}$$

Second-Order Stochastic Dominance

$$F_2(X;\eta) = \int_{-\infty}^{\eta} F_1(X;t)\,dt = \int_{\Omega} \max(0, \eta - X(\omega))\,P(d\omega) \;\text{ for all } \eta \in \mathbb{R}$$



$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_2(X;\eta) \le F_2(Y;\eta) \quad \text{for all } \eta \in \mathbb{R}$$

Characterization via Inverse Distribution Functions

First order dominance $\equiv$ Continuum of probability inequalities

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_{(-1)}(X;p) \geq F_{(-1)}(Y;p) \quad \text{for all} \quad 0 < p < 1$$

$$F_{(-1)}(X;p) = \inf\{\eta : F_1(X;\eta) \geq p\}$$

Absolute Lorenz function (Max Otto Lorenz, 1905)

$$F_{(-2)}(X;p) = \int_0^p F_{(-1)}(X;t)\,dt \quad \text{for} \quad 0 < p \leq 1,$$

$F_{(-2)}(X;0) = 0$ and $F_{(-2)}(X;p) = +\infty$ for $p \notin [0,1]$.

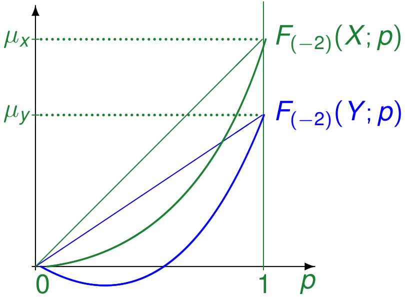
Lorenz function and Expected shortfall are Fenchel conjugates

$F_{(-2)}(X;\cdot) = [F_2(X;\cdot)]^*$ and $F_2(X;\cdot) = [F_{(-2)}(X;\cdot)]^*$

Second order dominance $\equiv$ Relation between Lorenz function

$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \text{for all} \quad 0 \leq p \leq 1.$

Characterization of Stochastic Dominance by Lorenz Functions



$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \text{for all} \quad 0 \leq p \leq 1.$$

Characterization by Rank Dependent Utility Functions

Rank dependent utility are introduced by Quiggin (1982), Schmeidler (1986–89), Yaari (1987).

$\mathscr{W}_1$ contains all continuous nondecreasing functions $w : [0,1] \to \mathbb{R}$.
$\mathscr{W}_2 \subset \mathscr{W}_1$ contains all concave subdifferentiable at 0 functions.

Theorem [DD, A. Ruszczyński, 2006]

(i) For any two random variables $X, Y \in \mathscr{L}_1(\Omega, \mathscr{F}, P)$ the relation $X \succeq_{(1)} Y$ holds if and only if for all $w \in \mathscr{W}_1$

$$\int_0^1 F_{(-1)}(X;p)\,dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\,dw(p) \tag{1}$$

(ii) $X \succeq_{(2)} Y$ holds true if and only if (1) is satisfied for all $w \in \mathscr{W}_2$.

Introduced by Dentcheva and Ruszczyński (2003)

$$\begin{aligned} &\min\ f(z) \\ &\text{subject to } G_i(z) \succeq_{(k_i)} Y_i, \quad i = 1..m \\ &z \in Z \end{aligned}$$

Y_i - benchmark random outcome

The dominance constraints reflect risk aversion

$G_i(z)$ is preferred over Y_i by all risk-averse decision makers with utility functions in the generator $\mathfrak{F}_{k_i}$.

Semi-infinite composite optimization

$\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ are Banach spaces, $\mathfrak{Y}$ is separable, T is a compact Hausdorff space

$$\begin{aligned} \min\ & \varphi(f(z)) \\ \text{s.t.}\ & \mathscr{G}(G(z), t) \le 0 \quad \text{for all} \quad t \in T \\ & z \in Z \end{aligned}$$

$f : \mathfrak{X} \to \mathfrak{Z}$, $G : \mathfrak{X} \to \mathfrak{Y}$ are continuously Fréchet differentiable; $\varphi : \mathfrak{Z} \to \mathbb{R}$ is convex, continuous; $\mathscr{G} : \mathfrak{Y} \times T \to \mathbb{R}$ is continuous, $\mathscr{G}(\cdot, t)$ is convex $\forall t \in T$
Problem $(\mathfrak{P}_2)$ with second order dominance constraint is obtained when $\mathfrak{Y} = \mathscr{L}_1(\Omega, \mathscr{F}, P)$, $X = G(z)$, and $\mathscr{G}$ is defined as follows:

$$\mathscr{G}(X, t) = \int_\Omega \max(0, t - X(\omega))\, P(d\omega) - v(t), \quad t \in [a, b] \subset \mathbb{R},$$

where $v(t) = \int_\Omega \max(0, t - Y(\omega))\, P(d\omega)$ for some $Y \in \mathscr{L}_1(\Omega, \mathscr{F}, P)$.

D. Dentcheva, A Ruszczyński: Composite semi-infinite optimization, Control and Cybernetics, 36 (2007) 3, 633-646.

Sets Defined by Dominance Relation

For all $k \geq 1$, Y - benchmark outcome in $\mathscr{L}_{k-1}(\Omega, \mathscr{F}, P)$, $[a, b] \subseteq \mathbb{R}$.

Acceptance sets $A_k(Y; [a,b]) = \{X \in \mathscr{L}_{k-1} : X \succeq_{(k)} Y \text{ in } [a,b]\}$

Theorem

The set $A_k(Y; [a,b])$ is *convex and closed* for all $[a, b]$, all Y, and $k \geq 2$. Its recession cone has the form

$$A_k^{\infty}(Y; [a,b]) = \left\{H \in \mathscr{L}_{k-1}(\Omega, \mathscr{F}, P) : H \geq 0 \ \text{ a.s. on } [a,b]\right\}$$

$A_1(Y; [a,b])$ is *closed* and $A_k(Y; [a,b]) \subseteq A_{k+1}(Y; [a,b]) \quad \forall k \geq 1$.
$A_k(Y; [a,b])$ is a cone pointed at Y if and only if Y is a constant in $[a, b]$.

Theorem [DD, A. Ruszczyński, 2004]

If $(\Omega, \mathscr{F}, P)$ is atomless, then $A_2(Y; \mathbb{R}) = \overline{\text{co}}\, A_1(Y; \mathbb{R})$ If $\Omega = \{1..N\}$, and $P[k] = 1/N$, then $A_2(Y; \mathbb{R}) = \text{co } A_1(Y; \mathbb{R})$

The result is not true for general probability spaces

Second Order Dominance Constraints

Given $Y \in \mathscr{L}_1(\Omega, \mathscr{F}, P)$ - benchmark random outcome

Primal Stochastic Dominance Constraints

$$(\mathfrak{P}_2) \qquad \begin{aligned} &\max f(z) \\ &\text{subject to } F_2(G(z);\eta) \le F_2(Y;\eta), \quad \forall\, \eta \in [a,b], \\ &z \in Z \end{aligned}$$

Inverse Stochastic Dominance Constraints

$$(\mathfrak{P}_{-2}) \qquad \begin{aligned} &\max f(z) \\ &\text{subject to } F_{(-2)}(G(z);p) \ge F_{(-2)}(Y;p), \quad \forall\, p \in [\alpha,\beta], \\ &z \in Z \end{aligned}$$

Z is a closed subset of a Banach space $\mathfrak{X}$, $[\alpha,\beta] \subset (0,1)$, $[a,b] \subset \mathbb{R}$
$G : \mathfrak{X} \to \mathscr{L}_1(\Omega, \mathscr{F}, P)$ is continuous and for P-almost all $\omega \in \Omega$ the functions $[G(\cdot)](\omega)$ are concave and continuous
$f : \mathfrak{X} \to \mathbb{R}$ is concave and continuous

Optimality Conditions Using von Neumann Utility Functions

The Lagrangian-like functional $L : \mathscr{Z} \times \mathfrak{F}_2([a,b]) \to \mathbb{R}$

$$L(z,u) := f(z) + \int_{\Omega} \left[u(G(z)) - u(Y) \right] P(d\omega)$$

$\mathfrak{F}_2([a,b])$ modified generator.

Uniform Dominance Condition (UDC) for problem ($\mathfrak{P}_2$)

A point $\tilde{z} \in Z$ exists such that $\inf_{\eta\in[a,b]} \left\{ F_2(Y;\eta) - F_2(G(\tilde{z});\eta) \right\} > 0.$

Theorem Assume UDC. If $\hat{z}$ is an optimal solution of ($\mathfrak{P}_2$) then $\hat{u} \in \mathfrak{F}_2([a,b])$ exists:

$$L(\hat{z},\hat{u}) = \max_{z\in Z} L(z,\hat{u}) \tag{2}$$

$$\int_{\Omega} \hat{u}\,(G(\hat{z}))\; P(d\omega) = \int_{\Omega} \hat{u}(Y)\, P(d\omega) \tag{3}$$

If for some $\hat{u} \in \mathfrak{F}_2([a,b])$ an optimal solution $\hat{z}$ of (2) satisfies the dominance constraints and (3), then $\hat{z}$ solves ($\mathfrak{P}_2$).

Optimality Conditions Using Rank Dependent Utility Function

Lagrangian-like functional $\Phi : \mathscr{Z} \times \mathscr{W}([\alpha, \beta]) \to \mathbb{R}$

$$\Phi(z, w) = f(z) + \int_0^1 F_{(-1)}(G(z); p)\, dw(p) - \int_0^1 F_{(-1)}(Y; p)\, dw(p)$$

$\mathscr{W}([\alpha, \beta])$ is the modified generator of the relaxed order

Uniform inverse dominance condition (UIDC) for $(\mathfrak{P}_{-2})$

$\exists \tilde{z} \in Z$ such that $\inf\limits_{p \in [\alpha, \beta]} \Big\{ F_{(-2)}(G(\tilde{z}); p) - F_{(-2)}(Y; p) \Big\} > 0.$

Theorem

Assume UIDC. If $\hat{z}$ is a solution of $(\mathfrak{P}_{-2})$, then $\hat{w} \in \mathscr{W}([\alpha, \beta])$ exists:

$$\Phi(\hat{z}, \hat{w}) = \max_{z \in Z} \Phi(z, \hat{w}) \tag{4}$$

$$\int_0^1 F_{(-1)}(G(\hat{z}); p)\, d\hat{w}(p) = \int_0^1 F_{(-1)}(Y; p)\, d\hat{w}(p) \tag{5}$$

If for some $\hat{w} \in \mathscr{W}([\alpha, \beta])$ and a solution $\hat{z}$ of (4) the dominance constraint and (5) are satisfied, then $\hat{z}$ is a solution of $(\mathfrak{P}_{-2})$.

Duality Relations to Utility Theories

The Dual Functionals

$D(u) = \sup_{z \in Z} L(z,u) \qquad \Psi(w) = \sup_{z \in Z} \Phi(z,w)$

The Dual Problems

$(\mathfrak{D}_2) \quad \min_{u \in \mathfrak{F}_2([a,b])} D(u) \qquad (\mathfrak{D}_{-2}) \quad \min_{w \in \mathscr{W}([\alpha,\beta])} \Psi(w).$

Theorem

Under UDC/UIDC, if problem $(\mathfrak{P}_{-2})$ resp. $(\mathfrak{P}_{-2})$ has an optimal solution, then the corresponding dual problem has an optimal solution and the same optimal value. The optimal solutions of the dual problem $(\mathfrak{D}_2)$ are the utility functions $\hat{u} \in \mathfrak{F}_2([a,b])$ satisfying (2)–(3) for an optimal solution $\hat{z}$ of problem $(\mathfrak{P}_2)$. The optimal solutions of $(\mathfrak{D}_{-2})$ are the rank dependent utility functions $\hat{w} \in \mathscr{W}([\alpha,\beta])$ satisfying (4)–(5) for an optimal solution $\hat{z}$ of problem $(\mathfrak{P}_{-2})$.

A measure of risk ϱ assigns to an uncertain outcome $X \in \mathscr{L}_p(\Omega, \mathscr{F}, P)$ a real value $\varrho(X)$ on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

A coherent measure of risk is a convex, positive homogeneous, p.w. monotone functional $\varrho : \mathscr{L}_1(\Omega, \mathscr{F}, P) \to \overline{\mathbb{R}}$ satisfying the $\varrho(X + a) = \varrho(X) - a$ for all $a \in \mathbb{R}$.

Theorem (Necessary Optimality Conditions):

Under the UIDC, if $\hat{z}$ is an optimal solution of $(\mathfrak{P}_{-2})$, then a risk measure $\hat{\varrho}$ and a constant $\kappa \geq 0$ exist such that $G(\hat{z})$ is also an optimal solution of the problem

$$\max_{z \in Z} \left\{ f(z) - \kappa \hat{\varrho}(G(z)) \right\} \quad \text{and} \tag{6}$$

$$\kappa \hat{\varrho}(G(\hat{z})) = \kappa \hat{\varrho}(Y). \tag{7}$$

If the dominance constraint is active, then condition (7) takes on the form $\hat{\varrho}(G(\hat{z})) = \hat{\varrho}(Y)$.

Extensions and Further Research Directions

- Non-convex problems Optimality conditions for problems with FSD constraints and problems with higher order dominance constraints with non-convex functions (DD, A Ruszczyński 2004, 2007)
- Multivariate orders based on scalarization of random vectors (DD, A Ruszczyński, 2007, T. Homem-de-Mello, S. Mehrotra, 2009)
- Dynamic orders Scalarizations as discounts for stochastic sequences; maximum prnciple; (DD, A Ruszczyński, 2008); Two-stage problems with dominance constraints on the recourse function (R. Schultz, F. Neise, R. Gollmer, U. Gotzes, D. Drapkin, 2007, 2009, 2010)
- Stochastic dominance efficiency in multi-objective optimization and its relations to dominance constraints (G. Mitra, C. Fabian, K. Darby-Dowman, D. Roman, 2006, 2009)
- Stability and sensitivity analysis (DD, R. Henrion, A Ruszczyński, 2007; Y. Liu, H. Xu, 2010; DD W. Römisch 2011)
  Robust Dominance Relation (DD, A Ruszczyński, 2010)

Problem and Assumptions

k-th Order Stochastic Dominance Constraints

$$(\mathfrak{P}_k) \qquad \begin{aligned} &\max\ f(z) \\ &\text{subject to } F_k(G(z,\xi);\eta) \le F_k(Y;\eta), \quad \forall\, \eta \in [a,b], \\ &\qquad z \in Z. \end{aligned}$$

$k \ge 2$, $Z \subset \mathbb{R}^n$ is nonempty closed convex, $Y \in \mathscr{L}_{k-1}(\Omega, \mathscr{F}, P)$
$f : \mathbb{R}^n \to \mathbb{R}$ is convex;
ξ is a random vector with closed convex support $\Xi \subset \mathbb{R}^s$;
$G : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}$ is continuous, concave w.r.to the first argument, satisfies the linear growth condition

$$|G(z,\xi)| \le C(B) \max\{1, \|\xi\|\} \quad (z \in B, \xi \in \Xi, \forall B \text{ bounded})$$

Theorem

The feasible set $\mathscr{Z}(\xi, Y)$ is closed and convex.

$$\mathscr{Z}(\xi, Y) = \{z \in Z :\ F_k(G(z,\xi),\eta) \le F_k(Y,\eta), \forall \eta \in \mathbb{R}\}$$

Rachev metrics on $\mathscr{L}_k(\Omega, \mathscr{F}, P)$

$$\mathbb{D}_{k,p}(X,Y) = \begin{cases} \left(\int_{\mathbb{R}} \left| F_k(X,\eta) - F_k(Y,\eta) \right|^p d\eta \right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty \\ \sup_{\eta \in \mathbb{R}} \left| F_k(X,\eta) - F_k(Y,\eta) \right| & \text{for } p = \infty \end{cases} \tag{8}$$

$\mathscr{L}_{k-1}(\Omega, \mathscr{F}, P)$-minimal distance ($k \ge 2$)

$$\ell_{k-1}(\xi, \tilde{\xi}) = \inf \left\{ \|\zeta - \tilde{\zeta}\|_{k-1} : \zeta \overset{d}{=} \xi,\ \tilde{\zeta} \overset{d}{=} \tilde{\xi} \right\}$$

and $\overset{d}{=}$ means equality in distribution.

Distance on $\mathscr{L}_{k-1}(\Omega, \mathscr{F}, P) \times \mathscr{L}_{k-1}(\Omega, \mathscr{F}, P)$

$$d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})) = \ell_{k-1}(\xi, \tilde{\xi}) + \mathbb{D}_{k,\infty}(Y, \tilde{Y}),$$

Quantitative Stability

Constraints qualification: k UDC

A point $\bar{z} \in Z$ exists such that $\min_{\eta \in [a,b]} (F_k(Y,\eta) - F_k(G(\bar{z},\xi),\eta)) > 0.$

$v(\xi, Y)$ is the optimal value of $(\mathfrak{P}_k)$ and $S(\xi, Y)$ is its solution set.
Consider nondecreasing growth function ψ:

$$\begin{aligned} \psi(\tau) &= \inf\{f(z) - v(\xi, Y) : d(z, S(\xi, Y)) \geq \tau,\, z \in \mathscr{Z}(\xi, Y)\} \\ \Theta(t) &= t + \psi^{-1}(2t) \quad (t \in \mathbb{R}_+). \end{aligned}$$

Theorem [DD and W. Römisch 2011]

Assume k UDC and let Z be compact and $G(z, \cdot)$ be Lipschitz for all $z \in Z$. Then constants $L > 0$ and $\delta > 0$ exist such that

$$d_{\mathrm{H}}(\mathscr{Z}(\xi, Y), \mathscr{Z}(\tilde{\xi}, \tilde{Y})) \leq L d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})),$$

$$|v(\xi, Y) - v(\tilde{\xi}, \tilde{Y})| \leq L\, d_k((\xi, Y), (\tilde{\xi}, \tilde{Y}))$$

$$\sup_{z \in S(\tilde{\xi}, \tilde{Y})} d(z, S(\xi, Y)) \leq \Theta\big(L\, d_k((\xi, Y), (\tilde{\xi}, \tilde{Y}))\big)$$

for all $(\tilde{\xi}, \tilde{Y})$ such that $d_k((\xi, Y), (\tilde{\xi}, \tilde{Y})) < \delta$.

Robust Dominance Relation

Uncertainty of the probability measure P_0

Assume that P_0 may vary in a set of probability measures $\mathscr{Q}$.

Consider $\mathscr{L}_p(\Omega, \mathscr{F}, P_0)$, $p \in [1, \infty)$ and

$$\mathscr{L}_q(\Omega, \mathscr{F}, P_0) \equiv \begin{cases} \text{measures } Q: & \text{absolutely continuous w.r.to } P_0 \\ & \text{with densities } dQ/dP_0 \in \mathscr{L}_q(\Omega, \mathscr{F}, P_0) \end{cases}$$

For any $X \in \mathscr{L}_p(\Omega, \mathscr{F}, P_0)$ and $Q \in \mathscr{L}_q(\Omega, \mathscr{F}, P_0)$

$$\langle Q, X \rangle = \int_\Omega X(\omega)\, Q(d\omega) = \int_\Omega X(\omega) \frac{dQ}{dP_0}(\omega)\, P_0(d\omega).$$

Let $\mathscr{Q} \subset \mathscr{L}_q(\Omega, \mathscr{F}, P_0)$ be a *convex, closed, and bounded set* of probability measures.

Definition

$X \in \mathscr{L}_p(\Omega, \mathscr{F}, P_0)$ dominates robustly $Y \in \mathscr{L}_p(\Omega, \mathscr{F}, P_0)$ over a set of probability measures $\mathscr{Q} \subset \mathscr{L}_q(\Omega, \mathscr{F}, P_0)$ if

$$\int_\Omega u(X) P(d\omega) \geq \int_\Omega u(Y) P(d\omega) \quad \forall\, u \in \mathscr{U},\ \forall\, P \in \mathscr{Q}.$$

Robust Preferences

Savage 1954, Gilboa and Schmeidler 1989,
F. Maccheroni, A. Rustichini and M. Marinacci, *Econometrica* 74 (2006)

Set of axioms imply the existence of a utility function $\bar{u} : \mathbb{R} \to \mathbb{R}$ and a set of measures $\mathscr{Q}$ such that

X is preferred over Y if and only if

$$\inf_{P \in \mathscr{Q}} \int_{\Omega} \bar{u}(X)\, P(d\omega) \geq \inf_{P \in \mathscr{Q}} \int_{\Omega} \bar{u}(Y)\, P(d\omega).$$

In the case, $\mathscr{U} = \{\bar{u}\}$, our definition implies

$$\inf_{P \in \mathscr{Q}} \Big\{ \int_{\Omega} [\bar{u}(X) - \bar{u}(Y)] P(d\omega) \Big\} \geq 0.$$

D. Dentcheva, A Ruszczynski: Robust stochastic dominance constraints, Mathematical Programming, 2010

Numerical Methods

- Large scale convex optimization methods for second order dominance constraints: applicable only to small problems
- Dual methods for SSD constraints (DD, Ruszczyński, 2005); (Rudolf, Ruszczyński 2006, Luedtke 2008).
- Subgradient Based Approximation Methods for SSD constraints with linear $G(\cdot)$ (Rudolf, Ruszczyński, 2006; Fabian, Mitra, and Roman, 2008)
- Combinatorial methods for FSD constraints (Rudolf, Noyan, Ruszczyński 2006) based on second order stochastic dominance relaxation ($\{X : X \succeq_{(2)} Y\} = \overline{\text{co}}\{X : X \succeq_{(1)} Y\}$)
- Subgradient methods based on quantile functions and conditional expectations (DD, Ruszczyński, 2010)
- Methods for two-stage problems with dominance constraints on the recourse (Schultz, Neise, Gollmer, Drapkin; DD and G. Martinez, 2011)
- Methods for multivariate linear dominance constraints (Homem-de-Mello, Mehrotra, 2009; Hu, Homem-de-Mello, and Mehrotra 2010, Armbruster and Luedtke 2010)
- Sample average approximation methods (Sun, Xu and Wang, 2011)

Applications

- Finance: portfolio optimization
- Electricity markets: portfolio of contracts and/or acceptability of contracts
- Inverse models and forecasting: Compare the forecast errors via stochastic dominance and design data collection for model calibration
- Network design: assign capacity to optimize network throughput
- Medicine: radiation therapy designs