Pricing Barrier Options with Square Root Constant Elasticity of Variance Process

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Abstract

The square root constant elasticity of variance (CEV) process has been paid little attention in previous research on valuation of barrier options. In this paper we derive analytical option pricing formulae of up-and-out options with this process using the eigenfunction expansion technique. We develop an efficient algorithm to compute numerical results from the formula. The numerical results are compared with the corresponding model prices under the Black-Scholes model. We find that the differences in the model prices between the square root CEV model and the Black-Scholes model can be significant as the time to maturity and volatility increase.

I. Introduction

European barrier options are path dependent options in which the existence of the options depends on whether the underlying asset price has touched a barrier level during the options' lifetime. They have emerged as significant products for hedging and investment in foreign exchange, equity and commodity markets since late 1980s, largely in the over-the-counter (OTC) markets. The pricing of barrier options has been studied in many literatures assuming the underlying asset price to follow a lognormal diffusion process, i.e. the Black-Scholes environment (Black and Scholes, 1973). Merton (1973) was the first to derive a closed-form solution for a down-and-out European call option. Other closed-form pricing formulae of single-barrier options were published by Rubinstein and Reiner (1991a, 1991b), Rich (1994), Henynen and Kat (1994) and Kwok, Wu and Yu (1998). The analytical valuation of double-barrier options was discussed in Kunitomo and Ikeda (1991), Geman and Yor (1996), Hui (1996, 1997) and Pelsser (1999).

The derivation of the option pricing formula for a square root CEV process as an alternative diffusion process for option valuation was first presented by Cox and Ross (1976). This process on valuation of barrier option has been paid little attention in previous research. Only recently, Boyle and Tian (1999) study this topic numerically using the trinomial tree method. However, it has never been studied analytically. The square root CEV process of stock price S dynamics can be expressed as

$$dS = \mu S dt + \sigma \sqrt{S} dZ$$

where dZ is a Weiner process. The equation shows that the instantaneous variance of the percentage price change is equal to σ^2/S and is a direct inverse function of the stock price. Several theoretical arguments imply an association between stock price and volatility. Black (1976) and Christie (1982) consider the effects of financial leverage on the variance of the stock. A fall in the stock price increases the debt-equity ratio of the firm; therefore both the riskiness and the variance of the stock increase. Black also proposes that a downturn in the business cycle might lead to an increase in the stock price volatility and hence to a fall in stock prices.

Empirical evidence has shown that the CEV process may be a better description of stock behavior than the more commonly used lognormal model. Schmalensee and Trippi (1978) find a strong negative relationship between stock price changes and changes in implied volatility after examining over a year of weekly data on six stocks. By applying the trading profits approach on 19,000 daily warrant price observations, Hauser and Lauterbach (1996) find that the square root CEV process offers better trading returns than using the Black-Scholes model. The superiority of the square root CEV process is strongest in out-of-the-money and longer time to expiration warrants. The results are consistent with the findings in Lauterbach and Schultz (1990). In view of the square root CEV process being a better candidate of describing the actual stock price behaviour than the Black-Scholes model, it is worthwhile to study the valuation of barrier options with this process.

In order to understand the differences between the square root CEV process and the lognormal process in pricing barrier options, in this paper we derive analytical pricing formulae of up-and-out options with the square root CEV process using the eigenfunction expansion technique. This technique is being widely used in solving physical science problems governed by partial differential equations and is discussed in Lewis (1998) about its applications in continuous-time finance. We develop an efficient algorithm to compute numerical results from the derived option pricing formulae. The numerical results are compared with the corresponding model prices under the Black-Scholes model. The scheme of this paper is as follows. In the following section we derive the square root CEV single-barrier option pricing formulae. Numerical implementation of the pricing formulae is shown in section III. The discussion of the numerical results is in section IV. In the last section we shall summarise our investigation.

II. Single-barrier square root CEV model

The single-barrier square root CEV model for an up-and-out European call option is described by the partial differential equation (Cox and Ross, 1976)

$$\frac{\partial P(S,\tau)}{\partial \tau} = \frac{1}{2}\sigma^2 S \frac{\partial^2 P(S,\tau)}{\partial S^2} + (r-d)\, S \frac{\partial P(S,\tau)}{\partial S} - rP(S,\tau) \tag{1}$$

with the absorbing boundary conditions at $S = 0$ and L, i.e. $P(0,\tau) = P(L,\tau) = 0$. P is the option value, S is the underlying price, τ is the time to maturity, σ^2/S is the instantaneous variance of the percentage price change, r is the risk-free interest rate and d is the dividend. To solve this partial differential equation, we use the usual trick, namely the method of separation of variables: $P(S,\tau) = u(S)\exp\left[-\left(r-d\right)S/\sigma^2\right]\exp\left(-\lambda\tau\right)$ for some positive constant λ. The function $u(S)$ is found to obey the ordinary differential equation

$$\frac{d^2u(S)}{dS^2} + \left(\frac{Q}{S} - E\right)u(S) = 0 \tag{2}$$

where $Q = -2\left(r-\lambda\right)/\sigma^2$ and $E = \left(r-d\right)^2/\sigma^4$. By a simple change of variables: $\rho = 2\sqrt{E}\;S$ and $\nu = Q/\left(2\sqrt{E}\right)$, Eq.(2) becomes

$$\frac{d^2u(\rho)}{d\rho^2} + \left(\frac{\nu}{\rho} - \frac{1}{4}\right)u(\rho) = 0 \quad . \tag{3}$$

This equation can be easily cast into the canonical form of the confluent hypergeometric equation (Slater, 1960)

$$\rho\frac{d^2G(\rho)}{d\rho^2} + \left(2-\rho\right)\frac{dG(\rho)}{d\rho} + \left(\nu-1\right)G(\rho) = 0 \tag{4}$$

where $G(\rho) = \left[u(\rho)/\rho\right]\exp\left(\rho/2\right)$, and the desired solution is simply given by the confluent hypergeometric function: $G(\rho) = {}_1F_1(1-\nu;2;\rho)$ where

$${}_1F_1(a;b;x) \equiv \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\cdot\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{a(a+1)(a+2)\cdot\cdots\cdot(a+n-1)}{b(b+1)(b+2)\cdot\cdots\cdot(b+n-1)}\cdot\frac{x^n}{n!} \quad . \tag{5}$$

[*Note*: By definition, $(a)_0 = (b)_0 = 1$.] It is obvious that $P(S,\tau) = \rho\;G(\rho)\;\exp\left(-\rho/2\right)$ $\exp\left[-\left(r-d\right)S/\sigma^2\right]\exp\left(-\lambda\tau\right)$ automatically satisfies the absorbing boundary condition at $S = 0$. Now, imposing the absorbing boundary condition at $S = L$: $P(L,\tau) = 0$, we can determine the admissible eigenvalues of λ and the corresponding solutions.

Without loss of generality, we concentrate on the cases with $r > d$, whose normalized basis functions are

$$P_m(S,\tau) = N_m\;\rho\;{}_1F_1(q_m;2;\rho)\;\exp(-\rho)\exp(-\lambda_m\tau) \quad , \quad m = 1,2,3,..... \tag{6}$$

with $q_m = 1-\nu_m = 1+(r-\lambda_m)/(r-d)$. These basis functions are orthogonal to each other with respect to the weighting function $C(S) \equiv \left[2/(\sigma^2S)\right]\;\exp\left[2\left(r-d\right)S/\sigma^2\right]$:

$$\int_0^L dS\;C(S)P_m(S,\tau)P_n(S,\tau) = 0 \quad , \quad m \neq n \quad . \tag{7}$$

The normalization constants N_m are given by

$$N_m = \left\{ \int_0^L dS\ C(S)\ \left[\rho\ {}_1F_1(q_m; 2; \rho)\ \exp(-\rho)\right]^2 \right\}^{-1/2} \quad . \tag{8}$$

In terms of these normalized basis functions, the option price $P(S,\tau)$ can then be expressed as a linear superposition:

$$P(S,\tau) = \sum_{m=0}^{\infty} \alpha_m P_m(S,\tau) \quad , \tag{9}$$

where the expansion coefficients α_m are determined by the inner product:

$$\alpha_m = \int_0^L dS\ C(S) P(S,0) P_m(S,0) \quad . \tag{10}$$

For a European up-and-out call option, the option price at maturity is simply given by $P(S,0) = \max(S - S_0, 0)$ for $0 \leq S < L$. Accordingly, Eq.(10) becomes

$$\alpha_m = \frac{N_m}{\sigma^2 \sqrt{E}} \int_{\rho_0}^{\rho_L} d\rho\ (\rho - \rho_0)\ {}_1F_1(q_m; 2; \rho) \tag{11}$$

where $\rho_0 = 2\sqrt{E}\ S_0$ and $\rho_L = 2\sqrt{E}\ L$. Making use of the relations: (Slater, 1960)

$$\begin{aligned} \frac{d}{dx}\left[x^b\ {}_1F_1(a; b+1; x)\right] &= b\ x^{b-1}\ {}_1F_1(a; b; x) \\ \frac{d}{dx}\left[{}_1F_1(a-1; b-1; x)\right] &= \frac{a-1}{b-1}\ {}_1F_1(a; b; x) \quad , \end{aligned} \tag{12}$$

the integral in Eq.(11) can be evaluated to yield:

$$\begin{aligned} \alpha_m = &\frac{N_m}{2\sigma^2\sqrt{E}}\left[\rho_L^2\ {}_1F_1(q_m; 3; \rho_L) - \rho_0^2\ {}_1F_1(q_m; 3; \rho_0)\right] - \\ &\frac{N_m \rho_0}{(q_m - 1)\sigma^2\sqrt{E}}\left[\ {}_1F_1(q_m - 1; 1; \rho_L) -\ {}_1F_1(q_m - 1; 1; \rho_0)\right] \quad . \end{aligned} \tag{13}$$

Hence, once the allowed values of λ_m are found, the option value $P(S,\tau)$ can be obtained readily.

Furthermore, for a European up-and-out put option, it is not difficult to show that the option price $\mathcal{P}(S,\tau)$ can be found by inspection as follows:

$$\mathcal{P}(S,\tau) = \mathcal{P}_0(S)\exp(-r\tau) + \sum_{m=0}^{\infty} \beta_m P_m(S,\tau) \tag{14}$$

where

$$
\begin{aligned}
\mathcal{P}_0(S) &= S_0 \frac{\sinh\left((\rho_L-\rho)/2\right)}{\sinh\left(\rho_L/2\right)} \exp\left(-\rho/2\right) \\
\beta_m &= \int_0^L dS\, C(S) \left[\mathcal{P}(S,0)-\mathcal{P}_0(S)\right] P_m(S,0) \quad .
\end{aligned} \tag{15}
$$

Obviously, the solution automatically satisfies the boundary conditions at $S=0$ and $S=L$: $\mathcal{P}(0,\tau)=S_0\exp(-r\tau)$ and $\mathcal{P}(L,\tau)=0$, as well as the final condition at maturity: $\mathcal{P}(S,0)=\max(S_0-S,0)$ for $0\le S<L$. With the help of the relation: (Slater, 1960)

$$
\frac{d}{dx}\left[\exp(-x)\ {}_1F_1(a;b;x)\right] = -\ \frac{b-a}{b}\ \exp(-x)\ {}_1F_1(a;b+1;x)\ \ , \tag{16}
$$

and those in Eq.(12), the integration in Eq.(15) can be easily performed to give

$$
\begin{aligned}
\beta_m \ =\ & \frac{N_m\rho_0}{2\sigma^2\sqrt{E}}\left[\frac{2}{q_m-1}\ {}_1F_1(q_m-1;1;\rho_0)-\rho_0\ {}_1F_1(q_m;3;\rho_0)\right]+ \\
& \frac{N_m\rho_0\exp(-\rho_L/2)}{2\sigma^2\sqrt{E}\ (q_m-1)\sinh(\rho_L/2)}\left[\ {}_1F_1(q_m-1;1;\rho_L)-\ {}_1F_1(q_m;1;\rho_L)\right]
\end{aligned} \tag{17}
$$

which in turn enables us to evaluate the option price $\mathcal{P}(S,\tau)$ readily. Finally, it should be noted that very similar procedures can be applied to tackle the cases with $r<d$.

In the following sections we shall present numerical results for a European up-and-out call option to illustrate the validity of our approach and to compare the option values priced under the Black-Scholes model and the square root CEV model.

III. Numerical implementation

In this section we discuss the algorithm of finding the eigenvalues λ_m and the speed of convergence of our series solutions.

1. Finding the eigenvalues λ_m

As suggested by Eq.(6), to determine the eigenvalues λ_m, we basically need to locate the roots q_m of the confluent hypergeometric function ${}_1F_1(q;2;\rho_L)$ along the line of constant ρ_L. Since the confluent hypergeometric function can never vanish for positive values of q and ρ_L, we can simply search for the possible roots only in the negative q region. With a little bit effort we can easily find the approximate location of the first root, i.e. the one closest to $q = 0$, by scanning the value of ${}_1F_1(q; 2; \rho_L)$. Then we can zoom into the exact location by either the Newton's method or the Quasi-Newton's method. As pointed out by Slater (1960), ${}_1F_1(q; 2; \rho_L)$ has only simple roots, and the separation between successive roots, i.e. $|q_{m+1} - q_m|$, is at least one. Hence, we can find the approximate location of other roots by inspecting the value of ${}_1F_1(-n; 2; \rho_L)$ for some non-negative integer n in accordance to the recurrence relation

$$\begin{aligned}(2+n)\cdot\ {}_1F_1(-n-1; 2; \rho_L) + (-2n-2+\rho_L)\cdot\ {}_1F_1(-n; 2; \rho_L) +& \\ n\cdot\ {}_1F_1(-n+1; 2; \rho_L) \;&=\; 0\ . \qquad (18)\end{aligned}$$

Provided that the confluent hypergeometric function changes sign as we increase n to $n+1$, there must be a root in the interval $[n, n+1]$. Again, we then use either the Newton's method or the Quasi-Newton's method to calculate the exact values of the roots. According to our calculations, this root-finding algorithm is indeed very efficient and accurate.

2. Convergence issues of the series solutions

In Table 1 we tabulate the number of basis functions required to obtain a converged series solution against various volatilities σ (Black-Scholes model) and stock prices for four different times to maturity ($\tau = 0.25,\ 0.5,\ 0.75,\ 1.0$). (Note that the value of σ to be used for the square root CEV model is adjusted to be $\sigma \equiv \sigma_{BS}\sqrt{S}$.) The other parameters used here are $r = 0.05$, $d = 0.0$, $S_0 = 20$ and $L = 26$. When the time to maturity is large (e.g. one year), the option value converges very quickly to the exact result; on the other hand, one needs to include more terms as the time to maturity gets smaller. For instance, the data indicates that for the worst case of $\tau = 0.25$ years with $\sigma^2_{BS} = 0.02$ and $S = 16$, the converged series solution is obtained with the use of about seventy basis functions; whilst in the most favourable case of $\tau = 1$ year with $\sigma^2_{BS} = 0.12$ and $S = 24$, only twelve basis functions are needed. More specifically, as shown in Figure 1, the number of terms needed to attain desired convergence is approximately proportional to the inverse of the square root of the factor $\sigma_{BS}^2\tau$. Furthermore, the convergence rate is slower for small S (see Figures 2 and 3) because the converged option value is pretty small and thus it needs more terms to cancel out the oscillation. Nevertheless, the convergence rate is basically rather rapid in most cases; Figures 4 and 5 show that the percentage error to the partial sum P_{n-1} of the first $n-1$ terms due to the nth term, i.e. $|P_n - P_{n-1}|/P$ (where P is the converged sum), roughly follows a Gaussian decay pattern. Hence, our series solutions can indeed provide an efficient and tractable valuation scheme for the single-barrier square root CEV option model.

IV. Discussion of numerical results

In order to compare option values priced under the Black-Scholes model and the square root CEV model, an up-and-out call option with strike $S_0 = 20$, barrier $L = 26$, risk free interest rate $r = 5\%$ and dividend $d = 0$, is priced using the formula obtained by the Black-Scholes model (see Rich (1994)) and Eq.(9) of the square root model respectively. The option values and deltas of the example calculated under the two models with different underlying price S from 16 to 24, time to maturity τ from 0.25 to 1, and σ^2 (Black-Scholes model) from 0.02 to 0.12, are illustrated in Table 2 and 3 respectively. The values of σ to be used for the square root model is adjusted to be $\sigma \equiv \sigma_{BS}\sqrt{S}$. The values of the deltas are measured as finite difference approximations to their continuous time equivalents.

In Table 2, the two models' prices show option values decrease with increases in volatility, time to maturity and underlying price. The increases in volatility, time to maturity and underlying price enhance the probability of hitting the barrier and therefore enhance the risk of option to be knocked out. It is a typical behavior of an up-and-out call.

According to the calculations performed by Beckers (1980) on regular call options, the square root model prices are higher than the Black-Scholes model prices for out-of-the-money options while the reverse is for in-the-money options. However, the model prices in Table 2 show that the square root model prices are generally higher than the Black-Scholes model prices for different underlying prices. For example in the barrier option with parameters of $S = 20$, $\sigma^2_{BS} = 0.04$ and time to maturity $\tau = 0.5$, its square root model price is 0.9946 which is 9% (0.0815 absolute) higher than its Black-Scholes model price 0.9131. For $S = 22$, the square root model price is 1.2480 which is 6% (0.0759 absolute) higher than its Black-Scholes model price 1.1721. The higher square root model prices can be explained by a lower knockout probability for a barrier option in which the underlying price is governed by the square root process. The instantaneous variance of the percentage price change in the square root model is equal to σ^2/S and decreases with the increase in the underlying price. This implies that the instantaneous variance of the percentage price change decreases as the price goes up and towards the barrier. As a result, the risk of the barrier option to be knocked out in the square root model is lower than that in the Black-Scholes model in which the instantaneous variance of the percentage price change is independent of the price.

For the barrier option with longer time to maturity $\tau = 1$, $S = 20$ and $\sigma^2_{BS} = 0.04$, the square root model price is 0.7681 which is 15% (0.1016 absolute) higher than the Black-Scholes model price 0.6665. Comparing with the model price differences of 9% (0.0815 absolute) for the barrier option with time to maturity $\tau = 0.5$, the differences in model prices increase as the time to maturity increases. The same observation is also found in higher volatility. For the barrier option with higher volatility $\sigma^2_{BS} = 0.08$, $S = 20$, time to maturity $\tau = 0.5$, the square root model price is 0.7325 which is 16% (0.0992 absolute) higher than the Black-Scholes model price 0.6333. These characteristics are also observed in square root model prices calculated by Beckers (1980) for regular options, in which the model price differences are more apparent as the time to maturity and volatility increase.

Table 3 shows the deltas of the up-and-out call. Both models show a typical characteristic of an up-and-out call that it has smaller deltas than a regular call and negative deltas near the barrier. Between the two models, the model deltas in the square root model are in general higher than that in the Black-Scholes model in absolute value terms.

It is more apparent for the at-the-money barrier options. By comparing the deltas of different underlying prices, the results also suggest that the model gammas in the square root model are higher than that in the Black-Scholes model.

In summary, we have shown that square root model prices are in general higher than the Black-Scholes model prices for the up-and-out call especially as the time to maturity and volatility increase. The numerical examples in Table 2 show that the differences in the model prices can be around 30%. They are similar to the results obtained by Boyle and Tian (1999) using the trinomial tree numerical method. If the superiority of the square root model being stronger in longer time to maturity warrants (Hauser and Lauterbach, 1996) is valid, the square root model could be important to make the model be a more attractive alternative formulation of equity barrier option pricing.

V. Conclusion

Barrier options are used extensively for hedging and investment in the OTC equity markets. More comparative option pricing and precise risk management is necessary. This paper provides the analytical pricing formulae for up-and-out options under the square root CEV process. We develop an efficient algorithm to compute the option pricing formulae. In the numerical example, the differences in the model prices between the square root CEV model and the Black-Scholes model can be significant as the time to maturity and volatility increase. In view of the square root CEV model being empirically considered to be a better candidate in equity option pricing than the traditional Black-Scholes model, more comparative pricing and precise risk management in equity barrier options can be achieved by using the square root CEV option valuation model. Further study of the valuation of other barrier options under the CEV process is necessary for understanding the implications of this process to other barrier options.

As a final remark, the eigenfunction expansion technique can be straightforwardly extended to tackle the case of a double-barrier option, in which an orthonormal basis consisting of the two independent solutions of the confluent hypergeometric equation in Eq.(4) is needed to represent the desired solution. Although the numerical implementation procedures may be a bit more complicated, yet they are basically similar to those of the up-and-out case. Nevertheless, for the case of down-and-out options, it can be easily shown that the expansion technique is no longer applicable due to the lack of an appropriate orthonormal basis of eigenfunctions (Slater, 1960).

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