The Eigenfunction Expansion Technique

Math 135B Supplementary Notes #3 March 16, 2005

Additional Details Continued:

Central to the eigenfunction expansion technique is the existence of a set of orthogonal eigenfunctions that can be used to construct solutions. For certain families of two-point boundary value problems there are theorems that prove the existence of sets of orthogonal eigenfunctions. One such family are Sturm-Liouville problems; linear second order equations of the form

$$-\frac{d(p(x)\,dy)}{dx^2} + q(x)\,y(x) \ = f(x) \text{ for } x \in [a,b] \tag{1}$$

with homogeneous boundary conditions

$$\begin{aligned} a_1 y(a) + a_2 \frac{dy}{dx}(a) &= 0 \\ b_1 y(b) + b_2 \frac{dy}{dx}(b) &= 0. \end{aligned}$$

The eigenfunction problem associated with Sturm-Liouville equations is

$$-\frac{d(p(x)\,d\phi)}{dx^2} + q(x)\,\phi(x) \ = \lambda\, r(x)\,\phi(x) \text{ for } x \in [a,b] \tag{2}$$

$$\begin{aligned} a_1 \phi(a) + a_2 \frac{d\phi}{dx}(a) &= 0 \\ b_1 \phi(b) + b_2 \frac{d\phi}{dx}(b) &= 0. \end{aligned}$$

where $r(x)$ is a strictly positive function. The introduction of $r(x)$ is a slight generalization of the eigenfunction problems that we have been working with in class.

If one assumes that $p(x)$, $q(x)$ and $r(x)$ are continuous and $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$ then the following facts can be proven

Fact #1: The eigenvalues, λ_k, of the eigenfunction problem (2) are real.

Fact #2: There is only one eigenfunction associated with each eigenvalue, e.g. the eigenvalues are "simple".

Fact #3: Eigenfunctions, $\phi_k(x)$, associated with distinct eigenvalues are orthogonal with respect to the inner product $\langle f,\ g\rangle = \int_a^b f(s)g(s)\,r(s)\ ds$. Specifically if $m \neq n$ then for eigenfunctions $\phi_m(x)$ and $\phi_n(x)$

$$\langle \phi_m,\ \phi_n \rangle = \int_a^b \phi_m(s)\phi_n(s)\, r(s)\ ds = 0.$$

Fact #4: The set of eigenfunctions $\{\phi_k\}$ are complete. If $f(x)$ is a function such $\|f\|_2 < \infty$, one can express $f(x)$ as

$$f(x) = \sum_{k=1}^{\infty} \gamma_k \phi_k(x).$$

with

$$\gamma_k = \frac{\langle f,\ \phi_k \rangle}{\langle \phi_k,\ \phi_k \rangle}$$

Here the equality of $f(x)$ and its eigenfunction expansion is in the L^2 norm, that is

$$\lim_{N \to \infty} \left\| f(x) - \sum_{k=1}^{N} \gamma_k \phi_k(x) \right\|_2 = 0$$

Fact #5: Other convergence theorems can be proven for the eigenfunction expansion. For example, if $f(x)$ is piecewise continuous with a finite number of discontinuites, then one can prove that for any $x \in [a, b]$

$$\lim_{N \to \infty} \sum_{k=1}^{N} \gamma_k \phi_k(x) = \frac{f(x^+) + f(x^-)}{2}$$

Thus, one has pointwise convergence at all values where $f(x)$ is continuous.

The theorems that prove the existence of orthogonal sets of eigenfunctions don't necessarily indicate how to obtain the eigenfunctions. In general it is a difficult problem to determine eigenfunctions and eigenvectors. However, for second order linear eigenfunction problems of the form

$$-\frac{d^2\phi}{dx^2} + Q(x)\,\frac{d\phi}{dx} + R(x)\,\phi = \lambda\,\phi \tag{3}$$

$$\begin{aligned} a_1\phi(a) + a_2\frac{d\phi}{dx}(a) &= 0 \\ b_1\phi(b) + b_2\frac{d\phi}{dx}(b) &= 0. \end{aligned} \tag{4}$$

there is a procedure that one can use. The starting point of the procedure is to recall two facts

(A) For any λ, every solution to the problem

$$-\frac{d^2\phi}{dx^2} + Q(x)\,\frac{d\phi}{dx} + R(x)\,\phi - \lambda\,\phi = 0 \tag{5}$$

can be expressed as

$$\phi(x) = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$$

where $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are two linearly independent solutions of (5) for a given value of λ.

(B) Two linearly independent solutions to (5) can be obtained by solving two initial value problems for (5) with linearly independent initial conditions. For example

$$-\frac{d^2y_1}{dx^2} + Q(x)\,\frac{dy_1}{dx} + R(x)\,y_1 - \lambda y_1 = 0 \qquad y_1(a) = 1 \quad \frac{dy_1}{dx}(a) = 0$$

$$-\frac{d^2y_2}{dx^2} + Q(x)\,\frac{dy_2}{dx} + R(x)\,y_2 - \lambda y_2 = 0 \qquad y_2(a) = 0 \quad \frac{dy_2}{dx}(a) = 1$$

Any eigenfunction can therefore be expressed as $\phi(x) = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$ for particular values of c_1, c_2 and λ. The values c_1, c_2 and λ associated with a specific eigenfunctions are determined by that requiring that the resulting function satisfy the boundary conditions (4).

Thus, the procedure consists of first forming two linearly independent solutions to (5) and then determining specific values of c_1, c_2 and λ so that the function $\phi(x) = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$ satisfies the boundary conditions. Note: eigenfunctions are only determined up to a multiplicative constant, so either c_1 or c_2 or some linear combination of both of them will remain unspecified.