Black-Litterman Portfolio Optimization: Recognizing uncertainty

Silvano Marchesi

Seminar Applied Portfolio Theory

silvanom@student.ethz.ch

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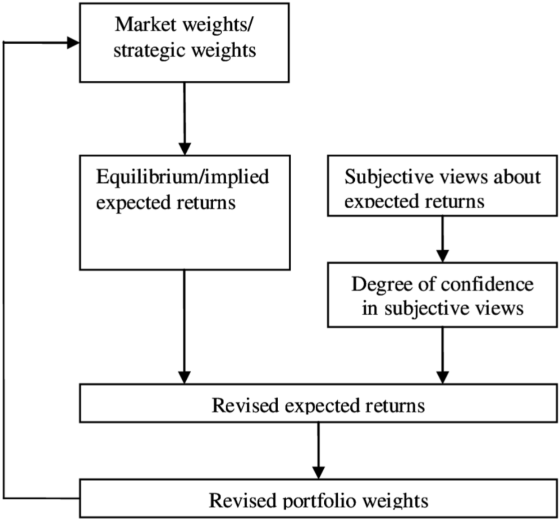
Overview

Black-Litterman Model

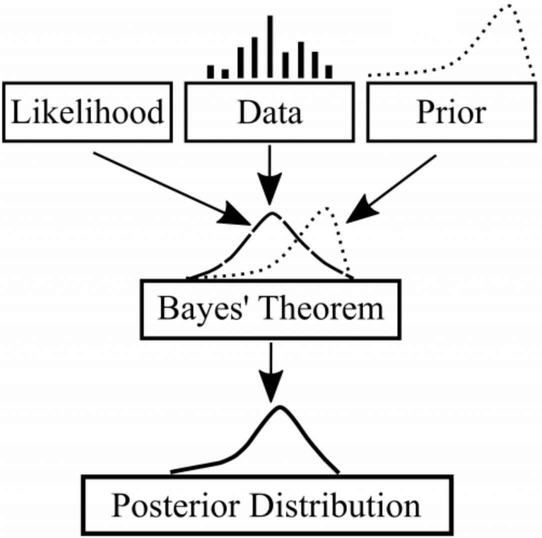
Motivation

- The approach is based on the recognition that there is no certainty when it comes to the prediction of financial markets
- The classical Markowitz-framework does not take uncertainty into account, is based solely on historical data and has many well-known drawbacks
- The Black-Litterman (BL) model addresses (some) of these flaws and tries to improve on them using a Bayesian three-step approach that allows for the inclusion of investor-specific views about the future
- In the BL framework expected returns are random variables!

Approach


Market weights/
strategic weights
Equilibrium/implied
expected returns
Subjective views about
expected returns
Degree of confidence
in subjective views
Revised expected returns
Revised portfolio weights

Visualization



Bayesian prior

Assume the market has the following attributes:

- n assets
- Expected return vector $\mu \in \mathbb{R}^{n \times 1}$
- Expected covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$

and let's assume also the market is in equilibrium and all investors hold the market portfolio, i.e. there exists a representative investor, equipped with a quadratic utility function:

$$\max_{w} \left(w'\mu + (1 - w'\mathbf{1})R_f - \frac{\lambda}{2} w'\Sigma w \right),$$

where λ is the average risk tolerance, w is the portfolio weight vector, R_f is the risk free rate.

Bayesian prior

One can derive the equilibrium excess returns (equilibrium risk premium) by inverting the FOC from the above optimization problem:

$$\underbrace{\mu - \mathbf{1}R_f}_{\Pi} = \lambda \Sigma w_m.$$

Multiply both sides with w and solve for λ:

$$\lambda = \frac{\mu_m}{\sigma_m^2},$$

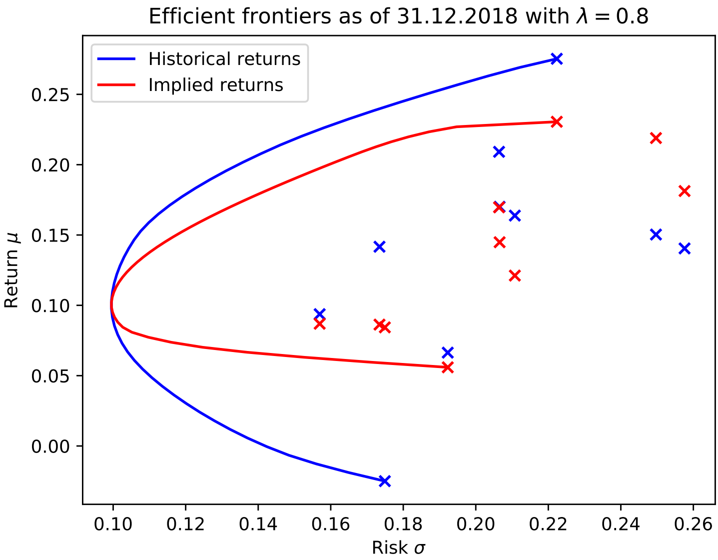
where μ_m is the market's excess return and σ_m^2 its variance.

Example

- Monthly US-stock data from Wharton Research Data Service
- As of 31.12.2018 select 10 stocks with highest market cap
- Lookback-window of 48 months
- Estimates are annualized
- $\lambda = 0.8$, average of annual S&P500 SR from 1934-2018
- $R_f = 0$

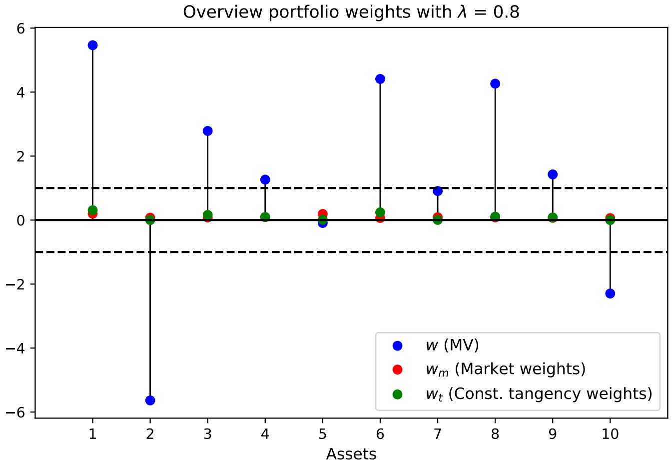
Permno	Stock	Market capitalization in USD	Market weight
10107	MICROSOFT CORP	779'673'563'830.00	0.1984
11850	EXXON CORP	288'703'299'330.00	0.0735
13407	FACEBOOK INC	314'939'267'940.00	0.0801
14542	GOOGLE INC	362'064'786'653.85	0.0921
14593	APPLE COMPUTER INC	748'539'127'973.98	0.1905
21936	PFIZER INC	252'317'733'750.00	0.0642
22111	JOHNSON & JOHNSON	346'109'260'900.00	0.0881
47896	JPMORGAN CHASE & CO	324'626'621'820.00	0.0826
55976	WAL MART STORES INC	270'624'969'000.00	0.0689
59408	BANK OF AMERICA CORP	241'821'814'080.00	0.0615

Example



Efficient frontiers as of 31.12.2018 with $\lambda = 0.8$
Historical returns
Implied returns
Return μ
Risk σ
0.25
0.20
0.15
0.10
0.05
0.00
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26

Example


Overview portfolio weights with λ = 0.8
6
4
2
0
−2
−4
−6
1
2
3
4
5
6
7
8
9
10
Assets
w (MV)
w_m (Market weights)
w_t (Const. tangency weights)

Incorporating Subjective Views

BL allows to efficiently and consistently translate proprietary analysis into views. One can specify the views in the following way:

- By assumption, each view is unique and uncorrelated with other views.
- Either the sum of weights in a view is zero (relative view) or is one (an absolute view).
- One does not require a view on all assets.

The views can be formalized as:

$$q = P\mu + \eta, \;\; \eta \sim \mathcal{N}(0, \Omega), \;\; P \in \mathbb{R}^{k \times n},$$

where $\Omega \in \mathbb{R}^{k \times k}$ is diagonal, i.e., no cross-information on views is assumed.

Example cont.

$$q = P\mu + \eta, \quad \eta \sim \mathcal{N}(0, \Omega), \quad P \in \mathbb{R}^{k \times n}, \Omega \in \mathbb{R}^{k \times k}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -0.2 & 0 & 0.2 & 0 & -1 & 0 \end{bmatrix}$$

$$q = \begin{bmatrix} -0.03 \\ 0.04 \\ 0.02 \\ 0.01 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

Combining Equilibrium Returns & subjective Views

We have

$$\widehat{\mu} = I\mu + \epsilon, \;\; \epsilon \sim \mathcal{N}(0, \tau\widehat{\Sigma}),$$

$$q = P\mu + \eta, \;\; \eta \sim \mathcal{N}(0, \Omega).$$

Let $y = (\widehat{\mu}, q)^T$, $X = (I, P^T)^T \in \mathbb{R}^{(k+n)\times n}$, and $u = (\epsilon, \eta)^T$.
Then $u \sim \mathcal{N}(0, \Psi)$ with

$$\Psi = \begin{pmatrix} \tau\widehat{\Sigma} & 0 \\ 0 & \Omega \end{pmatrix}.$$

Thus, one gets $y = X\mu + u$ and the corresponding GLS estimator:

$$\mu^c = (X^T\Psi^{-1}X)^{-1}X^T\Psi^{-1}y.$$

By plugging in the corresponding values, one gets the BL formula:

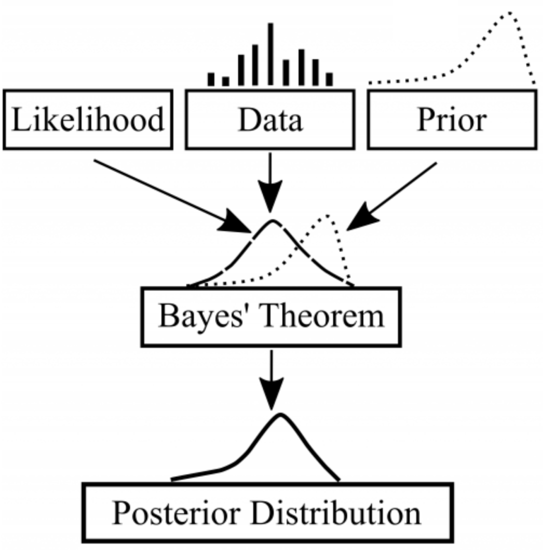
$$\mu^c = \left(\left(\tau\widehat{\Sigma}\right)^{-1} + P^T\Omega^{-1}P\right)^{-1}\left(\left(\tau\widehat{\Sigma}\right)^{-1}\widehat{\mu} + P^T\Omega^{-1}q\right)^{-1}.$$

Combining Equilibrium Returns & subjective Views

$$\mu^c = \left(\left(\tau \widehat{\Sigma} \right)^{-1} + P^T \Omega^{-1} P \right)^{-1} \left(\left(\tau \widehat{\Sigma} \right)^{-1} \widehat{\mu} + P^T \Omega^{-1} q \right)^{-1}$$

- First factor serves as normalization.
- Second factor balances between eq. returns $\widehat{\mu}$ and views P:
  - $(\tau\widehat{\Sigma})^{-1}$ serves as weighting/precision factor for the eq. returns.
  - $P^T\Omega^{-1}$ serves as weighting/precision factor of the views.
- Limit "no view" $P = 0$, then $\mu^c = \widehat{\mu}$.
- Limit "absolute confidence" $\Omega \to 0$, then $\mu^c = P^{-1}q$.
- Limit "no confidence" $\Omega \to \infty$, then $\mu^c = \widehat{\mu}$.
- Limit "no estimation error" $\tau \to 0$, then $\mu^c = \widehat{\mu}$.
- Limit "infinite estimation error" $\tau \to \infty$, then $\mu^c = P^{-1}q$.

Combining Equilibrium Returns & subjective Views

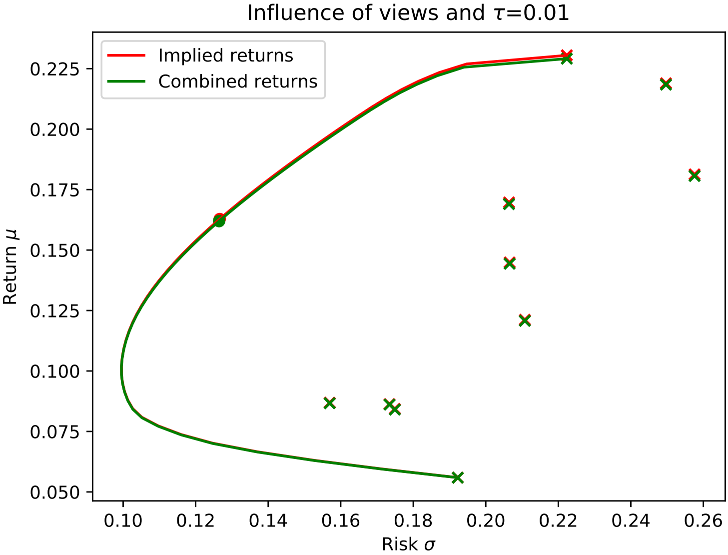


Combining Equilibrium Returns & subjective Views

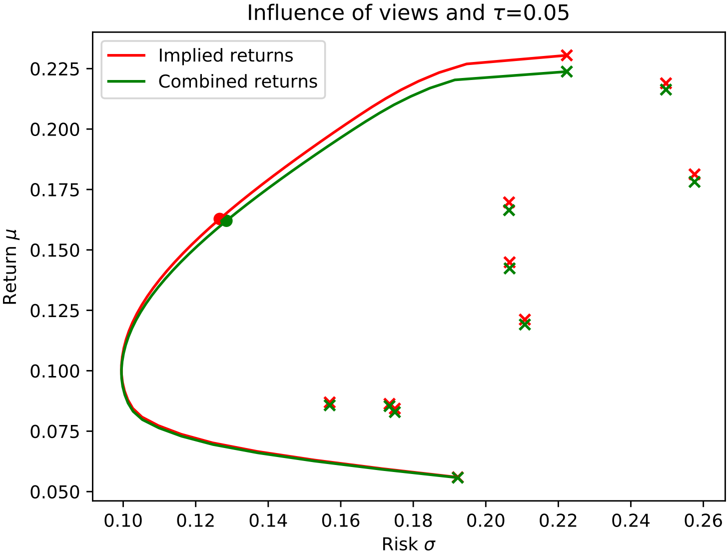
The third and last step is a Markowitz optimization, but with the adjusted mean and the given covariance structure. This gives the whole efficient frontier. To get the weights for the market portfolio with views, we can plug in

$$
\begin{aligned}
w^c &= \frac{1}{\lambda}\widehat{\Sigma}^{-1}\mu^c \\
&= \frac{1}{\lambda}\widehat{\Sigma}^{-1}\left(\widehat{\mu} + \tau\widehat{\Sigma}P^T\left(\tau P\widehat{\Sigma}P^T + \Omega\right)^{-1}(q - P\widehat{\mu})\right) \\
&= w + P^T\left(P\widehat{\Sigma}P^T + \frac{1}{\tau}\Omega\right)^{-1}\left(\frac{1}{\lambda}q - P\widehat{\Sigma}w\right).
\end{aligned}
$$

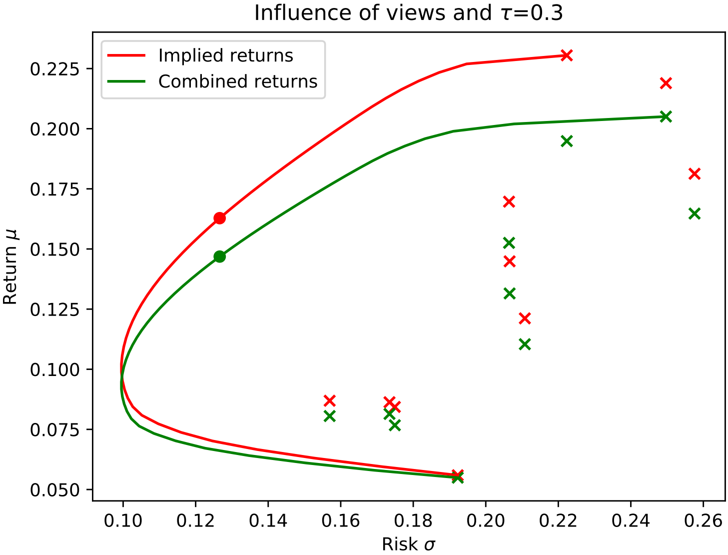
Parameter Variation



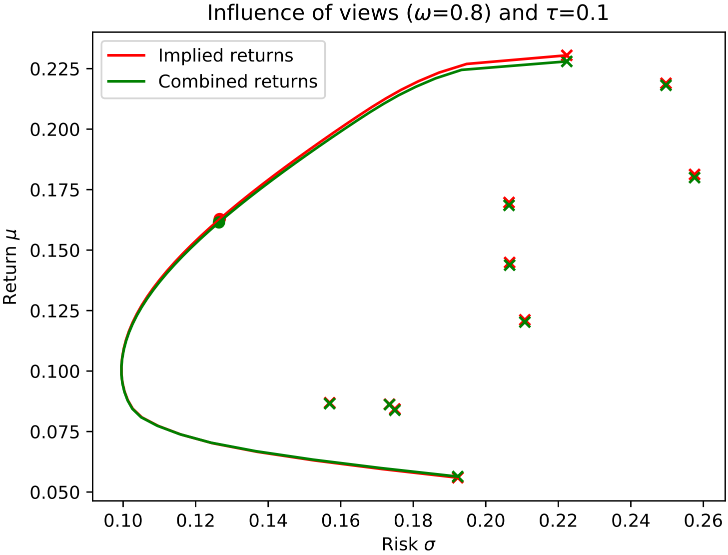
Parameter Variation



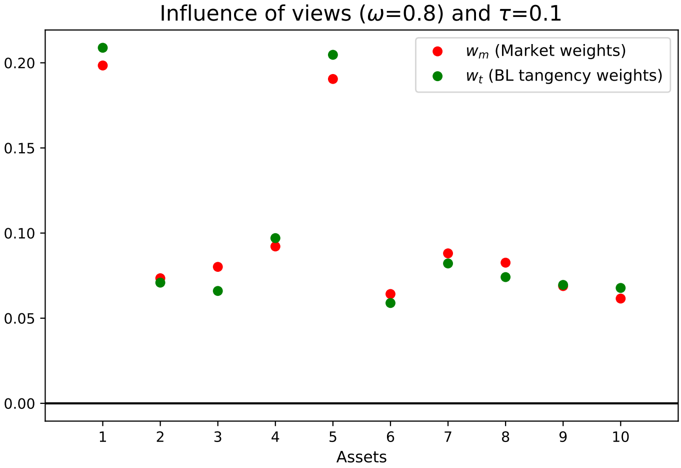
Parameter Variation



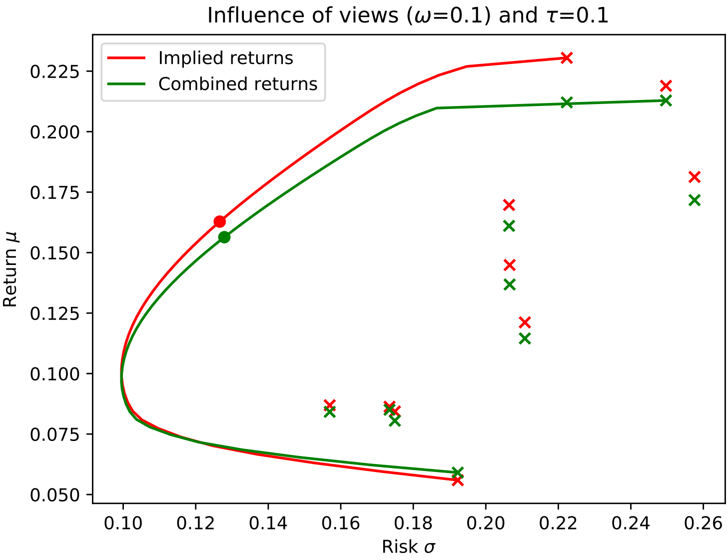
Parameter Variation



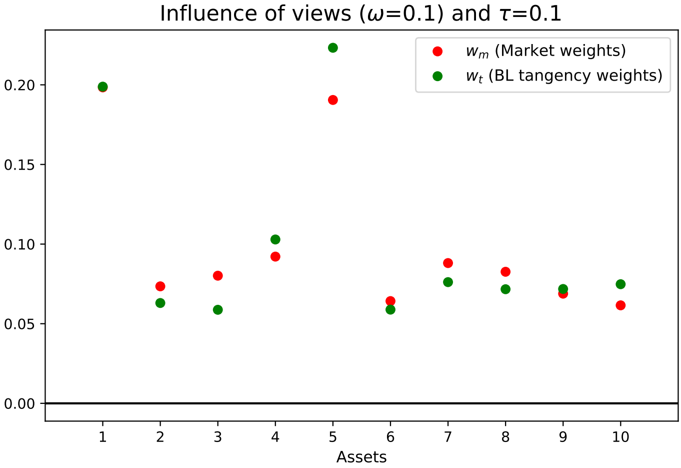
Parameter Variation



Parameter Variation

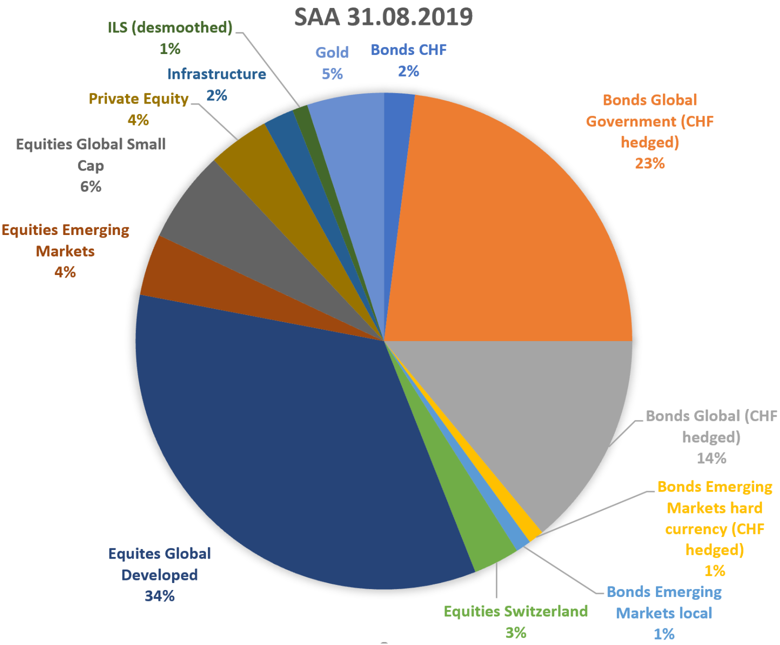


Parameter Variation



Computations

Strategic Asset Allocation (SAA) as of 31.08.2019



Backward Computation of the SAA

	31.03.2002	31.12.2001	31.01.1999	before
Bonds CHF	2.00%	2.02%	2.06%	2.20%
Bonds Global Government (CHF hedged)	23.00%	23.23%	23.71%	25.27%
Bonds Global (CHF hedged)	14.00%	14.14%	14.43%	15.38%
Bonds Emerging Markets hard currency (CHF hedged)	1.00%	1.01%	1.03%	1.10%
Bonds Emerging Markets local	1.00%	1.01%	1.03%	1.10%
Equities Switzerland	3.00%	3.03%	3.09%	3.30%
Equites Global Developed	34.00%	34.34%	35.05%	37.36%
Equities Emerging Markets	4.00%	4.04%	4.12%	4.40%
Equities Global Small Cap	6.00%	6.06%	6.19%	
Private Equity	4.00%	4.04%	4.12%	4.40%
Infrastructure	2.00%	2.02%		
ILS (desmoothed)	1.00%			
Gold	5.00%	5.05%	5.15%	5.49%
	100.00%	100.00%	100.00%	100.00%

Model Specification

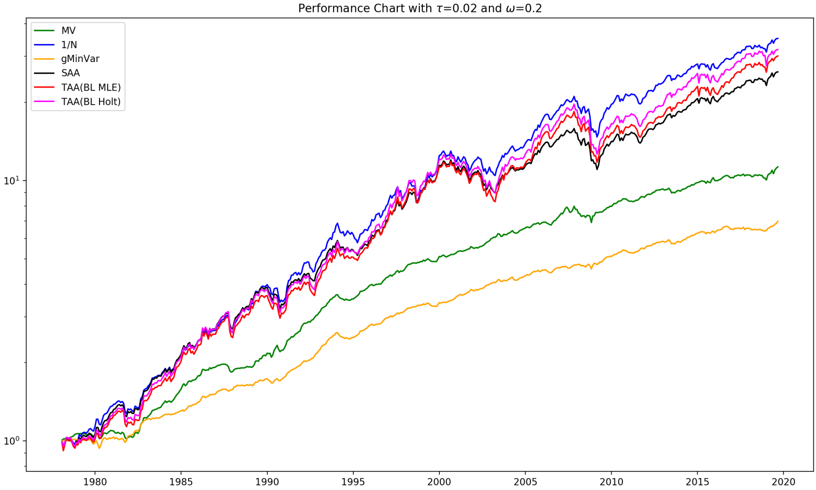
Parameter values:

- Rolling lookback-window of 48 months
- $\lambda = 4$ (calibrated to data)
- $\tau = 0.02$
- $\omega = 0.2$

Models for view-generation:

- Model 1: MLE (Historical) ($q = \mu$, $\omega_i = \sigma_i$)
- Model 2: HOLT Linear trend ($q = r_{t+h|t}$, ω)
  - $r_{t+h|t} = \ell_t + hb_t$
  - $\ell_t = \alpha r_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
  - $b_t = \beta(\ell_t - \ell_{t-1}) + (1-\beta)b_{t-1}$
  - $\alpha = 0.8, \beta = 0.2$

Results with $\tau = 0.02$ and $\omega = 0.2$



Results with $\tau = 0.02$ and $\omega = 0.2$

Portfolio	gMean p.a.	StdDev p.a.	Sharpe-Ratio	Skewness	Kurtosis	1%-Var	1%-ES
MV	0.0597	0.044	0.7628	-0.109	3.0089	-0.0346	-0.0384
gMinVar	0.0577	0.0323	0.6536	-0.1211	2.2915	-0.0208	-0.0271
1/N	0.0893	0.0945	0.6964	-0.5777	2.2508	-0.0646	-0.0915
SAA	0.0818	0.0896	0.6451	-0.4351	1.1562	-0.0588	-0.0784
TAA (BL MLE)	0.0856	0.1048	0.6008	-0.6681	1.8833	-0.0744	-0.1029
TAA (BL Holt)	0.0868	0.1061	0.6082	-0.671	1.9192	-0.0821	-0.1029

Conclusions

Conclusions

- We have discussed the well-known Black-Litterman model, laid focus on involving equilibrium-implied returns, incorporating investor specific views, identifying uncertainty and combining views
- Implied views from a benchmark can yield particularly valuable information
- If the implied views do not correspond with individual long-term view, it might be wise to take a second look at the benchmark
- Investors should be aware that expectations for returns are not point estimates and but rather random variables surrounded uncertainty
- The model gives investors a large degree of freedom in terms of the possibility to incorporate subjective views but also demands specification of many parameters

An illustrative example

[?]

References

Guangliang He and Robert Litterman, *The intuition behind black-litterman model portfolios*, Available at SSRN 334304 (1999).