Financial Optimization Models for Portfolio Asset Allocation

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Abstract. The purpose of this report is to compare the performances of three financial optimization models: (1) Mean-Variance Optimization, (2) Mean-Variance Optimization with cardinality constraints and short-selling prohibited and (3) the Black-Litterman model. Techniques 1 and 2 were designed to use the Fama-French three factor model for parameter estimates and technique 3 incorporated investor views. The performance of each optimization model was compared by running an investment simulation based on the weekly adjusted closing prices of twenty stocks from the S&P500, beginning in 2013 and ending in 2015. It was observed that the largest return was realized under the third optimization method, returning 90% (on time zero value) by the end of the simulation. The first and second portfolios returned 68% and 43% respectively.

1 Methodology

We motivate our work by formally stating our goal: we are interested in exploring the capabilities of financial models to construct an optimal portfolio of stocks. We turn to optimization models because we are interested in developing an efficient framework to justify our asset allocations to minimize risk of loss and ensure a confidence of gain.

The main optimization framework that we will develop in this project is a Mean–Variance optimization (MVO), which is a convex quadratic optimization problem. Applied to a portfolio of interest, MVO solves for an optimal set of asset weightings, subjected to the preferences of the investor. There is flexibility when it comes to the structure of an MVO and we shall explore specific models based on MVO in section 2.

The models presented in section 2 will be used to construct portfolios which will consist of up to twenty assets, all from the S&P500. The simulation is run from 2013 to 2015 based on the weekly adjusted prices of the twenty stocks. Since we are using historical data, we have the luxury of immediately comparing our results to realized returns.

The remainder of this section will be dedicated to developing our investment parameters for an MVO, namely our portfolio's Mean and Variance.

Portfolio Mean and Variance: Central to an MVO are the parameter we rely on. Suppose we are interesting in tracking up to n assets, each having the following properties

- a realized return of r_i,
- an expected return of $\mathbb{E}(r_i) = \mu_i$,
- a return variance of σ_i^2, and
- a return covariance of σ_{ij} with a different asset j

We can setup our optimization problem by establishing the decision variables x_i which correspond to the proportions of our wealth to invest in each asset i. With that, we can state the expected return μ_p and variance of our portfolio σ_p^2

$$\mu_p := \sum_{i=1}^{n} \mu_i x_i \tag{1}$$

$$\sigma_p^2 := \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \tag{2}$$

It is convenient to develop our parameters in matrix form, so we define $\mu \in \mathbb{R}^{n\times 1}$ to be the vector of expected returns, $\mathbf{x} \in \mathbb{R}^{n\times 1}$ to be the vector of optimized asset allocations and $Q \in \mathbb{R}^{n\times n}$ to be a symmetric covariance matrix. We express these quantities explicitly

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad Q = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{pmatrix}$$

The equations above provide the parameters for an MVO, but we need to justify how to arrive at parameter estimates. For such a task, we use a factor model.

Factor Model: A factor model attempts to explain the returns of asset by defining the returns as a function of several observed factors. Factor models can range in complexity being driven by different economic views which are considered important drivers for the return rate of an asset. The generic form of a k factor model is a regression model:

$$r_i := \alpha_i + \sum_{i=1}^{k} \beta_{ik} f_k + \epsilon_i \tag{3}$$

Where r_i is taken to be the rate of return of asset i. Note that we preserve randomness of the return r_i by noting that our factors may be noisy and ϵ_i represents idiosyncratic risk particular to the asset. Note that the following conditions are assumed when in an ideal environment

- The factors are independent from one another, $\sigma_{f_i,f_j} = 0 \quad \forall i \neq j$
- The factors are independent to idiosyncratic noise, $\sigma_{f_i,\epsilon_j} = 0 \quad \forall i, j$
- Idiosyncratic risk from one asset is independent to idiosyncratic risk from another, $\sigma_{\epsilon_i,\epsilon_j} = 0 \quad \forall i \neq j$
- Idiosyncratic risk is normally distributed with mean 0 and variance σ_{ϵ_i}, meaning $\epsilon_i \sim \mathcal{N}(0, \sigma_{\epsilon_i})$

For the sake of developing our parameter estimates, we will only be considering the last conditions. **We will not be assuming ideal conditions in our project**. We do this because, as we shall soon see, it often is the case that it is difficult to establish parameters which are independent of each other. With that, we proceed with our parameter estimates from the following

$$\mu_i := \mathbb{E}(r_i) \tag{4}$$

$$\sigma_i^2 := \mathbb{E}\big(r_i - \mathbb{E}(r_i)\big)^2 \tag{5}$$

$$\sigma_{ij} := \mathbb{E}\big(r_i - \mathbb{E}(r_i)\big)\big(r_j - \mathbb{E}(r_j)\big) \tag{6}$$

Where $\bar{f}_j$ is the geometric mean of the j^{th} factor. We note that our answers above greatly simply under the assumptions of an ideal environment. Again, it is easier to work factor models when in matrix form, so we express equations for $\mu \in \mathbb{R}^{n\times 1}$ and $S \in \mathbb{R}^{n\times n}$

$$\mu = \beta^T \mathbf{f} + \epsilon \tag{7}$$

$$S = \mathbf{x}^T Q \mathbf{x} + D \tag{8}$$

Where we have

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_n \end{pmatrix} \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \quad D = \begin{pmatrix} \sigma^2_{\epsilon_1} & \sigma^2_{\epsilon_{12}} & \dots & \sigma^2_{\epsilon_{1n}} \\ \sigma^2_{\epsilon_{21}} & \sigma^2_{\epsilon_2} & \dots & \sigma^2_{\epsilon_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{\epsilon_{n1}} & \sigma^2_{\epsilon_{n2}} & \dots & \sigma^2_{\epsilon_{nn}} \end{pmatrix}$$

Fama-French Three Factor Model: The Fama-French three-factor model will be what is used in this project for MVO parameter estimation. The model is as follows

$$r_{ip} - R_f := \alpha_i + \beta_{im}(f_m - R_f) + \beta_{is} SMB + \beta_{iv} HML \tag{9}$$

For which

- $r_{ip} - R_f$ is the excess return of asset i with respect to the risk free rate R_f during period p
- α_i is the intercept term, representing the absolute return
- β_{im} is the market risk factor loading, tracks how an asset moves with the market portfolio

- $f_m - R_f$ is a factor that tracks the excess return of the market portfolio
- β_{is} is the size risk factor loading, relates the performance of the stock's return to it's company's size
- SMB is a factor taken as the average return of a portfolio composed of small cap stocks minus the average return of large cap stocks
- β_{iv} is the value risk factor loading, relates the perceived value of the company's stock to performance
- HML is a factor computed as the average return of a portfolio composed of stocks with the highest Book-to-Market ratio minus the stocks with the lowest Book-to-Market ratio.

The Fama-French Three factor model does not satisfy the idea conditions for a factor model, since the factors may very well be dependent. So, it is justified to use equations 7 and 8 for parameter estimates. The last part to this section will summarize how to determine the factor loadings through a linear regression.

Linear Regression with the Fama-French Three Factor Model: The general factor model can be represented as a linear regression model. For this, we a dataset of size p containing known returns and factors for up to n assets. This project will perform the regression for $n = 20$ stocks based on $p = 52$ sets of returns and factors.

With the Fama-French Three Factor model, we will solve for the factor loading terms by writing equation 9 in the following form.

$$\mathbf{R} = X\beta + \epsilon \tag{10}$$

Here we have are

$$\mathbf{R} = \begin{pmatrix} r_{11} - R_f & r_{21} - R_f & \dots r_{n1} - R_f \\ r_{12} - R_f & r_{22} - R_f & \dots r_{n2} - R_f \\ \vdots & \vdots & \vdots \\ r_{1p} - R_f & r_{2p} - R_f & \dots r_{np} - R_f \end{pmatrix} \quad X = \begin{pmatrix} 1 & f_{m1} - R_F & SMB_1 & HML_1 \\ 1 & f_{m2} - R_F & SMB_2 & HML_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & f_{mp} - R_F & SMB_p & HML_p \end{pmatrix}$$

$$\beta = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_{11} & \beta_{21} & \dots & \beta_{n1} \\ \beta_{12} & \beta_{22} & \dots & \beta_{n2} \\ \beta_{13} & \beta_{23} & \dots & \beta_{n3} \end{pmatrix}$$

In which the matrix $X \in \mathbb{R}^{p\times 4}$ is our design matrix containing all of the regression information about our factors, $R \in \mathbb{R}^{p\times n}$ is the realized returns, $\beta \in \mathbb{R}^{4\times n}$ is the regression coefficients we wish to solve for and. Completing the linear regression (requiring that X is nonsingular) we calculate for β using the following closed form solution

$$\mathbf{b} = (X^T X)^{-1} X^T \mathbf{R} \tag{11}$$

Based on equation 11, we can determine our model error terms by taking

$$\epsilon = \mathbf{R} - X\mathbf{b} \tag{12}$$

Which provides a ready means to determine D in equation 8.

2 Portfolio Optimization Models

We use three optimization techniques based on the MVO introduced above. In this section, each model will be presented and briefly explained. The models are as follows:

1. The standard MVO model.
2. The cardinality–constrained MVO model.
3. The Black–Litterman model

2.1 Standard MVO Model

The MVO model we take as standard will seek to minimize portfolio risk subjected to the following constraints.

Return Constraint: We demand a *minimum* portfolio mean of $\mathbf{R}$, which will be taken as the average yearly return for each asset. This constraint ensures that we are adequately rewarded for the risk we assume.

Budget Constraint: We wish to use all of our available funds for asset allocation, so the sum of the proportions of our constituent assets should sum to unity.

Short-Selling Permitted: We do not prohibit taking on a short position in our portfolio. This means that the proportions of our assets are unbounded.

With these constraints, we present our standard MVO model as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T Q \mathbf{x} \\ st \quad & \mu^T \mathbf{x} \geq \mathbf{R} \\ & \mathbf{1}^T \mathbf{x} = 1 \\ & x_i \in \mathbb{R} \quad \forall i = 1 \ldots n \end{aligned}$$

Parameter Estimates: Most of section 1 was dedicated to explaining how parameters are estimated in an MVO formulation. Here we use this the Fama-French Three Factor model to estimate our expected returns μ and covariances Q.

2.2 Cardinality Constrained MVO Model

We build on the standard MVO model by introducing cardinality constraints to limit our exposures to all assets. We do this by designing our portfolio to hold a subset of cardinality k of the available assets. The result is a portfolio that is less diversified across the available assets. This may be desired if the investor is not confidence in the realization of their parameter estimates or if the investor believes the assets are strongly correlated. For this model, we have the following constraints.

Return Constraint: This will be the same as the standard MVO model. We demand a *minimum* portfolio mean of $\mathbf{R}$, which will be taken as the average yearly return for each asset.

Budget Constraint: This also will be the same as the standard MVO model. We want to ensure that we allocate all of our available funds.

Prohibiting Short-Selling: Here we impose that we may not take on a short position on any of our assets. Perhaps our risk aversion leads us away from shorting in fear of possible loss. This conservative approach strictly imposes bounds between 0 and 1 (inclusive) for all of our proportions.

Cardinality Constraint: We wish to hold up to a maximum of 12 stocks in our portfolio, of the available 20 at any given time. This constraint is the biggest change to our standard MVO model, transforming the optimization problem into a Mixed-Integer Quadratic Program (MIQP).

To accommodate this change, we need to add additional decision variables to $\mathbf{x}$. Like the standard MVO model, we have n decision variables for which $x_i \in \mathbb{R} \cap [0, 1]$ but now we need to introduce n additional binary variables $x_i^* = 0, 1$ which will indicate whether or not we are golding to commit to holding asset i.

Our new decision variable $x \in \mathbb{R}^{2n\times 1}$ looks like

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$$

And careful attention is required in setting up the optimization program since parameter estimates only apply to our continuous variables. We arrive at the formulation

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{x}^T \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{x} \\
st \quad & \begin{pmatrix} \mu \\ \mathbf{0} \end{pmatrix}^T \mathbf{x} \geq \mathbf{R} \\
& \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}^T \mathbf{x} = 1 \\
& \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}^T \mathbf{x} = 12 \\
& 0 \leq x_i \leq x^*_{i+n} \quad \forall i = 1 \dots n \\
& x_i \in \mathbb{R} \quad \forall i = 1 \dots n \\
& x_{n+i} = \{0, 1\} \forall i = 1 \dots n
\end{aligned}$$

Why $\boldsymbol{0 \leq x_i \leq x^*_{i+n}}$? We need a constraint to go along with our cardinality imposition to ensure that assets we wish to neglect from our portfolio are forced to a zero holding. For instance, if asset 2 is to be excluded, then $x^*_{22} = 0$ and by the nested inequality, $x_2 = 0$. If we wish to include the asset, the upper bound will be set to 1. So this constraint ensure both cardinality and our prohibition of short-selling.

Parameter Estimates: Just like our standard MVO model, we shall determine μ and Q from the Fama-French Three Factor model.

2.3 Black-Litterman Model

The Black-Litterman Model offers an alternative approach to providing parameter estimates for an MVO. Where in the previous models we relied solely on the Fama-French Three Factor model, the Black-Litterman model allows an investor to provide their own expected views on the performance of an assets returns and combine this with a market equilibrium position. The optimization constraints will be the same as in our standard MVO model, so we will dedicate most of this section to understanding the formulation of expected returns.

Under the Black-Litterman model, the expected return $\bar{\mu}$ which is adjusted for investor with r views concerning n is presented below

$$\bar{\mu} := \left(\tau Q^{-1} + P^T \Omega^{-1} P\right)^{-1} \left(\tau Q^{-1} \mathbf{\Pi} + P^T \Omega^{-1} \mathbf{q}\right) \tag{13}$$

Where

- $\tau \in \mathbb{R}$ is the covariance dampening factor, taken to be uniformly random between 0.01 and 0.05.
- $P \in \mathbb{R}^{r \times n}$ is what selects which assets are involved in a certain view.
- $\Omega \in \mathbb{R}^{r \times r}$ is a diagonal matrix portraying the investors certainties in each view
- $\mathbf{q} \in \mathbb{R}^{r \times 1}$ is the investors view vector
- $\mathbf{\Pi} \in \mathbb{R}^{n \times 1}$ is the asset return vector at market equilibrium

Several of these parameters are calculated deterministically, as follows

$$\mathbf{\Pi} := \frac{\mathbb{E}(r_m - R_f)}{\sigma_m^2} Q\mathbf{x_{mkt}} \tag{14}$$

$$\Omega_{ii} := \tau P_i Q P_i^T \quad \forall i = 1 \dots r \tag{15}$$

Here $\mathbf{P_i}$ is the i^{th} row of P, r_m is the return of the market portfolio and $\mathbf{x_{mkt}}$ is the vector of market-cap weightings. The market capitalization weighting of asset i with price c_i and shares outstanding N_i in a portfolio of n stocks is calculated as

$$x_{mkt,i} := \frac{p_i N_i}{\sum_{k=1}^{n} p_k N_k} \tag{16}$$

How to determine $\boldsymbol{\tau}$? Tau is not derived, rather, this parameter is used as a dampening coefficient for market variances so we concede that the true value is unknown. To account for this uncertainty, we take the following value for τ

$$\tau = \frac{0.05}{U} \tag{17}$$

$$U \sim unif[1, 5] \tag{18}$$

Given that U is a uniform draw, we take $\tau \sim unif[0.01, 0.05]$, in an attempt to cover the range of plausible values for τ.

All that is left to cover is our views vector $\mathbf{q}$ and the corresponding P matrix. These parameters are subjective and for the model that I will present will use six views, summarized below.

Company Size View: I contend that with this view, the ten largest cap stocks will under–perform our 10 smallest cap stocks by 2% per investment period. This view is dynamic, so our q_i vector entry and P_i column will change per investment period.

The inclusion of this view is meant to capture, in a very basic sense, how the size of a company influences its prospects for earnings. I contend that smaller companies are better able to return higher (more volatile) earnings owing to agility.

To update this view after each iteration, it is required to order our stocks in terms of market cap, and selecting ten column entries to assign as -1 (our large cap stocks) and ten entries to assign as 1 (our small cap stocks).

Relationship between Tech Stocks and Financials: Pertaining to the sensitivity in the relationship between the Information Technology and Financials Sector. If the combined average return of AAPL and IBM exceeds the combined average return of C, WFC and JPM, our view will be that tech stocks will outperform financial by an amount that is 25% higher than the difference between the average. If underperformed, the view will be 10% lower.

The inclusion of this view is meant to demonstrate how the Black–Litterman is able to accommodate relative performance metrics between different sectors. Considering this magnitude of this view, the economic interpretation can be be viewed as a strong correlation of the returns of tech stocks and financials.

This view is updated on a per period basis, and involves calculating the average returns of tech stocks and financial stocks. The row of P however will not change, as the view always pertains to the same stocks. What will get updated upon each period is the corresponding entry in q.

Momentum in Energy Stocks: This will make up our third and fourth view. Separately, if the average return of XOM and MRO is positive, the view predicts the average return will be 5% higher. If negative, 15% lower.

The goal of this view is to show that the Black-Litterman model is capable of providing views for individuals stocks, irrespective of asset class or sector. In this case, it is also meant to demonstrate a view with a larger downside return when compared to upside return. You may justify this by considering a period of poor returns is penalized more heavily than a period of good returns. I chose this view to be representative of Energy stocks which are subject to varying array of downside risks – a supply shortage in natural gas likely will have a prolonged effect on the company's performance.

Momentum in Telecommunication Stocks: This perspective will motivate our fifth and sixth view. Particularly, this will pertaining to a basic consideration of momentum in Telecommunications stocks. If the, over a given period, average return of T and Vz is positive, the view predicts the average return will be 10% over the next period higher for each of the stocks. If negative, then each stock will have an expected return that is 1% lower.

Including this view was done to mirror the previous views with the Energy stocks, but instead meant to demonstrate a large upside potential for return. This may or may not be representativeness of the telecommunications sector – the selection of these stocks were naive.

With an understanding with our views, we present our Black–Litterman model as the following optimization

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \frac{\lambda}{2}\mathbf{x}^T Q\mathbf{x} - \bar{\mu}^T\mathbf{x} \\
& \mathbf{1}^T\mathbf{x} = 1 \\
& x_i \in \mathbb{R} \quad \forall i = 1 \ldots n
\end{aligned}
$$

Budget Constraint: As in the case with all of our optimization models, we look to fully allocate our resources. Money sitting around isn't working for anyone.

Short-selling Permitted: As with our standard MVO model, we allow of portfolio to take on a short position on several assets.

Return Constraint? Upon first glance it appears that we are not demanding a minimum accepted return for assuming the risk of our position. This is true, and represents a significant difference between this model and the previous two that we have explored previously.

Note that our objective function looks different from that presented in our standard MVO and cardinality constrained models. The inclusion of the linear in the objective function changes the geometry of the problem. In this case, we are interested in minimizing the distance between the returns implied from the market and the returns implied from our investor views. This is why we adopt the coefficient λ preceding the quadratic term – this coefficient is called the risk aversion coefficient and is something that we have seen before

$$
\lambda := \frac{\mathbb{E}(r_m - R_f)}{\sigma_m^2}
$$

We can think of the risk aversion coefficient as providing a measure for the amount of return per unit risk assumed. We need this coefficient so that the difference we are optimizing is dimensionally consistent.

2.4 Transaction Costs

Noticeably absent in all of our optimization models is the cost to transact. The transactional costs varies depending on the investor – large institutional firms pay a reduced price per transaction when compared to a lay investor because institutions have larger trading volumes.

Depending on the motive, transaction costs may be included in the budget allocated for investing. This penalizes high frequency re-balancing of the portfolio as this reduces the potential gain from the portfolio investment strategy. Additional develops to the optimization framework could include constraints limiting the fees incurred from transaction costs

For this project, we shall consider having access to an express account without limit. We will model our transactional cost based on 0.5% of the trading volume at the stocks current price p_i. This is calculated

$$c_i = 0.005 p_i \Delta_i \tag{19}$$

Where $\Delta_i \in \mathbb{R}$ is size of the transaction. Our comparison of the three optimization models will include monitoring the incurred transactional costs on a per-period basis.

3 Results

Table 1 has the list of stocks tracked under each portfolio optimization method. Weekly adjusting closing prices were used from the start of 2012 to the end of 2015. The simulation began in 2013 and ran over the two years, with the portfolios being rebalanced every six months. The calibration of the factor model was done during the start of each investment period and used a years worth of prior weekly adjusted closing prices.

Shown immediately below is a plot which tracks the weekly value of each optimization model's investment portfolio. The value of the Black-Litterman model was observed to be drastically improved from the standard MVO model and the cardinality constrained MVO model, which performed similarly throughout the simulation.

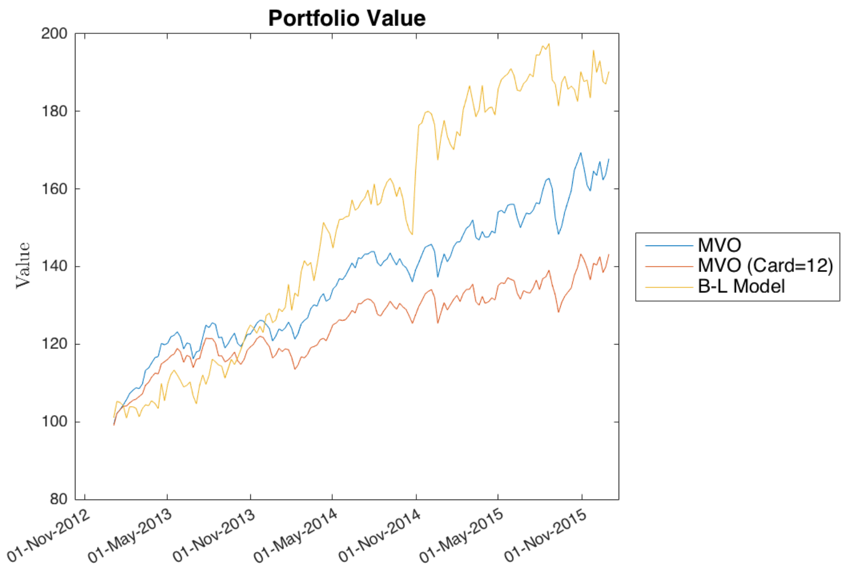

Fig. 1. Portfolio Performance of the Three Optimization Models

We see from figure 1 that in the long run, the BL outperformed the MVO and MVO-card models. We also note an increased performance owing to the permission of short-selling, as with MVO and BL. Additionally, MVO and MVO card performed very similar in terms of local trending.

Table 1. 20 Stocks from the S&P500 tracked

S&P500 Sector	**Company Tickers**		
Consumer Discretionary	F	MCD	DIS
Consumer Staples	KO	PEP	WMT
Financials	C	JPM	WFC
Healthcare	PFE	JNJ	
Industrials	CAT		
Energy	MRO	XOM	
Information Technology	AAPL	IBM	
Materials	NEM		
Utilities	ED		
Telecommunications	T	VZ	

Table 2. Weekly Portfolio Value Metrics

	MVO	**MVO-Card**	**BL**
Maximum Value (\$)	169	143	197
Week # for Maximum Value	148	148	138
Minimum Value (\$)	99.4	99.0	100
Week # for Minimum Value	1	1	0
Largest Weekly Gain (\$)	5.48	4.29	16.3
Week # for Largest Weekly Gain	121	151	95
Largest Weekly Loss (\$)	7.72	6.70	9.38
Week # for Largest Weekly Loss	139	102	138
Longest Winning Streak	8	13	6
Week # When Longest Winning Streak Ended	114	14	76
Longest Losing Streak	4	4	4
Week # When Longest Losing Streak Ended	51	36	36

Summarized in table 2 are some quick glance weekly portfolio value metrics. This information is useful in understanding how each portfolio model performs weekly. For instance, we see a similarity between MVO and MVO-Card. We also can infer information about the conditions and health of the market – we notice that our more aggressive portfolio models suffered their largest loss during near weeks 138 and 139, which suggests a period of market turmoil or downturn. To better see the underlying performance of each method, consider table 3 which summarizes average per-period rates of return and standard deviation.

Reported in table 3 are two average returns (r_p and r_a) and one standard deviation σ. Here, r_p is the average portfolio return over each period, based on the value of the portfolio at the beginning of each period. This is meant to provide an indication of how well an investor would have done had they invest money in the portfolio at the start of each period. For example, if an investor were to have put \$1 in the MVO portfolio at the beginning of period 2, they would expect to have \$1.05 by the end of period 2 (with $r_p = 5\%$).

Table 3. Per Period Rates of Return for our investment Models

	MVO			**MVO-Card**			**BL**		
	r_p	r_a	σ	r_p	r_a	σ	r_p	r_a	σ
Period 1	0.14	0.14	0.07	0.11	0.11	0.06	0.06	0.06	0.04
Period 2	0.05	0.22	0.02	0.04	0.18	0.02	0.11	0.18	0.06
Period 3	0.07	0.30	0.05	0.04	0.21	0.04	0.13	0.43	0.07
Period 4	0.00	0.42	0.02	-0.01	0.30	0.02	0.05	0.63	0.06
Period 5	0.06	0.49	0.03	0.04	0.33	0.02	0.05	0.82	0.03
Period 6	0.05	0.60	0.04	0.02	0.37	0.03	0.01	0.89	0.02

On the other hand, r_a provides an aggregate average return measured per period, which is based on the initial value of portfolio at the start of the investment simulation. This provides a measure to determine how well an investor would have done had they invest money (and kept their money) in the portfolio at the time 0. For instance, if an investor were to have put in \$1 in the BL portfolio at time 0, the average returns would dictate that this dollar would grow to \$1.89 by the end of 6 investment periods (with $r_a = 89\%$).

Lastly, the risk measure σ for each portfolio is the standard deviation of the portfolio's returns per period. It is not surprising that our most aggressive optimization model, being the BL model, consistently has the highest level of risk. Similarly our most conservative model, the MVO-Card, was the least volatile.

When it comes to evaluating volatility, we can calculate our portfolios per-period Value at Risk.

Value at Risk: Value at Risk (VaR) provides a measure to quantify the extent of potential loss. VaR estimates the maximum possible loss of a portfolio of assets over an interval of time and is associated with some probability. For instance if we report a 1 day 5% VaR of 100, this implies that the portfolio has a 0.05 probability of decreasing in value by more than 100. From this, we can infer our portfolio will suffer losses of at least 100 for every 1 day in 20.

VaR is a useful reporting tool to for portfolio risk management. We report VaR calculations on a per-period basis for the following probability levels: $p = 0.01$, $p = 0.05$, $p = 0.10$. For this, we use MATLAB's built in VaR function called *portvrisk*. Table 4 summarizes these VaR results.

Table 4. Per Period Rates of Return for our investment Models

	MVO			**MVO-Card**			**BL**		
p value	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
Period 1	3.7	0	0	3.5	0	0	2.4	0	0
Period 2	0	0	0	1	0	0	4.5	0	0
Period 3	6.9	1.9	0	7.6	4.0	2.1	6.3	0	0
Period 4	6.2	4.6	3.7	5.8	4.3	3.5	16.3	8.9	5.0
Period 5	0.8	0	0	0.8	0	0	5.8	1.4	0
Period 6	6.1	1.9	0	6.7	3.8	2.2	8.0	5.0	3.45

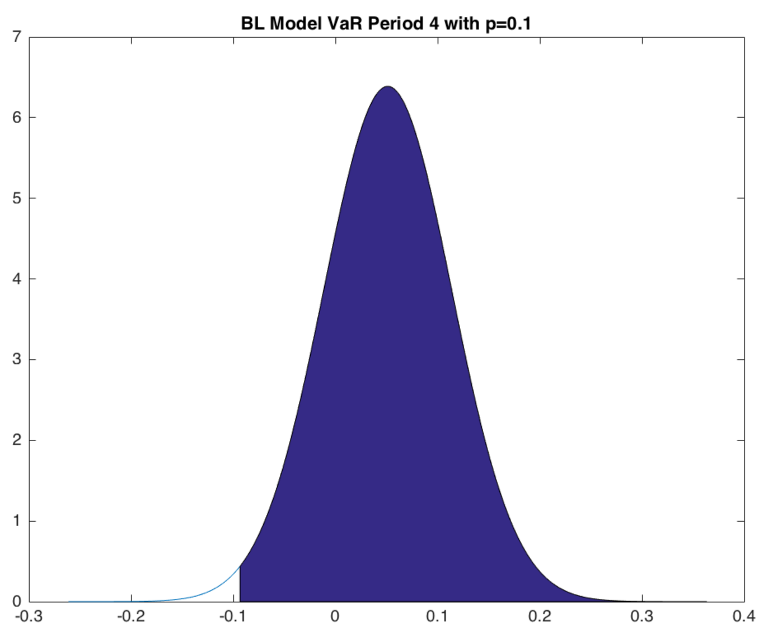

Fig. 2. VaR formulation for BL Model over Period 4 with p=0.10.

Shown in figure 2 is a graph of the normal density plot associated with the BL model over period 4 with $p = 0.1$. Note here that it is the unshaded tail region that corresponds to the maximum loss calculated to be 5. Which is to say, the shaded region of captures 90% of the area under the graph.

First note that our VaR values decrease for increasing values of p. This makes sense because as we widen our probability of loss, we should expect to see lower amounts of loss. Our VaR data nicely complements our understandings of our portfolios average returns and risk. To illustrate this, consider the row for period 4 in both tables 3 and 4. During this period, all portfolio models performed worse than in period 3. We see this from the decline in the average returns and the large VaR estimates.

Noting how our VaR values change over each investment period offers insight into how the assets in each portfolio are allocated. During periods when our VaR is high, we can infer that our portfolio may be over concentrated, aggressively short on some assets, or anything other potentially compromising position. For a picture of how our portfolios evolved over time, we appeal to an area plot for each method.

Shown in the following figures (3, 4 and 5) are area plots of the evolution of our asset allocations for each investment method.

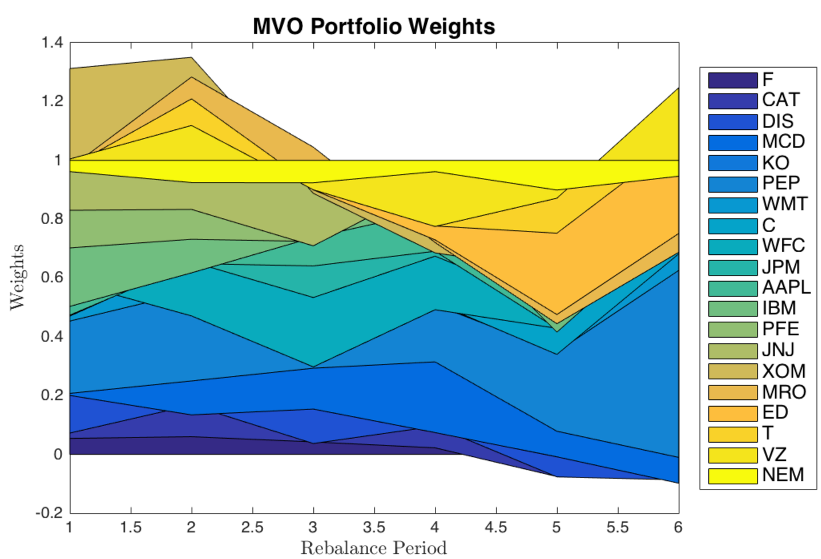

Fig. 3. Area Plot for MVO method

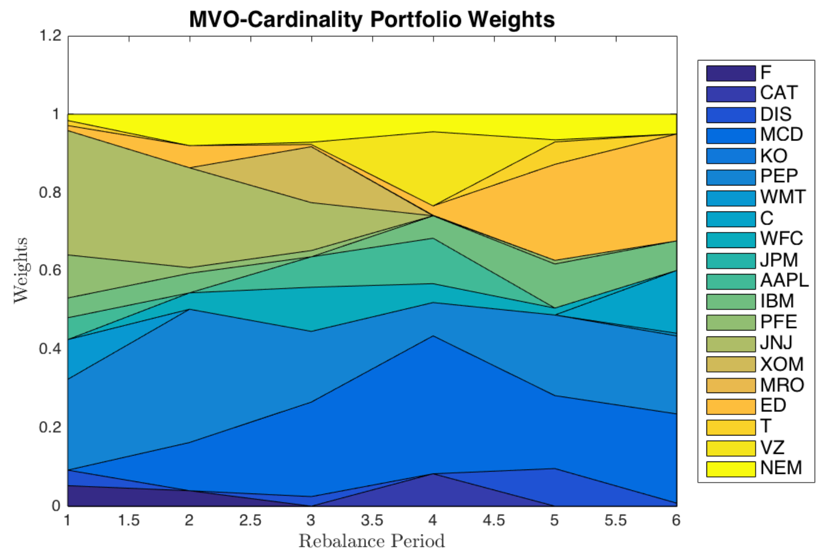

Fig. 4. Area Plot for MVO-Card method

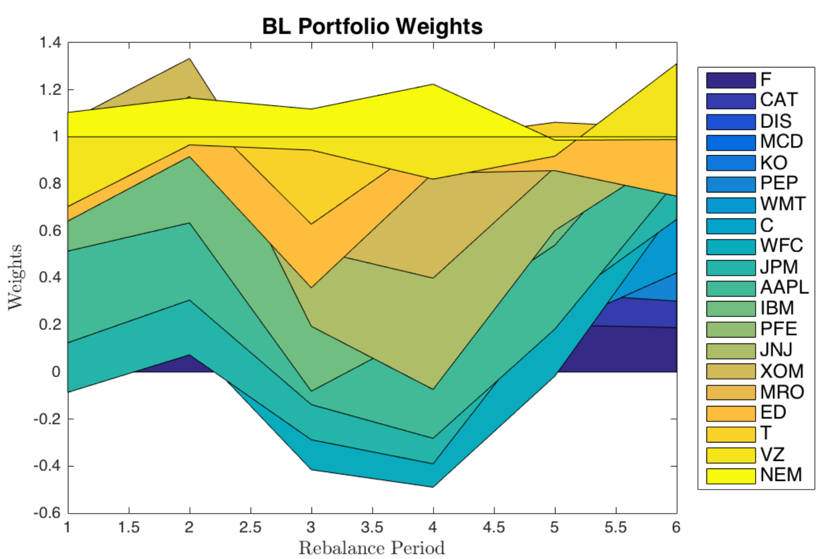

Fig. 5. Area Plot for BL method

Our area plots reveal that our portfolio's are relatively diverse over the entire investment period. Owing to our constraints, we are able to account for some of the risk from our MVO model and our BL model due to the permitted short positions. This more pronounced with the MVO model at the beginning of the investment period, and also observed heavily BL model towards the middle. The large short positions likely contributed to the improved returns from these portfolios.

There are several evolving positions which are consistent amongst all portfolios. For instance, all methods take on a long position on the stock NEM for the entirety of the investment period. Additionally, stocks such as F, CAT and DIS are all held in very low proportions. This is suggestive of these stocks being too volatile without having an adequate return to justify their holdings.

It is clear that among our models, MVO is more similar to MVO-Card. We see this in the area plots since evolutions in asset holdings tend to rise and fall in comparable amounts. BL on the other hand, regularly moves in contrast to MVO and MVO-Card. The most pronounced of this would be with with the stocks C, WFC, JMP, APPL and IBM. Recall that these stocks all pertain to an investor view that we adopted with the BL model. Among these stocks, our BL takes on a large short position, wheres MVO and MVO-Card go long. It seems that this position faired well for the BL model since it was the only model to boast a positive average return rate (despite the fact that this return sharply declined from the previous period).

When considering the evolution of our portfolios, we consider what effect this has on our transactional costs. Certainly we expect to be paying more under the BL model, since on a per-period basis, we are updating more of our asset allocations. Summarized in table 5 are the transactional fees incurred from rebalancing after each period.

Table 5. Incurred Transactional Costs Per Period

	MVO	**MVO-Card**	**BL**
Period 2	1.03	0.44	0.57
Period 3	1.36	0.53	2.07
Period 4	1.27	0.65	2.50
Period 5	1.50	0.75	2.34
Period 6	1.70	0.36	3.14

So it is indeed the case that we spend the most by far with the BL model. Additionally, it is reasonable that the cheapest is the MVO-Card method, since the cardinality constraint limits the size of the portfolio to up to 12 stocks. What is interesting to note with the BL model, is the more aggressive re-balancing strategy. This is influenced by the views of the BL model responding to an opportunity in the market conditions.

4 Next Steps and Improvements

This section will briefly highlight some areas which can be the subject for further study to improve on or expand the given models.

More Pricing Data: This project relied on weekly adjusted closing prices over the span of three years. It was shown that this provided a good measure of the the portfolio's performance. Since several parameters were based on these historical prices, it would be ideal to collect as many sample points as possible in order to try and best capture any patterns, trends and variations. This clearly comes at the trade-off of computational time, but over short time periods such as three years this should be easily managed.

One may also try to increase the available pricing data by increasing the calibration window. Recall that in this project, factor models were calibrated using the data from the previous year. This will achieve the task of increasing sample data, but this may not be appropriate. Sampling over a long period of time without justification introduces the possibility of working in "old" data that may not be relevant to the current market.

Improved Views: From a more throughout analysis of economic trends and emerging markets, one can attempt to construct investor views which are more robust are better able to capture or even predict market trends. The views adopted in this project were very simple, and primarily designed to demonstrate the capabilities of the BL model. BL views are well suited to be developed from a combination of methods ranging from a fundamental accounting analysis to generative machine learning models.

Expanding the Factor Model: The factor model used in this report was the Fama-French Three Factor model. This model offers an improved outlook on say the CAPM model since it includes more factors. If one can identify additional factors that are relevant, their inclusion into a regression model can improve the parameter estimates used in MVO formulations.

Reducing Transaction Costs: This was not of a primary concern for this project but suppose you as an investor are interested in paying as little as possible in transaction fees. This would directly impact how frequently and to what extent you re-balance your portfolio. If you contend that it is preferable to reduce your transaction costs (perhaps these costs come from your portfolio budget) then it is possible to include a transaction constraint in your MVO to ensure that your total transaction costs per period are capped at some upper threshold.

Shorter Investment Periods: Shorter investment periods correspond to increased opportunities to re-balance your portfolios. Clearly this comes at a cost from transaction fees, but if you are confident in your model's ability to adapt to the market, it may be preferable to have the freedom to quickly adjust the allocations of your assets. Pushed to the extreme, this motivates algorithmic trading based trading portfolios which are high-frequency in their re-balancing.

Modeling Changes in Outstanding Shares In this report, we assumed that the shares outstanding for each company was constant. This was a reasonable simplification to our simulation since the number of shares outstanding does not frequently change for a small group of stocks. The only significant actions that would alter the number of shares outstanding would be for instance, a stock split. To better accommodate this freedom, we could have incorporated the historical number of shares outstanding per company, as we did with the pricing data. This would have a provided a truer sense of the market performance of the optimization methods. Yet for comparisons, we expect the results to be similar.

Ideal Environment Assumption Note that in our factor mode, we assumed that the idiosyncratic risk between our associated were dependent. Here, we briefly assess the consequence of this assumption. Summarized in the following tables (6 and 7)are a quick summary of the portfolio statistics that would be associated with a simulation run under the assumption that the idiosyncratic risk of each asset were independent of each other.

We briefly note that the results from our simulation are improved when we assume that the idiosyncratic risk of our assets are dependent. This can be understood by the increase in overall volatility, representing increased opportunities to take advantage of observed correlations between assets. This noticeably improved the performances of our investment models which permitted short-selling.

Table 6. Weekly Portfolio Value Metrics With Independent Idiosyncratic Risk

	MVO	MVO-Card	BL
Maximum Value ($)	149	141	194
Week # for Maximum Value	157	157	138
Minimum Value ($)	98.9	98.5	100
Week # for Minimum Value	1	1	5
Largest Weekly Gain ($)	4.78	4.53	14.8
Week # for Largest Weekly Gain	151	151	95
Largest Weekly Loss ($)	5.64	5.80	9.24
Week # for Largest Weekly Loss	102	102	102
Longest Winning Streak	10	7	7
Week # When Longest Winning Streak Ended	11	148	136
Longest Losing Streak	6	5	5
Week # When Longest Losing Streak Ended	36	85	147

Table 7. Per Period Rates of Return for our investment Models With Independent Idiosyncratic Risk

	MVO			MVO-Card			BL		
	r_p	r_a	σ	r_p	r_a	σ	r_p	r_a	σ
Period 1	0.12	0.12	0.08	0.12	0.12	0.07	0.07	0.07	0.04
Period 2	0.05	0.21	0.03	0.04	0.20	0.02	0.10	0.18	0.05
Period 3	0.04	0.27	0.04	0.03	0.22	0.04	0.13	0.41	0.08
Period 4	-0.02	0.34	0.02	-0.01	0.31	0.02	0.05	0.62	0.05
Period 5	0.03	0.37	0.02	0.02	0.34	0.02	0.06	0.79	0.04
Period 6	0.04	0.41	0.03	0.01	0.35	0.03	0.03	0.90	0.03