DIRICHLET GENERATING FUNCTIONS AND FACTORIZATION IDENTITIES

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ABSTRACT. Just as ordinary power series generating functions are a powerful tool for establishing elementary identities in the theory of partitions, so Dirichlet series generating functions are appropriate to establish identities involving unordered factorizations of integers.

1. Introduction

The general field of additive number theory considers questions concerning representations of a given positive integer n as a *sum* of other integers. In particular, *partitions* treat the sums as unordered combinatorial objects, and *compositions* treat the sums as ordered. In this paper we consider analogous problems concerning representations of n as an unordered *product* of positive integers. Sometimes these problems appear in the literature with the description "factorisatio numerorum."

1.1. **Notation.** We use $H(n)$ to represent the number of factorizations of the positive integer n into factors in which the order of the factors distinguishes factorizations (in analogy with compositions for sums), and $P(n)$ to represent the number of factorization in which the order of factors is immaterial (in analogy with partitions). Note that the easy way to enumerate or list structures in which order is immaterial is to impose a canonical order. Just as partitions are often listed with summands in decreasing order, we will list parts in unordered factorizations in decreasing order. Sometimes it is also useful to explicitly keep track of the number of parts. We will introduce a second argument if this is desired, so that $H(n, k)$ (respectively $P(n, k)$) will mean the number of ordered (respectively, unordered) factorizations of n using exactly k factors. Of course, the convention is that all factors are integers greater than 1.

Write

$$n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}, \text{ with } 1 < p_1 < p_2 < \cdots < p_s, \tag{1}$$

to be the canonical factorization of n as a product of powers of distinct primes. Then many problems involving factorisatio numerorum depend only on the set of

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exponents in (1), $\{r_1, r_2, \cdots, r_s\}$. [5] developed the theory of compositions of "multipartite numbers" from this perspective, but Andrews suggests the more modern terminology "vector compositions" in [1], p. 57.

Factorizations have further connection to integer partitions as well as to set partitions. If $n = p^r$, we have that $H(n) = 2^{r-1}$ since each composition of the integer r corresponds to the exponents in an ordered factorization of n. Similarly $P(n) = p(r)$, where $p(r)$ counts partitions of the integer r. Secondly, if $n = p_1 p_2 \cdots p_s$ is square-free, then the number of unordered factorizations of n is given by the number of set partitions of the set of s prime factors of n. The corresponding number of ordered factorizations of n is given by the number of set partitions in which the order of subsets (but not the order of elements in each subset) matters. Set partitions are enumerated by the Bell numbers with the exponential generating function e^{e^z-1}, and ordered set partitions are enumerated by the ordered Bell numbers which have the exponential generating function $1/(2 - e^z)$.

2. Generating functions and the zeta function

The zeta function is central in generating functions for multiplicative problems. The analog to Euler's product representation of the zeta function,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \tag{2}$$

is the Dirichlet series generating function for $P(n)$:

$$\sum_{n=1}^{\infty} P(n)/n^s = \prod_{n=2}^{\infty} (1 - n^{-s})^{-1}. \tag{3}$$

Generating functions for factorizations with restrictions on the nature of the factors arise by restricting possible factors in the product, for example to odd numbers or squarefree numbers. Replace $(1 - n^{-s})^{-1}$ by $(1 + n^{-s})$ and the factorizations that are enumerated are restricted to have the factor n appearing at most once.

To build a Dirichlet generating function for $H(n)$, we first consider $H(n, k)$. We have

$$\sum_{n=2}^{\infty} H(n,k) n^{-s} = (\zeta(s) - 1)^k \tag{4}$$

and summing over k gives

$$\sum_{n=1}^{\infty} H(n) n^{-s} = (2 - \zeta(s))^{-1}. \tag{5}$$

These observations about Dirichlet series generating functions are well known (see, for example [2]). The authors of this paper have prepared a historical survey in [4] and a collection of implementations of algorithms for generating ordered and unordered factorizations in [3].

The situation is a little more complicated for unordered factorizations. For $Z(s) = \zeta(s) - 1$ we have the Dirichlet generating function for *ordered* factorizations with k parts just $Z(s)^k$. For unordered factorizations of n with k parts given as $P(n, k)$, we have

$$\sum_n P(n,2) n^{-s} = \frac{Z(s)^2}{2} + \frac{Z(2s)}{2},$$

$$\sum_n P(n,3)n^{-s} = \frac{Z(s)^3}{6} + \frac{Z(2s)Z(s)}{2} + \frac{Z(3s)}{3},$$

$$\sum_n P(n,4)n^{-s} = \frac{Z(s)^4}{24} + \frac{Z(2s)^2}{8} + \frac{Z(3s)Z(s)}{4} + \frac{Z(4s)}{4},$$

and in general careful bookkeeping gives the generating function for $P(n,k)$ as a sum over partitions of k.

Theorem 1. *The partition*

$$a_1 \cdot 1 + a_2 \cdot 2 + \ldots + a_n \cdot n \tag{6}$$

is associated with a term in the Dirichlet generating function whose numerator is

$$\prod_{i=1}^{n} Z(is)^{a_i} \tag{7}$$

and whose denominator is

$$\prod_{i=1}^{n} i^{a_i} a_i! \tag{8}$$

Hence

$$\sum_n P(n,k)n^{-s} = \sum_{k=a_1\cdot 1+a_2\cdot 2+\ldots+a_n\cdot n} \frac{Z(s)^{a_1} Z(2s)^{a_2} \ldots Z(ns)^{a_n}}{1^{a_1}\cdot a_1! 2^{a_2}\cdot a_2!\cdot \ldots \cdot n^{a_n}\cdot a_n!} \tag{9}$$

Proof By elementary principles the generating function $\sum_{n,k} P(n,k)n^{-s}t^k$ is

$$\prod_{n=2}^{\infty} \frac{1}{1-tn^{-s}} = \exp\left(\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} \frac{t^k}{k\cdot n^{ks}}\right) = \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} Z(ks)\right).$$

Now apply the identity for the generating function of the cycle index of the symmetric group, as in [6] Ex. 5.2.10, (5.30).

As a corollary we have a result about unit fractions:

Corollary 1.

$$\sum_{k=a_1\cdot 1+a_2\cdot 2+\ldots+a_n\cdot n} \frac{1}{1^{a_1}\cdot a_1! 2^{a_2}\cdot a_2!\cdot \ldots \cdot n^{a_n}\cdot a_n!} = 1 \tag{10}$$

This corollary can also be derived directly by considering cycle types of permutations, as in [7] Theorem 4.7.1.

3. Basic identities

By analogy with Euler's generating function approach to the partition identity that the number of partitions of n into odd parts is the same as the number of partitions of n into distinct parts, we may manipulate Dirichlet series to write

$$\prod_{n\geq 2}(1+n^{-s}) = \prod_{n\geq 2}\frac{1-n^{-2s}}{1-n^{-s}} = \prod_{\substack{n\geq 2 \\ n\neq m^2, m=1,2,3,\cdots}} \frac{1}{1-n^{-s}}. \tag{11}$$

Hence we obtain

Theorem 2. *The number of unordered factorizations of n into distinct parts equals the number of unordered factorizations of n into nonsquares.*

Table 1 gives the correspondence between factorizations for $n = 4096$.

Distinct	Non − square
4096	8 8 8 8
2 2048	2 2048
4 1024	2 2 32 32
8 512	8 512
2 4 512	2 2 2 512
16 256	2 2 2 2 2 2 2 2 2 2 2 2
2 8 256	2 2 2 2 2 2 2 2 2 8
32 128	32 128
2 16 128	2 2 2 2 2 128
4 8 128	2 2 8 128
2 32 64	2 8 8 32
4 16 64	2 2 2 2 2 2 8 8
2 4 8 64	2 2 2 8 8 8
8 16 32	2 2 2 2 8 32
2 4 16 32	2 2 2 2 2 2 2 32

TABLE 1. *Distinct and Non-Square Factorizations.*

Analogously, we have

Theorem 3. *The number of unordered factorizations of n into distinct parts in which no part is a kth power, for any fixed odd number $k \geq 3$, is equal to the number of unordered factorizations of n into parts that are not perfect squares or perfect kth powers.*

PROOF Start with the Dirichlet generating function for the left hand side.

$$\frac{\prod_{n=2}^{\infty}(1+n^{-s})}{\prod_{n=2}^{\infty}(1+n^{-ks})} = \prod_{n \text{ not a } k\text{th power}} (1+n^{-s}) = \prod_{n \text{ not a } k\text{th power}} \frac{1-n^{-2s}}{1-n^{-s}} = \prod_{\substack{n \text{ not a } k\text{th power} \\ \text{and not square}}} \frac{1}{1-n^{-s}}. \tag{12}$$

Theorem 4. *The number of unordered factorizations of n into distinct parts in which no part is a square, is equal to the difference between the number of unordered factorizations of n with an even number of square parts and the number of unordered factorizations of n with an odd number of square parts.*

Proof Start with the Dirichlet generating function for the left hand side.

$$\frac{\prod_{n=2}^{\infty}(1+n^{-s})}{\prod_{n=2}^{\infty}(1+n^{-2s})} = \prod_{n=2}^{\infty}\frac{1-n^{-2s}}{1-n^{-s}}\prod_{n=2}^{\infty}\frac{1}{(1+n^{-2s})} = \prod_{n \text{ not a square}}\frac{1}{(1-n^{-s})}\prod_{n \text{ square}}\frac{1}{(1+n^{-s})}. \tag{13}$$

We illustrate this theorem for $n = 72$. There are 4 factorizations of 72 into distinct and nonsquare parts, $12 \times 3 \times 2, 12 \times 6, 24 \times 3$ and 72. On the other hand there are 10 factorizations of 72 with an even number of square parts,

$3\times3\times2\times2\times2, 6\times3\times2\times2, 6\times6\times2, 8\times3\times3, 9\times4\times2, 12\times3\times2, 12\times6, 18\times2\times2, 24\times3, 72.$

Also there are $6 = 10 - 4$ factorizations of 72 with an odd number of square parts,

$$4 \times 3 \times 3 \times 2, 6 \times 4 \times 3, 9 \times 2 \times 2 \times 2, 9 \times 8, 18 \times 4, 36 \times 2.$$

We can also limit the repetitions of parts less restrictively.

Theorem 5. *The number of unordered factorizations of n with parts repeated at most k times equals the number of unordered factorizations of n into non $(k+1)$st powers.*

Explicit bijections can be developed to provide combinatorial proofs of these theorems. For the first theorem, given an unordered factorization into distinct parts, take square roots of any square parts to produce two new factors and continue to do this until the parts left are non-squares. This gives the corresponding factorization into non-squares. For example,

$$\begin{aligned} 36 \times 40 \times 81 \times 9 &= 6^2 \times 40 \times 3^4 \times 3^2 \\ &= 6 \times 6 \times 40 \times 3^2 \times 3^2 \times 3 \times 3 \\ &= 6 \times 6 \times 40 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \end{aligned}$$

This is the corresponding factorization into non-squares.

Conversely, consider a factorization into non-squares. Suppose a given part a occurs j times as a factor. Express j in binary form as $2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_r}$, $\alpha_1 > \alpha_2 > \ldots > \alpha_r \geq 0$. In the factorization replace the j factors $a \times a \times \ldots \times a$ by $a^{2^{\alpha_1}} \times a^{2^{\alpha_2}} \times \ldots \times a^{2^{\alpha_r}}$. After applying this process to each repeated factor one obtains a factorization into distinct parts. For example, we can reverse the process above to obtain

$$\begin{aligned} 6 \times 6 \times 40 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 &= 6^2 \times 40^1 \times 3^6 \\ &= 6^2 \times 40^1 \times 3^{4+2} \\ &= 36 \times 40 \times 3^4 \times 3^2 \\ &= 36 \times 40 \times 81 \times 9 \end{aligned}$$

The same idea has been used to establish that the number of partitions of n into odd parts is the number of partitions of n into distinct parts, and the partition theorem replacing "distinct parts" with "at most k parts" is analogous to the second theorem.

Theorem 6. *The number of unordered factorizations in which each part appears at least twice equals the number of unordered factorizations into squares or cubes.*

PROOF Consider the Dirichlet generating function of the left hand side:

$$\prod_{n=2}^{\infty}(1+n^{-2s}+n^{-3s}+\dots) = \prod_{n=2}^{\infty}\left(1+\frac{n^{-2s}}{1-n^{-s}}\right) = \prod_{n=2}^{\infty}\frac{1-n^{-s}+n^{-2s}}{1-n^{-s}} = \prod_{n=2}^{\infty}\frac{1+n^{-3s}}{1-n^{-2s}}. \tag{14}$$

The Dirichlet generating function of the right hand side is

$$\prod_{n=2}^{\infty}\frac{1}{1-n^{-2s}}\prod_{n\geq 2, n \text{ not a square}}\frac{1}{1-n^{-3s}}. \tag{15}$$

Now

$$\prod_{n \text{ not square}}\frac{1}{1-n^{-3s}} = \frac{\prod_{n\geq 2} 1/(1-n^{-3s})}{\prod_{n \text{ square}} 1/(1-n^{-3s})} = \prod_{n\geq 2}\frac{1}{1-n^{-3s}}\prod_{n\geq 2}(1-n^{-6s}) = \prod_{n\geq 2}(1+n^{-3s}) \tag{16}$$

as required.

Theorem 7. *The number of unordered factorizations of n with in which each part occurs a multiple of k times equals the number of unordered factorizations of n into kth powers.*

PROOF This is immediate from Dirichlet generating function identity

$$\prod_{n=2}^{\infty}(1+n^{-ks}+n^{-2ks}+\cdots) = \prod_{n=2}^{\infty}\frac{1}{1-(n^k)^{-s}}. \tag{17}$$

4. Factorizations with even parts and odd parts

Let $P_e(n)$ and $P_o(n)$ be the number of unordered factorizations of n into even and odd parts, respectively. Then

$$\sum_{n=1}^{\infty} P_e(n)/n^s = \prod_{n=1}^{\infty}(1-(2n)^{-s})^{-1}, \tag{18}$$

$$\sum_{n=1}^{\infty} P_o(n)/n^s = \prod_{n=2}^{\infty}(1-(2n-1)^{-s})^{-1}. \tag{19}$$

Theorem 8. *Let $n = 2^k m$ where $k \geq 0$ and m is odd. Then*

$$P(n) = \sum_{d|m} P(d)P_e(n/d). \tag{20}$$

PROOF Equating coefficients of n^{-s} in the identity

$$\sum_{n=1}^{\infty} P(n)/n^s = \sum_{n=1}^{\infty} P_o(n)/n^s \sum_{n=1}^{\infty} P_e(n)/n^s \tag{21}$$

yields

$$P(n) = \sum_{d|n} P_o(d)P_e(n/d). \tag{22}$$

Since $P_o(n) = P(n)$ for n odd, and $P_o(n) = 0$ for n even, the result follows.

Corollary 2. *Let $n = 2m$ where m is odd. Then*

$$P(n) = \sum_{d|m} P(d). \tag{23}$$

Let $n = 4m$ where m is odd. Then

$$P(n) = \sum_{d|m} P(d) + \frac{1}{2}\left(\sum_{\substack{d|m \\ d \text{ square}}} P(m/d) + \sum_{d|m} P(d)\tau(m/d) \right), \tag{24}$$

where $\tau(j)$ is the number of divisors of the number j.

Proof For the first part apply the above theorem for $k = 1$ together with $P_e(2j) = 1$ for any odd j. In the second case we have

$$P(4m) = \sum_{d|m} P(m/d)P_e(4d). \tag{25}$$

Now $P_e(4d) = 1 + \frac{\tau(d)}{2}$ if d is not a square and $P_e(4d) = 1 + \frac{\tau(d)+1}{2}$ if d is a square. Hence the result.

5. Series-product identities

We can also derive several series-product identities for unordered factorization Dirichlet series that are analogous to some well known series-product identities for integer partitions.

Let $P(n, k)$ denote the number of unordered factorizations of n with largest factor k. Such a factorization can be written as $k \times a$ where a represents a factorization of $\frac{n}{k}$ into factors $\leq k$. Thus

$$\sum_{n=1}^{\infty} P(n, k)n^{-s} = \frac{k^{-s}}{(1 - 2^{-s})(1 - 3^{-s}) \dots (1 - k^{-s})}. \tag{26}$$

Summing over all $k \geq 2$ gives all factorizations, whence

$$\prod_{n \geq 2} (1 - n^{-s})^{-1} = 1 + \sum_{k=2}^{\infty} \frac{k^{-s}}{(1 - 2^{-s})(1 - 3^{-s}) \dots (1 - k^{-s})}. \tag{27}$$

Similarly, if we consider the Dirichlet generating function for $P_d(n, k)$, the number of unordered factorizations with distinct factors and largest factor k, we obtain

$$\prod_{n \geq 2} (1 + n^{-s}) = 1 + \sum_{k=2}^{\infty} k^{-s}(1 + 2^{-s})(1 + 3^{-s}) \dots (1 + (k - 1)^{-s}). \tag{28}$$

Let $p(n, k)$ denote the number of unordered factorizations of n into primes with largest factor p_k. Such a factorization can be written as $p_k \times a$ where a represents a factorization of $\frac{n}{p_k}$ into prime factors $\leq p_k$. Thus

$$\sum_{n=1}^{\infty} p(n, k)n^{-s} = \frac{p_k^{-s}}{(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \dots (1 - p_k^{-s})}. \tag{29}$$

Summing over all $k \geq 1$ gives the unique prime factorization of every positive integer, whence

$$\zeta(s) = 1 + \sum_{k=1}^{\infty} \frac{p_k^{-s}}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\dots(1-p_k^{-s})}. \tag{30}$$

Similarly, if we consider the Dirichlet generating function for $p_d(n,k)$, the number of unordered factorizations with distinct prime factors and largest prime factor p_k, we obtain the unique prime factorization of every squarefree positive integer

$$\frac{\zeta(s)}{\zeta(2s)} = 1 + \sum_{k=1}^{\infty} p_k^{-s}(1+2^{-s})(1+3^{-s})(1+5^{-s})\dots(1+p_{k-1}^{-s}). \tag{31}$$

6. A factorization analogue of Euler's pentagonal number theorem?

It is also possible to interpret coefficients in the Dirichlet series expansion of $\prod_{n\geq 2}(1-n^{-s})$ as giving the difference between the number of unordered factorizations of n with an even number of factors, and the number of unordered factorizations with an odd number of factors.

Writing

$$\prod_{n\geq 2}(1-n^{-s}) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \tag{32}$$

it appears at first that $b(n) \in \{-1,0,1\}$ and thus one is tempted to look for an analog of Euler's pentagonal number theorem for $\prod_{n=1}^{\infty}(1-x^n)$. However, we can expand the Dirichlet series to see that eventually $|b(n)| \geq 2$:

$\prod_{n=2}^{\infty}(1-n^{-s}) = 1-2^{-s}-3^{-s}-4^{-s}-5^{-s}-7^{-s}-9^{-s}-11^{-s}+12^{-s}-13^{-s}-17^{-s}+$
$18^{-s}-19^{-s}+20^{-s}-23^{-s}+24^{-s}-25^{-s}+28^{-s}-29^{-s}+30^{-s}-31^{-s}+32^{-s}+36^{-s}-$
$37^{-s}+40^{-s}-41^{-s}+42^{-s}-43^{-s}+44^{-s}+45^{-s}-47^{-s}+48^{-s}-49^{-s}+50^{-s}+52^{-s}-$
$53^{-s}+54^{-s}+56^{-s}-59^{-s}+60^{-s}-61^{-s}+63^{-s}+66^{-s}-67^{-s}+68^{-s}+70^{-s}-71^{-s}+$
$72^{-s}-73^{-s}+75^{-s}+76^{-s}+78^{-s}-79^{-s}+80^{-s}-83^{-s}+84^{-s}+88^{-s}-89^{-s}+90^{-s}+$
$92^{-s}-97^{-s}+98^{-s}+99^{-s}+100^{-s}-101^{-s}+102^{-s}-103^{-s}+104^{-s}+105^{-s}-107^{-s}+$
$108^{-s}-109^{-s}+110^{-s}+112^{-s}-113^{-s}+114^{-s}+116^{-s}+117^{-s}-121^{-s}+124^{-s}+$
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$139^{-s}+140^{-s}+147^{-s}+148^{-s}-149^{-s}+150^{-s}-151^{-s}+152^{-s}+153^{-s}+154^{-s}+$
$156^{-s}-157^{-s}+162^{-s}-163^{-s}+164^{-s}+165^{-s}-167^{-s}-169^{-s}+170^{-s}+171^{-s}+$
$172^{-s}-173^{-s}+174^{-s}+175^{-s}+176^{-s}-179^{-s}-181^{-s}+182^{-s}+184^{-s}+186^{-s}+$
$188^{-s}+189^{-s}+190^{-s}-191^{-s}-193^{-s}+195^{-s}+196^{-s}-197^{-s}+198^{-s}-199^{-s}+$
$200^{-s}+204^{-s}+207^{-s}+208^{-s}+210^{-s}-211^{-s}+212^{-s}-216^{-s}+220^{-s}+222^{-s}-$
$223^{-s}+225^{-s}-227^{-s}+228^{-s}-229^{-s}+230^{-s}+231^{-s}+232^{-s}-233^{-s}+234^{-s}+$
$236^{-s}+238^{-s}-239^{-s}-240^{-s}-241^{-s}+242^{-s}+243^{-s}+244^{-s}+245^{-s}+246^{-s}+$
$248^{-s}+250^{-s}-251^{-s}+255^{-s}-257^{-s}+258^{-s}+260^{-s}+261^{-s}-263^{-s}+266^{-s}+$
$268^{-s}-269^{-s}-271^{-s}+272^{-s}+273^{-s}+275^{-s}+276^{-s}-277^{-s}+279^{-s}-281^{-s}+$
$282^{-s}-283^{-s}+284^{-s}+285^{-s}+286^{-s}-288^{-s}-289^{-s}+290^{-s}+292^{-s}-293^{-s}+$
$294^{-s}+296^{-s}+297^{-s}+304^{-s}+306^{-s}-307^{-s}+308^{-s}+310^{-s}-311^{-s}-313^{-s}+$
$315^{-s}+316^{-s}-317^{-s}+318^{-s}+322^{-s}+325^{-s}+328^{-s}+330^{-s}-331^{-s}+332^{-s}+$
$333^{-s}-336^{-s}-337^{-s}+338^{-s}+340^{-s}+342^{-s}+344^{-s}+345^{-s}-347^{-s}+348^{-s}-$
$349^{-s}+350^{-s}+351^{-s}-353^{-s}+354^{-s}+356^{-s}+357^{-s}-359^{-s}-2\times 360^{-s}+\dots$

In particular, if $n = p_1 p_2 \dots p_k$ where the p_i are distinct primes then $b(n) = \sum_{j=1}^{k} (-1)^j \left\{ {k \atop j} \right\}$, and [8] uses this to derive asymptotic estimates for $b(n)$. Here $\left\{ {k \atop j} \right\}$ represents a Stirling number of the second kind.

The property $b(n) \in \{-1, 0, 1\}$ does at least hold for infinitely many n. Let $\Omega(n)$ denote the total number of prime factors of an integer n. By considering the numbers whose exponents in their prime factorizations correspond to partitions of 1 up to 5 we see that

If $\Omega(n) \leq 4$, or if $\Omega(n) = 5$ and n is not squarefree, then $b(n) \in \{-1, 0, 1\}$.

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