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FORMULAS FOR THE NUMBER OF BINOMIAL COEFFICIENTS DIVISIBLE BY A FIXED POWER OF A PRIME

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ABSTRACT. Define $\theta_j(n)$ as the number of binomial coefficients $\binom{n}{s}$ divisible by exactly p^j. A formula for $\theta_2(n)$ is found, for all n, and formulas for $\theta_j(n)$ for $n=ap^k+bp^r$ and $n=c_1p^{k_1}+\cdots+c_mp^{k_m}$ $(k_1\geqq j,\ k_{i+1}-k_i\geqq j$ for $i=1,\cdots,m-1)$ are derived.

1. **Introduction.** Let p be a fixed prime and let $\theta_j(n)$ denote the number of binomial coefficients $\binom{n}{s}$ $(s=0,1,\cdots,n)$ divisible by exactly p^j.

If we put

$$n = c_0 + c_1 p + \cdots + c_r p^r \qquad (0 \leqq c_i < p) \tag{1.1}$$

it is well known [3] that

$$\theta_0(n) = (c_0 + 1)(c_1 + 1)\cdots(c_r + 1).$$

The evaluation of $\theta_j(n)$ for arbitrary j appears to be more difficult, however. Carlitz [1] has proved that

$$\theta_1(n) = \sum_{k=0}^{r-1}(c_0 + 1)\cdots(c_{k-1} + 1)(p - c_k - 1)c_{k+1}(c_{k+2} + 1)\cdots(c_r + 1)$$

and he has found formulas for $\theta_j(n)$ for the following values of n:

$$\begin{array}{ll} ap^r + bp^{r+1} & (0 \leqq a < p,\ 0 \leqq b < p), \\ b + ap + ap^2 + \cdots + ap^{r+j} & (0 < a < p,\ b = a \text{ or } a - 1). \end{array}$$

The writer [4] has considered this problem for $p=2$ and has found formulas for $\theta_j(n)$, $1\leqq j\leqq 4$, and for arbitrary j has evaluated $\theta_j(n)$ for a number of special values of n. These formulas are valid only for $p=2$, however.

In this paper we find formulas for $\theta_2(n)$ for all n and for $\theta_j(n)$ for the following values of n:

$$\begin{array}{ll} ap^k + bp^r & (0 < a < p,\ 0 < b < p,\ k < r), \\ c_1p^{k_1} + \cdots + c_mp^{k_m} & (0 < c_i < p,\ j \leqq k_1,\ j < k_{i+1} - k_i). \end{array}$$

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We shall use the following rule, which was proved by Kummer [**2**, p. 70]. Put

$$s = a_0 + a_1 p + \cdots + a_r p^r \qquad (0 \leqq a_i < p), \tag{1.2}$$

$$n - s = b_0 + b_1 p + \cdots + b_r p^r \qquad (0 \leqq b_i < p), \tag{1.3}$$

$$a_0 + b_0 = c_0 + \varepsilon_0 p,\ \varepsilon_0 + a_1 + b_1 = c_1 + \varepsilon_1 p, \cdots,$$
$$\varepsilon_{r-1} + a_r + b_r = c_r + \varepsilon_r p,$$

where each $\varepsilon_i=0$ or 1. Let N be the exponent of the highest power of p that divides $\binom{n}{s}$. Then we have $N=\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_r$.

2. **Evaluation of $\theta_2(n)$.** If n is given by (1.1) and s and $n-s$ are given by (1.2) and (1.3), it is clear that $N=2$ if and only if exactly two of the ε's are equal to 1, and $\varepsilon_r=0$. There are two possibilities. Either $\varepsilon_k=\varepsilon_{k+1}=1$ for some $0\leqq k\leqq r-2$ and all other ε's$=0$ or $\varepsilon_k=1$, $\varepsilon_m=1$ for some $0\leqq k\leqq r-3$, $k+2\leqq m\leqq r-1$ and all other ε's$=0$. In the first case we can take

$$a_k = c_k + 1, \cdots, p-1; \qquad a_{k+1} = c_{k+1}, \cdots, p-1;$$
$$a_{k+2} = 0, \cdots, c_{k+2} - 1.$$

So we have $(p-c_k-1)(p-c_{k+1})c_{k+2}$ choices and the remaining a's can be chosen in A_k ways, where

$$A_k = \left[\prod_{i=0}^{r} (c_i + 1)\right] \Big/ [(c_k + 1)(c_{k+1} + 1)(c_{k+2} + 1)]. \tag{2.1}$$

In the second case, we can take

$$a_k = c_k + 1, \cdots, p-1; \qquad a_{k+1} = 0, \cdots, c_{k+1} - 1,$$
$$a_m = c_m + 1, \cdots, p-1; \qquad a_{m+1} = 0, \cdots, c_{m+1} - 1,$$

so there are $(p-c_k-1)c_{k+1}(p-c_m-1)c_{m+1}$ choices. The remaining a's can be selected in $B_{k,m}$ ways, where

$$B_{k,m} = \left[\prod_{i=0}^{r} (c_i + 1)\right] \Big/ [(c_k + 1)(c_{k+1} + 1)(c_m + 1)(c_{m+1} + 1)]. \tag{2.2}$$

Thus we have

$$\begin{aligned} \theta_2(n) &= \sum_{k=0}^{r-2} (p - c_k - 1)(p - c_{k+1})c_{k+2}A_k \\ &\quad + \sum_{m=k+2}^{r-1} \sum_{k=0}^{r-3} (p - c_k - 1)c_{k+1}(p - c_m - 1)c_{m+1}B_{k,m}, \end{aligned} \tag{2.3}$$

where A_k and $B_{k,m}$ are defined by (2.1) and (2.2) respectively.

For example,

$$\theta_2(a + bp + cp^2) = (p - a - 1)(p - b)c,$$
$$\begin{aligned}\theta_2(a + bp + cp^2 + dp^3) = {} & (p - a - 1)(p - b)c(d + 1) \\ & + (a + 1)(p - b - 1)(p - c)d \\ & + (p - a - 1)b(p - c - 1)d.\end{aligned}$$

This method does not appear to be very practical for evaluating $\theta_j(n)$ for $j>2$.

3. **Special evaluations.** We can use Kummer's theorem to evaluate $\theta_j(ap^k+bp^r)$, where $r>k$, $0<a<p$, $0<b<p$. Suppose $k\geqq j$ and $r-k>j$. Then there are three ways to have exactly j of the ε's equal to 1:

(1) $\varepsilon_{r-j}=\varepsilon_{r-j+1}=\cdots=\varepsilon_{r-1}=1$, all other ε's$=0$;

(2) $\varepsilon_{k-j}=\varepsilon_{k-j+1}=\cdots=\varepsilon_{k-1}=1$, all other ε's$=0$;

(3) $\varepsilon_{k-m}=\varepsilon_{k-m+1}=\cdots=\varepsilon_{k-1}=1$, $\varepsilon_{r-h}=\cdots=\varepsilon_{r-1}=1$, $1\leqq m\leqq j-1$, $h=j-m$, all other ε's$=0$.

If s is given by (1.2), in the first case we can take

$$a_{r-j} = 1, \cdots, p - 1; \quad a_i = 0, \cdots, p - 1 \quad (r - j + 1 \leqq i \leqq r - 1);$$
$$a_r = 0, \cdots, b - 1; \quad a_k = 0, \cdots, a,$$

so there are $(p-1)p^{j-1}(a+1)b$ choices. Using similar reasoning in the other two cases, we have

$$\begin{aligned}\theta_j(ap^k + bp^r) = {} & (p - 1)p^{j-1}(a + 1)b \\ & + (p - 1)p^{j-1}a(b + 1) + (j - 1)(p - 1)^2p^{j-2}ab \\ & (k \geqq j, r > k + j).\end{aligned} \tag{3.1}$$

Similarly we have

$$\begin{aligned}&\theta_j(ap^k + bp^r) \\ &\quad = (p - 1)p^{j-1}(a + 1)b + k(p - 1)^2p^{j-2}ab \quad (k < j, r > k + j),\end{aligned} \tag{3.2}$$

$$= (p - a - 1)p^{j-1}b + k(p - 1)^2p^{j-2}ab \qquad (k < j, r = k + j), \tag{3.3}$$

$$\begin{aligned}&= (p - 1)p^{j-1}a(b + 1) + (p - 1)p^{j-2}(p - a)b \\ &\quad + (r - k - 1)(p - 1)^2p^{j-2}ab \qquad (k \geqq j, r < k + j),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&= (p - 1)p^{j-1}a(b + 1) + (p - a - 1)p^{j-1}b \\ &\quad + (j - 1)(p - 1)^2p^{j-2}ab \qquad (k \geqq j, r + k = j),\end{aligned} \tag{3.5}$$

$$\begin{aligned}&= (p - 1)p^{j-2}(p - a)b + (r - j)(p - 1)^2p^{j-2}ab \\ &\qquad (k < j, r < k + j, r \geqq j).\end{aligned} \tag{3.6}$$

We next evaluate $\theta_j(n)$ for

$$n = c_1 p^{k_1} + c_2p^{k_2} + \cdots + c_m p^{k_m} \qquad (k_1 \geqq j, k_{i+1} - k_i > j). \tag{3.7}$$

Using Kummer's theorem, we need to determine the number of ways we can have exactly j of the ε's equal to 1. Let $1 \leqq u \leqq m$ and choose u of the c_i's. Call them $c_{i_1}, \cdots, c_{i_u}$. Assign to each c_{i_w} a number t_w, $1 \leqq t_w$, such that $t_1+t_2+\cdots+t_u=j$. This can be done in $\binom{j-1}{u-1}$ ways, since there are $\binom{j-1}{u-1}$ different ways of distributing j nondistinct objects into u distinct cells with no cell left empty. We wish to have $\varepsilon_v=1$ $(v=i_w-h,\ 1\leqq h \leqq t_w,$ $1 \leqq w \leqq u)$ and all other ε's equal to 0. If s is given by (1.2) we can take

$$\begin{aligned} a_v &= 1, \cdots, p-1 && (v = i_w - t_w,\ 1 \leqq w \leqq u), \\ &= 0, \cdots, p-1 && (v = i_w - h,\ 1 \leqq h \leqq t_w - 1,\ 1 \leqq w \leqq u), \\ &= 0, \cdots, c_v - 1 && (v = i_w,\ 1 \leqq w \leqq u). \end{aligned}$$

Thus for a given u and a given selection $i_1, \cdots, i_u$, there are

$$\binom{j-1}{u-1}(p-1)^u p^{j-u} c_{i_1} \cdots c_{i_u}(c_1+1)\cdots(c_m+1)/(c_{i_1}+1)\cdots(c_{i_u}+1)$$

different ways to have j of the ε's equal to 1. Therefore

$$\theta_j(n) = \sum_{u=1}^{m} \binom{j-1}{u-1}(p-1)^u p^{j-u} E_u \tag{3.8}$$

where n is given by (3.7) and

$$E_u = \sum c_{i_1} \cdots c_{i_u}(c_1+1)\cdots(c_m+1)/[(c_{i_1}+1)\cdots(c_{i_u}+1)], \tag{3.9}$$

the sum being over all subsets $\{i_1, \cdots, i_u\}$ of $\{1, \cdots, m\}$ such that $i_1 < i_2 < \cdots < i_u$.

For example, if $n = ap^{k_1} + bp^{k_2} + cp^{k_3}$, $k_1 \geqq j$, $k_2 - k_1 > j$, $k_3 - k_2 > j$, then

$$\begin{aligned} \theta_j(n) = {} & (p-1)p^{j-1}[a(b+1)(c+1) + (a+1)b(c+1) + (a+1)(b+1)c] \\ & + (j-1)(p-1)^2 p^{j-2}[ab(c+1) + a(b+1)c + (a+1)bc] \\ & + \binom{j-1}{2}(p-1)^3 p^{j-3} abc. \end{aligned}$$

If n is given by (3.7) and $c_1 = c_2 = \cdots = c_k \equiv a$, then (3.8) becomes

$$\theta_j(n) = \sum_{u=1}^{m} \binom{j-1}{u-1}\binom{m}{u}(p-1)^u p^{j-u}(a+1)^{m-u} a^u. \tag{3.10}$$

If n is given by (3.7), except that $k_1 = j-1$ and $k_{i+1} - k_i = j$, we have, by an argument similar to the one above,

$$\theta_j(n) = p^{j-1}F + \sum_{u=2}^{m} \binom{j-1}{u-1}(p-1)^u p^{j-u} E_u, \tag{3.11}$$

where E_u is defined by (3.9) and

$$F = \sum_{i=1}^{m-1} (p - c_i - 1)c_{i+1}(c_1 + 1) \cdots (c_m + 1)/[(c_i + 1)(c_{i+1} + 1)].$$

If $k_1 \geqq j$ and $k_{i+1} - k_i = j$, then

$$\begin{aligned} \theta_j(n) &= (p-1)p^{j-1}c_1(c_2+1)\cdots(c_m+1) \\ &\quad + p^{j-1}F + \sum_{u=2}^{m} \binom{j-1}{u-1}(p-1)^u p^{j-u} E_u. \end{aligned} \tag{3.12}$$

If $k_1 = 0$, $k_{i+1} - k_i > j$, then

$$\theta_j(n) = \sum_{u=1}^{m} \binom{j-1}{u-1}(p-1)^u p^{j-u} G_u, \tag{3.13}$$

where

$$G_u = \sum c_{i_1} \cdots c_{i_u}(c_1 + 1) \cdots (c_m + 1)/[(c_{i_1} + 1) \cdots (c_{i_u} + 1)],$$

the sum being over all subsets $\{i_1, \cdots, i_u\}$ of $\{2, \cdots, m\}$ such that $i_1 < i_2 < \cdots < i_u$.

If n is given by (3.7) and $c_1 = c_2 = \cdots = c_m = a$, then (3.11) becomes

$$\begin{aligned} \theta_j(n) &= (m-1)a(p-a-1)p^{j-1}(a+1)^{m-2} \\ &\quad + \sum_{u=2}^{m} \binom{j-1}{u-1}\binom{m}{u}(p-1)^u p^{j-u}(a+1)^{m-u}a^u; \end{aligned} \tag{3.14}$$

(3.12) becomes

$$\begin{aligned} \theta_j(n) &= (p-1)p^{j-1}a(a+1)^{m-1} \\ &\quad + (m-1)(p-a-1)a(a+1)^{m-2}p^{j-1} \\ &\quad + \sum_{u=2}^{m} \binom{j-1}{u-1}\binom{m}{u}(p-1)^u p^{j-u}(a+1)^{m-u}a^u; \end{aligned} \tag{3.15}$$

(3.13) becomes

$$\theta_j(n) = \sum_{u=1}^{m-1} \binom{j-1}{u-1}\binom{m-1}{u}(p-1)^u p^{j-u}(a+1)^{m-u}a^u. \tag{3.16}$$

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