# ON THE PARTITION OF $J$-PARTITE NUMBERS †

*by* HANSRAJ GUPTA, F.N.I., *Panjab University, Chandigarh*

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ABSTRACT

This paper gives formulae for the number of partitions of $j$-partite numbers into exactly $k$ non-degenerate parts, distinct or otherwise. The recursion formulae obtained in the earlier part of the paper are suitable for computation. No use is made of generating functions.

1. Working on the lines of one of the author's papers (Gupta 1959), Om Prakash Gupta (1960) succeeded in obtaining formulae for the partitions of a $j$-partite number

$$N_j = (n_1, n_2, n_3, \ldots, n_j) = (n_i)$$

with each $n_i$ an integer $> 0$, into exactly $k$ parts distinct or otherwise, for values of $k \leqslant 6$. The results, he obtained, were enough to lead us to several conjectures. It was in an attempt to prove these that I was able to obtain a complete solution of the problem of partitioning $N_j$ into a fixed number of parts.

2. Notations.

All small letters including $\alpha$'s and $\beta$'s denote positive integers unless stated otherwise and $i$ runs from 1 to $j$.

$n^{(m)}$ stands for the product

$$n(n-1)(n-2) \ldots (n-m+1).$$

We say $N_j$ ($= (n_i)$) is greater than $M_j$ ($= (m_i)$) and write

$$N_j > M_j \text{ or } N_j - M_j > 0 \text{ or } M_j < N_j$$

when $n_i > m_i$ for every $i \leqslant j$.

We write $tM_j$ for $(tm_i)$.

When $M_j$ occurs as a part $t$ times in a partition, we indicate this by writing $M_j(t)$.

We say $t|N_j$ when $t|n_i$ for each $i \leqslant j$.

$p(N_j, k)$ denotes the number of partitions of $N_j$ into exactly $k$ summands and $q(N_j, k)$ the number of partitions of $N_j$ into exactly $k$ distinct parts. We take

$$p(N_j, 0) = 0 = q(N_j, 0)$$

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except when $N_j$ is null. In the exceptional case, we take

$$p(N_j, 0) = 1 = q(N_j, 0),$$

$$\lambda_t = \lambda_t(N_j) = 1 \text{ when } t \mid N_j$$

and

$$= 0 \text{ otherwise.}$$

We write

$$P_k \text{ or } P(N_j, k) \text{ for } k\,!\,p(N_j, k),$$

and

$$Q_k \text{ or } Q(N_j, k) \text{ for } k\,!\,q(N_j, k).$$

The number of those partitions of $N_j$ into exactly $k$ parts, which have just $t$ parts equal, $t > 1$, and all others distinct from each other and also from the $t$ parts, shall be denoted by $q_t(N_j, k)$ or simply $q^*(t)$. We shall refer to such partitions as the $t^*$-partitions of $N_j$, it being understood that the partitions are into exactly $k$ parts.

We define the operator $\sigma_r$ by the relation

$$\sigma_r f(N_j) = \sum_{0 < rM_j \leqslant N_j} f(N_j - rM_j)$$

$f(N_j)$ being any number-theoretic function defined for all $N_j$.

3. The recurrence formula for $q(N_j, k)$.

*Lemma*. The partitions of $N_j - tM_j$, where $M_j$ runs through all numbers for which $tM_j \leqslant N_j$; into $(k-t)$ distinct summands, provide for $t > 1$, all the $t^*$ and $(t+1)$ $*$-partitions of $N_j$ into exactly $k$ parts. For $t = 1$, these provide the partitions of $N_j$ into $k$ distinct parts, $k$ times over, besides the $q^*(2)$ $2^*$-partitions of $N_j$.

*Proof*. For a given $M_j$ and $t > 1$, $tM_j \leqslant N_j$; let

$$N_{1j}, N_{2j}, N_{3j}, \ldots, N_{k-t,j};\ t \leqslant k;$$

be any partition of $N_j - tM_j$ into $(k-t)$ distinct summands.
Then

$$M_j(t), N_{1j}, N_{2j}, \ldots, N_{k-t,j};\ t \leqslant k;$$

is a $t^*$-partition of $N_j$ when $M_j$ is distinct from each of the summands

$$N_{1j}, N_{2j}, \ldots, N_{k-t,j};$$

otherwise, it is a $(t+1)$ $*$-partition of $N_j$.

When $t = 1$, let $N_{1j}, N_{2j}, \ldots, N_{kj}$ be any partition of $N_j$ into exactly $k$ distinct summands.

Then from the partitions of $N_j - M_j$ into $(k-1)$ distinct parts as $M_j$ runs through all values $< N_j$, the partition

$$N_{1j}, N_{2j}, \ldots, N_{kj} \text{ of } N_j$$

will be obtained every time that $M_j$ passes through one of the values $N_{sj}$, $s = 1, 2, 3, \ldots, k$. The partitions of $N_j—M_j$ into $(k—1)$ distinct parts will give also all the 2*-partitions of $N_j$.

Thus, we have

$$kq(N_j, k)+q^*(2) = \sigma_1 q(N_j, k-1)$$
$$q^*(2)+q^*(3) = \sigma_2 q(N_j, k-2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$q^*(k-1)+q^*(k) = \sigma_{k-1} q(N_j, 1)$$

and finally

$$q^*(k) = \lambda_k = \sigma_k q(N_j, 0).$$

Hence

$$kq(N_j, k) = \sum_{s=1}^{k} (-1)^{s-1}\sigma_s q(N_j, k-s) \qquad \ldots \quad \ldots \quad \ldots \quad [A]$$

or what is the same thing

$$Q_k = \sum_{s=1}^{k} (-1)^{s-1}(k-1)^{(s-1)}\sigma_s Q_{k-s}; \qquad \ldots \quad \ldots \quad \ldots \quad [B]$$

where

$$Q_0 = q(N_j, 0).$$

In particular

$$Q_1 = \sigma_1 Q_0,$$
$$Q_2 = (\sigma_1^2-\sigma_2)Q_0,$$
$$Q_3 = (\sigma_1^3-3\sigma_1\sigma_2+2\sigma_3)Q_0.$$

4. The recurrence formula for $p(N_j, k)$.

Consider a fixed $M_j < N_j$.

Then $M_j$ occurs as a part in a partition of $N_j$ into $k$ summands, at least once in

$$p(N_j-M_j, k-1)$$

partitions; at least twice in

$$p(N_j-2M_j, k-2)$$

partitions; at least thrice in

$$p(N_j-3M_j, k-3)$$

partitions and so on; because if we take $M_j$ as a part $t$ times, we are left to partition $(N_j—tM_j)$ into $(k—t)$ parts.

Hence $M_j$ occurs as a part *exactly* once in

$$p(N_j-M_j, k-1)-p(N_j-2M_j, k-2)$$

partitions; exactly twice in

$$p(N_j-2M_j,\, k-2)-p(N_j-3M_j,\, k-3)$$

partitions and so on.

Thus if all the $p(N_j, k)$ partitions of $N_j$ into exactly $k$ parts were to be written out at length, the number of times $M_j$ will have to be written is exactly

$$p(N_j-M_j,\, k-1)+p(N_j-2M_j,\, k-2)+\ldots+p(N_j-rM_j,\, k-r),$$

where $r$ is the greatest integer $\leqslant k$ for which

$$N_j-rM_j \geqslant 0.$$

Letting $M_j$ vary now, the number of all the parts in the $p(N_j, k)$ partitions is given by

$$\sigma_1 p(N_j,\, k-1)+\sigma_2 p(N_j,\, k-2)+\ldots+\sigma_{k-1} p(N_j,\, 1)+\sigma_k p(N_j,\, 0).$$

Since the number of all the parts is also

$$= kp(N_j,\, k)$$

we must have

$$kp(N_j,\, k) = \sum_{s=1}^{k} \sigma_s p(N_j,\, k-s) \qquad [C]$$

or what is the same thing

$$P_k = \sum_{s=1}^{k} (k-1)^{(s-1)}\, \sigma_s P_{k-s} \qquad [D]$$

where $\qquad P_0 = p(N_j,\, 0).$

The rather surprising similarity between the recursion formulae for $p(N_j, k)$ and $q(N_j, k)$ shows that the expression for $p(N_j, k)$ will be the same as that for $q(N_j, k)$ with the minus signs in the latter replaced by plus signs. Thus

$$P_4 = (\sigma_1^4+6\sigma_1^2\sigma_2+3\sigma_2^2+8\sigma_1\sigma_3+6\sigma_4)P_0.$$

while

$$Q_4 = (\sigma_1^4-6\sigma_1^2\sigma_2+3\sigma_2^2+8\sigma_1\sigma_3-6\sigma_4)Q_0.$$

REMARK. The $k$-term formulae [A] and [C] are suitable for computation and are now being used by Om Prakash Gupta for the special case when $j = 2$.

5. The degree and the weight of an operator.

The degree of the operator

$$\sigma_{\alpha_1}^{\beta_1}\sigma_{\alpha_2}^{\beta_2}\ldots\sigma_{\alpha_t}^{\beta_t}$$

is defined as the sum

$$\sum_{r=1}^{t} \beta_r$$

and its weight as the sum

$$\sum_{r=1}^{t} \alpha_r\beta_r.$$

If

$$P_k = \phi_k(\sigma) \,.\, P_0,$$

and

$$Q_k = \psi_k(\sigma) \,.\, Q_0,$$

where $\phi_k(\sigma)$ and $\psi_k(\sigma)$ are polynomials in the operators $\sigma_1, \sigma_2, \ldots\ldots, \sigma_k$; then it can be readily shown that the weight of each term in these polynomials is $k$.

Moreover, in the expression for $Q_k$, the sign of any term for which the operator is of degree $(k-s)$ is given by $(-1)^s$. It would, therefore, be quite useful, sometimes, to group together terms of the same degree in the operators.

6. If we denote by $X_s$ the expression

$$\frac{x^s}{1-x^s}, |x| < 1;$$

and by $\{f(x)\}_{n_i}$, the coefficient of $x^{n_i}$ in the expansion of $f(x)$ in ascending powers of $x$, then

$$\sigma_t p(N_j, 0) = \lambda_t(N_j) = \prod_i (X_t)_{n_i}.$$

In general, it can be shown without difficulty, that

$$\sigma_{\alpha_1}^{\beta_1}\sigma_{\alpha_2}^{\beta_2}\ldots\sigma_{\alpha_t}^{\beta_t} p(N_j, 0) = \prod_i \left(X_{\alpha_1}^{\beta_1} X_{\alpha_2}^{\beta_2}\ldots X_{\alpha_t}^{\beta_t}\right)_{n_i}.$$

We make use of the fact that for all $s$ and $t$,

$$\sigma_s\sigma_t p(N_j, 0) = \sigma_s\lambda_t(N_j) = \prod_i (X_s X_t)_{n_i}.$$

The method for the computation of the value of

$$\left(X_{\alpha_1}^{\beta_1} X_{\alpha_2}^{\beta_2}\ldots X_{\alpha_t}^{\beta_t}\right)_{n_i} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad [E]$$

for given $n_i$ is well known (Cheema 1956).

We further notice that the coefficient of $x^{n_i}$ in [E] is a polynomial in $n_i$ of degree at least one less than $\sum_{r=1}^{t} \beta_r$. The coefficients of the terms in this polynomial depend not only on the values of $\alpha$'s and $\beta$'s but also on the residue class to which $n_i$ belongs with regard to the modulus given by the least common multiple of the $\alpha$'s. As an illustration, consider the coefficient of $x^{n_i}$ in $X_2 X_3^2$, i.e. in

$$\frac{x^8}{(1-x^2)(1-x^3)^2}. \qquad \qquad (6.1)$$

Suppose

$$n_i \equiv 5 \pmod{6}$$

then the coefficient of $x^{n_i}$ in the expansion of (6.1), i.e. in

$$(x^2+x^4+x^6+\ldots+x^{n_i-1})\left(x^6+2x^9+3x^{12}+\ldots+\frac{n_i-5}{3}.x^{n_i-2}\right),$$

is

$$\frac{n_i-5}{3}+\frac{n_i-11}{3}+\frac{n_i-17}{3}+\ldots+\frac{6}{3}$$
$$=\frac{1}{36}(n_i^2-4n_i-5).$$

While if

$$n_i \equiv 4 \pmod{6}$$

the coefficient is

$$\frac{n_i-7}{3}+\frac{n_i-13}{3}+\frac{n_i-19}{3}+\ldots+\frac{3}{3}$$
$$=\frac{1}{36}(n_i^2-8n_i+16).$$

In general, it can be shown that for large $n_i$, the coefficient of $x^{n_i}$ in the expansion of

$$X_{\alpha_1}^{\beta_1} X_{\alpha_2}^{\beta_2} \ldots X_{\alpha_t}^{\beta_t} X_1$$

is asymptotically equal to

$$n_i^b \Big/ \left\{b!\, \alpha_1^{\beta_1} \alpha_2^{\beta_2} \ldots \alpha_t^{\beta_t}\right\}$$

where

$$b = \sum_{r=1}^{t} \beta_r.$$

7. The expression for $P_k$ can be written in the form of a determinant. Thus

$$P_k = \begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 \ldots \sigma_{k-1} & \sigma_k \\ -(k-1) & \sigma_1 & \sigma_2 \ldots \sigma_{k-2} & \sigma_{k-1} \\ 0 & -(k-2) & \sigma_1 \ldots \sigma_{k-3} & \sigma_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots \sigma_1 & \sigma_2 \\ 0 & 0 & 0 \ldots -1 & \sigma_1 \end{vmatrix} P_0,$$

A similar formula can be given for $Q_k$.

If we replace each $\sigma$ by $x$ in the determinant for $P_k$, then the value of the determinant comes out to be

$$x(x+1)(x+2)\dots(x+k-1) \qquad \dots \qquad \dots \qquad \dots \quad (7.1)$$

Hence the sum of the coefficients of terms for which the operator is of degree $k-r$, $r \geqslant 0$, is the coefficient of $x^{k-r}$ in (7.1), i.e. it is the sum of the products of the first $(k-1)$ integers taken $r$ at a time.

For example, the coefficient of $\sigma_1^{k-2}\ \sigma_2$, for which the operator is of degree $(k-1)$ and which is the only term with an operator of this degree, will be the coefficient of $x^{k-1}$ in (7.1). This is readily seen to be $k(k-1)/2$.

From (7.1) it also follows that the sum of the coefficients of all the terms in $P_k$ is $k\,!$. Similarly, the sum of coefficients of the terms in $Q_k$ can be shown to be zero.

Since the weight of the operator for each term of $P_k$ is $k$, the number of terms in $P_k$ is the same as the number of unrestricted partitions of $k$.

8. Calculation of numerical coefficients of terms.

Let $N_k(T)$ denote the numerical coefficient of the term corresponding to the operator $T$ in $P_k$, where

$$T = \sigma_{\alpha_1}^{\beta_1}\sigma_{\alpha_2}^{\beta_2}\dots\sigma_{\alpha_t}^{\beta_t}$$

with $\alpha$'s all distinct and $\sum_{r=1}^{t} \alpha_r\beta_r = k$.

By inductive reasoning, we show that

$$N_k(T) = k\,!\Big/\prod_{r=1}^{t}\left\{\beta_r\,!\,\alpha_r^{\beta_r}\right\}$$

Let us assume that the result holds for all operators of weight $\leqslant (k-1)$. Then since

$$P_k = \sum_{s=1}^{k} (k-1)^{(s-1)}\sigma_s P_{k-s}\,;$$

$$N_k(T) = \sum_{r=1}^{t} (k-1)^{(\alpha_r-1)} N_{k-\alpha_r}\left(\frac{T}{\sigma_{\alpha_r}}\right),$$

$$= \sum_{r=1}^{t} (k-1)^{(\alpha_r-1)}\cdot\frac{(k-\alpha_r)\,!}{\prod_{h=1}^{t}\left\{\beta_h\,!\,\alpha_h^{\beta_h}\right\}\Big/\alpha_r\beta_r}\,;$$

$$= \frac{(k-1)\,!}{\prod_{h=1}^{t}\left\{\beta_h\,!\,\alpha_h^{\beta_h}\right\}}\sum_{r=1}^{t}\alpha_r\beta_r$$

$$= \frac{k\,!}{\prod_{h=1}^{t}\left\{\beta_h\,!\,\alpha_h^{\beta_h}\right\}}\,.$$

This completes the induction on $k$ and the result is proved.

It is noteworthy that the coefficient of each term in $P_k$ is a function of $k$ alone and is independent of $j$.

9. From the results of the preceding sections, it follows that (Wright 1956)

$$P_k = \sum_{\Sigma\alpha_r\beta_r = k} \left\{ \frac{k!}{\Pi\left\{\beta_r!\,\alpha_r^{\beta_r}\right\}} \prod_i \left(\Pi X_{\alpha_r}^{\beta_r}\right)_{n_i} \right\} \qquad \text{[F]}$$

and

$$Q_k = \sum_{\Sigma\alpha_r\beta_r = k} \left\{ (-1)^{k-\Sigma\beta_r} \cdot \frac{k!}{\Pi\left\{\beta_r!\,\alpha_r^{\beta_r}\right\}} \prod_i \left(\Pi X_{\alpha_r}^{\beta_r}\right)_{n_i} \right\}. \qquad \text{[G]}$$

Thus for $k = 6$, the eleven solutions of the equation $\Sigma\alpha_r\beta_r = k$ are

(1) $\alpha_1 = 1, \beta_1 = 6;$

(2) $\alpha_1 = 1, \beta_1 = 4;\ \alpha_2 = 2, \beta_2 = 1;$

(3) $\alpha_1 = 1, \beta_1 = 3;\ \alpha_2 = 3, \beta_2 = 1;$

(4) $\alpha_1 = 1, \beta_1 = 2;\ \alpha_2 = 2, \beta_2 = 2;$

(5) $\alpha_1 = 1, \beta_1 = 2;\ \alpha_2 = 4, \beta_2 = 1;$

(6) $\alpha_1 = 1, \beta_1 = 1;\ \alpha_2 = 2, \beta_2 = 1;\ \alpha_3 = 3, \beta_3 = 1;$

(7) $\alpha_1 = 2, \beta_1 = 3;$

(8) $\alpha_1 = 2, \beta_1 = 1;\ \alpha_2 = 4, \beta_2 = 1;$

(9) $\alpha_1 = 3, \beta_1 = 2;$

(10) $\alpha_1 = 1, \beta_1 = 1;\ \alpha_2 = 5, \beta_2 = 1;$

(11) $\alpha_1 = 6, \beta_1 = 1.$

Hence

$$\begin{aligned} Q_6 = {} & \frac{6!}{6!}\prod_i (X_1^6)_{n_i} - \frac{6!}{4!\,2}\prod_i (X_1^4X_2^1)_{n_i} + \frac{6!}{3!\,3}\prod_i (X_1^3X_3^1)_{n_i} \\ & + \frac{6!}{2!\,2!\,2^2}\prod_i (X_1^2X_2^2)_{n_i} - \frac{6!}{2!\,4}\prod_i (X_1^2X_4^1)_{n_i} \\ & - \frac{6!}{1\,.\,2\,.\,3}\prod_i (X_1^1X_2^1X_3^1)_{n_i} - \frac{6!}{3!\,2^3}\prod_i (X_2^3)_{n_i} \\ & + \frac{6!}{2\,.\,4}\prod_i (X_2^1X_4^1)_{n_i} + \frac{6!}{2!\,3^2}\prod_i (X_3^2)_{n_i} \\ & + \frac{6!}{1\,.\,5}\prod_i (X_1^1X_5^1)_{n_i} - \frac{6!}{6}\prod_i (X_6^1)_{n_i}. \end{aligned}$$

In $P_6$ all the above terms have positive signs.

10. It would be interesting to give the two highest degree terms in $P_k$. The first one is

$$\prod_{i=1}^{j} \left\{X_1^k\right\}_{n_i} = \prod_{i=1}^{j} \left\{\frac{x^k}{(1-x)^k}\right\}_{n_i},$$

$$= \prod_{i=1}^{j} \binom{n_i-1}{k-1};$$

and the second is

$$\frac{k!}{(k-2)!\,.\,2} \prod_{i=1}^{j} \left(X_1^{k-2} X_2^1\right)_{n_i}$$

$$= \frac{k(k-1)}{2} \prod_{i=1}^{j} \sum_{0 < 2m < n_i-1} \binom{n_i-2m-1}{k-3}. \quad \ldots \quad \ldots \quad (10.1)$$

Now

$$\sum_{0 < 2m < n_i-1} \binom{n_i-2m-1}{k-3}$$

is one of two polynomials in $n_i$ of degree $(k-2)$.

The value of the sigma in (10.1) lies between

$$\tfrac{1}{2}\binom{n_i-2}{k-2} \text{ and } \tfrac{1}{2}\binom{n_i-1}{k-2}$$

and is for large $n_i$, and fixed $k$,

$$\sim \tfrac{1}{2}\binom{n_i-\frac{3}{2}}{k-2}; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.2)$$

because

$$\sum_{0 < 2m < n_i-1} \binom{n_i-2m-1}{k-3} + \sum_{0 < 2m < n_i} \binom{n_i-2m}{k-3} = \binom{n_i-1}{k-2}$$

while

$$\sum_{0 < 2m < n_i-1} \binom{n_i-2m-1}{k-3} + \sum_{0 < 2m < n_i-2} \binom{n_i-2m-2}{k-3} = \binom{n_i-2}{k-2}.$$

Hence the second term is

$$\frac{k(k-1)}{2} \prod_{i=1}^{j} \left\{\frac{1}{2}\binom{n_i-\frac{3}{2}}{k-2} + O\left(n_i^{k-3}\right)\right\}.$$

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