MIS9

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1.

1. From equation 15.10 that:

$$A^T(AA^T+\lambda I)^{-1} = (A^TA+\lambda I)^{-1}A^T$$

$$\hat{x} = A^T(A^TA+\lambda I)^{-1}b$$

First, form $AA^T+\lambda I$ (m^2n flops)

Then, $LL^T = AA^T+\lambda I$ ($\frac{1}{3}m^3$ flops)

For $L^{-1}b$ and $L^{-T}L^{-1}b$, each costs n^2 flops

Last for $A^TL^{-T}L^{-1}b$ ($2mn$ flops)

So, the result based on first two steps, which is $O(m^2n+m^3)$

26

2.

2\. (a) The characteristic polynomial of C is $p(x) = \det(xI - C)$

$$\det(xI-C) = \begin{bmatrix} x & 0 & \cdots & 0 & c_0 \\ -1 & x & & 0 & c_1 \\ 0 & -1 & & 0 & c_2 \\ \vdots & & \ddots & & \\ 0 & 0 & & -1 & x+c_{n-1} \end{bmatrix}$$

$$= x \begin{bmatrix} x & 0 & \cdots & 0 & c_1 \\ -1 & x & \cdots & 0 & c_2 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 1 & x+c_{n-1} \end{bmatrix} + (-1)^{n+1} c_0 \begin{bmatrix} -1 & x & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

$$= x(x^{n-1} + c_{n-1}x^{n-2} + \cdots + c_2 x + c_1) + (-1)^{n+1} c_0 * (-1)^{n-1}$$

$$= x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

$$= x^n + \sum_{k=0}^{n-1} c_k x^k$$

(b) For C has n distinct eigenvalues, we have $\lambda_i^n = \sum_{k=0}^{n-1} c_k \lambda_i^k = 0 \rightarrow \lambda_i^n = -\sum_{k=0}^{n-1} c_k \lambda_i^k$

Then we choose the ith row of V

$$V_i^T = [1 \ \lambda_i \ \lambda_i^2 \ \cdots \ \lambda_i^{n-1}]$$

$$V_i^T C = [\lambda_i \ \cdots \ \lambda_i^{n-1} \ -\sum_{k=0}^{n-1} c_k \lambda_i^k$$

$$= [\lambda_i \ \lambda_i^2 \ \cdots \ \lambda_i^n]$$

$$= \lambda_i V_i^T$$

So. we can have $VC = \Lambda V$

3.

(a) For $C = F^* \Lambda F$, we have $FC = \Lambda F$,

and in the jth row. $f_j^T C = \lambda_j \cdot f_j^T$

$$\frac{1}{\sqrt{n}} \sum_{d=1}^{k-1} w^{(j-1)(d-1)} \cdot C_{n-k+d} + \sum_{d=k}^{n} w^{(j-1)(d-1)} C_{d-k+1}$$

As we know that $w = e^{\frac{-2\pi i}{n}}$. so $w^d = w^{d+n}$

and $\frac{1}{\sqrt{n}} \sum_{d=k}^{n+k-1} w^{(j-1)(d-1)} C_{d-k+1}$

Set d with $d-k+1$, we have

$\frac{1}{\sqrt{n}} w^{k-1} \sum_{d=1}^{n} w^{(j-1)(d-1)} C_d$. which is the kth entry of $f_j^T C$

Baced on $\Lambda = \mathrm{diag}(Fc)$.

$$\lambda_j = \sum_{d=1}^{n} w^{(j-1)(d-1)} C_d$$

Then we can have the kth entry of $\lambda_j f_j^T$ is

$$\frac{1}{\sqrt{n}} \lambda_j W^{k-1} = \frac{1}{\sqrt{n}} w^{k-1} \sum_{d=1} W^{(j-1)(d-1)} C_d$$

In the end of all $j, k = 1 \cdots\cdots n$.

$[FC]_{jk} = [\Lambda F]_{jk}$, so $C = F^* \Lambda F$.

(b) To solve $Cx = b$, we have $x = C^{-1} b$

, so $x = F^* \Lambda^{-1} F b$, which $\Lambda = \mathrm{Diag}(Fc)$

① Compute Fb via FFT $(n \log n)$

② Compute Fc via FFT $(n \log n)$

③ Compute $\Lambda^{-1} Fb$ (n)

④ Compute $F^* \Lambda^{-1} Fb$ $(n \log n)$

Therefore. the complexity is $O(n \log n)$

4.

4.

(a) Replace A with $U\Sigma V^T$ in $A^TA\hat{x}=A^Tb$

$$A^TA\hat{x}=V\Sigma U^TU\Sigma V^TV\Sigma^{-1}U^Tb$$
$$=V\Sigma\Sigma\Sigma^{-1}U^Tb$$
$$=V\Sigma U^Tb$$
$$=A^Tb$$

(b) For $\hat{x}$ that satisfies $A^TA\hat{x}=A^{-1}b$

$$\|Ax-b\|^2=\|Ax-A\hat{x}+A\hat{x}-b\|^2$$
$$=\|Ax-A\hat{x}\|^2-2(Ax-A\hat{x})^T(A\hat{x}-b)+\|A\hat{x}-b\|^2$$
$$=\|Ax-A\hat{x}\|^2+\|A\hat{x}-b\|^2-2(x-\hat{x})^T(A^TA\hat{x}-A^Tb)$$

As we know $A^TAx-A^Tb=0$ and $\|Ax-A\hat{x}\|^2 \geq 0$

So, $\|Ax-b\|^2 \geq \|A\hat{x}-b\|^2$, any $\hat{x}$ satifies $A^TA\hat{x}=A^Tb$ is an optimal solution that minimizes $\|Ax-b\|^2$.