MIS Homework 5

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Problem 7.8

Solution:

7.8

$$Ax + s = 0$$

$$1^T(Ax+s) = 0$$

$$(1^T A)x + (1^T s) = 0$$

A is the incidence matrix, and $1^T A$ is the row vector of columns sums of A, so $1^T A = 0$.

Thus we can have $1^T s = 0$.

$$coef = \frac{1}{\sqrt{n}} \cdot 1_n^T \alpha \qquad \frac{1}{\sqrt{n}}$$

$$coef = \frac{x^T d}{d^T d}$$

Problem 8.8

Solution:

8.8 For $y_i = \frac{c_1 + c_2 t_i + c_3 t_i^2}{1 + d_1 t_i + d_2 t_i^2}$, $i = 1 \dots k$

so we have $y_i = c_1 + c_2 t_i + c_3 t_i^2 - y_i d_1 t_i - y_i t_i^2$

$$A = \begin{bmatrix} 1, & t_1, & t_1^2 & -y_1 t_1, & -y_1 t_1^2 \\ 1, & t_2, & t_2^2 & -y_2 t_2 & -y_2 t_2^2 \\ & & \cdots & & \\ & & \cdots & & \\ 1 & t_k & t_k^2 & -y_k t_k & -y_k t_k^2 \end{bmatrix}$$

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ \cdots \\ y_k \end{bmatrix} \quad \text{and} \quad \theta = (c_1, c_2, c_3, d_1, d_2)$$

Problem 8.12

Solution:

8.12 (a) For $k \geq 1$, $b_k = \int_{-1}^{1} t^{k-1}\, dt = \begin{cases} \frac{2}{k} & k \text{ is odd} \\ 0 & k \text{ is even} \end{cases}$

So the quadrature rule has order d

$$w_1 t_1^{k-1} + w_2 t_2^{k-1} + \cdots + w_n t_n^{k-1} = b_k \quad k = 1, 2, \cdots d+1$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ & \cdots & & \\ t_1^d & t_2^d & \cdots & t_n^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \cdots \\ w_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_4 \end{bmatrix}$$

(b) For Trapezoid rule:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Simpson's rule:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix}$$

Simponson's $\frac{3}{8}$ rule:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\frac{1}{3} & \frac{1}{3} & 1 \\ 1 & \frac{1}{9} & \frac{1}{9} & 1 \\ -1 & -\frac{1}{27} & \frac{1}{27} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

Problem 10.16

Solution:

10.16 (a) $\mu = (A^T 1) \div n = \frac{(a_i^T 1)}{n}$

(b) $\tilde{A} = [a_1 \cdots a_k] - [\mu_1 1 \cdots \mu_k 1]$

$= A - 1\mu^T$

(c) $\Sigma_{ii} = \frac{1}{N}(\tilde{A}^T \tilde{A})_{ii} = \frac{1}{N}\tilde{a}_i^T \tilde{a}_i = \frac{1}{N}\|a_i\|^2$

$= std(a_i)^2$

$\Sigma_{ij} = \frac{1}{N}(\tilde{A}^T \tilde{A})_{ij} = \frac{1}{N}\tilde{a}_i^T \tilde{a}_j = \rho_{ij}\, std(a_i)\, std(a_j)$

$\rho_{ij} = \frac{1}{N}\frac{\tilde{a}_i^T \tilde{a}_j}{std(a_i)\, std(a_j)}$

(d) $Z = \tilde{A}\, diag\left(\frac{1}{std(a_1)} \cdots \frac{1}{std(a_k)}\right)$

$= (A - 1^T\mu)\, diag\left(\frac{1}{std(a_1)} \cdots \frac{1}{std(a_k)}\right)$

Problem 10.31

Solution:

10.31 (a). the total number of paths of length $\leq L$ is P_{ij}

$P = I + A + A^2 + \cdots + A^k$

(b) Calculate P when $k = 1, 2, \cdots$ and stop the first time when all entries of P are positive.

Problem 10.36

Solution:

10.36. (a)

$$Ax = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \cdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}$$

$$x^T A x = [x_1 \cdots x_n] \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j) \\ \cdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}$$

$$= \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

(b) $X^T (A^T) X = (AX)^T X = \langle AX . X \rangle$

$= \langle X . AX \rangle = X^T A X$

(c) $X^T ((A+A^T)/2) X$

$= \frac{X^T A X}{2} + \frac{X^T A^T X}{2}$

For problem b, we have $X^T A^T X = X^T A X$

So we have $X^T ((A+A^T)/2) X = X^T A X$

(d) $2x_1^2 - 3x_1 x_2 - x_2^2$

$= 2x_1^2 - \frac{3}{2} x_1 x_2 - \frac{3}{2} x_1 x_2 - x_2^2$

$$= [x_1 \; x_2] \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$= X^T A X$

$$A = \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem 11.11

Solution:

11.11 We have the 5 interpolation conditions

$$c_1 + c_2 + c_3 = 2(1 + d_1 + d_2)$$

$$c_1 + 2c_2 + 4c_3 = 5(1 + 2d_1 + 4d_2)$$

$$c_1 + 3c_2 + 9c_3 = 9(1 + 3d_1 + 9d_2)$$

$$c_1 + 4c_2 + 16c_3 = -(1 + 4d_1 + 16d_2)$$

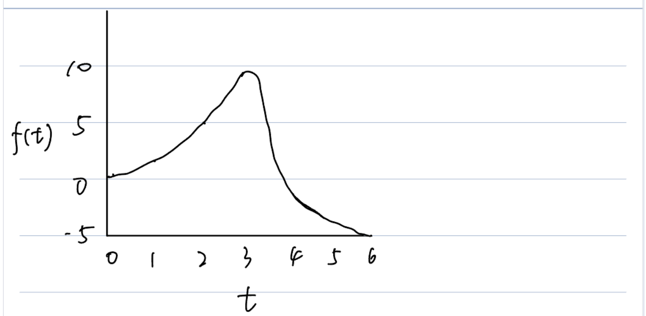
$$c_1 + 5c_2 + 25c_3 = -4(1 + 5d_1 + 25d_2)$$

This is a set of linear equations

$$\begin{bmatrix} 1 & 1 & 1 & -2 & -2 \\ 1 & 2 & 4 & -10 & -20 \\ 1 & 3 & 9 & -27 & -81 \\ 1 & 4 & 16 & 4 & 16 \\ 1 & 5 & 25 & 20 & 100 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \\ -1 \\ -4 \end{bmatrix}$$

Hence, $c_1 = 0.6296$ $c_2 = 0.6049$ $c_3 = -0.1975$

$d_1 = -0.5679$ $d_2 = 0.0864$



Problem 11.21

Solution:

11.21.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

where $b_k = \int_{-1}^{1} t^{k-1} dt = \begin{cases} \frac{2}{k} & t \text{ is odd} \\ 0 & t \text{ is even} \end{cases}$

The solution is $w_1 = 1.6375$ $w_2 = -0.6375$ $w_3 = -0.6375$ $w_4 = 1.6375$

For $f(t) = e^x$ $\alpha = 2.3504$ $\hat{\alpha} = 2.5157$