Blackjack is a card game that uses a deck of cards with J, Q, and K having a value of 10 and Aces having a value of either 1 or 11. If a hand has an Ace that counts as an 11, it is said to be "soft." Otherwise, the hand is said to be "hard." We make this distinction because having a soft hand allows more flexibility in player moves. The goal of the game is to get as close as possible to 21 without going over to beat the dealer's hand. Many rules pay 3:2, which means that if the player has a natural blackjack (10-valued card and Ace for their first two cards) and the dealer does not, then they get paid $3 for every $2 they bet [1].

Now we ask, how much does the player win or lose per game, on average? Here, we introduce the concept of the house edge, or the percentage of the player's initial bet that they lose after playing a game. A positive house edge means that on average, the player can expect to lose money on any bet [7]. Let's start with motivation for finding an optimal strategy. Suppose that you do not know anything about blackjack. Then, one strategy you may come up with is to follow the dealer, that is, hit under 17 and stand over 17. This could make sense since a total of 17+ is relatively high and hitting on 17+ gives you a high probability of busting. In this case, splits, surrenders, and doubles are not considered for simplicity.

To start, there are two factors that determine the player's probability of winning, being their current hand and the dealer's up-card. First, let us consider dealer probabilities. There are 6 possible outcomes for the dealer: 17, 18, 19, 20, 21, or dealer bust. We begin by finding the probabilities that the dealer ends on those totals or busts for every possible hand. For example, if the dealer's current hand is 17, then the probability they end with 17 is 1, and the probability of all other outcomes is 0. To calculate the probabilities of the outcomes for hands less than 17, we can consider the possible cards the dealer can draw. To simplify things, we can consider an infinite number of decks, so that cards of each rank will not be depleted. Then, in the case that the dealer has a 12, for example, drawing another card could bring them anywhere from 13 to 22. Notice that the probability the dealer lands on 17, 18, 19, 20, and 21 are all 1/13, and the probability that the dealer busts is 4/13. We can then find the probability of an outcome of 17 on a current total of 12, for example, by summing the probability that the dealer draws a 5 (ending on 17) and the probabilities that the dealer lands on 13 to 16, with those hands subsequently ending on 17. That is, let X be a random variable denoting the dealer's outcome and let Y be a random variable denoting the dealer's current total. Then,

$$\Pr(X = 17|Y = H12) = \frac{\Pr(X = 17|Y = H13) + \Pr(X = 17|Y = H14) + \cdots + \Pr(X = 17|Y = H17)}{13}.$$

The dealer's outcome is conditioned on their up-card. If we apply formulas of this type to every hand under 17, we can find the probability of all outcomes for every current hand and up-card [3].

Table 1. *Conditional Probabilities for Dealer Final Outcomes for every Dealer Up-card*

American rule - peek for blackjack AND the dealer doesn't have blackjack												
	Possible upcards ->		2	3	4	5	6	7	8	9	10	Ace
Outcome												
Bust			0.353608	0.373875	0.394468	0.416404	0.42315	0.262312	0.244741	0.228425	0.229785	0.166525
17			0.139809	0.135034	0.13049	0.122251	0.165438	0.368566	0.128567	0.119995	0.12071	0.188917
18			0.134907	0.130482	0.125938	0.122251	0.106267	0.137797	0.359336	0.119995	0.12071	0.188917
19			0.129655	0.125581	0.121386	0.1177	0.106267	0.078625	0.128567	0.350765	0.12071	0.188917
20			0.124026	0.120329	0.116485	0.113148	0.101715	0.078625	0.069395	0.119995	0.37071	0.188917
21			0.117993	0.1147	0.111233	0.108246	0.097163	0.074074	0.069395	0.060824	0.037376	0.077806
Total			1	1	1	1	1	1	1	1	1	1

M. Shackleford. "Blackjack Basic Strategy for Infinite Decks." YouTube, 2014, https://www.youtube.com/watch?v=jCF-Btu5ZCk.

Now, using these dealer probabilities, we can calculate the probability of winning and losing for every player move. For standing, this is simply the probability that the player's hand beats the dealer's hand. For example, let X denote whether the player wins by standing and let Y denote whether the player loses by standing. Then, from Table 1, we have

$$\Pr(X = 1|Total = 19 \cap Upcard = 2) = 0.353608 + 0.139809 + 0.134907 = 0.628324,$$

$$\Pr(Y = 1|Total = 19 \cap Upcard = 2) = 0.124026 + 0.117993 = 0.242019.$$

Notice that $\Pr(X = 1) + \Pr(Y = 1) \neq 1$, since there is also the possibility of a tie. Now, we can find the expected value of bets won, Z, as

$$E[Z|Total = t \cap Upcard = u] = \Pr(X = 1|Total = t \cap Upcard = u) - \Pr(Y = 1|Total = t \cap Upcard = u)\,.(1)$$

We can also find the probabilities of winning by hitting. For example, hitting on a hard 15 can bring the player anywhere from 16 to 25. Then, if the player lands on 16, they must hit again, since their total is under 17. Otherwise, they must stand, so we must reference the probabilities for winning by standing. We can calculate the expected value of bets won similarly to Equation (1). After finding the probabilities of every starting hand and up-card, we can find the expected value of playing one round, or the house edge [4]. Finally, we arrive at a house edge of 5.6746%.

Now, our goal is to reduce the house edge. The first step is to find the move between standing and hitting that maximizes the expected value for every combination of player hand and up-card. This optimization lowers the house edge to 2.4208%. Applying this process for the rest of the player moves, we can find the expected values for doubling, surrendering, and splitting. By finding the move that maximizes the expected value for every case, we can find the basic strategy. The house edge for a game following basic strategy is 0.4835% [4].

Table 2. *Basic Strategy for Hard Hands, S17 allowing Late Surrender*

	dealer upcard ->		2	3	4	5	6	7	8	9	10	Ace
player sheets (down)												
4			H	H	H	H	H	H	H	H	H	H
5			H	H	H	H	H	H	H	H	H	H
6			H	H	H	H	H	H	H	H	H	H
7			H	H	H	H	H	H	H	H	H	H
8			H	H	H	H	H	H	H	H	H	H
9			H	D	D	D	D	H	H	H	H	H
10			D	D	D	D	D	D	D	D	H	H
11			D	D	D	D	D	D	D	D	D	H
12			H	H	S	S	S	H	H	H	H	H
13			S	S	S	S	S	H	H	H	H	H
14			S	S	S	S	S	H	H	H	H	H
15			S	S	S	S	S	H	H	H	R	H
16			S	S	S	S	S	H	H	R	R	R
17			S	S	S	S	S	S	S	S	S	S
18			S	S	S	S	S	S	S	S	S	S
19			S	S	S	S	S	S	S	S	S	S
20			S	S	S	S	S	S	S	S	S	S
21			S	S	S	S	S	S	S	S	S	S

M. Shackleford. "Blackjack Basic Strategy for Infinite Decks." YouTube, 2014, https://www.youtube.com/watch?v=jCF-Btu5ZCk.

Now that we have discussed playing strategies, we may question whether betting strategies can increase player advantage and decrease the house edge. For example, we can consider the martingale strategy. Here, the player chooses an initial bet, doubling their bet after a loss and returning to their initial bet after a win. The idea is that after a sequence of losses of any length, if that sequence ends in a win, the player will win their initial bet back [2].

However, there are a few risks of using the martingale system. For instance, the martingale system is effective if the player plays until a win, but a win is never guaranteed under a limited number of rounds. There is a nonzero probability that the bet doubles to an amount that exceeds the player's bankroll. Secondly, the martingale strategy is not always an improvement over constant betting. For a simple example, let $p = 0.45$ be the probability of winning and $b = 6$ be the max number of times the player can double their initial bet of \$1. Here, the probability of winning is deliberately chosen to be less than 0.5 to account for the house edge. Then, the player can bet at most \$127. Let X be the amount earned after playing a streak. We have $E[X] = (-127)(0.55)^7 + (1)(1-(0.55)^7) = -0.9487$. Alternatively, let Y be the amount earned after placing a constant bet of \$1. Then we have $E[Y] = (-1)(0.55) + (1)(0.45) = -0.1$. Thus, the player can expect to lose more money by employing the martingale strategy than simply betting a constant amount. Only in the case where the probability of winning is greater than 0.5 is the martingale strategy more effective than constant betting.

To evaluate the martingale strategy for blackjack, we must first find the probability of winning and losing a game. From before, we can find the probabilities of winning and losing for standing and hitting on every possible player total against every possible up-card. Multiplying these probabilities with the probabilities of starting a game with these totals and summing the products, we can find the probability of winning or losing a game where the player may only hit or stand. In this case, the probability of winning is 0.4329 and the probability of losing is 0.4819.

Now, we may be interested in a context for the martingale strategy. For instance, suppose we want to find the probability that a player successfully doubles their bankroll of $1000 using an initial bet of $50. Recall that every streak that ends in a win earns the initial bet. This means that the player needs 20 consecutive successful streaks to double the initial $1000. Now, with a bankroll of $1000, the player can only afford to double their bet 3 times. However, if the player reaches $1550, they will be able to double their bet 4 times. Using the probabilities of success and failure for the hitting and standing only condition, we find that the loss rate is 0.52678. If the streaks are independent of each other, then the probability of doubling the initial bankroll is $p = (1-(0.52678)^4)^{11} * (1-(0.52678)^5)^9 = 0.28532$. In general, it seems that the larger the initial bet, the greater the probability of success. In fact, an initial bet of $1000 seems to be the best strategy, which makes sense because it is essentially a constant bet.

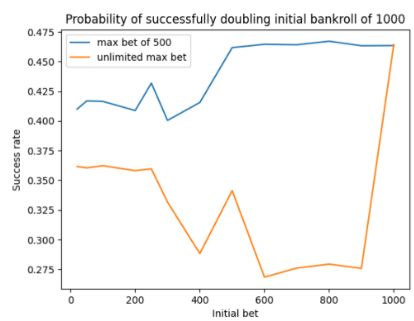
Table 3. *Probability of Successfully Doubling Initial Bankroll of $1000 for Different Sizes of Initial Bets*

	PHASE 1				PHASE 2			
	max bets 1	breakpoin	wins neede	prob 1	max bets 2	wins needed 2	prob 2	probability of success
bet size								
2	8	1022	11	0.936673	9	489	0.21656413	0.202849695
5	7	1275	55	0.536535	8	145	0.422158951	0.226503044
10	6	1270	27	0.558105	7	73	0.437625737	0.244241258
20	5	1260	13	0.58372	6	37	0.449683909	0.2624893
30	5	1890	30	0.288716	6	4	0.917226239	0.264818202
40	4	1240	6	0.618296	5	19	0.455300921	0.281510731
50	4	1550	11	0.414184	5	9	0.688876562	0.285321733
60	4	1860	15	0.300601	5	2	0.920516175	0.276708491
70	3	1050	1	0.85382	4	14	0.325680426	0.278072381
80	3	1200	3	0.622442	4	10	0.448739233	0.279313955
90	3	1350	4	0.531453	4	8	0.526738706	0.279936819
100	3	1500	5	0.453765	4	5	0.669880014	0.3039681
200	2	1400	2	0.52201	3	3	0.622441575	0.32492061
300	2		4	0.272494				0.272494245
400	1	1200	1	0.47322	2	2	0.522009813	0.247025302
500	1	1500	1	0.47322	2	1	0.722502466	0.341902366
600	1	1800	2	0.223937	2	0	1	0.22393684
700	1		2	0.223937	2			0.22393684
800	1		2	0.223937	2			0.22393684
900	1		2	0.223937	2			0.22393684
1000	1		1	0.47322	2			0.473219653

Notice that we used the hit and stand only strategy to calculate these probabilities. It would be difficult to calculate martingale success where doubling, surrendering, and splitting are allowed because these moves can change player bets. Additionally, the martingale strategy assumes that the player always wins their entire bet when they win. However, in the case of natural blackjack, the player wins 1.5 times their initial bet. Thus, the actual probabilities are higher than those shown in the table.

As for using the martingale strategy to find the probability of doubling the bankroll when doubling, surrendering, and splitting are allowed, we can use computer simulations. First, the success rate is maximized when the initial bet is the entire bankroll, which reflects our previous findings. Secondly, it seems that adding a maximum bet can improve the probability of success. For example, let the player start with an initial bet of $500. Then, if the player loses, they must double their bet to $1000, but they only have $500 remaining, so they fail. However, if there is a maximum bet of $500, then the player can continue playing and has another chance to rebound.

Figure 1. *Martingale Probability of Successfully Doubling Initial Bankroll of 1000, all Player Moves Allowed*



Let us introduce another strategy that may perform better than the martingale strategy. In fact, card counting strategies not only change how the player bets but also how they play. A deck that is high in high-valued cards (10 and Ace) is advantageous to the player. On the other hand, a deck that is high in low-valued cards (2 through 6) is advantageous for the dealer [5].

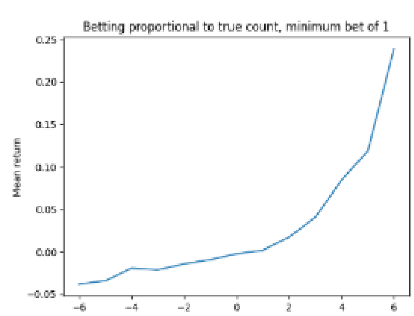
One of the most popular forms of card counting is called High-Low. As cards are revealed, the player updates the running count. If the card is a low card (2 through 6), the player adds 1, and if the card is a high card (10 or Ace), the player adds -1. The true count, which is the running count divided by the number of decks left in the shoe, is an approximation for the number of high cards left in the shoe. Card counters know to bet more when the true count is high and to bet less when the true count is low or negative. Additionally, as the true count changes, the optimal play also changes. For example, when the dealer shows an Ace, players can make a side bet called insurance where they place half of their original bet on that the dealer has a 10 as their hole card. If this is the case, the player gets paid 2:1, protecting them from dealer blackjack. Typically, insurance is not a good bet. To see why, let X be the amount won by taking insurance and let Y be the amount won by not taking insurance. We have $E[X] = \left(\frac{4}{13}\right)(0) + \left(\frac{9}{13}\right)\left(-\frac{1}{2}\right) = -0.3462$, and $E[Y] = \left(\frac{4}{13}\right)(-1) + \left(\frac{9}{13}\right)(0) = -0.3077$. However, the more 10-valued cards there are left in the shoe, the greater the probability of dealer blackjack, and the better taking insurance becomes. Playing with the "Illustrious 18" and "Fab 4" deviations, we can reduce the house edge to around 0.2759% [6].

Now, if we place a constant bet of $1 and track the house edge throughout a game, we find a linear relationship between the true count and the house edge. Notably, for true counts of 1 or greater, the house edge is overcome.

Figure 2. *Expected Bets Won by True Count with Constant Bet of $1*

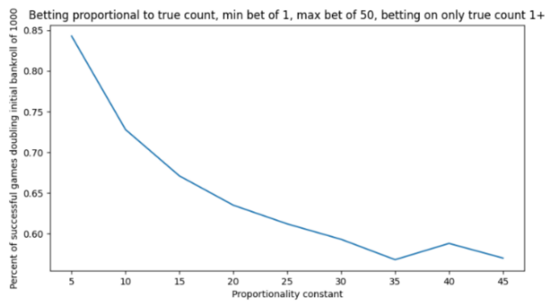
Let us consider betting proportional to the true count (Figure 3). We can make a few observations. First, the expected return for nonpositive true counts is the same as in Figure 2 since they use the bet. Else, the relationship between the true count and expected return is quadratic. Another consideration is that these averages were computed using simulations of 10 million rounds, meaning the result of any given round can vary wildly from the average.

Figure 3. *Expected Bets Won by Betting Proportional to True Count with Minimum Bet of $1*



Now consider our previous scenario of a player doubling their initial bankroll of $1000. We can calculate their probability of success using card counting and proportional betting. To maximize their probability of success, the player will only bet on positive true counts.

Figure 4. *Card Counting Probability of Successfully Doubling Initial Bankroll of 1000*



The proportionality constant is the factor by which the player bet changes by increasing true count. Thus, proportionality constants 0 and 50 both represent constant betting under these conditions. There are a few disadvantages to using card counting in this context. To start, if a player is joining a game, they will not know how many high-value or low-value cards have passed before joining the game, so they will not have an accurate true count. Secondly, this strategy is much slower than the martingale strategy. To start, the player only bets on positive true counts, meaning that their total does not change at all more than half of the time. Additionally, most of the time, the player is playing rounds with relatively low true counts of -3 to 3, which corresponds to smaller bets, especially for smaller proportionality constants. However, if the player has a larger initial bankroll available, it is possible to maintain a high probability of success while increasing the proportionality constant or max bet.

In conclusion, we can find an optimized playing strategy called basic strategy that maximizes the expected value of bets won in every case. Compared to a naïve strategy like following the dealer, the house edge is significantly reduced. We also found that the martingale betting strategy is not an effective strategy even compared to constant betting because the probability of winning a blackjack game is less than 0.5. For card counting, we found that under certain conditions, the player can overcome the house edge. Additionally, the strategy is relatively safe since the player has a high probability of successfully doubling their initial bankroll. However, it has the disadvantage of being hard to execute and time-consuming.

References

1. "Blackjack." *Bicycle*. https://bicyclecards.com/how-to-play/blackjack/ (Accessed 29 November 2023).
2. Hayes, A. "Martingale System: What It Is and How It Works in Investing." *Investopedia*. https://www.investopedia.com/terms/m/martingalesystem.asp (Accessed 29 November 2023).
3. Shackleford, M. "Blackjack Basic Strategy for Infinite Decks." *YouTube*, uploaded by Michael Shackleford, 11 August 2014, https://www.youtube.com/watch?v=jCF-Btu5ZCk.
4. Shackleford, M. "Blackjack house edge with infinite decks." *YouTube*, uploaded by Michael Shackleford, 12 August 2014, https://www.youtube.com/watch?v=wJsGnXgrGvg.
5. Shackleford, M. "Card Counting." *Wizard of Odds*. https://wizardofodds.com/games/blackjack/card-counting/introduction/ (Accessed 29 November 2023).
6. Shackleford, M. "Introduction to the High-Low Card Counting Strategy." *Wizard of Odds*. https://wizardofodds.com/games/blackjack/card-counting/high-low/ (Accessed 29 November 2023).
7. Shackleford, M. "What is the House Edge?" *Wizard of Odds*. https://wizardofodds.com/gambling/house-edge/ (Accessed 29 November 2023).