Notes on Quantum Field Theory

Itsuki Miyane

Last modified: June 6, 2024

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Chapter 1

Group and its representation theory

When we construct the field theory, we impose the theory of Lorentz invariance. One of the easiest ways is to use the field which transforms in a covariant manner with the spacetime Lorentz transformation[*1]. To construct field theories, the representation of the Lorentz group plays a crucial role. In this chapter, we will discuss the property of the Lorentz group and its representation theory.

1.1 Lorentz group

Here, I will give the formal definition of the Lorentz group:

> *Lorentz group* is the set of linear transformations
>
> $$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{1.1.1}$$
>
> which preserves the following inner product
>
> $$x^\mu y_\mu \equiv -x^0 y_0 + x^1 y_1 + x^2 y_2 + x^3 y_3. \tag{1.1.2}$$
>
> Such a transformation is similar to the orthogonal transformation $O(4)$ but one sign is flipped. In this sense, we will denote Lorentz group as $O(3,1)$.

In this text, I will use the metric convention as $\eta^{\mu\nu} = \text{diag}(-,+,+,+)$. From this definition, we can obtain the explicit conditions for the elements of the Lorentz group. Let x^μ, y^ν are transformed as

$$x^{\mu\prime} = \Lambda^\mu{}_\sigma x^\sigma,\ y^{\mu\prime} = \Lambda^\mu{}_\rho y^\rho. \tag{1.1.3}$$

Then the inner product should be

$$y^{\mu\prime} x_{\mu\prime} = y^{\mu\prime} \eta_{\mu\nu} x^{\nu\prime} = (\Lambda^\mu{}_\rho y^\rho)\eta_{\mu\nu}(\Lambda^\nu{}_\sigma x^\sigma) = y^\rho(\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma) x^\sigma. \tag{1.1.4}$$

By changing the dummy indices, we conclude that the $\Lambda \equiv (\Lambda^\mu{}_\nu)$ should satisfy

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma_\nu \ \text{ or } \ \eta = \Lambda^T \eta \Lambda \tag{1.1.5}$$

and, in this manner, we can write

$$SO(3,1) = \tag{1.1.6}$$

[*1] I guess it is not impossible to follow that method, though it is tough work. In that case, the Lorentz invariance is a miracle.

Appendix A

Seiberg-Witten Theory

Appendix B

Relationship between Supersymmetry and Morse theory

B.1 Introduction to supersymmetric quantum mechanics

Appendix C

Brief notes

C.1 Dirac operator and Spin complex

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