Relativity - Report 4

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(1) Let us consider the variation of the action

$$S_m = \int \mathrm{d}^4 \sqrt{-g} K(\phi, X) \tag{0.1}$$

where X denotes

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial[\mu] \phi \partial_\nu \phi. \tag{0.2}$$

It obeys

$$\begin{aligned} \delta S_m &= \int \mathrm{d}^4 \left[(\delta\sqrt{-g}) K + \sqrt{-g} (\delta K) \right] \\ &= -\frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left\{ g_{\mu\nu} K + K_X \partial_\mu \partial_\nu \phi \right\} \delta g^{\mu\nu} \end{aligned} \tag{0.3}$$

where $K_X \equiv \partial_X K(\phi, X)$. The definition of the energy-momentum tensor so far is

$$\delta S_m = -\frac{1}{2} \int \mathrm{d}^4 x \; \delta g^{\mu\nu} \sqrt{-g} T_{\mu\nu} \tag{0.4}$$

and by comparing this definition with the equation (0.3), we find

$$T_{\mu\nu} = g_{\mu\nu} K + K_X \partial_\mu \partial_\nu \phi \qquad (K_X \equiv \partial_X K(\phi, X)). \tag{0.5}$$

(2) The perturbed FLRW metric

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2A) & \partial_i B \\ \partial_i B & (1+2\psi)\delta_{ij} + 2\partial_i \partial_j E \end{pmatrix} \tag{0.6}$$

is obtained by metric perturbation. The energy-momentum tensor becomes

$$T^\mu_{\ \nu} = g^{\mu\rho} T_{\rho\nu} = \delta^\mu_\nu K(\phi, X) + g^{\mu\rho} K_X \partial_\rho \phi \partial_\nu \phi \tag{0.7}$$

from the previous result. By calculating the inverse matrix of the metric, we obtain*1

$$g^{\mu\nu} = \frac{1}{a^2(\eta)} \begin{pmatrix} 1-2A(\eta, \boldsymbol{x}) & \partial_x B & \partial_y B & \partial_z B \\ \partial_x B & 1-2(\psi+\partial_x^2 E) & -2\partial_x \partial_y E & -2\partial_x \partial_z E \\ \partial_y B & -2\partial_x \partial_y E & 1-2(\psi+\partial_y^2 E) & -2\partial_y \partial_z E \\ \partial_z B & -2\partial_x \partial_z E & -2\partial_y \partial_z E & 1-2(\psi+\partial_z^2 E) \end{pmatrix}. \tag{0.8}$$

*1 This result is obtained by purterbating $g^{\mu\nu} = g_0^{\mu\nu} + \delta g^{\mu\nu}$. This implies $g_0^{\mu\nu} = 0$ naively and $\delta g^{\mu\nu}$ should contribute to cancel out the perturbative parts in $g_{\mu\nu}$.

By perturbing $\phi(\eta, \boldsymbol{x})$ as $\phi(\eta) + \delta\phi(\eta, \boldsymbol{x})$[*2], we find $\delta T^0_{\ 0}$ as

$$\begin{aligned}\delta T^0_{\ 0} &= K(\phi + \delta\phi, X + X_\phi \delta\phi) + g^{0\rho} K_X(\phi + \delta\phi, X + X_\phi\delta\phi)\partial_\rho\phi\partial_0\phi \\ &= K + K_\phi\delta\phi + K_X X_\phi \delta\phi \\ &\quad + \frac{1}{a(\eta)}\left(K_X + K_{\phi X}\delta\phi + K_{XX}X_\phi\delta\phi\right)\left[\frac{2A}{a(\eta)}\phi' + \partial_i B \partial_i \phi\right]\phi'. \end{aligned} \tag{0.9}$$

Note that $g^{\mu\nu}$ has only the perturbative quantity, thus the contribution of its coefficient should be only the background. In the same way, we can evaluate $\delta T^0_{\ i}$ and $\delta T^i_{\ j}$. Anyway, we conclude that the result, though it is formal, is

$$\delta T^\mu_{\ \nu} = \delta^\mu_\nu \left(K + K_\phi\delta\phi + K_X X_\phi\delta\phi\right) + g^{\mu\rho}\left(K_X + K_{\phi X}\delta\phi + K_{XX}X_\phi\delta\phi\right)\partial_\rho\phi\partial_\nu\phi. \tag{0.10}$$

(3) (incomplete)
The perturbed Einstein equation is given by

$$\delta G^\mu_{\ \nu} = 8\pi G \delta T^\mu_{\ \nu} \tag{0.11}$$

in Lecture 14. In this case, by putting $K = -\Lambda$ ($=$ constant), the energy-momentum tensor becomes

$$\delta T^\mu_{\ \nu} = -\delta^\mu_\nu \Lambda - g^{\mu\rho}\Lambda\partial_\rho\phi\partial_\nu\phi \tag{0.12}$$

from the previous result.

[*2] I followed the instructions in the problem statement and omitted the bar on the background field.