Relativity Report 1

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1 Problem 1

The Christoffel symbol is defined as

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right) \tag{1.1}$$

in a coordinate system x^λ and transform into a system $\tilde{x}^{\lambda}$[*1]. The transformation laws of the coordinate are

$$g^{\mu\nu} = \frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\tilde{g}^{\rho\sigma} \tag{1.2}$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\lambda}. \tag{1.3}$$

Putting these relations into the definition (1.1), we obtain the transformation law

$$\begin{aligned}
\Gamma_{\mu\alpha\beta} &= \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right)\\
&= \frac{1}{2}\left\{\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial}{\partial \tilde{x}^\delta}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}g_{\lambda\gamma}\right) + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\gamma}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\lambda\delta}\right) - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\lambda}\left(\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\gamma\delta}\right)\right\}\\
&= \frac{1}{2}\left\{\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial}{\partial \tilde{x}^\delta}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\right)\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}g_{\lambda\gamma} + \frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\delta}\left(\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\right)g_{\lambda\gamma} + \frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^\delta}\right.\\
&\qquad + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\gamma}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\right)\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\lambda\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\gamma}\left(\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\right)g_{\lambda\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^\gamma}\\
&\qquad \left. - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\lambda}\left(\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\right)\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\gamma\delta} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\lambda}\left(\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\right)g_{\gamma\delta} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^\lambda}\right\}.
\end{aligned} \tag{1.4}$$

The waved terms in (1.4) become a twice of the Christoffel symbol:

$$\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^\gamma} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^\lambda} = 2\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\tilde{\Gamma}_{\lambda\gamma\delta}. \tag{1.5}$$

But we already find that there are additional terms in (1.4) and they violate the transformation law of the 3-rank covariant tensor. Thus we find the law as

$$\begin{aligned}
\Gamma_{\mu\alpha\beta} &= \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\tilde{\Gamma}_{\lambda\gamma\delta} + \frac{1}{2}\left\{\frac{\partial^2 \tilde{x}^\lambda}{\partial x^\beta \partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}g_{\lambda\gamma} + \frac{\partial^2 \tilde{x}^\gamma}{\partial x^\beta \partial x^\alpha}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}g_{\lambda\gamma}\right.\\
&\qquad \left. + \frac{\partial^2 \tilde{x}^\lambda}{\partial x^\alpha \partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\lambda\delta} + \frac{\partial^2 \tilde{x}^\delta}{\partial x^\alpha \partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}g_{\lambda\delta} - \frac{\partial \tilde{x}^\gamma}{\partial x^\mu x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\gamma\delta} - \frac{\partial^2 \tilde{x}^\delta}{\partial x^\mu \partial x^\beta}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}g_{\delta\gamma}\right\}\\
&= \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\tilde{\Gamma}_{\lambda\gamma\delta} + \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\alpha \partial x^\beta}g_{\lambda\gamma}
\end{aligned} \tag{1.6}$$

Note we freely change the dummy indices and cancel out equivalent terms.

[*1]In the problem statement, we should consider the transformation x^λ into a system "x'^λ" but I should apologize since we will use a different label "$\tilde{x}^\lambda$", though it is trivial.

2 Problem 2

From the previous problem, we obtain the transformation law

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}. \tag{2.1}$$

Since we already know the transformation laws, what we have to do is just to compute them. Thus the transformation should be

$$\begin{aligned}
\tilde{\nabla}_{\mu} \tilde{V}_{\nu} &\equiv \frac{\partial \tilde{V}_{\nu}}{\partial \tilde{x}^{\mu}} - \tilde{\Gamma}^{\lambda}{}_{\mu\nu} \tilde{V}_{\lambda} \\
&= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} V_{\beta} \right) - \left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \right) \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} \\
&= \cancel{\frac{\partial^2 x^{\beta}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}} V_{\beta} + \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \frac{\partial V_{\beta}}{\partial x^{\alpha}} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} - \cancel{\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}}} V_{\delta} \\
&= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \left(\frac{\partial V_{\beta}}{\partial x^{\alpha}} - \Gamma^{\gamma}{}_{\alpha\beta} V_{\gamma} \right) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}
\end{aligned} \tag{2.2}$$

and we finally attain

$$\tilde{\nabla}_{\mu} \tilde{V}_{\nu} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}. \tag{2.3}$$

3 Problem 3

The definition of the covariant derivative of two-rank tensor and as followings:

$$\nabla_\lambda T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho{}_{\mu\lambda} T_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} T_{\mu\rho}, \tag{3.1}$$

$$R^\mu{}_{\nu\lambda\rho} = \frac{\partial}{\partial x^\lambda}\Gamma^\mu{}_{\nu\rho} - \frac{\partial}{\partial x^\rho}\Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\alpha\lambda}\Gamma^\alpha{}_{\nu\rho} - \Gamma^\mu{}_{\alpha\rho}\Gamma^\alpha{}_{\nu\lambda}. \tag{3.2}$$

We already showed that $\nabla_\lambda A_\sigma$ is the 2-rank covariant tensor in the previous problem. Thus we should find $[\nabla_\mu, \nabla_\nu] T_{\lambda\rho}$ for a general two-rank tensor $T_{\mu\nu}$. Let us compute $\nabla_\mu \nabla_\nu T_{\lambda\rho}$ first. It is

$$\begin{aligned}
\nabla_\mu \nabla_\nu T_{\lambda\rho} &= \frac{\partial}{\partial x^\mu}\left(\nabla_\nu T_{\lambda\rho}\right) - {\color{gray}\Gamma^\alpha{}_{\nu\mu}(\nabla_\delta T_{\lambda\rho})} - \Gamma^\sigma{}_{\lambda\mu}(\nabla_\nu T_{\sigma\rho}) - \Gamma^\sigma{}_{\rho\mu}(\nabla_\nu T_{\lambda\sigma}) \\
&= {\color{gray}\frac{\partial^2 T_{\lambda\rho}}{\partial x^\mu \partial x^\nu}} - \frac{\partial \Gamma^\sigma{}_{\lambda\nu}}{\partial x^\mu} T_{\sigma\rho} - \Gamma^\sigma{}_{\lambda\nu}\frac{\partial T_{\sigma\rho}}{\partial x^\mu} - \frac{\partial \Gamma^\sigma{}_{\rho\nu}}{\partial x^\mu} T_{\lambda\sigma} - \Gamma^\sigma{}_{\rho\nu}\frac{\partial T_{\lambda\sigma}}{\partial x^\mu} - {\color{gray}\Gamma^\alpha{}_{\nu\mu}(\nabla_\delta T_{\lambda\rho})} \\
&\quad - \Gamma^\sigma{}_{\lambda\mu}\frac{\partial T_{\sigma\rho}}{\partial x^\nu} + \Gamma^\sigma{}_{\lambda\mu}\Gamma^\alpha{}_{\sigma\nu} T_{\alpha\rho} + \Gamma^\sigma{}_{\lambda\mu}\Gamma^\alpha{}_{\rho\nu} T_{\sigma\alpha} \\
&\quad - \Gamma^\sigma{}_{\rho\mu}\frac{\partial T_{\lambda\sigma}}{\partial x^\nu} + \Gamma^\sigma{}_{\rho\mu}\Gamma^\alpha{}_{\lambda\nu} T_{\alpha\sigma} + \Gamma^\sigma{}_{\rho\mu}\Gamma^\alpha{}_{\sigma\nu} T_{\lambda\alpha}.
\end{aligned}$$

Note that grayed terms vanish from contributions of $\nabla_\nu \nabla_\mu T_{\lambda\rho}$. When we evaluate $\nabla_\mu \nabla_\nu T_{\lambda\rho}$, we just need to flip the indices as $\mu \leftrightarrow \nu$ and we get

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] T_{\lambda\rho} &= \underline{-\frac{\partial \Gamma^\sigma{}_{\lambda\nu}}{\partial x^\mu} T_{\sigma\rho}} - \cancel{\Gamma^\sigma{}_{\lambda\nu}\frac{\partial T_{\sigma\rho}}{\partial x^\mu}} - \underline{\frac{\partial \Gamma^\sigma{}_{\rho\nu}}{\partial x^\mu} T_{\lambda\sigma}} - \cancel{\Gamma^\sigma{}_{\rho\nu}\frac{\partial T_{\lambda\sigma}}{\partial x^\mu}} - \cancel{\Gamma^\sigma{}_{\lambda\mu}\frac{\partial T_{\sigma\rho}}{\partial x^\nu}} \\
&\quad + \underline{\Gamma^\sigma{}_{\lambda\mu}\Gamma^\alpha{}_{\sigma\nu} T_{\alpha\rho}} + {\color{green}\Gamma^\sigma{}_{\lambda\mu}\Gamma^\alpha{}_{\rho\nu} T_{\sigma\alpha}} - \cancel{\Gamma^\sigma{}_{\rho\mu}\frac{\partial T_{\lambda\sigma}}{\partial x^\nu}} + {\color{red}\Gamma^\sigma{}_{\rho\mu}\Gamma^\alpha{}_{\lambda\nu} T_{\alpha\sigma}} + \underline{\Gamma^\sigma{}_{\rho\mu}\Gamma^\alpha{}_{\sigma\nu} T_{\lambda\alpha}} \\
&\quad + \underline{\frac{\partial \Gamma^\sigma{}_{\lambda\mu}}{\partial x^\nu} T_{\sigma\rho}} + \cancel{\Gamma^\sigma{}_{\lambda\mu}\frac{\partial T_{\sigma\rho}}{\partial x^\nu}} + \underline{\frac{\partial \Gamma^\sigma{}_{\rho\mu}}{\partial x^\nu} T_{\lambda\sigma}} + \cancel{\Gamma^\sigma{}_{\rho\mu}\frac{\partial T_{\lambda\sigma}}{\partial x^\nu}} + \cancel{\Gamma^\sigma{}_{\lambda\nu}\frac{\partial T_{\sigma\rho}}{\partial x^\mu}} \\
&\quad - \underline{\Gamma^\sigma{}_{\lambda\nu}\Gamma^\alpha{}_{\sigma\mu} T_{\alpha\rho}} - {\color{red}\Gamma^\sigma{}_{\lambda\nu}\Gamma^\alpha{}_{\rho\mu} T_{\sigma\alpha}} + \cancel{\Gamma^\sigma{}_{\rho\nu}\frac{\partial T_{\lambda\sigma}}{\partial x^\mu}} - {\color{green}\Gamma^\sigma{}_{\rho\nu}\Gamma^\alpha{}_{\lambda\mu} T_{\alpha\sigma}} - \underline{\Gamma^\sigma{}_{\rho\nu}\Gamma^\alpha{}_{\sigma\mu} T_{\lambda\alpha}} \\
&= \underline{\left(\frac{\partial \Gamma^\sigma{}_{\lambda\mu}}{\partial x^\nu} - \frac{\partial \Gamma^\sigma{}_{\lambda\nu}}{\partial x^\mu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\sigma{}_{\alpha\nu} + \Gamma^\alpha{}_{\lambda\nu}\Gamma^\sigma{}_{\alpha\mu}\right) T_{\sigma\rho}} \\
&\quad + \underline{\left(\frac{\partial \Gamma^\sigma{}_{\rho\mu}}{\partial x^\nu} - \frac{\partial \Gamma^\sigma{}_{\rho\nu}}{\partial x^\mu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\sigma{}_{\alpha\nu} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\sigma{}_{\alpha\mu}\right) T_{\lambda\sigma}} \\
&= R^\sigma{}_{\lambda\nu\mu} T_{\sigma\rho} + R^\sigma{}_{\rho\nu\mu} T_{\lambda\sigma}.
\end{aligned} \tag{3.3}$$

By substituting $T_{\lambda\rho}$ with $\nabla_\lambda A_\rho$, the required result realized.

4 Problem 4

Let us show the following formula[*2]:

$$\frac{\partial}{\partial x^\mu}\sqrt{-g} = \sqrt{-g}\Gamma^\lambda{}_{\mu\lambda}. \tag{4.1}$$

Assuming this relation, we immediately reach the answer

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}A^\lambda\right) &= \frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^\lambda}\sqrt{-g}\right)A^\lambda + \frac{\partial A^\lambda}{\partial x^\lambda} \\ &= \Gamma^\sigma{}_{\lambda\sigma}A^\kappa + \frac{\partial A^\lambda}{\partial x^\lambda} = \nabla_\lambda A^\lambda. \end{aligned}\tag{4.2}$$

So what is left is to prove the relation (4.1).

Proof. We will use the relation

$$\ln g = \operatorname{Tr}\ln g \tag{4.3}$$

and take the derivative to x^λ. Then we get

$$\frac{\partial}{\partial x^\lambda}\ln g = \frac{\partial}{\partial x^\lambda}\operatorname{Tr}\ln g \tag{4.4}$$

and by computing both sides carefully, we can obtain

$$\frac{1}{g}\frac{\partial g}{\partial x^\lambda} = g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}. \tag{4.5}$$

Note that we use the relation

$$\frac{\partial}{\partial x^\lambda}\ln g_{\alpha\beta} = (g^{-1})_{\alpha\gamma}\frac{\partial g_{\gamma\beta}}{\partial x^\lambda} \tag{4.6}$$

when we derivate the RHS of (4.4). Thus we find

$$\begin{aligned}\frac{\partial}{\partial x^\lambda}\sqrt{-g} &= -\frac{\partial g}{\partial x^\lambda}\cdot\frac{1}{2}\frac{1}{\sqrt{-g}} \\ &= \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\end{aligned} \tag{4.7}$$

by using (4.5). On the other hand, the Christoffel symbol is defined as (1.1) and contracting the indices, the relation holds

$$\Gamma^\lambda{}_{\mu\lambda} = \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \tag{4.8}$$

and finally we attain (4.1). ■

[*2]The exponential of matrix A satisfies

$$\det e^A = e^{\operatorname{Tr} A}$$

and taking $A \equiv \ln g$, we find (4.1).