Notes on Quantum Field Theory

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Contents

Chapter 1

Group and its representation theory

When we construct the field theory, we impose the theory of Lorentz invariance. One of the easiest ways is to use the field which transforms in a covariant manner with the spacetime Lorentz transformation[*1]. To construct field theories, the representation of the Lorentz group plays a crucial role. In this chapter, we will discuss the property of the Lorentz group and its representation theory.

1.1 Lorentz group

Here, I will give the formal definition of the Lorentz group:

> *Lorentz group* is the set of linear transformations
>
> $$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{1.1.1}$$
>
> which preserves the following inner product
>
> $$x^\mu y_\mu \equiv -x^0 y_0 + x^1 y_1 + x^2 y_2 + x^3 y_3. \tag{1.1.2}$$
>
> Such a transformation is similar to the orthogonal transformation $O(4)$ but one sign is flipped. In this sense, we will denote the Lorentz group as $O(3,1)$.

In this text, I will use the metric convention as $\eta^{\mu\nu} = \mathrm{diag}(-,+,+,+)$. From this definition, we can obtain the explicit conditions for the elements of the Lorentz group. Let x^μ, y^ν are transformed as

$$x^{\mu\prime} = \Lambda^\mu{}_\sigma x^\sigma,\ y^{\mu\prime} = \Lambda^\mu{}_\rho y^\rho. \tag{1.1.3}$$

Then the inner product should be

$$y^{\mu\prime} x_{\mu\prime} = y^{\mu\prime}\eta_{\mu\nu}x^{\nu\prime} = (\Lambda^\mu{}_\rho y^\rho)\eta_{\mu\nu}(\Lambda^\nu{}_\sigma x^\sigma) = y^\rho(\Lambda^\mu{}_\rho \eta_{\mu\nu}\Lambda^\nu{}_\sigma)x^\sigma. \tag{1.1.4}$$

By changing the dummy indices, we conclude that the $\Lambda \equiv (\Lambda^\mu{}_\nu)$ should satisfy

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma}\Lambda^\sigma_\nu \ \text{ or } \ \eta = \Lambda^T \eta \Lambda \tag{1.1.5}$$

and, in this manner, we can write

$$SO(3,1) = \tag{1.1.6}$$

[*1] I guess it is not impossible to follow that method, though it is tough work. In that case, the Lorentz invariance is a miracle.

Appendix A

Seiberg-Witten Theory

Appendix B

Relationship between Supersymmetry and Morse theory

B.1 Introduction to supersymmetric quantum mechanics

Appendix C

Brief notes

C.1 Dirac operator and Index theorem

This is the summary of the presentation at QFT seminar in Wathematica.

C.1.1 Dirac operator and Spin complex

Let $\Delta(M)$ the section of the spin bundle $S(M)$, i.e. $\Delta(M) = \Gamma(M, S(M))$. We assume that the dimension of the manifold M is even integer $m = 2l$. The spin group is generated by $2l$ Dirac matrices $\{\gamma^k\}_{k=1,\cdots,2l}$ which satisfy

$$\gamma^{\alpha\dagger} = \gamma^\alpha,\ \{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta} \tag{C.1.1}$$

and we can define

$$\gamma^{m+1} \equiv i\gamma^1\gamma^2\cdots\gamma^{2l}. \tag{C.1.2}$$

Note that we assume that the metric has the Euclidean signature. When we consider the representation of the gamma matrices, we take so that γ^{m+1} is diagonal

$$\gamma^{m+1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.1.3}$$

A Dirac operator $\psi \in \Delta(M)$ is not the irreducible representation of the $Spin(m)$ and we can obtain them separating $\Delta(M)$ according to the eigenvalue of γ^{m+1}, called *chirality*. Since γ^{m+1} has eigenvalues ± 1, we divide

$$\Delta(M) = \Delta^+(M) \oplus \Delta^-(M). \tag{C.1.4}$$

The projection operator$P_\pm$ is given by

$$P_\pm \equiv \frac{1 \pm \gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.1.5}$$

and we write

$$\psi^+ = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M)\ ,\ \psi^- = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \Delta^-(M). \tag{C.1.6}$$

The *Dirac operator* is given by

$$i\nabla\!\!\!/\psi \equiv i\gamma^\mu\partial_\mu\psi. \tag{C.1.7}$$

Note that we omitted the spin connection for simplicity.

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