[Report 4]

Note: Write the details of reaching the final answers in English. The deadline for this report is 18:00, on 12 June, 2024. Upload an electric file on the Moodle system. The forms of the report can be a PDF, word, or scanned electric file with handwriting.

Solve the following problems.

(1) Consider the matter action given by

$$S_m = \int \mathrm{d}^4x \sqrt{-g}\, K(\phi, X)\,, \tag{0.1}$$

where g is the determinant of metric tensor $g_{\mu\nu}$, and K is a function of a scalar field ϕ and its kinetic energy $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. This scalar field is called k-essence. Derive the corresponding energy-momentum tensor $T_{\mu\nu}$ in a covariant form.

(2) On the spatially flat FLRW background, the perturbed line element containing four scalar perturbations A, B, ψ, and E are given by

$$\mathrm{d}s^2 = a^2(\eta)\left\{-(1+2A)\mathrm{d}\eta^2 + 2\partial_i B \mathrm{d}\eta \mathrm{d}x^i + [(1+2\psi)\delta_{ij} + 2\partial_i\partial_j E]\mathrm{d}x^i \mathrm{d}x^j\right\}\,, \tag{0.2}$$

where $a(\eta)$ is a scale factor depending on the conformal time η. For the matter sector, we consider a scalar field given by the action (0.1). We decompose the scalar field ϕ into the background and perturbed parts, as $\phi = \bar{\phi}(\eta) + \delta\phi(\eta, \boldsymbol{x})$. For the line element (0.2), derive the nonvanishing components of the perturbed mixed energy-momentum tensor $\delta T^{\mu}{}_{\nu}$ of k-essence at linear order. You can omit a bar from $\bar{\phi}(\eta)$ for simplicity.

(3) We consider the Universe dominated by the energy densities of dark matter and dark energy. Ignoring the contributions of vector and tensor perturbations, the spacetime geometry is given by the perturbed line element (0.2) containing four scalar perturbations. We deal with dark matter as a pressureless perfect fluid ($P = 0$ and $\delta P = 0$ without the anisotropic stress) characterized by the density perturbation $\delta\rho$ and the momentum perturbation $\delta q = a\rho(v + B)$, where ρ is the background dark matter density, v is the velocity potential. Dark energy is treated as a cosmological constant Λ (which corresponds to $K = -\Lambda$ in the language of k-essence). We define the gauge-invariant perturbations

$$\Phi = \psi - \mathcal{H}(E' - B)\,, \qquad \delta\rho_m = \delta\rho - 3\frac{\mathcal{H}}{a}\delta q\,, \tag{0.3}$$

where a prime represents the derivative with respect to conformal time η, and $\mathcal{H} = a'/a$. From the (00) and (0i) components of perturbed Einstein equations, derive the relation between Φ and $\delta\rho_m$.

(4) Let us continue to consider the physical situation explained in (3). We define the gauge-invariant dark matter density contrast

$$\delta = \frac{\delta\rho_m}{\rho}\,, \tag{0.4}$$

and work in the Newtonian gauge where the two gravitational potentials Ψ and Φ are identical to A and ψ, respectively. From the perturbed continuity equations, derive the second-order differential equation for the linear perturbation δ by using the result of (3), the relation $\Psi = -\Phi$, and background equations of motion.

During the matter era dominated by the dark matter density, find the scale-factor dependence of δ.