Relativity - Report 2

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*In this report, I use Mathematica for heavy calculations. I will omit "massive" computations, such as the derivation of Christoffel symbols and covariant derivatives, etc.

(1) The background line element

$$\mathrm{d}s^2 = -\left(1-\frac{2\mu}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{0.1}$$

implies the metric is obtained as

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-2\mu/r\right) & 0 & 0 & 0 \\ 0 & \left(1-2\mu/r\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{0.2}$$

Thus, we find the inverse

$$g^{\mu\nu} = \begin{pmatrix} -(1-2\mu/r)^{-1} & 0 & 0 & 0 \\ 0 & 1-2\mu/r & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix} \tag{0.3}$$

and Christoffel symbols are given by

$$\begin{array}{lll} \Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{\mu/r}{1-2\mu/r}, & \Gamma^r{}_{tt} = \frac{\mu(1-2\mu/r)}{r^2}, & \Gamma^r{}_{rr} = -\frac{\mu}{r^2(1-2\mu/r)}, \\ \Gamma^r{}_{\theta\theta} = -r\left(1-\frac{2\mu}{r}\right), & \Gamma^r{}_{\varphi\varphi} = -r\left(1-\frac{2\mu}{r}\right)\sin^2\theta, & \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \\ \Gamma^\theta{}_{\varphi\varphi} = -\cos\theta\sin\theta, & \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r}, & \Gamma^\varphi{}_{\varphi\theta} = \Gamma^\varphi{}_{\theta\varphi} = \frac{1}{\tan\theta} \end{array} \tag{0.4}$$

and otherwise are zero. By using these results, we get the Klein-Gordon equation as

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi &= g^{\mu\nu}(\partial_\mu\partial_\nu\Phi - \Gamma^\rho{}_{\mu\nu}\partial_\rho\Phi) \\ &= \frac{2}{r}\left(1-\frac{\mu}{r}\right)\frac{\partial\Phi}{\partial r} + \left(1-\frac{2\mu}{r}\right)\frac{\partial^2\Phi}{\partial r^2} - \frac{1}{1-2\mu/r}\frac{\partial^2\Phi}{\partial t^2} \\ &\quad + \frac{1}{r^2}\left[\frac{1}{\tan\theta}\frac{\partial\Phi}{\partial\theta} + \frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\theta^2}\right]. \end{aligned} \tag{0.5}$$

We will insert the expression

$$\Phi(x) = \frac{1}{r}\phi(t,r)Y_{lm}(\theta,\varphi) \tag{0.6}$$

where $Y_{lm}(\theta.\varphi)$ is a spherical harmonics which satisfies the following relation:

$$\frac{1}{r^2}\left[\frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\theta^2}\right]Y_{lm}(\theta,\varphi) = -\frac{l(l+1)}{r^2}Y_{lm}(\theta,\varphi). \tag{0.7}$$

Putting the expansion (0.6) into (0.5), each term becomes

$$\frac{2}{r}\left(1-\frac{\mu}{r}\right)\frac{\partial \Phi}{\partial r} = \frac{2}{r}\left(1-\frac{\mu}{r}\right)\left[-\frac{\phi}{r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}\right]Y_{lm} \tag{0.8}$$

$$\left(1-\frac{2\mu}{r}\right)\frac{\partial^2 \Phi}{\partial r^2} = \left(1-\frac{2\mu}{r}\right)\left(\frac{2\phi}{r^3}-\frac{2}{r^2}\frac{\partial \phi}{\partial r}+\frac{1}{r}\frac{\partial^2 \phi}{\partial r^2}\right)Y_{lm} \tag{0.9}$$

$$\frac{1}{1-2\mu/r}\frac{\partial^2 \Phi}{\partial t^2} = \frac{1}{r(1-2\mu/r)}\frac{\partial^2 \phi}{\partial t^2}Y_{lm}. \tag{0.10}$$

Merging two equalities (0.8), (0.9), the sum becomes

$$\begin{aligned}
&\frac{2}{r}\left(1-\frac{\mu}{r}\right)\left[-\frac{\phi}{r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}\right]+\left(1-\frac{2\mu}{r}\right)\left(\frac{2\phi}{r^3}-\frac{2}{r^2}\frac{\partial \phi}{\partial r}+\frac{1}{r}\frac{\partial^2 \phi}{\partial r^2}\right)\\
&= -\frac{2\mu}{r^4}\phi+\frac{2\mu}{r^3}\frac{\partial \phi}{\partial r}+\frac{1}{r}\frac{\partial^2 \phi}{\partial r^2}-\frac{2\mu}{r^2}\frac{\partial^2 \phi}{\partial r^2}\\
&= -\frac{2\mu}{r^4}\phi+\frac{1}{r}\frac{\partial}{\partial r}\left[\left(1-\frac{2\mu}{r}\right)\frac{\partial \phi}{\partial r}\right]
\end{aligned} \tag{0.11}$$

where we omit the overall factor Y_{lm}. Thus we obtain equality as

$$-\frac{2\mu}{r^4}\phi+\frac{1}{r}\frac{\partial}{\partial r}\left[\left(1-\frac{2\mu}{r}\right)\frac{\partial \phi}{\partial r}\right]-\frac{1}{r(1-2\mu/r)}\frac{\partial^2 \phi}{\partial t^2}-\frac{l(l+1)}{r^3}\phi=0 \tag{0.12}$$

and finally

$$\frac{\partial^2 \phi}{\partial t^2}-\left(1-\frac{2\mu}{r}\right)\frac{\partial}{\partial r}\left[\left(1-\frac{2\mu}{r}\right)\frac{\partial \phi}{\partial r}\right]+\left(1-\frac{2\mu}{r}\right)\left[\frac{2\mu}{r^3}+\frac{l(l+1)}{r^2}\right]\phi=0. \tag{0.13}$$

Thus, the effective potential is given by

$$V(r) = \left(1-\frac{2\mu}{r}\right)\left[\frac{2\mu}{r^3}+\frac{l(l+1)}{r^2}\right]. \tag{0.14}$$

(2) If we assume $l \gg 1$, we can rewrite the potential

$$V(r) \sim l(l+1)\left(\frac{1}{r^2}-\frac{2\mu}{r^3}\right) \tag{0.15}$$

effectively and this potential has the extremum at

$$r_m = 3\mu. \tag{0.16}$$

Let us consider the radius of a photon sphere r_p. It has already been given in Lecture 7 as

$$r_p = 3\mu. \tag{0.17}$$

Therefore, the result is, of course, $r_m = r_p$.

References

[1] R. M. Wald, *General Relativity*, University of Chicago Press, Chicago (1984).

[2] "*Klein Gordon equation in Schwarzschild spacetime (spherical harmonic mode expansion)*", StackExchange. (Last access: May 12, 2024)

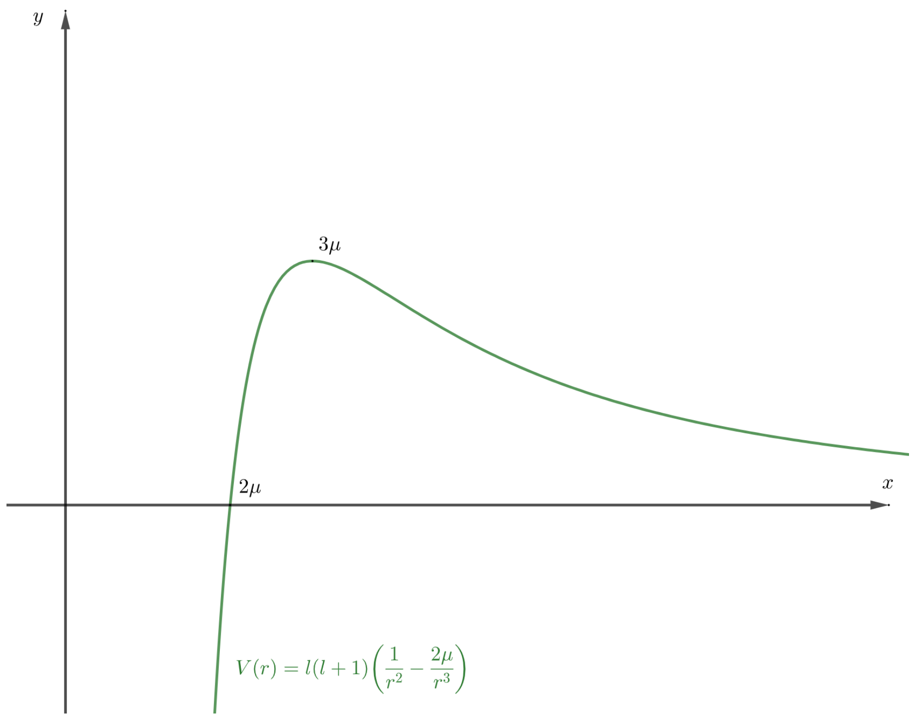

Figure 0.1: Potential $V(r)$ in the limit $l \gg 1$