Relativity Report 2

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*In this report, I use Mathematica for heavy calculations.

(1) The background line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{0.1}$$

implies the metric is obtained as

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-2\mu/r\right) & 0 & 0 & 0 \\ 0 & \left(1-2\mu/r\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{0.2}$$

Thus, we find the inverse

$$g^{\mu\nu} = \begin{pmatrix} -(1-2\mu/r)^{-1} & 0 & 0 & 0 \\ 0 & 1-2\mu/r & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix} \tag{0.3}$$

and Christoffel symbols are given by

$$\begin{array}{lll} \Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{\mu/r}{1-2\mu/r}, & \Gamma^r{}_{tt} = \frac{\mu(1-2\mu r)}{r^2}, & \Gamma^r{}_{rr} = -\frac{\mu}{r^2(1-2\mu r)}, \\ \Gamma^r{}_{\theta\theta} = -r\left(1-\frac{2\mu}{r}\right), & \Gamma^r{}_{\varphi\varphi} = -r\left(1-\frac{2\mu}{r}\right)\sin^2\theta, & \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \\ \Gamma^\theta{}_{\varphi\varphi} = -\cos\theta\sin\theta, & \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r}, & \Gamma^\varphi{}_{\varphi\theta} = \Gamma^\varphi{}_{\theta\varphi} = \frac{1}{\tan\theta} \end{array} \tag{0.4}$$

and otherwise are zero. By using these results, we get the Klein-Gordon equation as

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi &= g^{\mu\nu}(\partial_\mu\partial_\nu\Phi + \Gamma^\rho{}_{\mu\nu}\partial_\rho\Phi) \\ &= \end{aligned} \tag{0.5}$$

References

[1] R. M. Wald, *General Relativity*, University of Chicago Press, Chicago (1984).

[2] Klein Gordon equation in Schwarzschild spacetime (spherical harmonic mode expansion), StackExchange. (Last access: May 12, 2024)