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Anomalies on orbifolds

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Abstract

We review the form of the chiral anomaly on an S^1/Z_2 orbifold with chiral boundary conditions. The 4-divergence of the higher-dimensional current evaluated at a given point in the extra dimension is proportional to the probability of finding the chiral zero mode there. Nevertheless the anomaly, appropriately defined as the five-dimensional divergence of the current, lives entirely on the orbifold fixed planes and is independent of the shape of the zero mode. We show how to obtain these results in a simple way in terms of the properties of the Kaluza–Klein modes of the orbifold.

1. Introduction

Theories involving spatial dimensions beyond the four of ordinary experience have long been of interest to physicists. Especially interesting are higher-dimensional theories on orbifolds [1,2], spaces obtained from compact extra dimensions by dividing by a discrete symmetry. Such a compact theory may be viewed at low energies as an effective 4-dimensional theory involving a spectrum of Kaluza–Klein particles. For an orbifold compactification, the resulting theory may be chiral and the corresponding gauge theory anomalous [3]. To ensure consistency of the orbifold gauge theory, the chiral anomalies must cancel. It is therefore interesting to ask what the conditions for anomaly cancellation look like from the effective theory point of view. In particular, does cancellation of the 4-dimensional anomaly in the effective theory imply conservation of the corresponding 5-dimensional current? This is reasonable in the anomaly's avatar as an infrared phenomenon. If not, additional restrictions on the low energy theory beyond conventional anomaly cancellation might be required.

How could a 5-dimensional chiral anomaly, a failure of 5-dimensional current conservation, remain after 4-dimensional anomaly cancellation? If the orbifold theory has an anomaly throughout the bulk, satisfying anomaly cancellation conditions in the effective 4-dimensional theory would in general not suffice to ensure bulk anomaly cancellation. Even if the 5-dimensional anomaly is localized at the orbifold fixed points, if this anomaly depends on details of the bulk physics, such as the wave functions of the Kaluza–Klein modes, 4-dimensional anomaly cancellation would again be insufficient to ensure anomaly cancellation.

The local nature of higher-dimensional anomaly cancellation plays an important role in modern lat-

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tice gauge theories [4] and is amply demonstrated in M-theory.[3] In this note, we give a simple demonstration in the effective field theory language that while orbifold compactification can introduce anomalous charge non-conservation, it is localized at the orbifold fixed points. We compute the full 5-dimensional divergence of the 5-dimensional current. The 4-dimensional divergence gives a contribution to the anomaly that is non-zero in the bulk and depends on the shape of the zero modes. However, the 5-divergence has an additional piece that is the extra-dimensional derivative of the extra component of the current, and this piece precisely cancels the anomaly in the bulk. The complete anomaly is independent of the details of the bulk physics. For example, for a 5-dimensional fermion coupled to an external gauge potential $A_C(x, x_4)$ on an S^1/Z_2 orbifold with fixed points at $x_4 = 0$ and $x_4 = L$ we will find[4]

$$\partial_C J^C(x, x_4) = \frac{1}{2}[\delta(x_4) + \delta(x_4 - L)]\mathcal{Q}, \qquad (1.1)$$

where J^C is the 5-dimensional current and $\mathcal{Q}$ is just the usual 4-dimensional chiral anomaly for a charged Dirac fermion in an external gauge potential $A_\mu(x, x_4)$:

$$\mathcal{Q} = \frac{1}{16\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}. \qquad (1.2)$$

This expression has no dependence on the details of the Kaluza–Klein mode decomposition. This implies that cancellation of the 4-dimensional anomaly is sufficient to eliminate the 5-dimensional anomaly.

2. The 4-dimensional anomaly

We begin with a brief review of a free Dirac fermion of mass m in four dimensions coupled to an external gauge potential $A^\mu(x)$. The classical equations of motion for the fermion lead to the naïve equation for the divergence of the axial current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$:

$$\partial_\mu j_5^\mu + 2im\bar{\psi}\gamma^5\psi = 0. \qquad (2.1)$$

However the corresponding operator equation in the quantum theory receives a famous correction, the anomaly:

$$\partial_\mu j_5^\mu + 2im\bar{\psi}\gamma^5\psi = Q \qquad (2.2)$$

with Q given by (1.2).

It is worthwhile to consider the expectation value of this equation in the presence of the external gauge potential. Note that a non-zero expectation value for the divergence of the current would require a pole at $p^2 = 0$ in the expectation value of the current itself. For a massive fermion there is no state to produce such a pole, and consequently the expectation value of the divergence of the axial current must vanish. The operator equation (2.2) then implies

$$2im\langle\bar{\psi}\gamma^5\psi\rangle = Q. \qquad (2.3)$$

Since this theory involves only an external gauge field we are free to contemplate the above equations for the case of a single Weyl fermion rather than a Dirac fermion. In this case the fermion is massless, and the mass term in (2.2) vanishes. Also the anomaly of the 2-component Weyl fermion is half that of a 4-component Dirac fermion:

$$\text{(Weyl fermion)} \quad \partial_\mu j^\mu = \frac{1}{2}Q. \qquad (2.4)$$

For the Weyl fermion the expectation value of the divergence of the current does not vanish — the massless chiral fermion is exactly the state needed to produce the pole in the expectation value of the axial current:

$$\text{(Weyl fermion)} \quad \langle\partial_\mu j^\mu\rangle = \frac{1}{2}Q. \qquad (2.5)$$

In our subsequent discussion we will encounter a theory with many fermions with various masses M_i and chiral charges q_i, interacting with gauge potentials with complicated non-diagonal chiral couplings to the fermions A_{ij}^μ. The generalization of (2.2) is straightforward:

$$\partial_\mu J_5^\mu + 2i\bar{\psi}M\gamma^5\psi = \frac{1}{2}\mathcal{Q}, \qquad (2.6)$$

where $\mathcal{Q}$ is the trace of the chiral charge with the external gauge fields:

$$\mathcal{Q} = \frac{1}{16\pi^2}\,\mathrm{tr}\, qF\widetilde{F}. \qquad (2.7)$$

[3] See [5,6]. For recent work, see, for example, [7] and references therein.

[4] We use a capital Roman letter like C to refer generically to all dimensions, $C = 0, 1, 2, 3$ or 4, and a Greek letter like μ to refer specifically to the usual four dimensions.

A non-zero expectation value of the divergence of the current still requires a pole in the current itself. The arguments of Coleman–Grossman [8] show that only massless modes can produce such a pole, and consequently

$$\langle \partial_\mu J_5^\mu \rangle = \frac{1}{32\pi^2}\,\mathrm{tr}(P_0 q P_0 F P_0 \widetilde{F}), \tag{2.8}$$

where P_0 is the projector onto the massless sector of the theory.

We will see that the essential difference in the emergence of the anomaly for massless and massive fermions is reflected in the structure of the 5-dimensional anomaly. Only zero modes associated with massless fermions in the effective 4-dimensional theory produce anomalies in the 4-dimensional divergence. However, massive modes can still affect the anomaly through terms like (2.3).

3. Chiral fermions in five dimensions

Our starting point is a chiral orbifold model with a 4-component fermion field in five dimensions coupled to a classical gauge field. In order to avoid unnecessary clutter in our equations, we will consider only the Abelian example — the extension to the non-Abelian case is straightforward. The extra dimension, x_4, is in the interval $[0, L]$ and the mass depends on x_4. The action is

$$S = \int dx \int_0^L dx_4\, \bar{\psi}\big(i\not{D} - i\gamma_4 D_4 - m(x_4)\big)\psi, \tag{3.1}$$

where

$$\not{D} = \gamma^\mu D_\mu, \qquad D_C = \partial_C + iA_C \tag{3.2}$$

with A_C a classical gauge potential. This theory has a naïvely conserved current

$$J^C(x, x_4) = \bar{\psi}(x, x_4)\gamma^C \psi(x, x_4). \tag{3.3}$$

The Dirac matrix γ_4 is related to what is conventionally called γ_5 by a factor of i,

$$\gamma_4 = -i\gamma_5. \tag{3.4}$$

The key to the model is the orbifold construction that restricts the physical region in the extra dimension to the interval $[0, L]$. To implement this we extend the fields to functions on the doubled interval $x_4 \in [0, 2L)$, and then impose

$$\psi(x, x_4) = \psi(x, 2L + x_4) = \gamma_5\,\psi(x, -x_4), \tag{3.5}$$

$$A_\mu(x, x_4) = A_\mu(x, 2L + x_4) = A_\mu(x, -x_4), \tag{3.6}$$

$$A_4(x, x_4) = A_4(x, 2L + x_4) = -A_4(x, -x_4). \tag{3.7}$$

In order for the action to be well-defined the mass function must satisfy

$$m(x_4) = m(2L + x_4) = -m(-x_4). \tag{3.8}$$

It is through the boundary condition (3.5) that chirality enters into the theory. Specifically, (3.5)–(3.7) imply that the boundaries of the physical region, $x_4 = 0, L$, are fixed-points of the orbifold. If we decompose ψ into chiral components, $\psi_\pm$, where

$$\psi = \psi_+ + \psi_-, \qquad \gamma_5\psi_\pm = \pm\psi_\pm, \tag{3.9}$$

then (3.5)–(3.7) are equivalent to defining all the fields on a circle, $x_4 \in [0, 2L)$ with $2L$ identified with 0, but with ψ_+, and A_μ symmetric about the fixed points $x_4 = 0, L$, and ψ_- and A_4 antisymmetric.

The classical Lagrangian (3.1) and the orbifold boundary conditions (3.5)–(3.7) are invariant under gauge transformations of the form

$$\psi(x, x_4) \to e^{i\phi(x, x_4)}\psi(x, x_4), \tag{3.10}$$

$$A_\mu(x, x_4) \to A_\mu(x, x_4) - \partial_\mu\phi(x, x_4), \tag{3.11}$$

$$A_4(x, x_4) \to A_4(x, x_4) - \partial_4\phi(x, x_4) \tag{3.12}$$

provided ϕ satisfies

$$\phi(x, x_4) = \phi(x, 2L + x_4) = \phi(x, -x_4). \tag{3.13}$$

The orbifold boundary conditions (3.5)–(3.7) give rise to a massless chiral fermion, and we expect that this gauge symmetry is then broken by the chiral anomaly. We seek the precise form of the anomaly. In particular, we are interested in the "shape" of the anomaly in the extra dimension. Because we have kept the mass function $m(x_4)$ arbitrary except for the boundary conditions, we can get very different shapes for the wave function of the chiral zero mode in the extra dimension. If the anomaly depends on the shape of the zero mode, and therefore on the mass function, this would make it difficult to cancel the anomaly in the 5-dimensional theory. We will see explicitly that this does not happen.

4. Calculation of the anomaly

We begin by choosing a gauge[5] in which $A_4 = 0$. In this gauge the KK mode wave functions are independent of the gauge fields. To decompose the fermion field into KK modes, we define the functions $\xi_M^{\pm}(x_4)$ satisfying

$$[-\partial_4 + m(x_4)]\xi_M^{-}(x_4) = M\xi_M^{+}(x_4),$$
$$[\partial_4 + m(x_4)]\xi_M^{+}(x_4) = M\xi_M^{-}(x_4) \tag{4.1}$$

with $M \geqslant 0$ for ξ_M^+ and $M > 0$ for ξ_M^-. The $\xi_M^{\pm}(x_4)$ can be chosen real and, respectively, symmetric and antisymmetric about the fixed points $x_4 = 0$ and $x_4 = L$. The Ms are the masses of the KK modes and the $\xi_M^{\pm}(x_4)$ are their wave functions. They form an orthogonal basis for the functions on the circle $[0, 2L)$, respectively, symmetric and antisymmetric about the fixed points. Alternatively they form an orthogonal basis for the functions on $[0, L]$ satisfying, respectively, Neumann and Dirichlet boundary conditions at the fixed points. We normalize them such that

$$\int_0^L dx_4\, \xi_M^+(x_4)\xi_{M'}^+(x_4) = \int_0^L dx_4\, \xi_M^-(x_4)\xi_{M'}^-(x_4) = \delta_{MM'}. \tag{4.2}$$

The orbifold boundary conditions ensure that no zero mode appears in the ξ_M^-. It is nevertheless convenient to introduce $\xi_0^- \equiv 0$ which will allow us to treat the plus and minus modes more symmetrically. This mode clearly does not satisfy (4.2), but we will never encounter a formula which involves the norm of this function. Now we expand the fermion fields in the ξ_Ms

$$\psi_{\pm}(x, x_4) = \sum_M \psi_{M\pm}(x)\xi_M^{\pm}(x_4), \tag{4.3}$$

where

$$\psi_{M\pm}(x) = \int_0^L dx_4\, \xi_M^{\pm}(x_4)\psi_{\pm}(x, x_4). \tag{4.4}$$

[5] The gauge transformations (3.10)–(3.12) and the boundary conditions (3.5)–(3.7) guarantee that such a gauge exists.

Inserting (4.3) into the action, (3.1), gives

$$S = \int dx \int_0^L dx_4 \times \Bigg(\sum_M \big[\bar\psi_{M+}(x)\xi_M^+(x_4) + \bar\psi_{M-}(x)\xi_M^-(x_4)\big] \times \big(i\partial\!\!\!/ - A\!\!\!/ - \gamma_5\partial_4 - m(x_4)\big) \times \sum_{M'} \big[\psi_{M'+}(x)\xi_{M'}^+(x_4) + \psi_{M'-}(x)\xi_{M'}^-(x_4)\big]\Bigg) \tag{4.5}$$

$$= \int dx \Bigg(\sum_M \bar\psi_M(x)(i\partial\!\!\!/ - M)\psi_M(x) - \sum_{M,M'} \bar\psi_{M'+}(x)A\!\!\!/^{+}_{M'M}(x)\psi_{M+}(x) - \sum_{M,M'} \bar\psi_{M'-}(x)A\!\!\!/^{-}_{M'M}(x)\psi_{M-}(x)\Bigg), \tag{4.6}$$

where

$$\psi_M(x) = \psi_{M+}(x) + \psi_{M-}(x) \tag{4.7}$$

is a 4-component 4-dimensional Dirac field for each $M > 0$ and equal to ψ_0^+ for $M = 0$, and

$$A^{\mu\pm}_{M'M}(x) = \int_0^L dx_4\, \xi_{M'}^{\pm}(x_4)\xi_M^{\pm}(x_4)A^{\mu}(x, x_4). \tag{4.8}$$

Using this mode decomposition the components of the current J^C become

$$J^{\mu}(x, x_4) = \sum_{M',M} \big(\xi_{M'}^+(x_4)\xi_M^+(x_4)\bar\psi_{M'+}(x)\gamma^{\mu}\psi_{M+}(x) + \xi_{M'}^-(x_4)\xi_M^-(x_4)\bar\psi_{M'-}(x)\gamma^{\mu}\psi_{M-}(x)\big), \tag{4.9}$$

$$J^4(x, x_4) = \sum_{M',M} \big(\xi_{M'}^+(x_4)\xi_M^-(x_4)\bar\psi_{M'+}(x)i\gamma_5\psi_{M-}(x) + \xi_{M'}^-(x_4)\xi_M^+(x_4)\bar\psi_{M'-}(x)i\gamma_5\psi_{M+}(x)\big). \tag{4.10}$$

We can write all this in a useful matrix notation by collecting the $\psi_M(x)$ into a column vector $\Psi(x)$, the gauge potentials into matrices $\mathcal{A}^{\mu\pm}$ with matrix elements $A^{\mu\pm}_{M'M}$ and introducing the mass matrix $\mathcal{M}$ with matrix elements $M\delta_{M'M}$. Then

$$S = \int dx\, \overline{\Psi}(x)(i\partial\!\!\!/ - A\!\!\!/ - \mathcal{M})\Psi(x), \tag{4.11}$$

where

$$A\!\!\!/ = A\!\!\!/^{+} P_+ + A\!\!\!/^{-} P_- \tag{4.12}$$

and as usual

$$P_\pm = \frac{1 \pm \gamma_5}{2}. \tag{4.13}$$

Evidently, this is a theory of a single massless Weyl fermion and an infinite tower of Dirac fermions, interacting with classical gauge field $\mathcal{A}^\mu$ with a complicated matrix coupling in the flavor space of the 4-dimensional fermion fields. Note that these couplings are chiral.

A similar matrix notation simplifies the current. Define

$$\Xi(x_4) = \Xi^+(x_4)P_+ + \Xi^-(x_4)P_-, \tag{4.14}$$

with $\Xi^\pm(x_4)$ the matrices with matrix elements

$$\big[\Xi^\pm(x_4)\big]_{M'M} = \xi^\pm_{M'}(x_4)\xi^\pm_M(x_4), \tag{4.15}$$

and

$$\Omega(x_4) = \Omega^+(x_4)P_+ + \Omega^-(x_4)P_-, \tag{4.16}$$

with $\Omega^\pm(x_4)$ the matrices with matrix elements

$$\big[\Omega^\pm(x_4)\big]_{M'M} = \xi^\mp_{M'}(x_4)\xi^\pm_M(x_4). \tag{4.17}$$

In this notation the 5-dimensional current (3.3) is

$$J^\mu(x, x_4) = \overline{\Psi}(x)\gamma^\mu \Xi(x_4)\Psi(x),$$
$$J^4(x, x_4) = \overline{\Psi}(x) i\gamma_5 \Omega(x_4)\Psi(x). \tag{4.18}$$

Classically, this current is conserved. This is obvious from the starting point, but we can see it directly in this notation by separately computing the derivative of J^4 and the 4-dimensional divergence of J^μ. The derivative of J^4 is

$$\partial_4 J^4(x, x_4) = \overline{\Psi}(x) i\gamma_5 \partial_4 \Omega(x_4)\Psi(x) \tag{4.19}$$

and using the equations for the KK wave functions gives

$$\partial_4\Omega^+(x_4) = -\mathcal{M}\Xi^+(x_4) + \Xi^-(x_4)\mathcal{M}, \tag{4.20}$$
$$\partial_4\Omega^-(x_4) = \mathcal{M}\Xi^-(x_4) - \Xi^+(x_4)\mathcal{M}. \tag{4.21}$$

The 4-dimensional divergence is

$$\begin{aligned}
\partial_\mu J^\mu(x, x_4) &= \big(\partial_\mu \overline{\Psi}(x)\big)\gamma^\mu \Xi(x_4)\Psi(x) \\
&\quad + \overline{\Psi}(x)\gamma^\mu \Xi(x_4)\big(\partial_\mu \Psi(x)\big) \\
&= i\overline{\Psi}(x)\mathcal{M}\big(\Xi^+(x_4)P_+ + \Xi^-(x_4)P_-\big)\Psi(x) \\
&\quad - i\overline{\Psi}(x)\big(\Xi^+(x_4)P_- + \Xi^-(x_4)P_+\big)\mathcal{M}\Psi(x)
\end{aligned} \tag{4.22}$$

which exactly cancels the contribution of (4.19) and gives current conservation, at least at the classical level.

The cancellation of the A^μ dependence in (4.22) is not obvious in this notation, but follows because of the completeness of the ξ functions. We will need this result below, so we will pause to discuss it here. Consider, for example, the matrix product

$$\Xi^-(x_4)\mathcal{A}^{\mu-}(x). \tag{4.23}$$

The important point is that we can write

$$\Xi^-(x_4)\mathcal{A}^{\mu-}(x) = A^\mu(x, x_4)\Xi^-(x_4). \tag{4.24}$$

To see this, look at the $M'M$ matrix element (both M' and M non-zero because we are looking at Ξ^-),

$$\begin{aligned}
&\sum_{M''>0} \xi^-_{M'}(x_4)\xi^-_{M''}(x_4)\int_0^L dx'_4\, \xi^-_{M''}\big(x'_4\big)\xi^-_M\big(x'_4\big)A^\mu\big(x, x'_4\big) \\
&\quad = \xi^-_{M'}(x_4)\xi^-_M(x_4)A^\mu(x, x_4),
\end{aligned} \tag{4.25}$$

where (4.25) follows because the $\xi^-_M(x_4)$ are complete on the space of functions that vanish at the boundaries, and the product $\xi^-_M(x_4)A^\mu(x, x_4)$ is such a function. Similar arguments can be used to show that

$$\mathcal{A}^{\mu-}(x)\Xi^-(x_4) = A^\mu(x, x_4)\Xi^-(x_4) \tag{4.26}$$

and

$$\begin{aligned}
\Xi^+(x_4)\mathcal{A}^{\mu+}(x) &= \mathcal{A}^{\mu+}(x)\Xi^+(x_4) \\
&= A^\mu(x, x_4)\Xi^+(x_4).
\end{aligned} \tag{4.27}$$

Because of (4.24), (4.26) and (4.27), the A^μ dependence cancels in the 4-dimensional divergence (4.22) and the 5-dimensional current is classically conserved.

Quantum mechanically, because (4.11) is just a 4-dimensional gauge theory (with an admittedly peculiar gauge field) we know that the current has an anomalous divergence that is simply the trace of the chiral charge in the current with the square of the gauge field strength. Thus the anomaly is equal to

$$\frac{1}{32\pi^2}\operatorname{tr}\big(\Xi^+(x_4)\mathcal{F}^+(x)\widetilde{\mathcal{F}}^+(x) - \Xi^-(x_4)\mathcal{F}^-(x)\widetilde{\mathcal{F}}^-(x)\big), \tag{4.28}$$

where $\mathcal{F}^{\mu\nu\pm}(x) = \partial^\mu\mathcal{A}^{\nu\pm}(x) - \partial^\nu\mathcal{A}^{\mu\pm}(x)$. But now arguments precisely analogous to those leading to (4.24) imply that we can rewrite (4.28) as

$$\frac{1}{32\pi^2}F(x,x_4)\widetilde{F}(x,x_4)\operatorname{tr}\big(\Xi^+(x_4) - \Xi^-(x_4)\big). \tag{4.29}$$

It remains to calculate the trace in (4.29). Using (4.15) we can write

$$\operatorname{tr}\big(\Xi^+(x_4) - \Xi^-(x_4)\big) = \sum_{M\geqslant 0}\xi_M^+(x_4)^2 - \sum_{M>0}\xi_M^-(x_4)^2. \tag{4.30}$$

To evaluate this, consider

$$\Delta(x_4,y_4) \equiv \sum_{M\geqslant 0}\xi_M^+(x_4)\xi_M^+(y_4) - \sum_{M>0}\xi_M^-(x_4)\xi_M^-(y_4) \tag{4.31}$$

which reduces to (4.30) if we set $y_4 = x_4$. Because the $\xi_M^-(y_4)$ functions are antisymmetric at $y_4 = 0$ while the $\xi_M^+(y_4)$ are symmetric, we can write

$$\Delta(x_4,-y_4) = \sum_{M\geqslant 0}\xi_M^+(x_4)\xi_M^+(y_4) + \sum_{M>0}\xi_M^-(x_4)\xi_M^-(y_4). \tag{4.32}$$

But taken together, the ξ_M^+ and ξ_M^- are a complete set of functions with periodic boundary conditions on the interval $[0,2L)$, so we can write [6]

$$\Delta(x_4,-y_4) = 2\sum_N \delta(x_4 - y_4 - 2NL). \tag{4.33}$$

[6] The ξs are however still normalized on the half-circle, and are then $\sqrt{\text{twice}}$ as large as the eigenfunctions properly normalized on the circle.

Thus

$$\Delta(x_4,x_4) = 2\sum_N\delta(2x_4 - 2NL) = \sum_N\delta(x_4 - NL). \tag{4.34}$$

Note that these delta functions defined on the physical interval $[0,L]$ satisfy

$$\int_0^L \delta(x_4)f(x_4) = \frac{1}{2}f(0), \tag{4.35}$$

$$\int_0^L \delta(x_4 - L)f(x_4) = \frac{1}{2}f(L). \tag{4.36}$$

Restricting to the interval $[0,L]$, we have the final result that the anomaly is

$$\partial_C J^C = \frac{1}{2}[\delta(x_4) + \delta(x_4 - L)]\mathcal{Q}. \tag{4.37}$$

This result is very gratifying. It shows that there is no anomaly in the 5-dimensional bulk and that the anomaly on the orbifold fixed points is entirely independent of the shape of the modes. This implies that the cancellation of the 4-dimensional anomaly is sufficient to eliminate the 5-dimensional anomaly.

Also note that the anomaly appears "split" between the two fixed points — if we integrate over the extra dimension we pick up one-half of the anomaly of a chiral mode from $x_4 = 0$ and one-half from $x_4 = L$. Again this is independent of the shape of the chiral zero mode.

Using (2.8), we can compute the matrix element of $\partial_\mu J^\mu$. Only the chiral zero mode contributes, so

$$\langle\partial_\mu J^\mu(x,x_4)\rangle = \frac{1}{2}\mathcal{Q}_0\xi_0^+(x_4)^2, \tag{4.38}$$

where $\mathcal{Q}_0$ is just $\mathcal{Q}$ calculated with gauge potential $A_{00}^{\mu+}$. Thus the 4-divergence of the current has a matrix element varying in the bulk as the square of the zero mode wave function, but this variation is precisely canceled by the rest of the 5-divergence, to produce (4.37).

5. Anomaly cancellation

We close this brief note with an example of anomaly cancellation. Because the anomaly is independent of the bulk physics, cancellation of anomalies is also straightforward. If we have a collection of 5-dimensional fermions, all that is required is that the zero modes form an anomaly-free representation of the low-energy 4-dimensional gauge group. These zero modes may have completely different wave functions in the extra dimension. Our analysis in Sections 3 and 4 shows that the 5-dimensional anomaly is independent of the details and cancels if the 4-dimensional low energy theory is anomaly free. As an extreme example, consider a theory with 2 fermions: Ψ and X with charge $+1$ and -1 and piecewise constant mass terms $m_\Psi(x_4) = -m_X(x_4) = m$ for $0 < x_4 < L$ and satisfying the boundary condition (3.8). The zero modes ψ_0, χ_0 have charge $+1, -1$ and therefore the 4-dimensional low energy theory is anomaly free. But for large mL, these zero modes are concentrated at opposite boundaries. For large $mL > 0$, the zero mode ψ_0 is concentrated near $x_4 = 0$ while the zero mode χ_0 is concentrated near $x_4 = L$ (and vice versa for $m < 0$). In the limit $mL \to \infty$, the non-zero modes are arbitrarily heavy and the zero modes live entirely on the separate fixed points at $x_4 = 0$ and $x_4 = L$. Nevertheless, our general analysis shows that the 5-dimensional anomalies must cancel for all m.

For simplicity we will show how this works for a 4-dimensional gauge potential $A^\mu(x)$ which is constant in x_4, which corresponds to turning on only the gauge field zero mode. In this case $A^\mu_{00}(x) = A^\mu(x)$. Here we will discuss in detail only the extreme limit, $mL \to \infty$ and show how our result for the form of the anomaly can be interpreted in terms of familiar 4-dimensional results.

First consider the contribution of Ψ to the anomaly. As $mL \to \infty$ the square of the properly normalized zero mode wave function goes to

$$\psi_0^+(x_4)^2 = 2\delta(x_4) \tag{5.1}$$

and the 5-dimensional theory has a chiral fermion bound to the fixed point $x_4 = 0$ [9]. From (4.38) we see that the 4-divergence is exactly what we expect from a single chiral fermion localized at $x_4 = 0$:

$$\left\langle \partial_\mu J^\mu_\psi(x, x_4)\right\rangle = \frac{1}{2}\mathcal{Q}\, 2\delta(x_4). \tag{5.2}$$

Since the full anomaly (4.37) is always evenly split between the two fixed points, the contribution from $\partial_4 J^4$ must then be

$$\left\langle \partial_4 J^4_\psi\right\rangle = -\frac{1}{2}\mathcal{Q}\delta(x_4) + \frac{1}{2}\mathcal{Q}\delta(x_4 - L). \tag{5.3}$$

This has a natural effective theory interpretation. Since the fermion in the bulk is massive, we should integrate it out. This results in a Chern–Simons term in the bulk effective action [10], whose gauge variation resides entirely at the boundaries $x_4 = 0, L$, reproducing (5.3). The full anomaly is the sum of the 4-dimensional divergence and the variation of the Chern–Simons term, and this sum is evenly split between the two fixed points at $x_4 = 0$ and L.

For X, as $mL \to \infty$ the square of the properly normalized zero mode wave function goes to

$$\chi_0^+(x_4)^2 = 2\delta(x_4 - L), \tag{5.4}$$

the 5-dimensional theory has a chiral fermion bound to the fixed point $x_4 = L$, and the 4-divergence of the current has the form

$$\left\langle \partial_\mu J^\mu_\chi(x, x_4)\right\rangle = -\frac{1}{2}\mathcal{Q} 2\delta(x_4 - L). \tag{5.5}$$

The full anomaly (4.37) is always evenly split between the two fixed points, and thus the contribution from $\partial_4 J^4$ must be the same as that in (5.3),

$$\left\langle \partial_4 J^4_\chi\right\rangle = -\frac{1}{2}\mathcal{Q}\delta(x_4) + \frac{1}{2}\mathcal{Q}\delta(x_4 - L). \tag{5.6}$$

Again, this comes from the variation of a Chern–Simons term in the bulk effective action. Adding (5.2), (5.3), (5.5) and (5.6), we see explicitly the cancellation of the 5-dimensional anomaly.

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