Relativity - Report 3

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(1) If we put $\theta = \pi/2$, the quantity Σ becomes r^2 and the line element is obtained as

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \frac{4\mu ac}{r}\mathrm{d}t\mathrm{d}\varphi + \frac{r^2}{\Delta}\mathrm{d}r^2 + \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)\mathrm{d}\varphi^2 \tag{0.1}$$

and the metric also is as

$$g_{\mu\nu} = \begin{pmatrix} -c^2(1-2\mu/r) & 0 & -2\mu ac/r \\ 0 & r^2/\Delta & 0 \\ -2\mu ac/r & 0 & r^2 + a^2 + 2\mu a^2/r \end{pmatrix} \tag{0.2}$$

where Δ still remain $r^2 - 2\mu r + a^2$. We obtain its inverse

$$g^{\mu\nu} = \begin{pmatrix} -\dfrac{r^3 + a^2(r+2\mu)}{c^2 r(a^2 + r^2 - 2\mu r)} & 0 & -\dfrac{2a\mu}{a^2cr + cr^3 - 2cr^2\mu} \\ 0 & \dfrac{a^2 + r^2 - 2r\mu}{r^2} & 0 \\ -\dfrac{2a\mu}{a^2cr + cr^3 - 2cr^2\mu} & 0 & \dfrac{r - 2\mu}{a^2 + r^3 - 2r^2\mu} \end{pmatrix} \tag{0.3}$$

and $\dot{t}$ and $\dot{\varphi}$ are, then, immediately derived as

$$\begin{aligned} \dot{t} &= g^{tt}p_t + g^{t\varphi}p_\varphi \\ &= \frac{ckr^3 - 2ah\mu + a^2ck(r+2\mu)}{cr(a^2 + r(r-2\mu))}, \end{aligned} \tag{0.4}$$

$$\begin{aligned} \dot{\varphi} &= g^{\varphi t}p_t + g^{\varphi\varphi}p_\varphi \\ &= \frac{hr - 2h\mu + 2ack\mu}{a^2r + r^3 - 2r^2\mu}. \end{aligned} \tag{0.5}$$

(2) What we need to do is just insert the inverse metric which we already obtained (0.3) into

$$\begin{aligned} g^{\mu\nu}p_\mu p_\nu &= g^{tt}p_t^2 + g^{\varphi\varphi}p_\varphi^2 + 2g^{t\varphi}p_t p_\varphi + g^{rr}p_r^2 \\ &= g^{tt}\cdot(-kc^2)^2 + g^{\varphi\varphi}\cdot h^2 + 2g^{t\varphi}\cdot(-kc^2)\cdot h + g_{rr}\dot{r}^2. \end{aligned} \tag{0.6}$$

Note that we use the fact that p_r is obtained by lowering the indices $\dot{r}$, i.e. $p_r = g_{rr}\dot{r}$. Putting the inverse matrix components $g^{\mu\nu}$ and organizing the equations, we will get the effective potential as

$$V_{\mathrm{eff}}(r) = \frac{h^2 - a^2c^2(k^2-1)}{2r^2} - \frac{(h-ack)^2\mu}{r^3} - \frac{c^2\mu}{r}. \tag{0.7}$$

It is obvious that $V_{\mathrm{eff}}(r)$ satisfies the relation

$$\frac{1}{2}\dot{r}^2 + V_{\mathrm{eff}}(r) = \frac{1}{2}c^2(k^2-1) \tag{0.8}$$

obtained from $g^{\mu\nu}p_\mu p_\nu = -c^2$ since it was defined to satisfy that equation.

(3) Since we will analyze the circular motion, we can put $\dot{r}$ to zero in the previous result. Therefore the last result becomes

$$\frac{h^2 - a^2c^2(k^2-1)}{2}u^2 - x^2\mu u^3 - c^2\mu u = \frac{1}{2}c^2(k^2-1) \tag{0.9}$$

and solving to x^2, we find

$$x^2 = \frac{1}{2\mu u^3}c^2(k^2-1) + \frac{c^2}{u^2} + \frac{1}{2\mu u}\left[h^2 - a^2c^2(k^2-1)\right]. \tag{0.10}$$

We still use x^2 when we derive k and h and insert the solution

$$x = -\sqrt{\frac{1}{2\mu u^3}c^2(k^2-1) + \frac{c^2}{u^2} + \frac{1}{2\mu u}\left[h^2 - a^2c^2(k^2-1)\right]}(<0) \tag{0.11}$$

to the obtained expression at the final step. To evaluate two values k and h, we should prepare one more relation for k and h. Since we are assuming the circular motion, its orbits should be stabilized at the potential minimum. So, to express such a situation, we will consider the extremum condition[*1]

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}u} = \left\{h^2 - a^2c^2(k^2-1)\right\}u - 3x^2\mu u^2 - c^2\mu = 0 \tag{0.12}$$

and solve (0.12) with (0.9) to k and h. Then we finally obtain

$$k = -\frac{\sqrt{c^2(1-\mu u) + \mu u^3 x}}{c} \tag{0.13}$$

$$h = -\sqrt{\frac{\mu\left(c^2\left(1-a^2u^2\right) + u^2x\left(a^2\,u^2+3\right)\right)}{u}} \tag{0.14}$$

with x in (0.11).

Note that x remains in the result. I regard x as a constant since I guess so from the problem statement.

(4) To evaluate the innermost stable circular orbit, we should derive a second-order derivative

$$\frac{\mathrm{d}^2V_{\text{eff}}}{\mathrm{d}r^2} = \frac{3\left(h^2 - a^2c^2\left(k^2-1\right)\right)}{r^4} - \frac{12\mu(h-ack)^2}{r^5} - \frac{2c^2\mu}{r^3} \tag{0.15}$$

and take it to zero. This equality decides the orbit radius r. To make coefficients just depend on μ and a, we should vanish k and h by using the previous result. Thus, inserting the last results (0.13), (0.14) and (0.11) and organizing messy expression, we can find the equality for r as

$$\mu r^2\left(r(r-6\mu)(r-3\mu) - a^2(7\mu+3r)\right) - 6\sqrt{\mu^3 r^3\left(a^3 + ar(r-2\mu)\right)^2} = 0. \tag{0.16}$$

References

[1] Chapter 22 Geodesic motion in Kerr spacetime. (Last accessed: May 29, 2024)

[2] Kerr Geometry and Rotating Black Hole. (Last accessed: May 29, 2024)

[*1] It is obvious that the extremum condition for V_{eff} does not depend on the variables r, u. It means that

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} = 0 \iff \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}u} = 0.$$

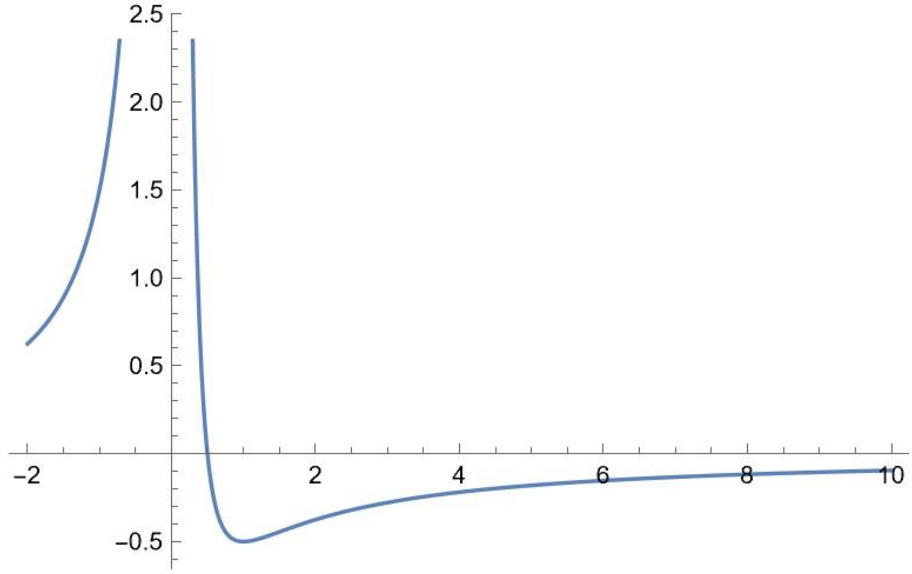

Figure 0.1: One example of the effective potential V_{eff}