[Report 2]

Note: Write the details of reaching the final answers in English. The deadline for this report is 18:00, on 22 May, 2024. Upload an electric file on the Moodle system. The forms of the report can be a PDF, word, or scanned electric file with handwriting.

The Schwarzschild black hole is described by the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,, \tag{0.1}$$

where μ is a constant. On this background, we consider a massless scalar field Φ obeying the Klein-Gordon equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0\,, \tag{0.2}$$

where $g^{\mu\nu}$ is the metric tensor, and ∇_μ is the covariant derivative operator. On the background spacetime (0.1), the scalar field can be decomposed in the form

$$\Phi = \frac{1}{r}\phi(t,r)Y_{lm}(\theta,\varphi)\,, \tag{0.3}$$

where ϕ is a function of time t and radial distance r, and $Y_{lm}(\theta,\varphi)$ is a spherical harmonics with the integers l and m in the range $l \geq |m|$.

(1) Show that ϕ obeys the partial differential equation

$$\frac{\partial^2\phi}{\partial t^2} - \left(1 - \frac{2\mu}{r}\right)\frac{\partial}{\partial r}\left[\left(1 - \frac{2\mu}{r}\right)\frac{\partial\phi}{\partial r}\right] + V(r)\phi = 0\,, \tag{0.4}$$

and derive the potential $V(r)$.

(2) In the eikonal limit $l \gg 1$, obtain the distance $r = r_m$ at which the potential $V(r)$ has an extremum outside the event horizon ($r > 2\mu$). Derive the relation between r_m and the radius r_p of a photon sphere around the Schwarzschild black hole.