

# Relativity Report 1

Itsuki Miyane ID: 5324A057-8

Last modified: April 28, 2024

## Contents

<b>1</b>	<b>Problem 1</b>	<b>2</b>
<b>2</b>	<b>Problem 2</b>	<b>3</b>
<b>3</b>	<b>Problem 3</b>	<b>4</b>
<b>4</b>	<b>Problem 4</b>	<b>5</b>

# 1 Problem 1

The Christoffel symbol is defined as

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2} \left( \frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right) \quad (1.1)$$

in a coordinate system  $x^\lambda$  and transform into a system  $\tilde{x}^{\lambda*1}$ . The transformation laws of the coordinate are

$$g^{\mu\nu} = \frac{\partial x^\mu}{\partial \tilde{x}^\rho} \frac{\partial x^\nu}{\partial \tilde{x}^\sigma} \tilde{g}^{\rho\sigma} \quad (1.2)$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\lambda}. \quad (1.3)$$

Putting these relations into the definition (1.1), we obtain the transformation law

$$\begin{aligned} \Gamma_{\mu\alpha\beta} &= \frac{1}{2} \left( \frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right) \\ &= \frac{1}{2} \left\{ \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial}{\partial \tilde{x}^\delta} \left( \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} g_{\lambda\gamma} \right) + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\gamma} \left( \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} g_{\lambda\delta} \right) - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\lambda} \left( \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} g_{\gamma\delta} \right) \right\} \\ &= \frac{1}{2} \left\{ \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial}{\partial \tilde{x}^\delta} \left( \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \right) \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} g_{\lambda\gamma} + \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\delta} \left( \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \right) g_{\lambda\gamma} + \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^\delta} \right. \\ &\quad + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\gamma} \left( \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \right) \frac{\partial \tilde{x}^\delta}{\partial x^\beta} g_{\lambda\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\gamma} \left( \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \right) g_{\lambda\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^\gamma} \\ &\quad \left. - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\lambda} \left( \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \right) \frac{\partial \tilde{x}^\delta}{\partial x^\beta} g_{\gamma\delta} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\lambda} \left( \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \right) g_{\gamma\delta} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^\lambda} \right\}. \quad (1.4) \end{aligned}$$

The waved terms  $\sim$  in (1.4) become a twice of the Christoffel symbol:

$$\frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^\gamma} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^\lambda} = 2 \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \tilde{\Gamma}_{\lambda\gamma\delta}. \quad (1.5)$$

But we already find that there are additional terms in (1.4) and they violate the transformation law of the 3-rank covariant tensor. Thus we find the law as

$$\begin{aligned} \Gamma_{\mu\alpha\beta} &= \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \tilde{\Gamma}_{\lambda\gamma\delta} + \frac{1}{2} \left\{ \frac{\partial^2 \tilde{x}^\lambda}{\partial x^\beta \partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} g_{\lambda\gamma} + \frac{\partial^2 \tilde{x}^\gamma}{\partial x^\beta \partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} g_{\lambda\gamma} \right. \\ &\quad \left. + \frac{\partial^2 \tilde{x}^\lambda}{\partial x^\alpha \partial x^\mu} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} g_{\lambda\delta} + \frac{\partial^2 \tilde{x}^\delta}{\partial x^\alpha \partial x^\beta} \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} g_{\lambda\delta} - \frac{\partial \tilde{x}^\gamma}{\partial x^\mu} \frac{\partial \tilde{x}^\delta}{\partial x^\alpha} \frac{\partial \tilde{x}^\lambda}{\partial x^\beta} g_{\gamma\delta} - \frac{\partial^2 \tilde{x}^\delta}{\partial x^\mu \partial x^\beta} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} g_{\delta\gamma} \right\} \\ &= \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\beta} \tilde{\Gamma}_{\lambda\gamma\delta} + \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} \frac{\partial^2 \tilde{x}^\gamma}{\partial x^\alpha \partial x^\beta} g_{\lambda\gamma} \end{aligned} \quad (1.6)$$

Note we freely change the dummy indices and cancel out equivalent terms.

---

<sup>\*1</sup>In the problem statement, we should consider the transformation  $x^\lambda$  into a system " $x'^\lambda$ " but I should apologize since we will use a different label " $\tilde{x}^\lambda$ ", though it is trivial.

## 2 Problem 2

From the previous problem, we obtain the transformation law

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}. \quad (2.1)$$

Since we already know the transformation laws, what we have to do is just to compute them. Thus the transformation should be

$$\begin{aligned} \tilde{\nabla}_{\mu} \tilde{V}_{\nu} &\equiv \frac{\partial \tilde{V}_{\nu}}{\partial \tilde{x}^{\mu}} - \tilde{\Gamma}^{\lambda}_{\mu\nu} \tilde{V}_{\lambda} \\ &= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial}{\partial x^{\alpha}} \left( \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} V_{\beta} \right) - \left( \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \right) \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} \\ &= \cancel{\frac{\partial^2 x^{\beta}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} V_{\beta}} + \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \frac{\partial V_{\beta}}{\partial x^{\alpha}} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}_{\alpha\beta} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} - \cancel{\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta}} \\ &= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \left( \frac{\partial V_{\beta}}{\partial x^{\alpha}} - \Gamma^{\gamma}_{\alpha\beta} V_{\gamma} \right) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta} \end{aligned} \quad (2.2)$$

and we finally attain

$$\tilde{\nabla}_{\mu} \tilde{V}_{\nu} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}. \quad (2.3)$$

### 3 Problem 3

The definition of the covariant derivative of two-rank tensor and as followings:

$$\nabla_\lambda T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\lambda} - \Gamma_{\mu\lambda}^\rho T_{\rho\nu} - \Gamma_{\nu\lambda}^\rho T_{\mu\rho}, \quad (3.1)$$

$$R^\mu_{\nu\lambda\rho} = \frac{\partial}{\partial x^\lambda} \Gamma^\mu_{\nu\rho} - \frac{\partial}{\partial x^\rho} \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\alpha\lambda} \Gamma^\alpha_{\nu\rho} - \Gamma^\mu_{\alpha\rho} \Gamma^\alpha_{\nu\lambda}. \quad (3.2)$$

We already showed that  $\nabla_\lambda A_\sigma$  is the 2-rank covariant tensor in the previous problem. Thus we should find  $[\nabla_\mu, \nabla_\nu]T_{\lambda\rho}$  for a general two-rank tensor  $T_{\mu\nu}$ . Let us compute  $\nabla_\mu \nabla_\nu T_{\lambda\rho}$  first. It is

$$\begin{aligned} \nabla_\mu \nabla_\nu T_{\lambda\rho} &= \frac{\partial}{\partial x^\mu} (\nabla_\nu T_{\lambda\rho}) - \Gamma_{\nu\mu}^\alpha (\nabla_\delta T_{\lambda\rho}) - \Gamma_{\lambda\mu}^\sigma (\nabla_\nu T_{\sigma\rho}) - \Gamma_{\rho\mu}^\sigma (\nabla_\nu T_{\lambda\sigma}) \\ &= \frac{\partial^2 T_{\lambda\rho}}{\partial x^\mu \partial x^\nu} - \frac{\partial \Gamma_{\lambda\nu}^\sigma}{\partial x^\mu} T_{\sigma\rho} - \Gamma_{\lambda\nu}^\sigma \frac{\partial T_{\sigma\rho}}{\partial x^\mu} - \frac{\partial \Gamma_{\rho\nu}^\sigma}{\partial x^\mu} T_{\lambda\sigma} - \Gamma_{\rho\nu}^\sigma \frac{\partial T_{\lambda\sigma}}{\partial x^\mu} - \Gamma_{\nu\mu}^\alpha (\nabla_\delta T_{\lambda\rho}) \\ &\quad - \Gamma_{\lambda\mu}^\sigma \frac{\partial T_{\sigma\rho}}{\partial x^\nu} + \Gamma_{\lambda\mu}^\sigma \Gamma_{\sigma\nu}^\alpha T_{\alpha\rho} + \Gamma_{\lambda\mu}^\sigma \Gamma_{\rho\nu}^\alpha T_{\sigma\alpha} \\ &\quad - \Gamma_{\rho\mu}^\sigma \frac{\partial T_{\lambda\sigma}}{\partial x^\nu} + \Gamma_{\rho\mu}^\sigma \Gamma_{\lambda\nu}^\alpha T_{\alpha\sigma} + \Gamma_{\rho\mu}^\sigma \Gamma_{\sigma\nu}^\alpha T_{\lambda\alpha}. \end{aligned}$$

Note that grayed terms vanish from contributions of  $\nabla_\nu \nabla_\mu T_{\lambda\rho}$ . When we evaluate  $\nabla_\mu \nabla_\nu T_{\lambda\rho}$ , we just need to flip the indices as  $\mu \leftrightarrow \nu$  and we get

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]T_{\lambda\rho} &= -\frac{\partial \Gamma_{\lambda\nu}^\sigma}{\partial x^\mu} T_{\sigma\rho} - \cancel{\Gamma_{\lambda\nu}^\sigma \frac{\partial T_{\sigma\rho}}{\partial x^\mu}} - \frac{\partial \Gamma_{\rho\nu}^\sigma}{\partial x^\mu} T_{\lambda\sigma} - \cancel{\Gamma_{\rho\nu}^\sigma \frac{\partial T_{\lambda\sigma}}{\partial x^\mu}} - \cancel{\Gamma_{\nu\mu}^\alpha \frac{\partial T_{\sigma\rho}}{\partial x^\nu}} \\ &\quad + \Gamma_{\lambda\mu}^\sigma \Gamma_{\sigma\nu}^\alpha T_{\alpha\rho} + \Gamma_{\lambda\mu}^\sigma \Gamma_{\rho\nu}^\alpha T_{\sigma\alpha} - \cancel{\Gamma_{\rho\mu}^\sigma \frac{\partial T_{\lambda\sigma}}{\partial x^\nu}} + \Gamma_{\rho\mu}^\sigma \Gamma_{\lambda\nu}^\alpha T_{\alpha\sigma} + \Gamma_{\rho\mu}^\sigma \Gamma_{\sigma\nu}^\alpha T_{\lambda\alpha} \\ &\quad + \frac{\partial \Gamma_{\lambda\mu}^\sigma}{\partial x^\nu} T_{\sigma\rho} + \cancel{\Gamma_{\lambda\mu}^\sigma \frac{\partial T_{\sigma\rho}}{\partial x^\nu}} + \frac{\partial \Gamma_{\rho\mu}^\sigma}{\partial x^\nu} T_{\lambda\sigma} + \Gamma_{\rho\mu}^\sigma \frac{\partial T_{\lambda\sigma}}{\partial x^\nu} + \cancel{\Gamma_{\nu\mu}^\alpha \frac{\partial T_{\sigma\rho}}{\partial x^\mu}} \\ &\quad - \Gamma_{\lambda\nu}^\sigma \Gamma_{\sigma\mu}^\alpha T_{\alpha\rho} - \Gamma_{\lambda\nu}^\sigma \Gamma_{\rho\mu}^\alpha T_{\sigma\alpha} + \cancel{\Gamma_{\rho\nu}^\sigma \frac{\partial T_{\lambda\sigma}}{\partial x^\mu}} - \Gamma_{\rho\nu}^\sigma \Gamma_{\lambda\mu}^\alpha T_{\alpha\sigma} - \cancel{\Gamma_{\rho\nu}^\sigma \Gamma_{\sigma\mu}^\alpha T_{\lambda\alpha}} \\ &= \left( \frac{\partial \Gamma_{\lambda\mu}^\sigma}{\partial x^\nu} - \frac{\partial \Gamma_{\lambda\nu}^\sigma}{\partial x^\mu} + \Gamma_{\lambda\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\lambda\nu}^\alpha \Gamma_{\alpha\mu}^\sigma \right) T_{\sigma\rho} \\ &\quad + \left( \frac{\partial \Gamma_{\rho\mu}^\sigma}{\partial x^\nu} - \frac{\partial \Gamma_{\rho\nu}^\sigma}{\partial x^\mu} + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma + \Gamma_{\rho\nu}^\alpha \Gamma_{\alpha\mu}^\sigma \right) T_{\lambda\sigma} \\ &= R^\sigma_{\lambda\nu\mu} T_{\sigma\rho} + R^\sigma_{\rho\nu\mu} T_{\lambda\sigma}. \end{aligned} \quad (3.3)$$

By substituting  $T_{\lambda\rho}$  with  $\nabla_\lambda A_\rho$ , the required result realized.

## 4 Problem 4

Let us show the following formula<sup>\*2</sup>:

$$\frac{\partial}{\partial x^\mu} \sqrt{-g} = \sqrt{-g} \Gamma_{\mu\lambda}^\lambda. \quad (4.1)$$

Assuming this relation, we immediately reach the answer

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\lambda} (\sqrt{-g} A^\lambda) &= \frac{1}{\sqrt{-g}} \left( \frac{\partial}{\partial x^\lambda} \sqrt{-g} \right) A^\lambda + \frac{\partial A^\lambda}{\partial x^\lambda} \\ &= \Gamma_{\lambda\sigma}^\sigma A^\lambda + \frac{\partial A^\lambda}{\partial x^\lambda} = \nabla_\lambda A^\lambda. \end{aligned} \quad (4.2)$$

So what is left is to prove the relation (4.1).

**Proof.** We will use the relation

$$\ln g = \text{Tr} \ln g \quad (4.3)$$

and take the derivative to  $x^\lambda$ . Then we get

$$\frac{\partial}{\partial x^\lambda} \ln g = \frac{\partial}{\partial x^\lambda} \text{Tr} \ln g \quad (4.4)$$

and by computing both sides carefully, we can obtain

$$\frac{1}{g} \frac{\partial g}{\partial x^\lambda} = g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}. \quad (4.5)$$

Note that we use the relation

$$\frac{\partial}{\partial x^\lambda} \ln g_{\alpha\beta} = (g^{-1})_{\alpha\gamma} \frac{\partial g_{\gamma\beta}}{\partial x^\lambda} \quad (4.6)$$

when we derivate the RHS of (4.4). Thus we find

$$\begin{aligned} \frac{\partial}{\partial x^\lambda} \sqrt{-g} &= -\frac{\partial g}{\partial x^\lambda} \cdot \frac{1}{2} \frac{1}{\sqrt{-g}} \\ &= \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \end{aligned} \quad (4.7)$$

by using (4.5). On the other hand, the Christoffel symbol is defined as (1.1) and contracting the indices, the relation holds

$$\Gamma_{\mu\lambda}^\lambda = \frac{1}{2} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \quad (4.8)$$

and finally we attain (4.1). ■

---

<sup>\*2</sup>The exponential of matrix  $A$  satisfies

$$\det e^A = e^{\text{Tr} A}$$

and taking  $A \equiv \ln g$ , we find (4.1).