Report 1

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1 Problem 1*

The Christoffel symbol is defined as

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) \tag{1.1}$$

in a coordinate system x^λ and transform into a system $\tilde{x}^\lambda$*[1]. The transformation laws of the coordinate are

$$g^{\mu\nu} = \frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\tilde{g}^{\rho\sigma} \tag{1.2}$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\lambda}. \tag{1.3}$$

Putting these relations into the definition (1.1), we obtain the transformation law

$$\begin{aligned}
\Gamma_{\mu\alpha\beta} &= \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) \\
&= \frac{1}{2}\left\{\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial}{\partial \tilde{x}^\delta}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}g_{\lambda\gamma}\right) + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\gamma}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\lambda\delta}\right) - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\lambda}\left(\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\gamma\delta}\right)\right\} \\
&= \frac{1}{2}\left\{\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial}{\partial \tilde{x}^\delta}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\right)\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}g_{\lambda\gamma} + \frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\delta}\left(\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\right)g_{\lambda\gamma} + \underset{\sim\sim\sim}{\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^\delta}}\right. \\
&\quad + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\gamma}\left(\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\right)\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\lambda\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\gamma}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\left(\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\right)g_{\lambda\delta} + \underset{\sim\sim\sim}{\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^\gamma}} \\
&\quad \left. - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\lambda}\left(\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\right)\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\gamma\delta} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial}{\partial \tilde{x}^\lambda}\left(\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\right)g_{\gamma\delta} - \underset{\sim\sim\sim}{\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^\lambda}}\right\}.
\end{aligned} \tag{1.4}$$

The waved terms $\sim\sim\sim$ in (1.4) become a twice of the Christoffel symbol:

$$\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^\delta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^\gamma} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^\lambda} = 2\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\tilde{\Gamma}_{\lambda\gamma\delta}. \tag{1.5}$$

But we know that there are additional terms in (1.4) and they violate the transformation law of the 3-rank covariant tensor. Thus we find the law as

$$\begin{aligned}
\Gamma_{\mu\alpha\beta} &= \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\tilde{\Gamma}_{\lambda\gamma\delta} + \frac{1}{2}\left\{\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial x^\epsilon}{\partial \tilde{x}^\delta}\left(\frac{\partial^2 \tilde{x}^\lambda}{\partial x^\epsilon \partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} + \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\epsilon \partial x^\alpha}\right)g_{\lambda\gamma}\right. \\
&\qquad + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial x^\epsilon}{\partial \tilde{x}^\gamma}\left(\frac{\partial^2 \tilde{x}^\lambda}{\partial x^\mu \partial x^\epsilon}\frac{\partial \tilde{x}^\delta}{\partial x^\beta} + \frac{\partial^2 \tilde{x}^\delta}{\partial x^\mu \partial x^\epsilon}\frac{\partial \tilde{x}^\lambda}{\partial x^\beta}\right)g_{\lambda\delta} \\
&\qquad \left. - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial x^\epsilon}{\partial \tilde{x}^\lambda}\left(\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\epsilon \partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta} + \frac{\partial \tilde{x}^\gamma}{\partial x^\beta}\frac{\partial^2 \tilde{x}^\delta}{\partial x^\epsilon \partial x^\alpha}\right)g_{\gamma\delta}\right\} \\
&= \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}\tilde{\Gamma}_{\lambda\gamma\delta} + \frac{\partial \tilde{x}^\delta}{\partial x^\beta}\frac{\partial x^\epsilon}{\partial \tilde{x}^\delta}\frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\epsilon \partial x^\alpha}g_{\lambda\gamma} \\
&\qquad + \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha}\frac{\partial x^\epsilon}{\partial \tilde{x}^\gamma}\frac{\partial^2 \tilde{x}^\lambda}{\partial x^\mu \partial x^\epsilon}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\lambda\delta} - \frac{\partial \tilde{x}^\lambda}{\partial x^\mu}\frac{\partial x^\epsilon}{\partial \tilde{x}^\lambda}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\epsilon \partial x^\alpha}\frac{\partial \tilde{x}^\delta}{\partial x^\beta}g_{\gamma\delta}.
\end{aligned} \tag{1.6}$$

(I think this result is somewhat strange.)

*[1] In the problem statement, we should consider the transformation x^λ into a system "x'^λ" but I should apologize since we will use a different label "$\tilde{x}^\lambda$", though it is trivial.

2 Problem 2

From the previous problem, we obtain the transformation law

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}. \tag{2.1}$$

Since we already know the transformation laws, what we have to do is just to compute them. Thus the transformation should be

$$\begin{aligned}
\tilde{\nabla}_{\mu} \tilde{V}_{\nu} &\equiv \frac{\partial \tilde{V}_{\nu}}{\partial \tilde{x}^{\mu}} - \tilde{\Gamma}^{\lambda}{}_{\mu\nu} \tilde{V}_{\lambda} \\
&= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} V_{\beta} \right) - \left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \right) \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} \\
&= \cancel{\frac{\partial^2 x^{\beta}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}} V_{\beta} + \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \frac{\partial V_{\beta}}{\partial x^{\alpha}} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} - \cancel{\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}}} V_{\delta} \\
&= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \left(\frac{\partial V_{\beta}}{\partial x^{\alpha}} - \Gamma^{\gamma}{}_{\alpha\beta} V_{\gamma} \right) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}
\end{aligned} \tag{2.2}$$

and we finally attain

$$\tilde{\nabla}_{\mu} \tilde{V}_{\nu} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}. \tag{2.3}$$

3 Problem 3*

4 Problem 4

Let us show the following formula:

$$\frac{\partial}{\partial x^\mu}\sqrt{-g} = \sqrt{-g}\Gamma^\lambda{}_{\mu\lambda}. \tag{4.1}$$

Assuming this relation, we immediately reach the answer

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}A^\lambda\right) &= \frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^\lambda}\sqrt{-g}\right)A^\lambda + \frac{\partial A^\lambda}{\partial x^\lambda} \\ &= \Gamma^\sigma{}_{\lambda\sigma}A^\kappa + \frac{\partial A^\lambda}{\partial x^\lambda} = \nabla_\lambda A^\lambda. \end{aligned} \tag{4.2}$$

So what is left is to prove the relation (4.1).

Proof. We will use the relation

$$\ln\det g = \mathrm{Tr}\ln g \tag{4.3}$$

and take the derivative to x^λ. ■