

Relativity - Report 4

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(1) Let us consider the variation of the action

$$S_m = \int d^4 \sqrt{-g} K(\phi, X) \quad (0.1)$$

where X denotes

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_{[\mu} \phi \partial_{\nu]} \phi. \quad (0.2)$$

It obeys

$$\begin{aligned} \delta S_m &= \int d^4 [(\delta \sqrt{-g}) K + \sqrt{-g} (\delta K)] \\ &= -\frac{1}{2} \int d^4 x \sqrt{-g} \{g_{\mu\nu} K + K_X \partial_\mu \partial_\nu \phi\} \delta g^{\mu\nu} \end{aligned} \quad (0.3)$$

where $K_X \equiv \partial_X K(\phi, X)$. The definition of the energy-momentum tensor so far is

$$\delta S_m = -\frac{1}{2} \int d^4 x \delta g^{\mu\nu} \sqrt{-g} T_{\mu\nu} \quad (0.4)$$

and by comparing this definition with the equation (0.3), we find

$$T_{\mu\nu} = g_{\mu\nu} K + K_X \partial_\mu \partial_\nu \phi \quad (K_X \equiv \partial_X K(\phi, X)). \quad (0.5)$$

(2) The perturbed FLRW metric

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2A) & \partial_i B \\ \partial_i B & (1+2\psi)\delta_{ij} + 2\partial_i \partial_j E \end{pmatrix} \quad (0.6)$$

is obtained by metric perturbation. The energy-momentum tensor becomes

$$T^\mu_\nu = g^{\mu\rho} T_{\rho\nu} = \delta^\mu_\nu K(\phi, X) + g^{\mu\rho} K_X \partial_\rho \phi \partial_\nu \phi \quad (0.7)$$

from the previous result. By calculating the inverse matrix of the metric, we obtain^{*1}

$$g^{\mu\nu} = \frac{1}{a^2(\eta)} \begin{pmatrix} 1-2A(\eta, \mathbf{x}) & \partial_x B & \partial_y B & \partial_z B \\ \partial_x B & 1-2(\psi+\partial_x^2 E) & -2\partial_x \partial_y E & -2\partial_x \partial_z E \\ \partial_y B & -2\partial_x \partial_y E & 1-2(\psi+\partial_y^2 E) & -2\partial_y \partial_z E \\ \partial_z B & -2\partial_x \partial_z E & -2\partial_y \partial_z E & 1-2(\psi+\partial_z^2 E) \end{pmatrix}. \quad (0.8)$$

^{*1}This result is obtained by perturbing $g^{\mu\nu} = g_0^{\mu\nu} + \delta g^{\mu\nu}$. This implies $g_0^{\mu\nu} = 0$ naively and $\delta g^{\mu\nu}$ should contribute to cancel out the perturbative parts in $g_{\mu\nu}$.

By perturbing $\phi(\eta, \mathbf{x})$ as $\phi(\eta) + \delta\phi(\eta, \mathbf{x})$ ^{*2}, we find δT_0^0 as

$$\begin{aligned}\delta T_0^0 &= K(\phi + \delta\phi, X + X_\phi \delta\phi) + g^{0\rho} K_X(\phi + \delta\phi, X + X_\phi \delta\phi) \partial_\rho \phi \partial_0 \phi \\ &= K + K_\phi \delta\phi + K_X X_\phi \delta\phi \\ &\quad + \frac{1}{a(\eta)} (K_X + K_{\phi X} \delta\phi + K_{XX} X_\phi \delta\phi) \left[\frac{2A}{a(\eta)} \phi' + \partial_i B \partial_i \phi \right] \phi'.\end{aligned}\quad (0.9)$$

Note that $g^{\mu\nu}$ has only the perturbative quantity, thus the contribution of its coefficient should be only the background. In the same way, we can evaluate δT_i^0 and δT_j^i . Anyway, we conclude that the result, though it is formal, is

$$\delta T_\nu^\mu = \delta_\nu^\mu (K + K_\phi \delta\phi + K_X X_\phi \delta\phi) + g^{\mu\rho} (K_X + K_{\phi X} \delta\phi + K_{XX} X_\phi \delta\phi) \partial_\rho \phi \partial_\nu \phi. \quad (0.10)$$

(3) (incomplete)

The perturbed Einstein equation is given by

$$\delta G_\nu^\mu = 8\pi G \delta T_\nu^\mu \quad (0.11)$$

in Lecture 14. In this case, by putting $K = -\Lambda$ (= constant), the energy-momentum tensor becomes

$$\delta T_\nu^\mu = -\delta_\nu^\mu \Lambda - g^{\mu\rho} \Lambda \partial_\rho \phi \partial_\nu \phi \quad (0.12)$$

from the previous result.

^{*2}I followed the instructions in the problem statement and omitted the bar on the background field.