[Report 3]

Note: Write the details of reaching the final answers in English. The deadline for this report is 18:00, on 29 May, 2024. Upload an electric file on the Moodle system. The forms of the report can be a PDF, word, or scanned electric file with handwriting.

We discuss the motion of a massive particle around a steadily rotating body with mass M. The space-time geometry around such a rotating body is described by the Kerr line element

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2\mu r}{\Sigma}\right)\mathrm{d}t^2 - \frac{4\mu a c r \sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta\mathrm{d}\varphi^2\,, \quad (0.1)$$

where c is the speed of light, $\mu = GM/c^2$ with G being the gravitational constant, a is a positive constant, and

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2\mu r + a^2\,. \quad (0.2)$$

We confine ourselves to the particle motion in the equatorial plane

$$\theta = \frac{\pi}{2}\,. \quad (0.3)$$

(1) The four-velocity of a unit-mass particle at space-time point $x^\mu = (t, r, \theta, \varphi)$ is given by $p^\mu = \dot{x}^\mu$, where a dot represents the derivative with respect to the proper time τ. On the Kerr background (0.1), there are two conserved quantities p_t and p_φ along the particle trajectory. We denote these conserved values as $p_t = -kc^2$ and $p_\varphi = h$. Express $\dot{t}$ and $\dot{\varphi}$ by using k, h, μ, a, and r.
(2) The massive particle satisfies the relation

$$g^{\mu\nu}p_\mu p_\nu = -c^2\,, \quad (0.4)$$

where $g^{\mu\nu}$ is the metric tensor. One can write this equation in the form

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff}(r) = \frac{1}{2}c^2(k^2 - 1)\,. \quad (0.5)$$

Derive the effective potential $V_{\rm eff}(r)$ as a function of r by using k, h, μ, a.
(3) Let us consider the circular motion of the massive particle. Defining the variables $x = h - ack$ and $u = 1/r$, obtain the solution to x^2 corresponding to the circular motion. For the solution with $x < 0$, express the values of k and h by using μ, a, and u.
(4) For the innermost stable circular orbit, obtain the equation for the radius r to be satisfied. Use μ and a for the coefficients of this equation.