Relativity - Report 4

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(1) Let us consider the variation of the action

$$S_m = \int \mathrm{d}^4 \sqrt{-g} K(\phi, X) \tag{0.1}$$

where X denotes

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_{[}\mu] \phi \partial_\nu \phi. \tag{0.2}$$

It obeys

$$\begin{aligned} \delta S_m &= \int \mathrm{d}^4 \left[(\delta\sqrt{-g}) K + \sqrt{-g}(\delta K) \right] \\ &= -\frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left\{ g_{\mu\nu} K + K_X \partial_\mu \partial_\nu \phi \right\} \delta g^{\mu\nu} \end{aligned} \tag{0.3}$$

where $K_X \equiv \partial_X K(\phi, X)$. The definition of the energy-momentum tensor so far is

$$\delta S_m = -\frac{1}{2} \int \mathrm{d}^4 x \, \delta g^{\mu\nu} \sqrt{-g} T_{\mu\nu} \tag{0.4}$$

and by comparing this definition with the equation (0.3), we find

$$T_{\mu\nu} = g_{\mu\nu} K + K_X \partial_\mu \phi \partial_\nu \phi \qquad (K_X \equiv \partial_X K(\phi, X)). \tag{0.5}$$

(2) The perturbed FLRW metric

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2A) & \partial_i B \\ \partial_i B & (1+2\psi)\delta_{ij} + 2\partial_i \partial_j E \end{pmatrix} \tag{0.6}$$

is obtained by metric perturbation. The energy-momentum tensor becomes

$$T^\mu_\nu = g^{\mu\rho} T_{\rho\nu} = \delta^\mu_\nu K(\phi, X) + g^{\mu\rho} K_X \partial_\rho \phi \partial_\nu \phi \tag{0.7}$$

from the previous result. By calculating the inverse matrix of the metric, we obtain[*1]

$$g^{\mu\nu} = \frac{1}{a^2(\eta)} \begin{pmatrix} 1 - 2A(\eta, \boldsymbol{x}) & \partial_x B & \partial_y B & \partial_z B \\ \partial_x B & 1 - 2(\psi + \partial_x^2 E) & -2\partial_x \partial_y E & -2\partial_x \partial_z E \\ \partial_y B & -2\partial_x \partial_y E & 1 - 2(\psi + \partial_y^2 E) & -2\partial_y \partial_z E \\ \partial_z B & -2\partial_x \partial_z E & -2\partial_y \partial_z E & 1 - 2(\psi + \partial_z^2 E) \end{pmatrix}. \tag{0.8}$$

To compute δT^μ_ν, let me consider the variation $\phi \to \phi(\eta) + \delta\phi(\eta, \boldsymbol{x})$, i.e., we will calculate

$$T^\mu_\nu = \delta^\mu_\nu K(\phi + \delta\phi, X + \delta X) + g^{\mu\rho} K_X(\phi + \delta\phi, X + \delta X) \partial_\mu(\phi + \delta\phi) \partial_\nu(\phi + \delta\phi) \tag{0.9}$$

[*1] This result is obtained by purterbating $g^{\mu\nu} = g_0^{\mu\nu} + \delta g^{\mu\nu}$. This implies $g_0^{\mu\nu} = 0$ naively and $\delta g^{\mu\nu}$ should contribute to cancel out the perturbative parts in $g_{\mu\nu}$.

and take the first order of the perturbative term and identify it as δT^μ_ν. When we evaluate the above value, we should be careful with the fact that the background field does not depend on the space coordinates, and the derivative to that direction vanishes.

Thus we obtain

$$\delta X = \frac{\phi'(\delta\phi' - A\phi')}{a^2}, \tag{0.10}$$

$$K = \frac{K_X\phi'(\delta\phi' - A\phi')}{a^2} + \delta\phi K_\phi, \tag{0.11}$$

$$K_X = \frac{K_{XX}\phi'(\delta\phi' - A\phi')}{a^2} + \delta\phi K_{X\phi} \tag{0.12}$$

and we can get (0.9) as

$$\delta T^0_{\ 0} = \frac{K_{XX}{\phi'}^3(A\phi' - \delta\phi')}{a^4} + \frac{\phi'(AK_X\phi' + \delta\phi'(-K_X) - \delta\phi K_{X\phi}\phi')}{a^2} + \delta\phi K_\phi \tag{0.13}$$

$$\delta T^{ii} = \frac{K_X\phi'(\delta\phi' - \phi' A)}{a^2} + K_\phi\delta\phi \tag{0.14}$$

$$\delta T^i_{\ 0} = \frac{1}{a^2}\left\{\partial_i(\delta\phi) + \phi'\partial_i B\right\} \tag{0.15}$$

$$\delta T^0_{\ i} = -\frac{1}{a^2}K_X\partial_i(\delta\phi)\delta\phi' \tag{0.16}$$

where $i = 1, 2, 3$.

(It looks peculiar to me since the result δT^μ_ν is not symmetric form.)

(3) (incomplete)