Relativity - Report 4

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(1) Let us consider the variation of the action

$$S_m = \int \mathrm{d}^4 \sqrt{-g} K(\phi, X) \tag{0.1}$$

where X denotes

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_{[\mu]} \phi \partial_\nu \phi. \tag{0.2}$$

It obeys

$$\begin{aligned} \delta S_m &= \int \mathrm{d}^4 \left[(\delta\sqrt{-g}) K + \sqrt{-g} (\delta K) \right] \\ &= -\frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left\{ g_{\mu\nu} K + K_X \partial_\mu \partial_\nu \phi \right\} \delta g^{\mu\nu} \end{aligned} \tag{0.3}$$

where $K_X \equiv \partial_X K(\phi, X)$. The definition of the energy-momentum tensor so far is

$$\delta S_m = -\frac{1}{2} \int \mathrm{d}^4 x \, \delta g^{\mu\nu} \sqrt{-g} T_{\mu\nu} \tag{0.4}$$

and by comparing this definition with the equation (0.3), we find

$$T_{\mu\nu} = g_{\mu\nu} K + K_X \partial_\mu \phi \partial_\nu \phi \qquad (K_X \equiv \partial_X K(\phi, X)). \tag{0.5}$$

(2) The perturbed FLRW metric

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2A) & \partial_i B \\ \partial_i B & (1+2\psi)\delta_{ij} + 2\partial_i \partial_j E \end{pmatrix} \tag{0.6}$$

is obtained by metric perturbation. The energy-momentum tensor becomes

$$T^\mu_{\ \nu} = g^{\mu\rho} T_{\rho\nu} = \delta^\mu_\nu K(\phi, X) + g^{\mu\rho} K_X \partial_\rho \phi \partial_\nu \phi \tag{0.7}$$

from the previous result. By calculating the inverse matrix of the metric, we obtain[*1]

$$g^{\mu\nu} = \frac{1}{a^2(\eta)} \begin{pmatrix} 1 - 2A(\eta, \boldsymbol{x}) & \partial_x B & \partial_y B & \partial_z B \\ \partial_x B & 1 - 2(\psi + \partial_x^2 E) & -2\partial_x \partial_y E & -2\partial_x \partial_z E \\ \partial_y B & -2\partial_x \partial_y E & 1 - 2(\psi + \partial_y^2 E) & -2\partial_y \partial_z E \\ \partial_z B & -2\partial_x \partial_z E & -2\partial_y \partial_z E & 1 - 2(\psi + \partial_z^2 E) \end{pmatrix}. \tag{0.8}$$

To compute $\delta T^\mu_{\ \nu}$, let me consider the variation $\phi \to \phi(\eta) + \delta\phi(\eta, \boldsymbol{x})$, i.e., we will calculate

$$T^\mu_{\ \nu} = \delta^\mu_\nu K(\phi + \delta\phi, X + \delta X) + g^{\mu\rho} K_X(\phi + \delta\phi, X + \delta X) \partial_\mu(\phi + \delta\phi) \partial_\nu(\phi + \delta\phi) \tag{0.9}$$

[*1] This result is obtained by purterbating $g^{\mu\nu} = g_0^{\mu\nu} + \delta g^{\mu\nu}$. This implies $g_0^{\mu\nu} = 0$ naively and $\delta g^{\mu\nu}$ should contribute to cancel out the perturbative parts in $g_{\mu\nu}$.

and take the first order of the perturbative term and identify it as δT^{μ}_{ν}. When we evaluate the above value, we should be careful with the fact that the background field does not depend on the space coordinates, and the derivative to that direction vanishes.

Thus we obtain

$$\delta X = \frac{\phi'(\delta\phi' - A\phi')}{a^2}, \tag{0.10}$$
$$K = \frac{K_X\phi'(\delta\phi' - A\phi')}{a^2} + \delta\phi K_\phi, \tag{0.11}$$
$$K_X = \frac{K_{XX}\phi'(\delta\phi' - A\phi')}{a^2} + \delta\phi K_{X\phi} \tag{0.12}$$

and we can get (0.9) as

$$\delta T^0_{\ 0} = \frac{K_{XX}{\phi'}^3(A\phi' - \delta\phi')}{a^4} + \frac{\phi'(AK_X\phi' + \delta\phi'(-K_X) - \delta\phi K_{X\phi}\phi')}{a^2} + \delta\phi K_\phi \tag{0.13}$$
$$\delta T^{ii} = \frac{K_X\phi'(\delta\phi' - \phi' A)}{a^2} + K_\phi\delta\phi \tag{0.14}$$
$$\delta T^i_{\ 0} = \frac{1}{a^2}\left\{\partial_i(\delta\phi) + \phi'\partial_i B\right\} \tag{0.15}$$
$$\delta T^0_{\ i} = -\frac{1}{a^2}K_X\partial_i(\delta\phi)\delta\phi' \tag{0.16}$$

where $i = 1, 2, 3$.

(3) The perturbative Einstein equation is given as

$$3\mathcal{H}(\psi' - \mathcal{H}A) - \nabla^2\psi + \mathcal{H}\nabla^2(E' - B) = 4\pi G a^2\delta\rho \tag{0.17}$$
$$\psi' - \mathcal{H}A = 4\pi G a\delta q \tag{0.18}$$

in this course. In this case, we will use the notation as follows:

$$K = -\Lambda, \tag{0.19}$$
$$\delta q = a\rho(v + B), \tag{0.20}$$
$$\Phi = \psi - \mathcal{H}(E' - B), \tag{0.21}$$
$$\delta\rho_m = \delta\rho - 3\frac{\mathcal{H}}{a}\delta q. \tag{0.22}$$

The first setup is irrelevant at this moment.

For the first step, insert (0.18) into (0.17) and organize some terms. We immediately find

$$\nabla^2(\psi - \mathcal{H}(E' - B)) = 4\pi G a\left\{3\mathcal{H}\delta q - a\delta\rho\right\}. \tag{0.23}$$

We notice the foremost term corresponds with Φ. In addition, we try to put (0.22) into the $\delta\rho$ on the right-hand side. Thus we obtain slick equality

$$\nabla^2\Phi + 4\pi G a^2\delta\rho_m = 0. \tag{0.24}$$

(4) Let me confirm the given relation again:

$$\delta \equiv \frac{\delta\rho_m}{\rho}, \quad \Psi = A, \quad \Phi = \psi, \quad \Phi + \Phi = 0. \tag{0.25}$$

Naively, we just put that relation to the previous result (0.24) and attain the equality

$$-\nabla^2 A + 4\pi G a^2 \rho\delta = 0. \tag{0.26}$$

We will try to evaluate undetermined factors ρ and A. From the problem 2, we obtain $T^0_{\ 0} = -\rho$ as

$$-\rho = K - \frac{K_X \phi'^2}{a^2} + \frac{K_{XX}\phi'^3(A\phi' - \delta\phi')}{a^4} + \frac{\phi'(AK_X\phi' + \delta\phi'(-K_X) - \delta\phi K_{X\phi}\phi')}{a^2} + \delta\phi K_\phi \tag{0.27}$$

up to the first order of the perturbation. In this case, since we set $K = -\Lambda$, the derivation of K, such as $K_X, K_{X\phi}$, etc., vanishes. Thus we obtain the relation

$$\rho = \Lambda. \tag{0.28}$$

Let us figure out A in terms of δ. To accomplish this, we compute (0.18) and (0.22). These relations imply equality

$$-(1+\mathcal{H})A = 4\pi G a^2 \Lambda(v + B). \tag{0.29}$$

On the other hand, we evaluate (0.22) and the relation

$$\delta = \frac{\delta\rho}{\Lambda} - 3\mathcal{H}(v + B) \tag{0.30}$$

holds. From the result of problem 2, we find $\delta T^0_{\ 0} = -\delta\rho = 0$. Putting this equation into (0.29) and vanishing the factor $v + B$, we conclude A as

$$A = \frac{4\pi a^2 \Lambda}{3\mathcal{H}(1+\mathcal{H})}\delta \tag{0.31}$$

and we can organize the equation (0.29) for

$$\nabla^2\left[\frac{\delta}{3\mathcal{H}(1+\mathcal{H})}\right] = \delta. \tag{0.32}$$