

# Relativity - Report 3

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- (1) If we put  $\theta = \pi/2$ , the quantity  $\Sigma$  becomes  $r^2$  and the line element is obtained as

$$ds^2 = -c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \frac{4\mu ac}{r} dt d\varphi + \frac{r^2}{\Delta} dr^2 + \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right) d\varphi^2 \quad (0.1)$$

and the metric also is as

$$g_{\mu\nu} = \begin{pmatrix} -c^2(1 - 2\mu/r) & 0 & -2\mu ac/r \\ 0 & r^2/\Delta & 0 \\ -2\mu ac/r & 0 & r^2 + a^2 + 2\mu a^2/r \end{pmatrix} \quad (0.2)$$

where  $\Delta$  still remain  $r^2 - 2\mu r + a^2$ . We obtain its inverse

$$g^{\mu\nu} = \begin{pmatrix} -\frac{r^3 + a^2(r + 2\mu)}{c^2 r(a^2 + r^2 - 2\mu r)} & 0 & -\frac{2a\mu}{a^2 cr + cr^3 - 2cr^2\mu} \\ 0 & \frac{a^2 + r^2 - 2r\mu}{r^2} & 0 \\ -\frac{2a\mu}{a^2 cr + cr^3 - 2cr^2\mu} & 0 & \frac{r - 2\mu}{a^2 + r^3 - 2r^2\mu} \end{pmatrix} \quad (0.3)$$

and  $\dot{t}$  and  $\dot{\varphi}$  are, then, immediately derived as

$$\begin{aligned} \dot{t} &= g^{tt} p_t + g^{t\varphi} p_\varphi \\ &= \frac{ckr^3 - 2ah\mu + a^2 ck(r + 2\mu)}{cr(a^2 + r(r - 2\mu))}, \end{aligned} \quad (0.4)$$

$$\begin{aligned} \dot{\varphi} &= g^{\varphi t} p_t + g^{\varphi\varphi} p_\varphi \\ &= \frac{hr - 2h\mu + 2ack\mu}{a^2 r + r^3 - 2r^2\mu}. \end{aligned} \quad (0.5)$$

- (2) What we need to do is just insert the inverse metric which we already obtained (0.3) into

$$\begin{aligned} g^{\mu\nu} p_\mu p_\nu &= g^{tt} p_t^2 + g^{\varphi\varphi} p_\varphi^2 + 2g^{t\varphi} p_t p_\varphi + g^{rr} p_r^2 \\ &= g^{tt} \cdot (-kc^2)^2 + g^{\varphi\varphi} \cdot h^2 + 2g^{t\varphi} \cdot (-kc^2) \cdot h + g_{rr} \dot{r}^2. \end{aligned} \quad (0.6)$$

Note that we use the fact that  $p_r$  is obtained by lowering the indices  $\dot{r}$ , i.e.  $p_r = g_{rr} \dot{r}$ . Putting the inverse matrix components  $g^{\mu\nu}$  and organizing the equations, we will get the effective potential as

$$V_{\text{eff}}(r) = \frac{h^2 - a^2 c^2 (k^2 - 1)}{2r^2} - \frac{(h - ack)^2 \mu}{r^3} - \frac{c^2 \mu}{r}. \quad (0.7)$$

It is obvious that  $V_{\text{eff}}(r)$  satisfies the relation

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2} c^2 (k^2 - 1) \quad (0.8)$$

obtained from  $g^{\mu\nu} p_\mu p_\nu = -c^2$  since it was defined to satisfy that equation.

- (3) Since we will analyze the circular motion, we can put  $\dot{r}$  to zero in the previous result. Therefore the last result becomes

$$\frac{h^2 - a^2 c^2 (k^2 - 1)}{2} u^2 - x^2 \mu u^3 - c^2 \mu u = \frac{1}{2} c^2 (k^2 - 1) \quad (0.9)$$

and solving to  $x^2$ , we find

$$x^2 = \frac{1}{2\mu u^3} c^2 (k^2 - 1) + \frac{c^2}{u^2} + \frac{1}{2\mu u} [h^2 - a^2 c^2 (k^2 - 1)]. \quad (0.10)$$

We still use  $x^2$  when we derive  $k$  and  $h$  and insert the solution

$$x = -\sqrt{\frac{1}{2\mu u^3} c^2 (k^2 - 1) + \frac{c^2}{u^2} + \frac{1}{2\mu u} [h^2 - a^2 c^2 (k^2 - 1)]} (< 0) \quad (0.11)$$

to the obtained expression at the final step. To evaluate two values  $k$  and  $h$ , we should prepare one more relation for  $k$  and  $h$ . Since we are assuming the circular motion, its orbits should be stabilized at the potential minimum. So, to express such a situation, we will consider the extremum condition<sup>\*1</sup>

$$\frac{dV_{\text{eff}}}{du} = \{h^2 - a^2 c^2 (k^2 - 1)\} u - 3x^2 \mu u^2 - c^2 \mu = 0 \quad (0.12)$$

and solve (0.12) with (0.9) to  $k$  and  $h$ . Then we finally obtain

$$k = -\frac{\sqrt{c^2(1 - \mu u) + \mu u^3 x}}{c} \quad (0.13)$$

$$h = -\sqrt{\frac{\mu(c^2(1 - a^2 u^2) + u^2 x(a^2 u^2 + 3))}{u}} \quad (0.14)$$

with  $x$  in (0.11).

I regard  $x$  as a constant.

- (4) To evaluate the innermost stable circular orbit, we should derive a second-order derivative

$$\frac{d^2 V_{\text{eff}}}{dr^2} = \frac{3(h^2 - a^2 c^2 (k^2 - 1))}{r^4} - \frac{12\mu(h - ack)^2}{r^5} - \frac{2c^2 \mu}{r^3} \quad (0.15)$$

and take it to zero. This equality decides the orbit radius  $r$ . To make coefficients just depend on  $\mu$  and  $a$ , we should vanish  $k$  and  $h$  by using the previous result. Thus, inserting the last results (0.13), (0.14) and (0.11) and organizing messy expression, we can find the equality for  $r$  as

$$\mu r^2 (r(r - 6\mu)(r - 3\mu) - a^2(7\mu + 3r)) - 6\sqrt{\mu^3 r^3 (a^3 + ar(r - 2\mu))^2} = 0. \quad (0.16)$$

## References

- [1] [Chapter 22 Geodesic motion in Kerr spacetime](#). (Last accessed: May 29, 2024)  
 [2] [Kerr Geometry and Rotating Black Hole](#). (Last accessed: May 29, 2024)

<sup>\*1</sup>It is obvious that the extremum condition for  $V_{\text{eff}}$  does not depend on the variables  $r, u$ . It means that

$$\frac{dV_{\text{eff}}}{dr} = 0 \iff \frac{dV_{\text{eff}}}{du} = 0.$$

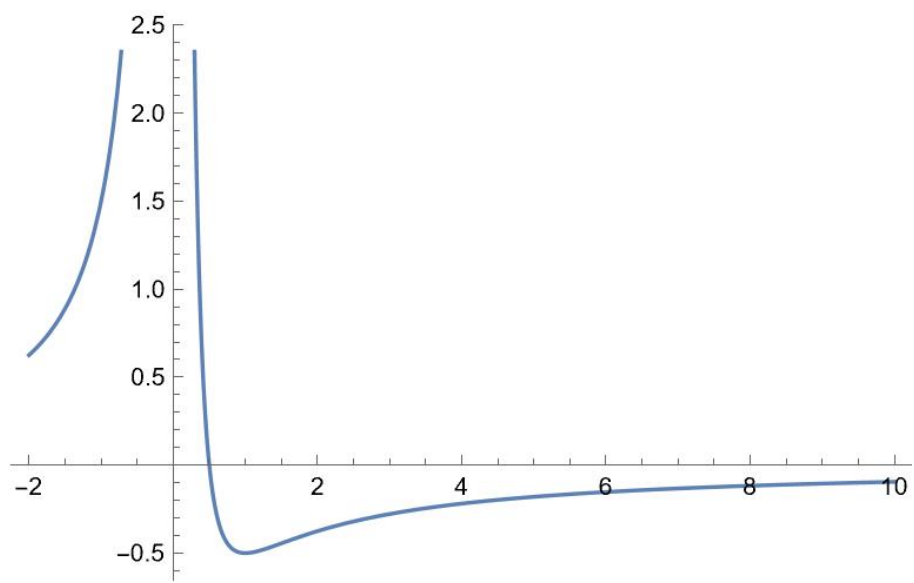


Figure 0.1: One example of the effective potential  $V_{\text{eff}}$