[Report 1]

Note: Write the details of reaching the final answers in English. The deadline for this report is 18:00, on 8 May, 2024. Upload an electric file on the Moodle system. The forms of the report can be a PDF, word, or scanned electric file with handwriting.

Solve the following problems.

(1) For a coordinate system x^λ with the metric tensor $g_{\alpha\beta}$, the Christoffel symbol is defined by

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{2} g^{\nu\lambda} \left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda} \right) , \tag{0.1}$$

where $g_{\alpha\beta,\lambda} = \partial g_{\alpha\beta}/\partial x^\lambda$. We also introduce

$$\Gamma_{\mu\alpha\beta} = g_{\mu\nu} \Gamma^\nu_{\alpha\beta} \, . \tag{0.2}$$

Under the coordinate transformation $x^\mu \to x'^\mu$, consider how $\Gamma_{\mu\alpha\beta}$ is transformed and show that $\Gamma_{\mu\alpha\beta}$ is not a 3-rank covariant tensor.

(2) Show that the covariant derivative $\nabla_\nu V_\mu$ of a 1-rank tensor V_μ (vector field) is the 2-rank covariant tensor.

(3) Prove the following equality

$$[\nabla_\mu, \nabla_\nu] \nabla_\lambda A_\rho = R^\sigma{}_{\lambda\nu\mu} \nabla_\sigma A_\rho + R^\sigma{}_{\rho\nu\mu} \nabla_\lambda A_\sigma \, , \tag{0.3}$$

where A_ρ is a covariant vector field and $R^\sigma{}_{\lambda\nu\mu}$ is a Riemann tensor.

(4) For a contravariant vector field A^λ, show that the following relation holds

$$\nabla_\lambda A^\lambda = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} A^\lambda \right)_{,\lambda} \, , \tag{0.4}$$

where g is a determinant of the metric tensor $g_{\mu\nu}$.