Report 1

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1 Problem 1*

The Christoffel symbol is defined as

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right) \tag{1.1}$$

in a coordinate system x^{λ} and transform into a system $\tilde{x}^{\lambda}$[*1]. The transformation laws of the coordinate are

$$g^{\mu\nu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\rho}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\sigma}}\tilde{g}^{\rho\sigma} \tag{1.2}$$

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial}{\partial \tilde{x}^{\lambda}}. \tag{1.3}$$

Putting these relations into the definition (1.1), we obtain the transformation law

$$\begin{aligned}
\Gamma_{\mu\alpha\beta} &= \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right) \\
&= \frac{1}{2}\left\{\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial}{\partial \tilde{x}^{\delta}}\left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}g_{\lambda\gamma}\right) + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{x}^{\gamma}}\left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}g_{\lambda\delta}\right) - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial}{\partial \tilde{x}^{\lambda}}\left(\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}g_{\gamma\delta}\right)\right\} \\
&= \frac{1}{2}\left\{\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial}{\partial \tilde{x}^{\delta}}\left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\right)\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}g_{\lambda\gamma} + \frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial}{\partial \tilde{x}^{\delta}}\left(\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\right)g_{\lambda\gamma} + \frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^{\delta}}\right. \\
&\qquad + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{x}^{\gamma}}\left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\right)\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}g_{\lambda\delta} + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{x}^{\gamma}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\left(\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\right)g_{\lambda\delta} + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^{\gamma}} \\
&\qquad \left. - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial}{\partial \tilde{x}^{\lambda}}\left(\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\right)\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}g_{\gamma\delta} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{x}^{\lambda}}\left(\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\right)g_{\gamma\delta} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^{\lambda}}\right\}.
\end{aligned} \tag{1.4}$$

The waved terms in (1.4) become a twice of the Christoffel symbol:

$$\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial g_{\lambda\gamma}}{\partial \tilde{x}^{\delta}} + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial g_{\lambda\delta}}{\partial \tilde{x}^{\gamma}} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial g_{\gamma\delta}}{\partial \tilde{x}^{\lambda}} = 2\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\tilde{\Gamma}_{\lambda\gamma\delta}. \tag{1.5}$$

But we know that there are additional terms in (1.4) and they violate the transformation law of the 3-rank covariant tensor. Thus we find the law as

$$\begin{aligned}
\Gamma_{\mu\alpha\beta} &= \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\tilde{\Gamma}_{\lambda\gamma\delta} + \frac{1}{2}\left\{\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial x^{\epsilon}}{\partial \tilde{x}^{\delta}}\left(\frac{\partial^2 \tilde{x}^{\lambda}}{\partial x^{\epsilon}\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial^2 \tilde{x}^{\gamma}}{\partial x^{\epsilon}\partial x^{\alpha}}\right)g_{\lambda\gamma}\right. \\
&\qquad + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial x^{\epsilon}}{\partial \tilde{x}^{\gamma}}\left(\frac{\partial^2 \tilde{x}^{\lambda}}{\partial x^{\mu}\partial x^{\epsilon}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}} + \frac{\partial^2 \tilde{x}^{\delta}}{\partial x^{\mu}\partial x^{\epsilon}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\beta}}\right)g_{\lambda\delta} \\
&\qquad \left. - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial x^{\epsilon}}{\partial \tilde{x}^{\lambda}}\left(\frac{\partial^2 \tilde{x}^{\gamma}}{\partial x^{\epsilon}\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}} + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\beta}}\frac{\partial^2 \tilde{x}^{\delta}}{\partial x^{\epsilon}\partial x^{\alpha}}\right)g_{\gamma\delta}\right\} \\
&= \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\tilde{\Gamma}_{\lambda\gamma\delta} + \frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}\frac{\partial x^{\epsilon}}{\partial \tilde{x}^{\delta}}\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial^2 \tilde{x}^{\gamma}}{\partial x^{\epsilon}\partial x^{\alpha}}g_{\lambda\gamma} \\
&\qquad + \frac{\partial \tilde{x}^{\gamma}}{\partial x^{\alpha}}\frac{\partial x^{\epsilon}}{\partial \tilde{x}^{\gamma}}\frac{\partial^2 \tilde{x}^{\lambda}}{\partial x^{\mu}\partial x^{\epsilon}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}g_{\lambda\delta} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial x^{\epsilon}}{\partial \tilde{x}^{\lambda}}\frac{\partial^2 \tilde{x}^{\gamma}}{\partial x^{\epsilon}\partial x^{\alpha}}\frac{\partial \tilde{x}^{\delta}}{\partial x^{\beta}}g_{\gamma\delta}.
\end{aligned} \tag{1.6}$$

(I think this result is somewhat strange.)

[*1] In the problem statement, we should consider the transformation x^{λ} into a system "x'^{λ}" but I should apologize since we will use a different label "$\tilde{x}^{\lambda}$", though it is trivial.

2 Problem 2

From the previous problem, we obtain the transformation law

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}. \tag{2.1}$$

Since we already know the transformation laws, what we have to do is just to compute them. Thus the transformation should be

$$\begin{aligned}
\tilde{\nabla}_{\mu} \tilde{V}_{\nu} &\equiv \frac{\partial \tilde{V}_{\nu}}{\partial \tilde{x}^{\mu}} - \tilde{\Gamma}^{\lambda}{}_{\mu\nu} \tilde{V}_{\lambda} \\
&= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} V_{\beta} \right) - \left(\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} + \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \right) \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} \\
&= \cancel{\frac{\partial^2 x^{\beta}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}} V_{\beta} + \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \frac{\partial V_{\beta}}{\partial x^{\alpha}} - \frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \Gamma^{\gamma}{}_{\alpha\beta} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}} V_{\delta} - \cancel{\frac{\partial \tilde{x}^{\lambda}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \cdot \frac{\partial x^{\delta}}{\partial \tilde{x}^{\lambda}}} V_{\delta} \\
&= \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \left(\frac{\partial V_{\beta}}{\partial x^{\alpha}} - \Gamma^{\gamma}{}_{\alpha\beta} V_{\gamma} \right) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}
\end{aligned} \tag{2.2}$$

and we finally attain

$$\tilde{\nabla}_{\mu} \tilde{V}_{\nu} = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \nabla_{\alpha} V_{\beta}. \tag{2.3}$$

3 Problem 3*

4 Problem 4

Let us show the following formula[*2]:

$$\frac{\partial}{\partial x^\mu}\sqrt{-g} = \sqrt{-g}\Gamma^\lambda{}_{\mu\lambda}. \tag{4.1}$$

Assuming this relation, we immediately reach the answer

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}A^\lambda\right) &= \frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^\lambda}\sqrt{-g}\right)A^\lambda + \frac{\partial A^\lambda}{\partial x^\lambda} \\ &= \Gamma^\sigma{}_{\lambda\sigma}A^\kappa + \frac{\partial A^\lambda}{\partial x^\lambda} = \nabla_\lambda A^\lambda. \end{aligned}\tag{4.2}$$

So what is left is to prove the relation (4.1).

Proof. We will use the relation

$$\ln g = \mathrm{Tr}\ln g \tag{4.3}$$

and take the derivative to x^λ. Then we get

$$\frac{\partial}{\partial x^\lambda}\ln g = \frac{\partial}{\partial x^\lambda}\mathrm{Tr}\ln g \tag{4.4}$$

and by computing both sides carefully, we can obtain

$$\frac{1}{g}\frac{\partial g}{\partial x^\lambda} = g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}. \tag{4.5}$$

Note that we use the relation

$$\frac{\partial}{\partial x^\lambda}\ln g_{\alpha\beta} = (g^{-1})_{\alpha\gamma}\frac{\partial g_{\gamma\beta}}{\partial x^\lambda} \tag{4.6}$$

when we derivate the RHS of (4.4). Thus we find

$$\begin{aligned}\frac{\partial}{\partial x^\lambda}\sqrt{-g} &= -\frac{\partial g}{\partial x^\lambda}\cdot\frac{1}{2}\frac{1}{\sqrt{-g}} \\ &= \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\end{aligned} \tag{4.7}$$

by using (4.5). On the other hand, the Christoffel symbol is defined as (1.1) and contracting the indices, the relation holds

$$\Gamma^\lambda{}_{\mu\lambda} = \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \tag{4.8}$$

and finally we attain (4.1). ■

[*2]The exponential of matrix A satisfies

$$\det e^A = e^{\mathrm{Tr}\,A}$$

and taking $A \equiv \ln g$, we find (4.1).