Adaptive Filtering - Theory and Applications

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Introduction

Estimation Techniques

Several techniques to solve estimation problems.

- Classical Estimation

  Maximum Likelihood (ML), Least Squares (LS), Moments, etc.

- Bayesian Estimation

  Minimum MSE (MMSE), Maximum A Posteriori (MAP), etc.

Linear Estimation

Frequently used in practice when there is a limitation in computational complexity – *Real-time operation*

Linear Estimators

Simpler to determine: depend on the first two moments of data

- Statistical Approach – Optimal Linear Filters
  - Minimum Mean Square Error
  - Require second order statistics of signals

- Deterministic Approach – Least Squares Estimators
  - Minimum Least Squares Error
  - Require handling of a data observation matrix

Limitations of Optimal Filters and LS Estimators

- Statistics of signals may not be available or cannot be accurately estimated
- There may not be available time for statistical estimation (real-time)
- Signals and systems may be non-stationary
- Memory required may be prohibitive
- Computational load may be prohibitive

Iterative Solutions

- Search the optimal solution starting from an initial guess
- Iterative algorithms are based on classical optimization algorithms
- Require reduced computational effort per iteration
- Need several iterations to converge to the optimal solution
- These methods form the basis for the development of adaptive algorithms
- Still require the knowledge of signal statistics

Adaptive Filters

Usually approximate iterative algorithms and:

- Do not require previous knowledge of the signal statistics
- Have a small computational complexity per iteration
- Converge to a neighborhood of the optimal solution

Adaptive filters are good for:

- Real-time applications, when there is no time for statistical estimation
- Applications with nonstationary signals and/or systems

Properties of Adaptive Filters

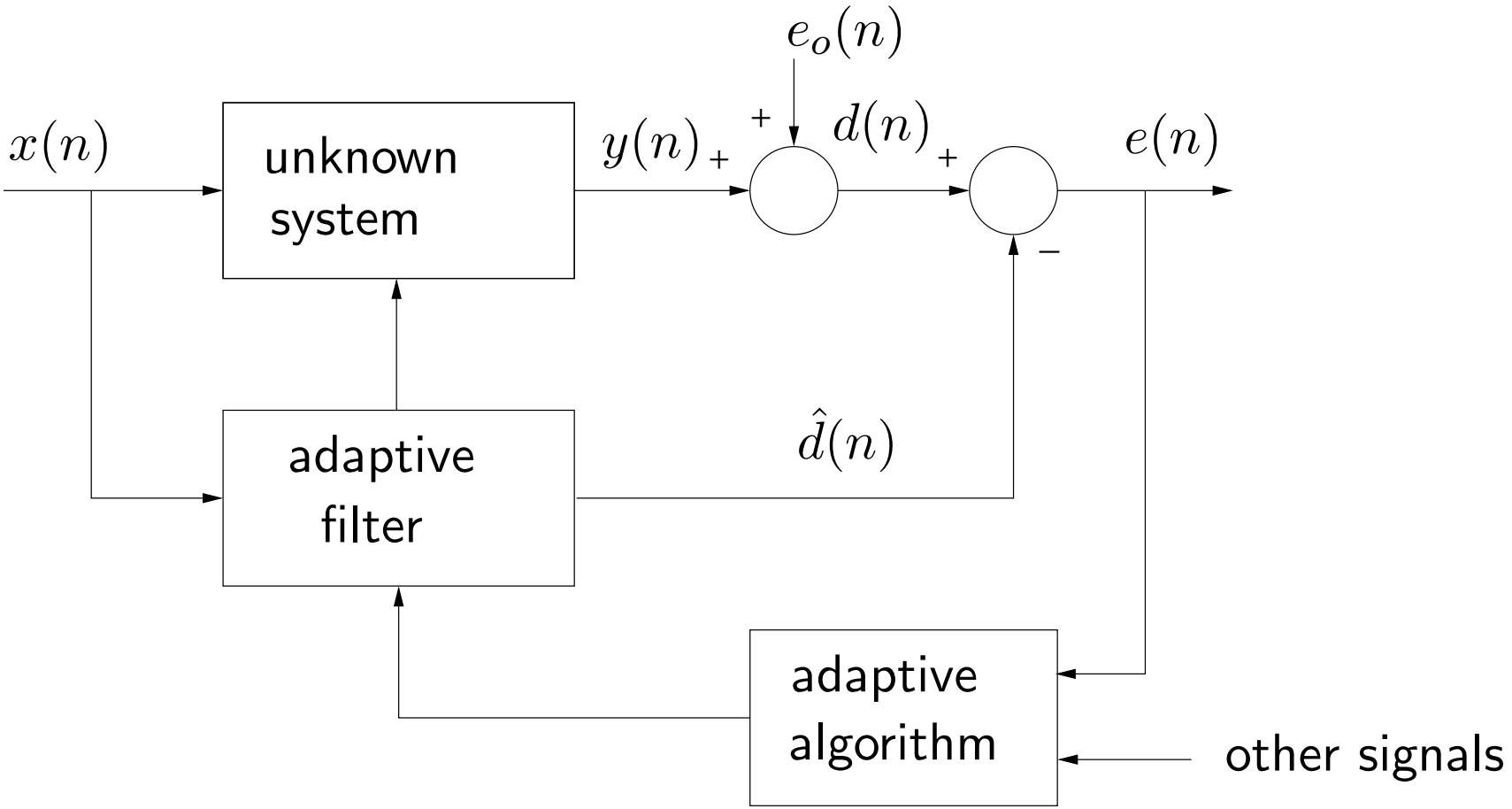
- They can operate satisfactorily in unknown and possibly time-varying environments without user intervention
- They improve their performance during operation by learning statistical characteristics from current signal observations
- They can track variations in the *signal operating environment* (SOE)

Adaptive Filtering Applications

Basic Classes of Adaptive Filtering Applications

- System Identification
- Inverse System Modeling
- Signal Prediction
- Interference Cancelation

System Identification



Applications – System Identification

Channel Estimation

- Communications systems
- Objective: model the channel to design distortion compensation
- $x(n)$: training sequence

Plant Identification

- Control systems
- Objective: model the plant to design a compensator
- $x(n)$: training sequence

Echo Cancellation

- Telephone systems and VoIP
- Echo caused by network impedance mismatches or acoustic environment
- Objective: model the echo path impulse response
- $x(n)$: transmitted signal
- $d(n)$: echo + noise

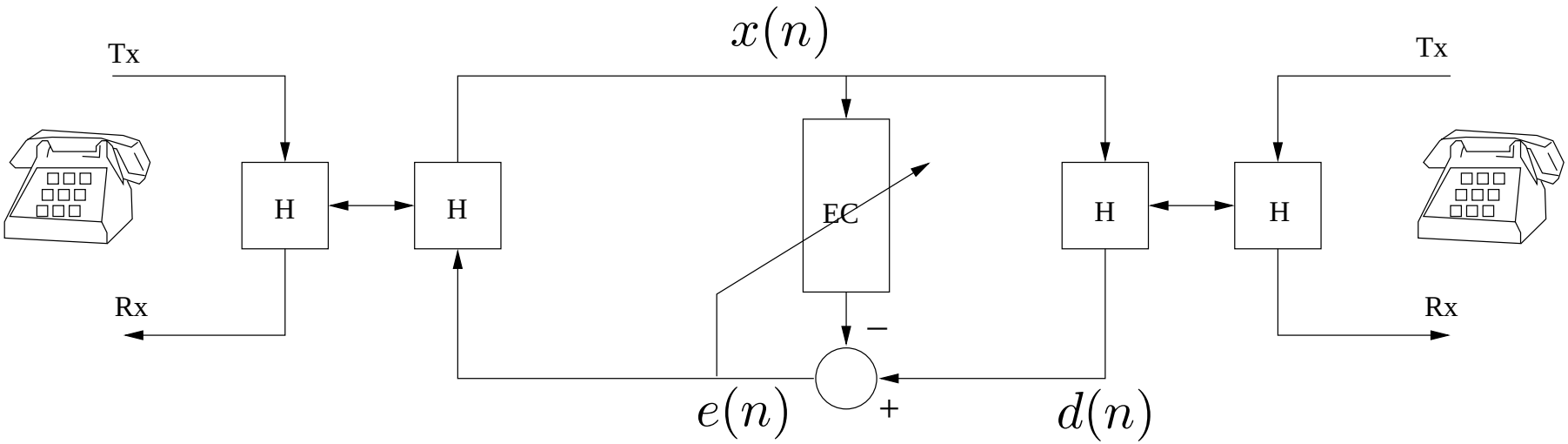
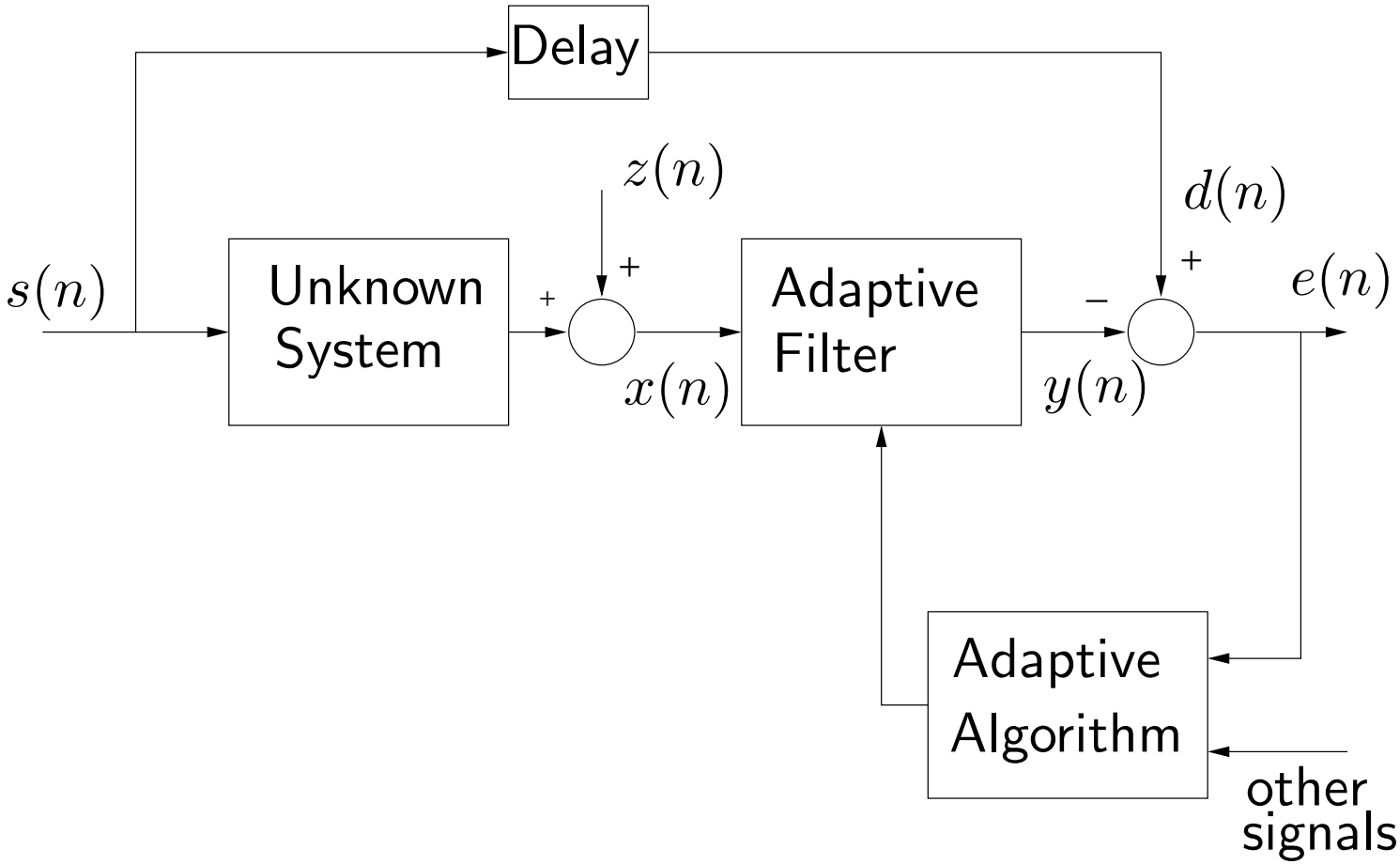


Figure: Network Echo Cancellation

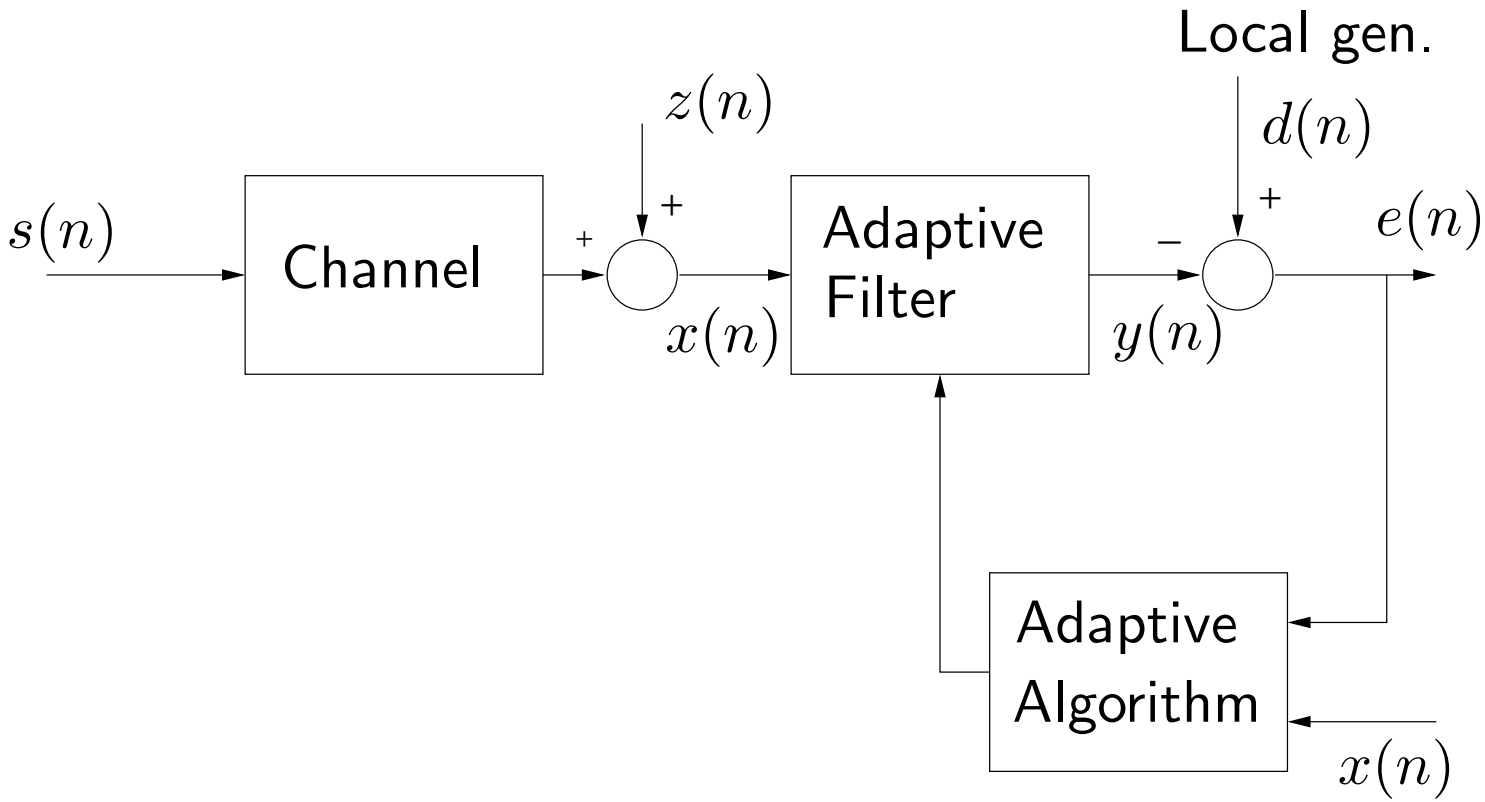
Inverse System Modeling

- Adaptive filter attempts to estimate unknown system's inverse
- Adaptive filter input usually corrupted by noise
- Desired response $d(n)$ may not be available



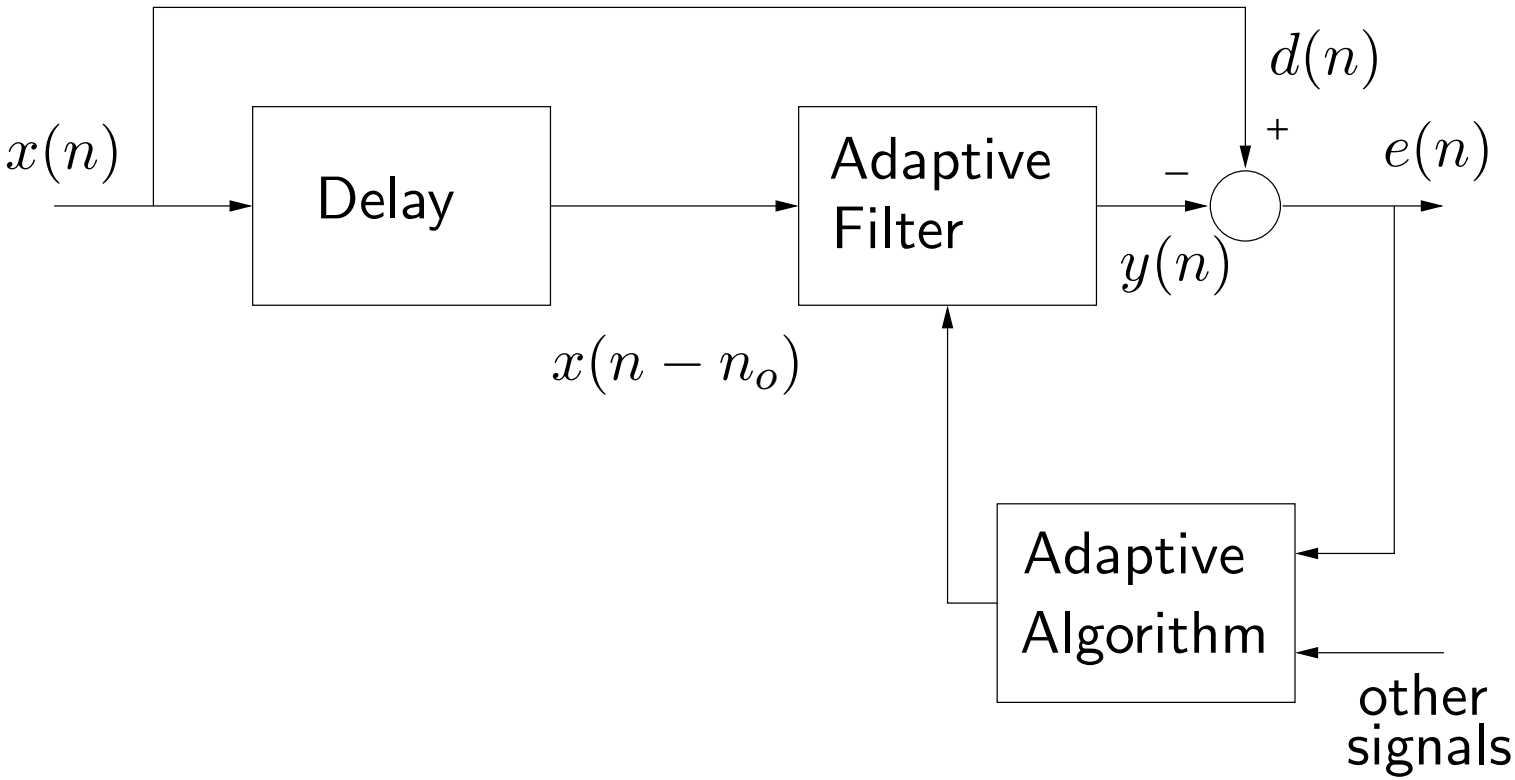
Applications – Inverse System Modeling

Channel Equalization



- Objective: reduce intersymbol interference
- Initially – training sequence in $d(n)$
- After training: $d(n)$ generated from previous decisions

Signal Prediction

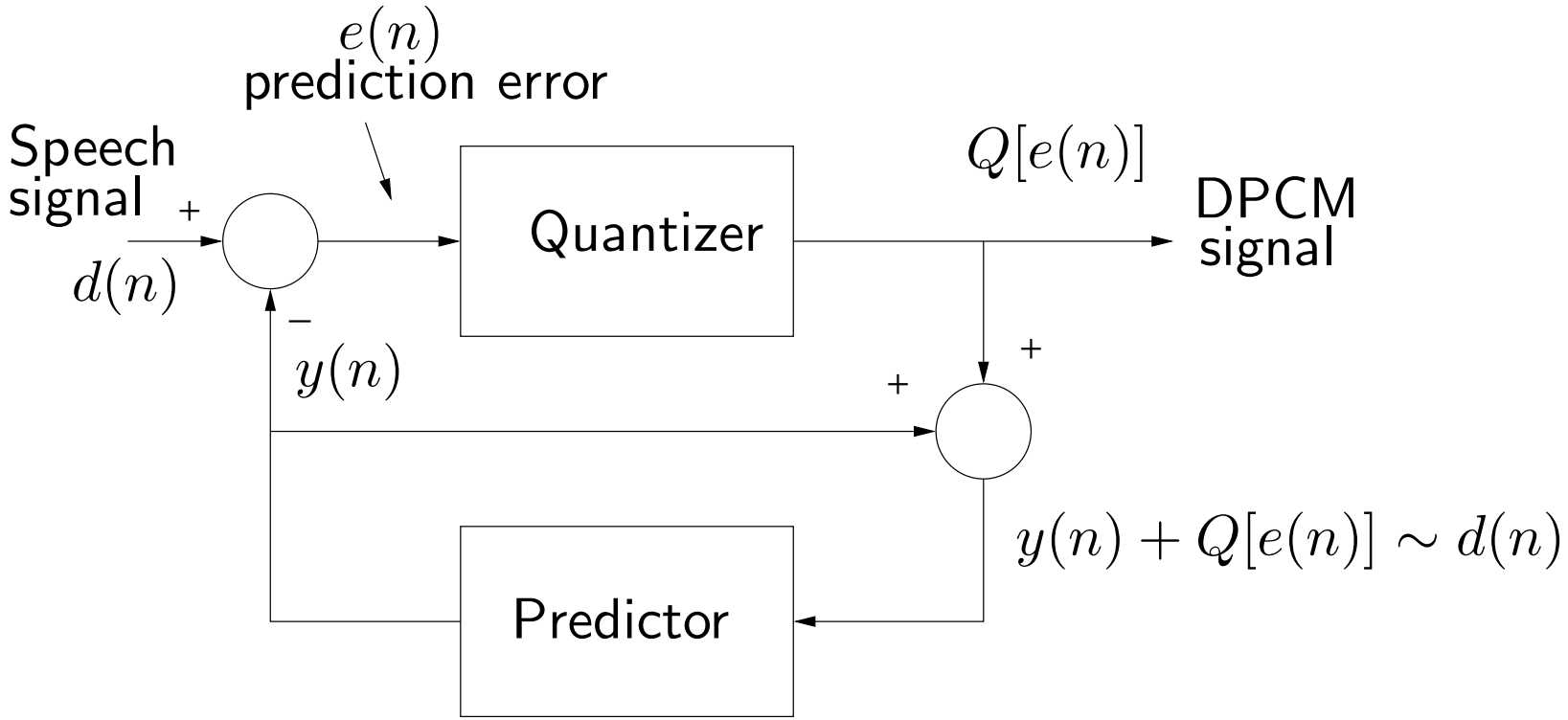


- most widely used case – forward prediction
- signal $x(n)$ to be predicted from samples $\{x(n - n_o), x(n - n_o - 1), \ldots, x(n - n_o - L)\}$

Application – Signal Prediction

DPCM Speech Quantizer - Linear Predictive Coding

- Objective: Reduce speech transmission bandwidth
- Signal transmitted all the time: quantization error
- Predictor coefficients are transmitted at low rate

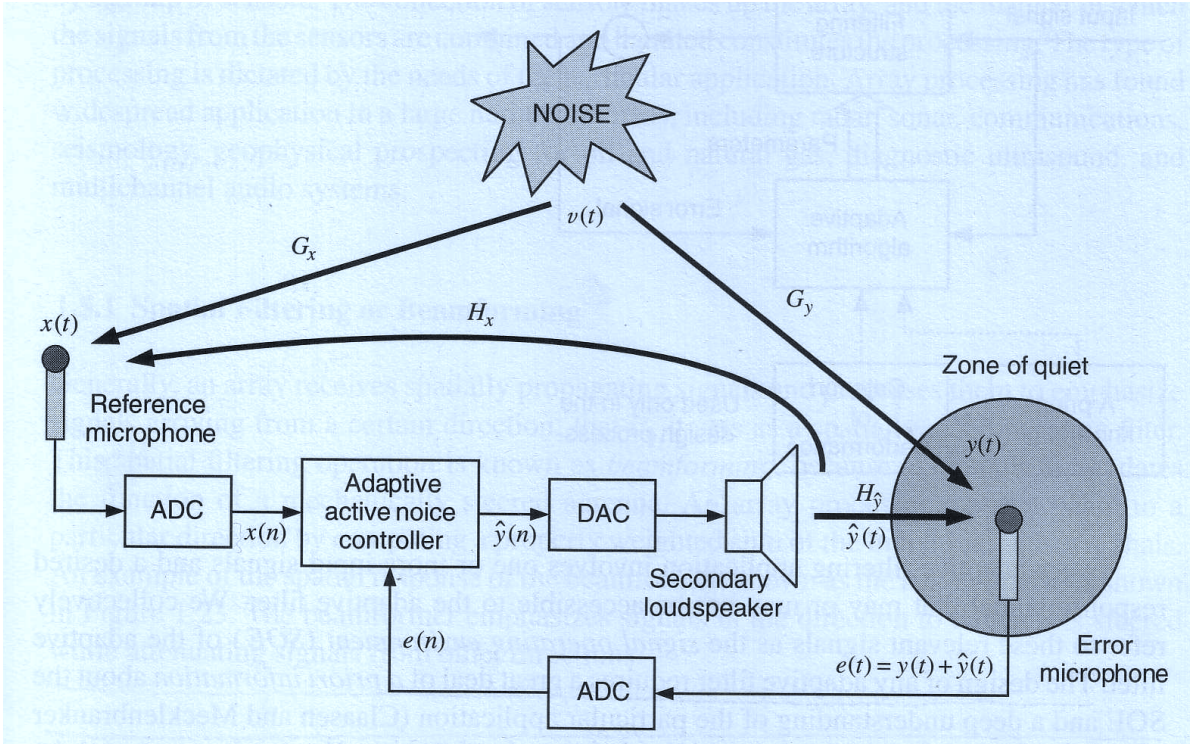


Interference Cancelation

- One or more sensor signals are used to remove interference and noise
- Reference signals correlated with the inteference should also be available
- Applications:
  - array processing for radar and communications
  - biomedical sensing systems
  - active noise control systems

Application – Interference Cancelation

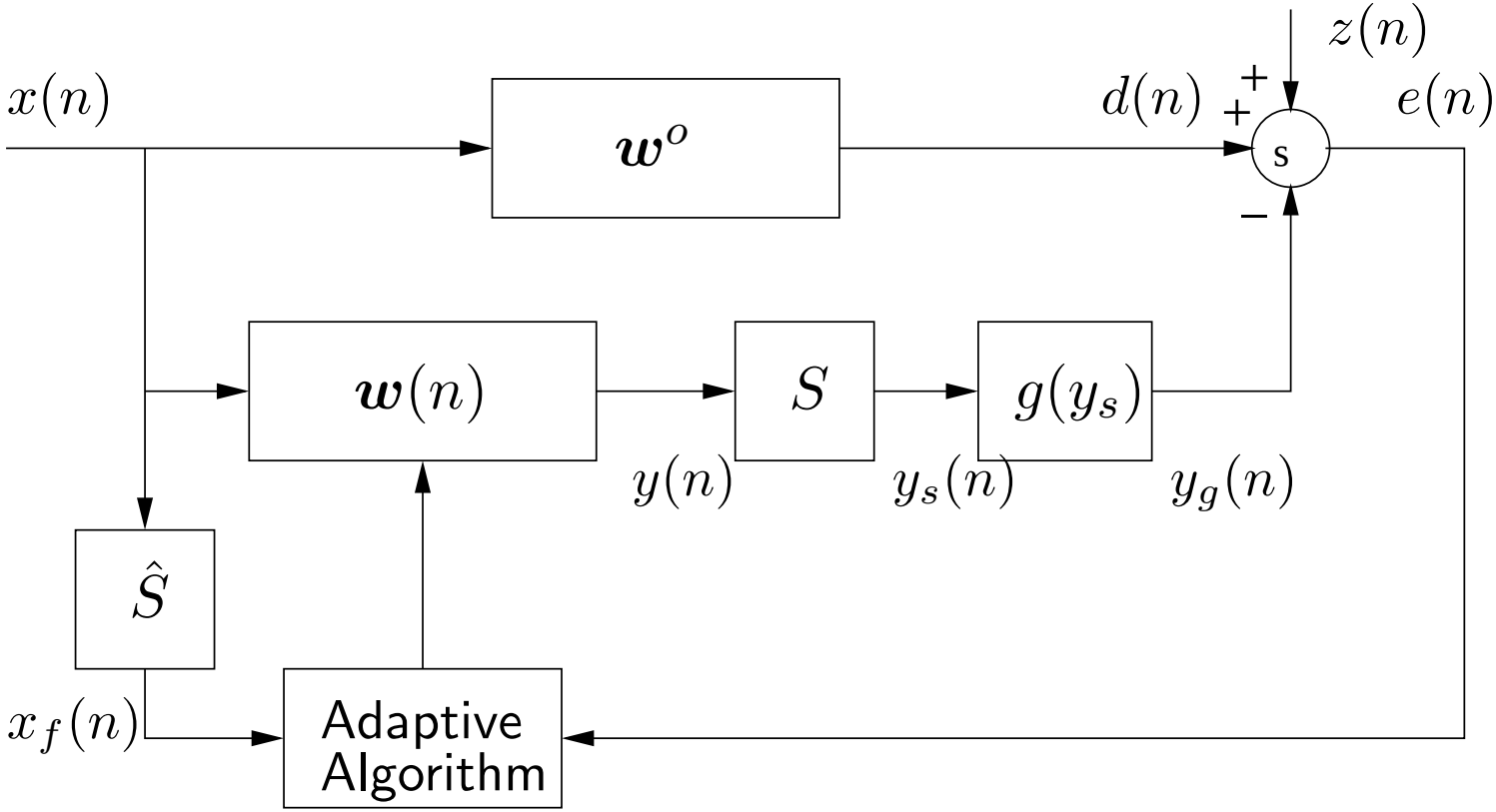
Active Noise Control



Ref: D.G. Manolakis, V.K. Ingle and S.M. Kogon, *Statistical and Adaptive Signal Processing, 2000.*

- Cancelation of acoustic noise using destructive interference
- Secondary system between the adaptive filter and the cancelation point is unavoidable
- Cancelation is performed in the acoustic environment

Active Noise Control – Block Diagram



Adaptive Filtering Principles

Adaptive Filter Features

Adaptive filters are composed of three basic modules:

- Filtering strucure
  - Determines the output of the filter given its input samples
  - Its weights are periodically adjusted by the adaptive algorithm
  - Can be linear or nonlinear, depending on the application
  - Linear filters can be FIR or IIR
- Performance criterion
  - Defined according to application and mathematical tractability
  - Is used to derive the adaptive algorithm
  - Its value at each iteration affects the adaptive weight updates
- Adaptive algorithm
  - Uses the performance criterion value and the current signals
  - Modifies the adaptive weights to improve performance
  - Its form and complexity are function of the structure and of the performance criterion

Signal Operating Environment (SOE)

Comprises all informations regarding the properties of the signals and systems

- Input signals
- Desired signal
- Unknown systems

If the SOE is nonstationary

- Aquisition or convergence mode: from start until close to best performance
- Tracking mode: readjustment following SOE's time variations

Adaptation can be

- Supervised – desired signal is available ➪ $e(n)$ can be evaluated
- Unsupervised – desired signal is unavailable

Performance Evaluation

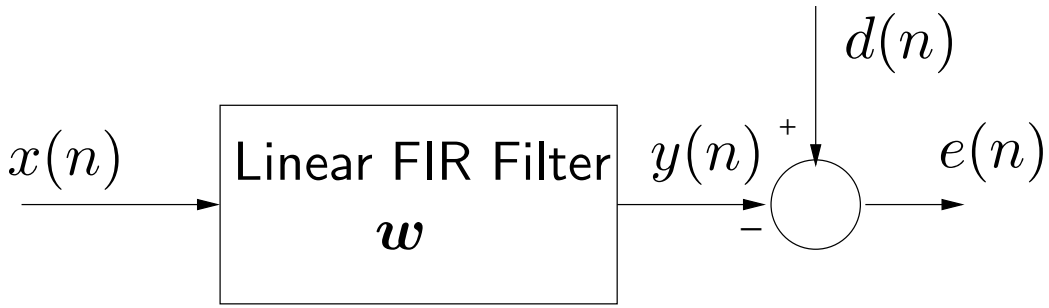
- Convergence rate
- Misadjustment
- Tracking
- Robustness (disturbances and numerical)
- Computational requirements (operations and memory)
- Structure
  - facility of implementation
  - performance surface
  - stability

Optimum versus Adaptive Filters in Linear Estimation

Conditions for this study

- Stationary SOE
- Filter structure is transversal FIR
- All signals are real valued
- Performance criterion: Mean-square error $E[e^2(n)]$

The Linear Estimation Problem



$$J_{ms} = E[e^2(n)]$$

The Linear Estimation Problem

$$\boldsymbol{x}(n) = [x(n), x(n-1), \cdots, x(n-N+1)]^T$$

$$y(n) = \boldsymbol{x}^T(n)\boldsymbol{w}$$

$$e(n) = d(n) - y(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}$$

$$J_{ms} = E[e^2(n)] = \sigma_d^2 - 2\boldsymbol{p}^T\boldsymbol{w} + \boldsymbol{w}^T\boldsymbol{R}_{xx}\boldsymbol{w}$$

where

$$\boldsymbol{p} = E[\boldsymbol{x}(n)d(n)]; \quad \boldsymbol{R}_{xx} = E[\boldsymbol{x}(n)\boldsymbol{x}^T(n)]$$

Normal Equations

$$\boldsymbol{R}_{xx}\boldsymbol{w}^o = \boldsymbol{p} \quad \Rightarrow \quad \boldsymbol{w}^o = \boldsymbol{R}_{xx}^{-1}\boldsymbol{p} \quad \text{for } \boldsymbol{R}_{xx} > 0$$

$$J_{ms_{min}} = \sigma_d^2 - \boldsymbol{p}^T\boldsymbol{R}_{xx}^{-1}\boldsymbol{p}$$

What if $d(n)$ is nonstationary?

$$\boldsymbol{x}(n) = [x(n), x(n-1), \cdots, x(n-N+1)]^T$$

$$y(n) = \boldsymbol{x}^T(n)\boldsymbol{w}(n)$$

$$e(n) = d(n) - y(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)$$

$$J_{ms}(n) = E[e^2(n)] = \sigma_d^2(n) - 2\boldsymbol{p}(n)^T\boldsymbol{w}(n) + \boldsymbol{w}^T(n)\boldsymbol{R}_{xx}\boldsymbol{w}(n)$$

where

$$\boldsymbol{p}(n) = E[\boldsymbol{x}(n)d(n)]; \quad \boldsymbol{R}_{xx} = E[\boldsymbol{x}(n)\boldsymbol{x}^T(n)]$$

Normal Equations

$$\boldsymbol{R}_{xx}\boldsymbol{w}^o(n) = \boldsymbol{p}(n) \quad \Rightarrow \quad \boldsymbol{w}^o(n) = \boldsymbol{R}_{xx}^{-1}\boldsymbol{p}(n) \quad \text{for } \boldsymbol{R}_{xx} > 0$$

$$J_{ms_{min}}(n) = \sigma_d^2(n) - \boldsymbol{p}^T(n)\boldsymbol{R}_{xx}^{-1}\boldsymbol{p}(n)$$

Optimum Filters versus Adaptive Filters

Optimum Filters

- Compute $\boldsymbol{p}(n) = E[\boldsymbol{x}(n)d(n)]$
- Solve $\boldsymbol{R}_{xx}\boldsymbol{w}^o = \boldsymbol{p}(n)$
- Filter with $\boldsymbol{w}^o(n)$ ➪ $y(n) = \boldsymbol{x}^T(n)\boldsymbol{w}^o(n)$

Nonstationary SOE: Optimum filter determined for each value of n

Adaptive Filters

- Filtering: $y(n) = \boldsymbol{x}^T(n)\boldsymbol{w}(n)$
- Evaluate error: $e(n) = d(n) - y(n)$
- Adaptive algorithm:

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \Delta\boldsymbol{w}[\boldsymbol{x}(n), e(n)]$$

$\Delta\boldsymbol{w}(n)$ is chosen so that $\boldsymbol{w}(n)$ is close to $\boldsymbol{w}^o(n)$ for n large

Characteristics of Adaptive Filters

- Search for the optimum solution on the performance surface
- Follow principles of optimization techniques
- Implement a recursive optimization solution
- Convergence speed may depend on initialization
- Have stability regions
- Steady-state solution fluctuates about the optimum
- Can track time varying SOEs better than optimum filters
- Performance depends on the performance surface

Iterative Solutions for the Optimum Filtering Problem

Performance (Cost) Functions

- Mean-square error ➪ $E[e^2(n)]$ (Most popular)
  Adaptive algorithms: Least-Mean Square (LMS), Normalized LMS (NLMS), Affine Projection (AP), Recursive Least Squares (RLS), etc.
- Regularized MSE

$$J_{rms} = E[e^2(n)] + \alpha \|\boldsymbol{w}(n)\|^2$$

  Adaptive algorithm: leaky least-mean square (leaky LMS)
- ℓ_1 norm criterion

$$J_{\ell_1} = E[|e(n)|]$$

  Adaptive algorithm: Sign-Error

Performance (Cost) Functions – continued

- Least-mean fourth (LMF) criterion

$$J_{LMF} = E[e^4(n)]$$

  Adaptive algorithm: Least-Mean Fourth (LMF)

- Least-mean-mixed-norm (LMMN) criterion

$$J_{LMMN} = E[\alpha e^2(n) + \frac{1}{2}(1-\alpha)e^4(n)]$$

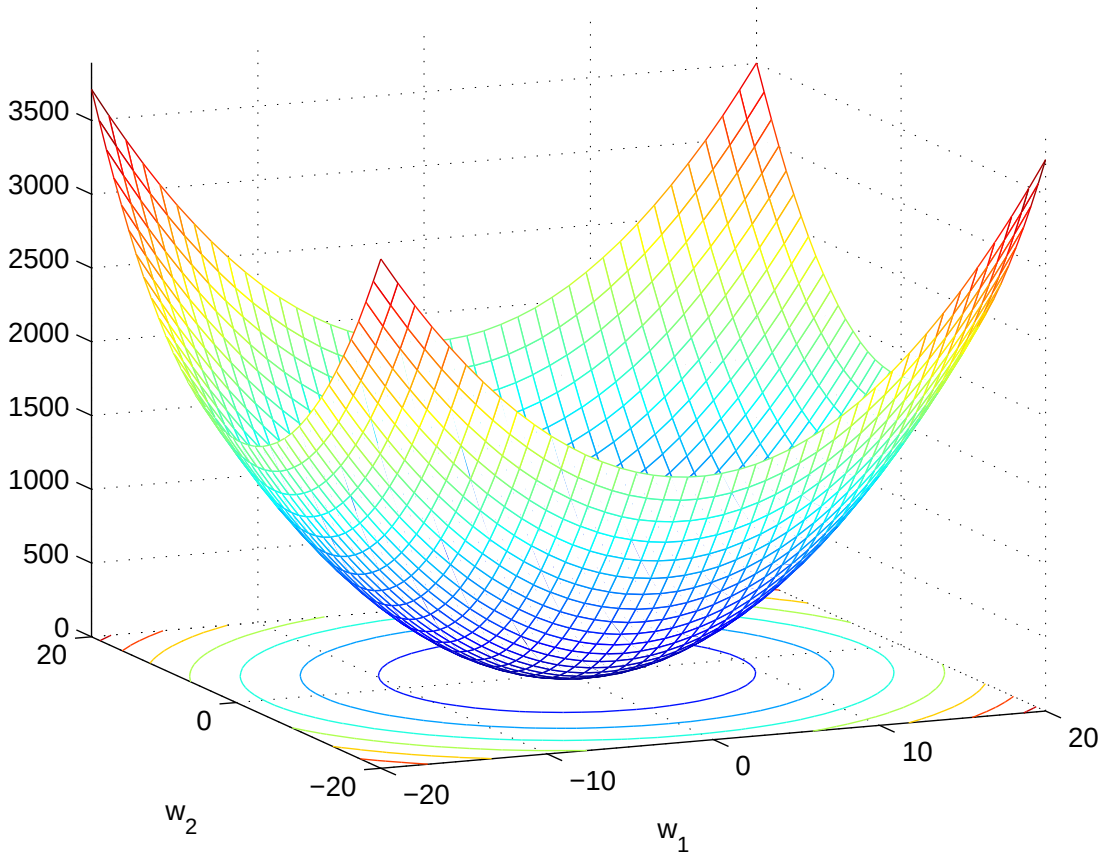
  Adaptive algorithm: Least-Mean-Mixed-Norm (LMMN)

- Constant-modulus criterion

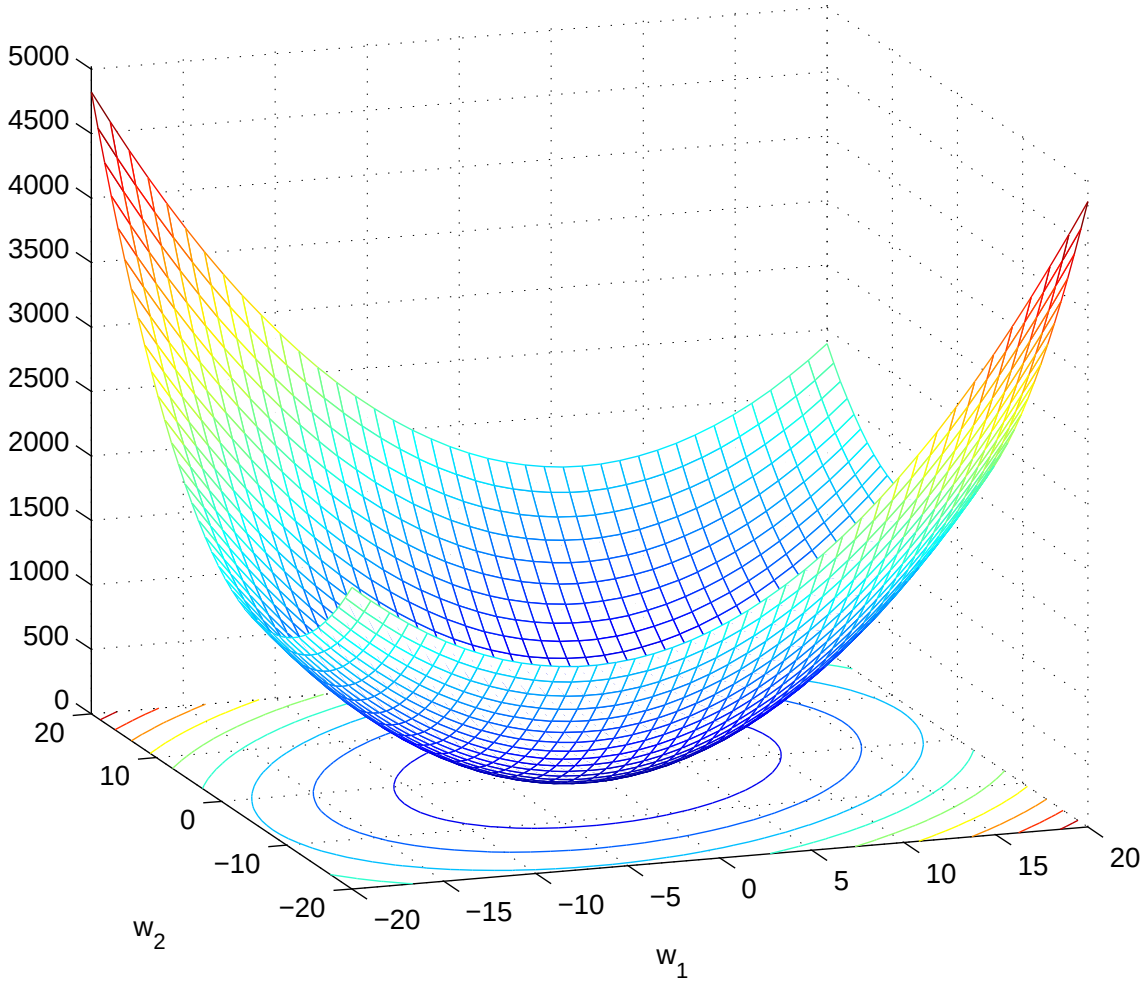
$$J_{CM} = E[\left(\gamma - |\boldsymbol{x}^T(n)\boldsymbol{w}(n)|^2\right)^2]$$

  Adaptive algorithm: Constant-Modulus (CM)

MSE Performance Surface – Small Input Correlation



MSE Performance Surface – Large Input Correlation



The Steepest Descent Algorithm – Stationary SOE

Cost Function

$$J_{ms}(n) = E[e^2(n)] = \sigma_d^2 - 2\boldsymbol{p}^T\boldsymbol{w}(n) + \boldsymbol{w}^T(n)\boldsymbol{R}_{xx}\boldsymbol{w}(n)$$

Weight Update Equation

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu\boldsymbol{c}(n)$$

μ: step-size
$\boldsymbol{c}(n)$: correction term (determines direction of $\Delta\boldsymbol{w}(n)$)
Steepest descent adjustment:

$$\boldsymbol{c}(n) = -\nabla J_{ms}(n) \quad \Rightarrow \quad J_{ms}(n+1) \leq J_{ms}(n)$$

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\boldsymbol{p} - \boldsymbol{R}_{xx}\boldsymbol{w}(n)]$$

Weight Update Equation About the Optimum Weights

Weight Error Update Equation

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\boldsymbol{p} - \boldsymbol{R}_{xx}\boldsymbol{w}(n)]$$

Using $\boldsymbol{p} = \boldsymbol{R}_{xx}\boldsymbol{w}^o$

$$\boldsymbol{w}(n+1) = (\boldsymbol{I} - \mu\boldsymbol{R}_{xx})\boldsymbol{w}(n) + \mu\boldsymbol{R}_{xx}\boldsymbol{w}^o$$

Weight error vector: $\boldsymbol{v}(n) = \boldsymbol{w}(n) - \boldsymbol{w}^o$

$$\boxed{\boldsymbol{v}(n+1) = (\boldsymbol{I} - \mu\boldsymbol{R}_{xx})\boldsymbol{v}(n)}$$

- Matrix $\boldsymbol{I} - \mu\boldsymbol{R}_{xx}$ must be *stable* for convergence ($|\lambda_i| < 1$)
- Assuming convergence, $\lim_{n\to\infty} \boldsymbol{v}(n) = \boldsymbol{0}$

Convergence Conditions

$$\boldsymbol{v}(n+1) = (\boldsymbol{I} - \mu \boldsymbol{R}_{xx})\boldsymbol{v}(n); \quad \boldsymbol{R}_{xx} \quad \text{positive definite}$$

Eigen-decomposition of $\boldsymbol{R}_{xx}$

$$\boldsymbol{R}_{xx} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$$

$$\boldsymbol{v}(n+1) = (\boldsymbol{I} - \mu \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T)\boldsymbol{v}(n)$$

$$\boldsymbol{Q}^T\boldsymbol{v}(n+1) = \boldsymbol{Q}^T\boldsymbol{v}(n) - \mu \boldsymbol{\Lambda}\boldsymbol{Q}^T\boldsymbol{v}(n)$$

Defining $\tilde{\boldsymbol{v}}(n+1) = \boldsymbol{Q}^T\boldsymbol{v}(n+1)$

$$\boxed{\tilde{\boldsymbol{v}}(n+1) = (\boldsymbol{I} - \mu \boldsymbol{\Lambda})\tilde{\boldsymbol{v}}(n)}$$

Convergence Properties

$$\tilde{\boldsymbol{v}}(n+1) = (\boldsymbol{I} - \mu\boldsymbol{\Lambda})\tilde{\boldsymbol{v}}(n)$$

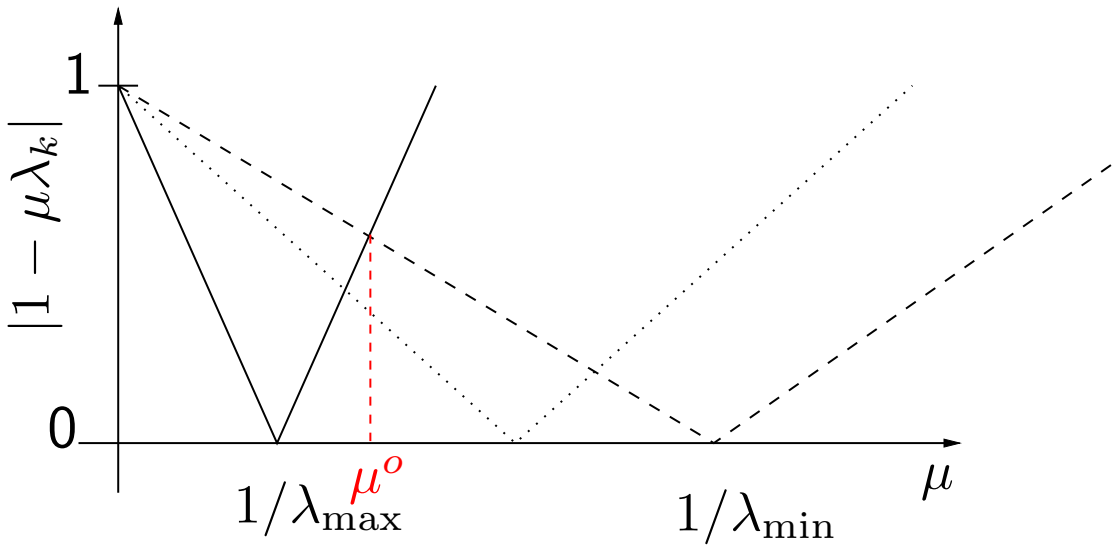
- $\tilde{v}_k(n+1) = (1-\mu\lambda_k)\tilde{v}_k(n),\ k = 1, \ldots, N$
- $\tilde{v}_k(n) = (1-\mu\lambda_k)^n \tilde{v}_k(0)$
- Convergence modes
  - monotonic if $\quad 0 < 1 - \mu\lambda_k < 1$
  - oscillatory if $\quad -1 < 1 - \mu\lambda_k < 0$
- Convergence if $|1 - \mu\lambda_k| < 1 \Rightarrow \boxed{0 < \mu < \frac{2}{\lambda_{\max}}}$

Optimal Step-Size

$$\tilde{v}_k(n) = (1 - \mu\lambda_k)^n \tilde{v}_k(0); \text{convergence modes: } 1 - \mu\lambda_k$$

- $\max |1 - \mu\lambda_k|$: slowest mode
- $\min |1 - \mu\lambda_k|$: fastest mode

Optimal Step-Size – continued



$$\min_{\mu} \max\{|1-\mu\lambda_k|\}$$

$$1-\mu^o\lambda_{\min} = -(1-\mu^o\lambda_{\max})$$

$$\boxed{\mu^o = \frac{2}{\lambda_{\max}+\lambda_{\min}}}$$

Optimal slowest modes: $\pm\frac{\rho-1}{\rho+1}$; $\rho = \frac{\lambda_{\max}}{\lambda_{\min}}$

The Learning Curve – $J_{ms}(n)$

$$J_{ms}(n) = J_{ms_{\min}} + \overbrace{\tilde{\boldsymbol{v}}^T(n)\boldsymbol{\Lambda}\tilde{\boldsymbol{v}}(n)}^{\text{Excess MSE}} = J_{ms_{\min}} + \sum_{k=1}^{N} \lambda_k \tilde{v}_k^2(n)$$

Since $\tilde{v}_k(n) = (1-\mu\lambda_k)^n \tilde{v}_k(0)$,

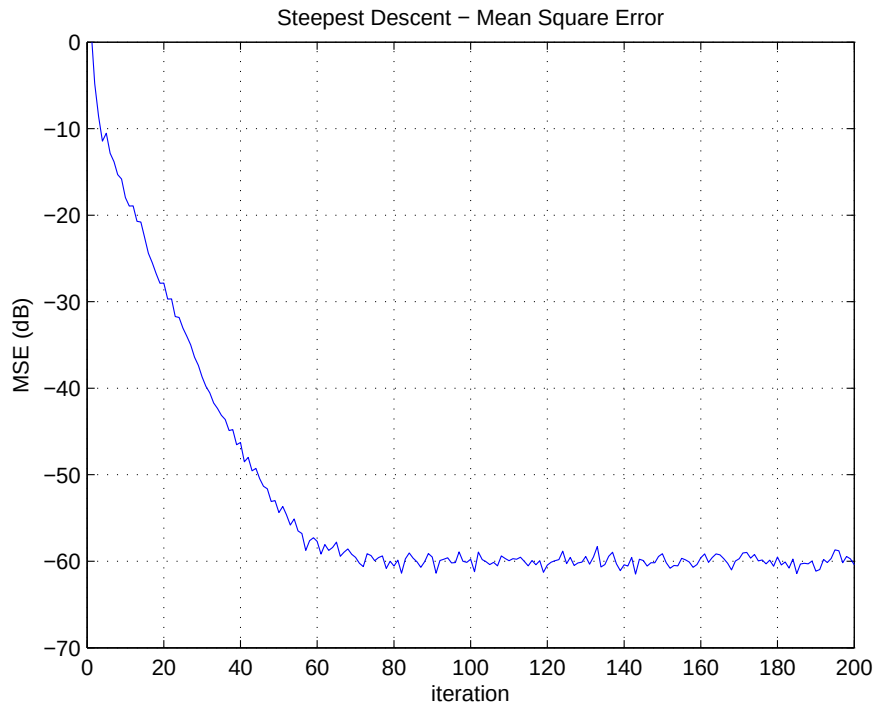
$$J_{ms}(n) = J_{ms_{\min}} + \sum_{k=1}^{N} \lambda_k (1-\mu\lambda_k)^{2n} \tilde{v}_k^2(0)$$

- $\lambda_k(1-\mu\lambda_k)^2 > 0$ ➪ monotonic convergence
- Stability limit is again $0 < \mu < \frac{2}{\lambda_{\max}}$
- $J_{ms}(n)$ converges faster than $\boldsymbol{w}(n)$
- Algorithm converges faster as $\rho = \lambda_{\max}/\lambda_{\min} \to 1$

Simulation Results

$$x(n) = \alpha\, x(n-1) + v(n)$$

Input: White noise ($\rho = 1$)



Input: AR(1), $\alpha = 0.7$ ($\rho = 5.7$)

- Linear system identification - FIR with 20 coefficients
- Step-size $\mu = 0.3$
- Noise power $\sigma_v^2 = 10^{-6}$

The Newton Algorithm

- Steepest descent: linear approx. of J_{ms} about the operating point
- Newton's method: Quadratic approximation of J_{ms}

Expanding $J_{ms}(\boldsymbol{w})$ in Taylor's series about $\boldsymbol{w}(n)$,

$$\begin{aligned} J_{ms}(\boldsymbol{w}) \sim & J_{ms}[\boldsymbol{w}(n)] + \nabla^T J_{ms}[\boldsymbol{w} - \boldsymbol{w}(n)] \\ & + \frac{1}{2}[\boldsymbol{w} - \boldsymbol{w}(n)]^T \boldsymbol{H}(n)[\boldsymbol{w} - \boldsymbol{w}(n)] \end{aligned}$$

Diferentiating w.r.t. $\boldsymbol{w}$ and equating to zero at $\boldsymbol{w} = \boldsymbol{w}(n+1)$,

$$\nabla J_{ms}[\boldsymbol{w}(n+1)] = \nabla J_{ms}[\boldsymbol{w}(n)] + \boldsymbol{H}[\boldsymbol{w}(n)][\boldsymbol{w}(n+1) - \boldsymbol{w}(n)] = \boldsymbol{0}$$

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) - \boldsymbol{H}^{-1}[\boldsymbol{w}(n)] \nabla J_{ms}[\boldsymbol{w}(n)]}$$

The Newton Algorithm – continued

- $\nabla J_{ms}[\boldsymbol{w}(n)] = -2\boldsymbol{p} + 2\boldsymbol{R}_{xx}\boldsymbol{w}(n)$
- $\boldsymbol{H}(\boldsymbol{w}(n)) = 2\boldsymbol{R}_{xx}$

Thus, adding a step-size control,

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) - \mu\boldsymbol{R}_{xx}^{-1}[-\boldsymbol{p} + \boldsymbol{R}_{xx}\boldsymbol{w}(n)]}$$

- Quadratic surface ➪ conv. in *one* iteration for $\mu = 1$ ☺
- Requires the determination of $\boldsymbol{R}_{xx}^{-1}$ ☹
- Can be used to derive simpler adaptive algorithms
- When $\boldsymbol{H}(n)$ is close to singular ➪ regularization

$$\tilde{\boldsymbol{H}}(n) = 2\boldsymbol{R}_{xx} + 2\varepsilon\boldsymbol{I}$$

Basic Adaptive Algorithms

Least Mean Squares (LMS) Algorithm

- Can be interpreted in different ways
- Each interpretation helps understanding the algorithm behavior
- Some of these interpretations are related to the steepest descent algorithm

LMS as a Stochastic Gradient Algorithm

- Suppose we use the estimate $J_{ms}(n) = E[e^2(n)] \simeq e^2(n)$
- The estimated gradient vector becomes

$$\hat{\nabla} J_{ms}(n) = \frac{\partial e^2(n)}{\partial \boldsymbol{w}(n)} = 2e(n)\frac{\partial e(n)}{\partial \boldsymbol{w}(n)}$$

- Since $e(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)$,

$$\hat{\nabla} J_{ms}(n) = -2e(n)\boldsymbol{x}(n) \qquad (\textit{stochastic gradient})$$

and, using the steepest descent weight update equation,

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu e(n)\boldsymbol{x}(n)} \qquad \text{(LMS weight update)}$$

LMS as a Stochastic Estimation Algorithm

- $\nabla J_{ms}(n) = -2\boldsymbol{p} + 2\boldsymbol{R}_{xx}\boldsymbol{w}(n)$
- Stochastic estimators

$$\hat{\boldsymbol{p}} = d(n)\boldsymbol{x}(n) \qquad \hat{\boldsymbol{R}}_{xx} = \boldsymbol{x}(n)\boldsymbol{x}^T(n)$$

Then,

$$\hat{\nabla} J_{ms}(n) = -2d(n)\boldsymbol{x}(n) + 2\boldsymbol{x}(n)\boldsymbol{x}^T(n)\boldsymbol{w}(n)$$

- Using $\hat{\nabla} J_{ms}(n)$ is the steepest descent weight update,

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu e(n)\boldsymbol{x}(n)}$$

LMS – A Solution to a Local Optimization

- Error expressions
  $e(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)$ (*a priori* error)
  $\epsilon(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n+1)$ (*a posteriori* error)
- We want to maximize $|\epsilon(n) - e(n)|$ with $|\epsilon(n)| < |e(n)|$

$$\epsilon(n) - e(n) = -\boldsymbol{x}^T(n)\Delta\boldsymbol{w}(n)$$
$$\Delta\boldsymbol{w}(n) = \boldsymbol{w}(n+1) - \boldsymbol{w}(n)$$

- Expressing $\Delta\boldsymbol{w}(n)$ as $\Delta\boldsymbol{w}(n) = \Delta\tilde{\boldsymbol{w}}(n)e(n)$

$$\epsilon(n) - e(n) = -\boldsymbol{x}^T(n)\Delta\tilde{\boldsymbol{w}}(n)e(n)$$

- For max $|\epsilon(n) - e(n)|$ ➪ $\Delta\tilde{\boldsymbol{w}}(n)$ in the direction of $\boldsymbol{x}(n)$

- ➪ $\Delta\tilde{\boldsymbol{w}}(n) = \mu\boldsymbol{x}(n)$ and

$$\Delta\boldsymbol{w}(n) = \mu\boldsymbol{x}(n)e(n)$$

and

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu e(n)\boldsymbol{x}(n)}$$

- As

$$\epsilon(n) - e(n) = -\boldsymbol{x}^T(n)\Delta\boldsymbol{w}(n) = -\mu\boldsymbol{x}^T(n)\boldsymbol{x}(n)e(n)$$

$|\epsilon(n)| < |e(n)|$ requires $|1 - \mu\boldsymbol{x}^T(n)\boldsymbol{x}(n)| < 1$, or

$$\boxed{0 < \mu < \frac{2}{\|\boldsymbol{x}(n)\|^2}} \quad \textit{(stability region)}$$

Observations - LMS Algorithm

- LMS is a noisy approximation of the steepest descent algorithm
- The gradient estimate is unbiased
- The errors in the gradient estimate lead to $J_{ms_{ex}}(\infty) \neq 0$
- Vector $\boldsymbol{w}(n)$ is now random
- Steepest descent properties are no longer guaranteed
  ➪ *LMS analysis required*
- The instantaneous estimates allow tracking without redesign

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The Normalized LMS Algorithm – NLMS

- Most Employed adaptive algorithm in real-time applications
- Like LMS has different interpretations (even more)
- Alleviates a drawback of the LMS algorithm

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu e(n)\boldsymbol{x}(n)$$

  - If amplitude of $\boldsymbol{x}(n)$ is large ➪ Gradient noise amplification
  - Sub-optimal performance when σ_x^2 varies with time (for instance, speech)

NLMS – A Solution to a Local Optimization Problem

- Error expressions
  $e(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)$ (*a priori* error)
  $\epsilon(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n+1)$ (*a posteriori* error)
- We want to maximize $|\epsilon(n) - e(n)|$ with $|\epsilon(n)| < |e(n)|$

$$\epsilon(n) - e(n) = -\boldsymbol{x}^T(n)\Delta\boldsymbol{w}(n) \qquad (A)$$
$$\Delta\boldsymbol{w}(n) = \boldsymbol{w}(n+1) - \boldsymbol{w}(n)$$

- For $|\epsilon(n)| < |e(n)|$ we impose the restriction

$$\epsilon(n) = (1-\mu)e(n), \quad |1-\mu| < 1$$

$$\Rightarrow \quad \epsilon(n) - e(n) = -\mu e(n) \qquad (B)$$

- For max $|\epsilon(n) - e(n)| \Rightarrow \Delta\boldsymbol{w}(n)$ in the direction of $\boldsymbol{x}(n)$

$$\Rightarrow \Delta\boldsymbol{w}(n) = k\boldsymbol{x}(n) \qquad (C)$$

- Using (A), (B) and (C),

$$k = \mu \frac{e(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n)}$$

and

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu \frac{e(n)\boldsymbol{x}(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n)}} \quad \text{(NLMS weight update)}$$

NLMS – Solution to a Constrained Optimization Problem

- Error sequence

$$y(n) = \boldsymbol{x}^T(n)\boldsymbol{w}(n) \quad \text{(estimate of } d(n)\text{)}$$
$$e(n) = d(n) - y(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n) \quad \text{(estimation error)}$$

- Optimization (principle of minimal disturbance)

$$\begin{aligned} \text{Minimize} \quad & \|\Delta\boldsymbol{w}(n)\|^2 = \|\boldsymbol{w}(n+1) - \boldsymbol{w}(n)\|^2 \\ \text{subject to:} \quad & \boldsymbol{x}^T(n)\boldsymbol{w}(n+1) = d(n) \end{aligned}$$

This problem can be solved using the method of Lagrange multipliers

- Using Lagrange multipliers, we minimize

$$f[\boldsymbol{w}(n+1)] = \|\boldsymbol{w}(n+1) - \boldsymbol{w}(n)\|^2 + \lambda[d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n+1)]$$

- Differentiating w.r.t. $\boldsymbol{w}(n+1)$ and equating the result to zero,

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \frac{1}{2}\lambda\boldsymbol{x}(n) \qquad (*)$$

Using this result in $\boldsymbol{x}^T(n)\boldsymbol{w}(n+1) = d(n)$ yields

$$\lambda = \frac{2e(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n)}$$

- Using this result in (*) yields

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu\frac{e(n)\boldsymbol{x}(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n)}}$$

NLMS as an Orthogonalization Process

- Conditions for the analysis
  - $e(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)$
  - $\boldsymbol{w}^o$ is the optimal solution (Wiener solution)
  - $d(n) = \boldsymbol{x}^T(n)\boldsymbol{w}^o$ (no noise)
  - $\boldsymbol{v}(n) = \boldsymbol{w}(n) - \boldsymbol{w}^o$ (weight error vector)
- Error signal

$$\begin{aligned} e(n) &= d(n) - \boldsymbol{x}^T(n)[\boldsymbol{v}(n) + \boldsymbol{w}^o] \\ &= \boldsymbol{x}^T(n)\boldsymbol{w}^o - \boldsymbol{x}^T(n)[\boldsymbol{v}(n) + \boldsymbol{w}^o] \\ &= -\boldsymbol{x}^T(n)\boldsymbol{v}(n) \end{aligned}$$

$$e(n) = -\boldsymbol{x}^T(n)\boldsymbol{v}(n)$$

- *Interpretation:* To minimize the error $\boldsymbol{v}(n)$ should be orthogonal to all input vectors
  Restriction: We have only one vector $\boldsymbol{x}(n)$
- *Iterative solution:*
  We can subtract from $\boldsymbol{v}(n)$ its component in the direction of $\boldsymbol{x}(n)$ at each iteration
- If there are N adaptive coefficients, $\boldsymbol{v}(n)$ could be reduced to zero after N orthogonal input vectors
- Iterative projection extraction
  ➪ Gram-Schmidt orthogonalization

- Recursive orthogonalization

$$\begin{aligned} n=0: &\quad \boldsymbol{v}(1) = \boldsymbol{v}(0) - \text{proj. of } \boldsymbol{v}(0) \text{ onto } \boldsymbol{x}(0) \\ n=1: &\quad \boldsymbol{v}(2) = \boldsymbol{v}(1) - \text{proj. of } \boldsymbol{v}(1) \text{ onto } \boldsymbol{x}(1) \\ \vdots: &\quad \vdots \\ n+1: &\quad \boldsymbol{v}(n+1) = \boldsymbol{v}(n) - \text{proj. of } \boldsymbol{v}(n) \text{ onto } \boldsymbol{x}(n) \end{aligned}$$

- Projection of $\boldsymbol{v}(n)$ onto $\boldsymbol{x}(n)$

$$\mathcal{P}_{\boldsymbol{x}(n)}[\boldsymbol{v}(n)] = \left\{\boldsymbol{x}(n)[\boldsymbol{x}^T(n)\boldsymbol{x}(n)]^{-1}\boldsymbol{x}^T(n)\right\}\boldsymbol{v}(n)$$

- Weight update equation

$$\begin{aligned} \boldsymbol{v}(n+1) &= \boldsymbol{v}(n) - \mu \overbrace{\underbrace{[\boldsymbol{x}^T(n)\boldsymbol{x}(n)]^{-1}}_{\text{scalar}}\boldsymbol{x}^T(n)\boldsymbol{v}(n)}^{-e(n)}\boldsymbol{x}(n) \\ &= \boldsymbol{v}(n) + \mu\frac{e(n)\boldsymbol{x}(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n)} \end{aligned}$$

NLMS has its owns problems!

- NLMS solves the LMS gradient error amplification problem, but ...
- What happens if $\|\boldsymbol{x}(n)\|^2$ gets too small?
- One needs to add some regularization

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu \frac{e(n)\boldsymbol{x}(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n) + \varepsilon} \quad (\varepsilon\text{−NLMS})$$

ε-NLMS – Stochastic Approximation of Regularized Newton

- Regularized Newton Algorithm

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon \boldsymbol{I} + \boldsymbol{R}_{xx}]^{-1}[\boldsymbol{p} - \boldsymbol{R}_{xx}\boldsymbol{w}(n)]$$

- Instantaneous estimates

$$\hat{\boldsymbol{p}} = \boldsymbol{x}(n)d(n)$$
$$\hat{\boldsymbol{R}}_{xx} = \boldsymbol{x}(n)\boldsymbol{x}^T(n)$$

- Using these estimates

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon \boldsymbol{I} + \boldsymbol{x}(n)\boldsymbol{x}^T(n)]^{-1}\boldsymbol{x}(n) \overbrace{[d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)]}^{e(n)}$$

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon \boldsymbol{I} + \boldsymbol{x}(n)\boldsymbol{x}^T(n)]^{-1}\boldsymbol{x}(n)e(n)$$

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon \boldsymbol{I} + \boldsymbol{x}(n)\boldsymbol{x}^T(n)]^{-1}\boldsymbol{x}(n)e(n)$$

- Inversion of $\varepsilon \boldsymbol{I} + \boldsymbol{x}(n)\boldsymbol{x}^T(n)$?
- Matrix Inversion Formula

$$[\boldsymbol{A} + \boldsymbol{BCD}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}[\boldsymbol{C}^{-1} + \boldsymbol{DA}^{-1}\boldsymbol{B}]^{-1}\boldsymbol{DA}^{-1}$$

- Thus,

$$[\varepsilon \boldsymbol{I} + \boldsymbol{x}(n)\boldsymbol{x}^T(n)]^{-1} = \varepsilon^{-1}\boldsymbol{I} - \frac{\varepsilon^{-2}}{1 + \varepsilon^{-1}\boldsymbol{x}^T(n)\boldsymbol{x}(n)}\boldsymbol{x}(n)\boldsymbol{x}^T(n)$$

- Post-multiplying both sides by $\boldsymbol{x}(n)$ and rearranging

$$[\varepsilon \boldsymbol{I} + \boldsymbol{x}(n)\boldsymbol{x}^T(n)]^{-1}\boldsymbol{x}(n) = \frac{\boldsymbol{x}(n)}{\varepsilon + \boldsymbol{x}^T(n)\boldsymbol{x}(n)}$$

and

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu\frac{e(n)\boldsymbol{x}(n)}{\boldsymbol{x}^T(n)\boldsymbol{x}(n) + \varepsilon}$$

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The Affine Projection Algorithm

- Consider the NLMS algorithm but using $\{\boldsymbol{x}(n), \boldsymbol{x}(n-1), \ldots, \boldsymbol{x}(n-P)\}$

$$\text{Minimize} \quad \|\Delta \boldsymbol{w}(n)\|^2 = \|\boldsymbol{w}(n+1) - \boldsymbol{w}(n)\|^2$$

$$\text{subject to:} \begin{cases} \boldsymbol{x}^T(n)\boldsymbol{w}(n+1) = d(n) \\ \boldsymbol{x}^T(n-1)\boldsymbol{w}(n+1) = d(n-1) \\ \quad \vdots \\ \boldsymbol{x}^T(n-P)\boldsymbol{w}(n+1) = d(n-P) \end{cases}$$

- Observation matrix

$$\boldsymbol{X}(n) = [\boldsymbol{x}(n), \boldsymbol{x}(n-1), \cdots, \boldsymbol{x}(n-P)]$$

- Desired signal vector

$$\boldsymbol{d}(n) = [d(n), d(n-1), \ldots, d(n-P)]$$

- Error vector

$$\boldsymbol{e}(n) = [e(n), e(n-1), \ldots, e(n-P)] = \boldsymbol{d}(n) - \boldsymbol{X}^T(n)\boldsymbol{w}(n)$$

- Vector of the constraint errors

$$\boldsymbol{e}_c(n) = \boldsymbol{d}(n) - \boldsymbol{X}^T(n)\boldsymbol{w}(n+1)$$

- Using Lagrange multipliers, we minimize

$$f[\boldsymbol{w}(n+1)] = \|\boldsymbol{w}(n+1) - \boldsymbol{w}(n)\|^2 + \boldsymbol{\lambda}^T[\boldsymbol{d}(n) - \boldsymbol{X}^T(n)\boldsymbol{w}(n+1)]$$

- Differentiating w.r.t. $\boldsymbol{w}(n+1)$ and equating the result to zero,

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \frac{1}{2}\boldsymbol{X}(n)\boldsymbol{\lambda} \qquad (*)$$

Using this result in $\boldsymbol{X}^T(n)\boldsymbol{w}(n+1) = \boldsymbol{d}(n)$ yields

$$\boldsymbol{\lambda} = 2[\boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{e}(n)$$

- Using this result in (*) yields

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu\boldsymbol{X}(n)[\boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{e}(n)}$$

Affine Projection
Solution to the Underdetermined Least-Squares Problem

- We want minimize

$$\|\boldsymbol{e}_c(n)\|^2 = \|\boldsymbol{d}(n) - \boldsymbol{X}^T(n)\boldsymbol{w}(n+1)\|^2$$

  where $\boldsymbol{X}^T(n)$ is $(P+1) \times N$ with $(P+1) < N$
- Thus, we look for the least-squares solution of the underdetermined system

$$\boldsymbol{X}^T(n)\boldsymbol{w}(n+1) = \boldsymbol{d}(n)$$

- The solution is

$$\boldsymbol{w}(n+1) = \left[\boldsymbol{X}^T(n)\right]^+ \boldsymbol{d}(n) = \boldsymbol{X}(n)[\boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{d}(n)$$

- Using $\boldsymbol{d}(n) = \boldsymbol{e}(n) + \boldsymbol{X}^T(n)\boldsymbol{w}(n)$ yields

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu\boldsymbol{X}(n)[\boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{e}(n)}$$

Observations:

- Order of the AP algorithm: $P+1$ ($P=0 \rightarrow$ NLMS)
- Convergence speed increases with P (but not linearly)
- Computational complexity increases with P (not linearly)
- If $\boldsymbol{X}^T(n)\boldsymbol{X}(n)$ is close to singular, use $\boldsymbol{X}^T(n)\boldsymbol{X}(n)+\varepsilon\boldsymbol{I}$
- The scalar error of NLMS becomes a vector error in AP (except for $\mu=1$)

Affine Projection – Stochastic Approximation of Regularized Newton

- Regularized Newton Algorithm

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\tilde{\varepsilon}\boldsymbol{I} + \boldsymbol{R}_{xx}]^{-1}[\boldsymbol{p} - \boldsymbol{R}_{xx}\boldsymbol{w}(n)]$$

- Estimates using time window statistics

$$\hat{\boldsymbol{p}} = \frac{1}{P+1}\sum_{k=n-P}^{n}\boldsymbol{X}(k)d(k) = \frac{1}{P+1}\boldsymbol{X}(n)\boldsymbol{d}(n)$$

$$\hat{\boldsymbol{R}}_{xx} = \frac{1}{P+1}\sum_{k=n-P}^{n}\boldsymbol{X}(k)\boldsymbol{X}^T(k) = \frac{1}{P+1}\boldsymbol{X}(n)\boldsymbol{X}^T(n)$$

- Using these estimates with $\tilde{\varepsilon} = \varepsilon/(P+1)$,

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon\boldsymbol{I} + \boldsymbol{X}(n)\boldsymbol{X}^T(n)]^{-1}\boldsymbol{X}(n)\overbrace{[\boldsymbol{d}(n) - \boldsymbol{X}^T(n)\boldsymbol{w}(n)]}^{\boldsymbol{e}(n)}$$

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon\boldsymbol{I} + \boldsymbol{X}(n)\boldsymbol{X}^T(n)]^{-1}\boldsymbol{X}(n)\boldsymbol{e}(n)$$

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu[\varepsilon \boldsymbol{I} + \boldsymbol{X}(n)\boldsymbol{X}^T(n)]^{-1}\boldsymbol{X}(n)\boldsymbol{e}(n)$$

- Using the Matrix Inversion Formula

$$[\varepsilon \boldsymbol{I} + \boldsymbol{X}(n)\boldsymbol{X}^T(n)]^{-1} = \boldsymbol{X}(n)[\varepsilon \boldsymbol{I} + \boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}$$

and

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu \boldsymbol{X}(n)[\varepsilon \boldsymbol{I} + \boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{e}(n)}$$

Affine Projection Algorithm as a Projection onto an Affine Subspace

- Conditions for the analysis
  - $\boldsymbol{e}(n) = \boldsymbol{d}(n) - \boldsymbol{X}^T(n)\boldsymbol{w}(n)$
  - $\boldsymbol{d}(n) = \boldsymbol{X}^T(n)\boldsymbol{w}^o$ defines the optimal solution in the least-squares sense
  - $\boldsymbol{v}(n) = \boldsymbol{w}(n) - \boldsymbol{w}^o$ (weight error vector)
- Error vector

$$\begin{aligned}\boldsymbol{e}(n) &= \boldsymbol{d}(n) - \boldsymbol{X}^T(n)[\boldsymbol{v}(n) + \boldsymbol{w}(n+1)] \\ &= \boldsymbol{X}^T(n)\boldsymbol{w}(n+1) - \boldsymbol{X}^T(n)[\boldsymbol{v}(n) + \boldsymbol{w}(n+1)] \\ &= -\boldsymbol{X}^T(n)\boldsymbol{v}(n)\end{aligned}$$

$$\boldsymbol{e}(n) = -\boldsymbol{X}^T(n)\boldsymbol{v}(n)$$

- *Interpretation:* To minimize the error $\boldsymbol{v}(n)$ should be orthogonal to all input vectors
  Restriction: We are going to use only $\{\boldsymbol{x}(n), \boldsymbol{x}(n-1), \ldots, \boldsymbol{x}(n-P)\}$
- *Iterative solution:*
  We can subtract from $\boldsymbol{v}(n)$ its projection onto the range of $\boldsymbol{X}(n)$ at each iteration

$$\boldsymbol{v}(n+1) = \boldsymbol{v}(n) - \boldsymbol{P}_{\boldsymbol{X}(n)}\boldsymbol{v}(n) \quad \text{(Proj. onto an affine subspace)}$$

- Using $\boldsymbol{X}^T(n)\boldsymbol{v}(n) = -\boldsymbol{e}(n)$

$$\begin{aligned}\boldsymbol{v}(n+1) &= \boldsymbol{v}(n) - \left\{\boldsymbol{X}(n)[\boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{X}^T(n)\right\}\boldsymbol{v}(n) \\ &= \boldsymbol{v}(n) + \mu\boldsymbol{X}(n)[\boldsymbol{X}^T(n)\boldsymbol{X}(n)]^{-1}\boldsymbol{e}(n) \quad \text{(AP algorithm)}\end{aligned}$$

Pseudo Affine Projection Algorithm

- Major problem with the AP algorithm
  For $\mu \neq 1$, $e(n) \rightarrow \boldsymbol{e}(n)$ ➪ large computational complexity
- The Pseudo-AP algorithm replaces the input $x(n)$ with its P-th order autoregressive prediction

$$\hat{\boldsymbol{x}}(n) = \sum_{k=1}^{P} a_k \boldsymbol{x}(n-k)$$

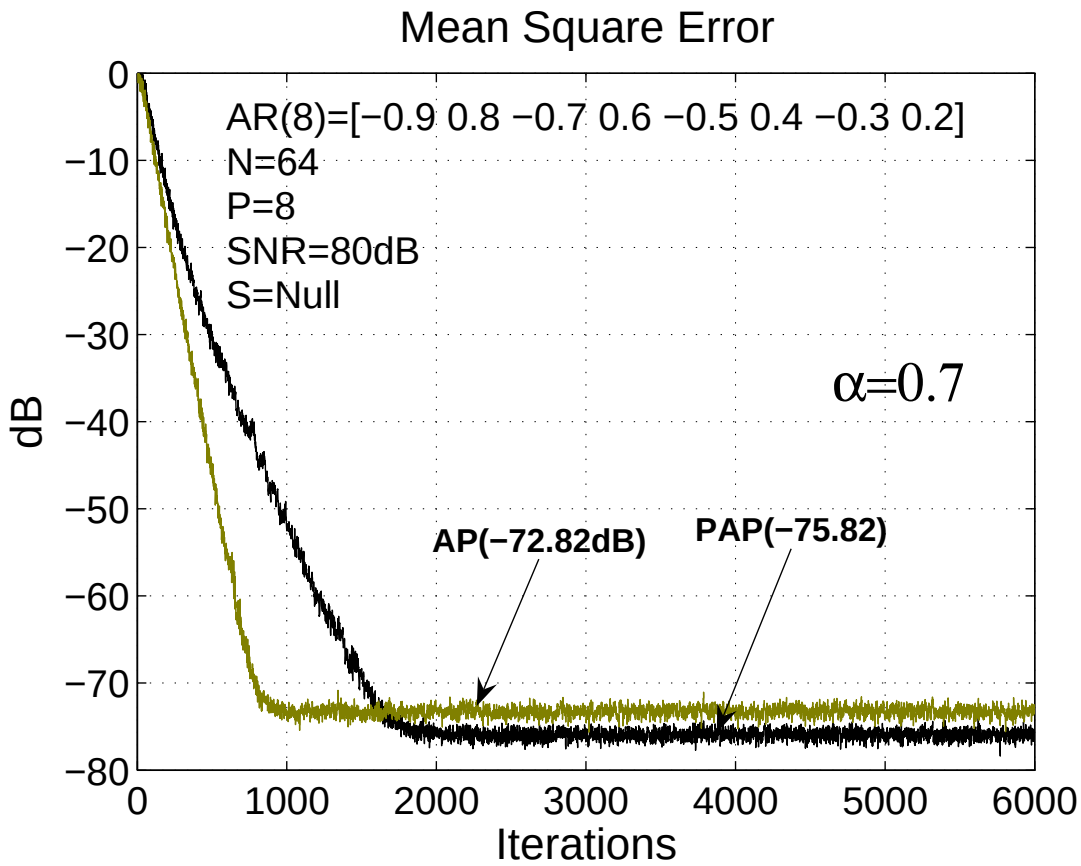
- Using the least-squares solution for $\boldsymbol{a} = [a_1, a_2, \ldots, a_P]^T$,

$$\boldsymbol{a} = [\boldsymbol{X}_p^T(n)\boldsymbol{X}_p(n)]^{-1}\boldsymbol{X}_p^T(n)\boldsymbol{x}(n), \quad \boldsymbol{X}_p(n) = [\boldsymbol{x}(n-1), \ldots, \boldsymbol{x}(n-P)]$$

- Now, we subtract from $\boldsymbol{x}(n)$ its projections onto the last P input vectors

$$\begin{aligned}\boldsymbol{\phi}(n) &= \boldsymbol{x}(n) - \boldsymbol{X}_p(n)\boldsymbol{a}(n) \\ &= \boldsymbol{x}(n) - \left\{\boldsymbol{X}_p(n)[\boldsymbol{X}_p^T(n)\boldsymbol{X}_p(n)]^{-1}\boldsymbol{X}_p^T(n)\right\}\boldsymbol{x}(n) \\ &= (\boldsymbol{I} - \boldsymbol{P}_P)\boldsymbol{x}(n); \qquad (\boldsymbol{P}_P\text{: projection matrix})\end{aligned}$$

Example – Pseudo-AP *versus* AP



- Using ϕ as the new input vector for the NLMS algorithm,

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu \frac{\boldsymbol{\phi}(n)}{\boldsymbol{\phi}^T(n)\boldsymbol{\phi}(n)} e(n)}$$

- It can be shown that Pseudo-AP is identical to AP for an AR input and $\mu = 1$
- Otherwise, Pseudo-AP is different from AP
- Simpler to implement than AP for $\mu \neq 1$
- Can lead even to better steady-state results than AP for AR inputs
- For AR inputs: NLMS with input "orthogonalization"

Some Research Results

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Deterministic Algorithms

Recursive Least Squares Algorithm (RLS)

- Based on a deterministic philosophy
- Designed for the present realization of the input signals
- Least squares method adapted for real time processing of temporal series
- Convergence speed is not strongly dependent on the input statistics

RLS – Problem Definition

- Define

$$\boldsymbol{x}_o(n) = [x(n), x(n-1), \ldots, x(0)]^T$$
$$\boldsymbol{x}(n) = [x(n), x(n-1), \ldots, x(n-N+1)]^T \quad \text{input to the } N\text{-tap filter}$$

- Desired signal $d(n)$
- Estimator of $d(n)$: $y(n) = \boldsymbol{x}^T(n)\boldsymbol{w}(n)$
- Cost function – Squared Error

$$J_{ls}(n) = \sum_{k=0}^{n} e^2(k) = \sum_{k=0}^{n} [d(k) - \boldsymbol{x}^T(k)\boldsymbol{w}(n)]^2$$
$$= \boldsymbol{e}^T(n)\boldsymbol{e}(n), \quad \boldsymbol{e}(n) = [e(n), e(n-1), \ldots, e(0)]^T$$

In vector form

$$\boldsymbol{d}(n) = [d(n), d(n-1), \ldots, d(0)]^T$$

$$\boldsymbol{y}(n) = [y(n), y(n-1), \ldots, y(0)]^T, \quad y(k) = \boldsymbol{x}^T(k)\boldsymbol{w}(n)$$

$$\boldsymbol{y}(n) = \sum_{k=0}^{N-1} w_k(n)\boldsymbol{x}_o(n-k) = \boldsymbol{\Theta}(n)\boldsymbol{w}(n)$$

$$\boldsymbol{\Theta}(n) = \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-N+1) \\ x(n-1) & x(n-2) & \cdots & x(n-N) \\ \vdots & \vdots & \vdots & \vdots \\ x(0) & x(-1) & \cdots & x(-N+1) \end{bmatrix}$$

$$= [\boldsymbol{x}_o(n), \boldsymbol{x}_o(n-1), \ldots \boldsymbol{x}_o(n-N+1)]$$

$$\boxed{\boldsymbol{e}(n) = \boldsymbol{d}(n) - \boldsymbol{\Theta}(n)\boldsymbol{w}(n)}$$

$$\boldsymbol{e}(n) = \boldsymbol{d}(n) - \boldsymbol{\Theta}(n)\boldsymbol{w}(n)$$

and we minimize

$$J_{ls}(n) = \|\boldsymbol{e}(n)\|^2$$

- Normal Equations

$$\boldsymbol{\Theta}^T(n)\boldsymbol{\Theta}(n)\boldsymbol{w}(n) = \boldsymbol{\Theta}^T(n)\boldsymbol{d}(n)$$

$$\tilde{\boldsymbol{R}}(n)\boldsymbol{w}(n) = \tilde{\boldsymbol{p}}(n)$$

- Alternative representation for $\tilde{\boldsymbol{R}}$ and $\tilde{\boldsymbol{p}}$

$$\tilde{\boldsymbol{R}}(n) = \boldsymbol{\Theta}^T(n)\boldsymbol{\Theta}(n) = \sum_{k=0}^{n} \boldsymbol{x}(k)\boldsymbol{x}^T(k)$$

$$\tilde{\boldsymbol{p}}(n) = \boldsymbol{\Theta}^T(n)\boldsymbol{d}(n) = \sum_{k=0}^{n} \boldsymbol{x}(k)d(k)$$

RLS – Observations

- $\boldsymbol{w}(n)$ remains fixed for $0 \leq k \leq n$ to determine $J_{ls}(n)$
- Optimum vector $\boldsymbol{w}(n)$ is $\boldsymbol{w}^o(n) = \tilde{\boldsymbol{R}}^{-1}(n)\tilde{\boldsymbol{p}}$
- The algorithm has growing memory
- In an adaptive implementation,

$$\boldsymbol{w}^o(n) = \tilde{\boldsymbol{R}}^{-1}(n)\tilde{\boldsymbol{p}}$$

  must be solved for each iteration n
- Problems for an adaptive implementation
  - Dificulties with nonstationary signals (infinite data window)
  - Computational complexity

RLS - Forgetting the Past

- Modified cost function

$$J_{ls}(n) = \sum_{k=0}^{n} \lambda^{n-k} e^2(k) = \sum_{k=0}^{n} \lambda^{n-k} [d(k) - \boldsymbol{x}^T(k)\boldsymbol{w}(n)]^2$$
$$= \boldsymbol{e}^T(n)\boldsymbol{\Lambda}\boldsymbol{e}(n), \quad \boldsymbol{\Lambda} = \text{diag}[1, \lambda, \lambda^2, \ldots, \lambda^n]$$

- Modified Normal Equations

$$\boldsymbol{\Theta}^T(n)\boldsymbol{\Lambda}\boldsymbol{\Theta}(n)\boldsymbol{w}(n) = \boldsymbol{\Theta}^T(n)\boldsymbol{\Lambda}\boldsymbol{d}(n)$$

$$\hat{\boldsymbol{R}}(n)\boldsymbol{w}(n) = \hat{\boldsymbol{p}}(n)$$

where

$$\hat{\boldsymbol{R}}(n) = \boldsymbol{\Theta}^T(n)\boldsymbol{\Lambda}\boldsymbol{\Theta}(n) = \sum_{k=0}^{n} \lambda^{n-k} \boldsymbol{x}(k)\boldsymbol{x}^T(k)$$

$$\hat{\boldsymbol{p}}(n) = \boldsymbol{\Theta}^T(n)\boldsymbol{\Lambda}\boldsymbol{d}(n) = \sum_{k=0}^{n} \lambda^{n-k} \boldsymbol{x}(k)d(k)$$

RLS – Recursive Updating

- Correlation Matrix

$$\begin{aligned}\hat{\boldsymbol{R}}(n) &= \sum_{k=0}^{n} \lambda^{n-k} \boldsymbol{x}(k)\boldsymbol{x}^T(k) \\ &= \sum_{k=0}^{n-1} \lambda^{n-k} \boldsymbol{x}(k)\boldsymbol{x}^T(k) + \boldsymbol{x}(n)\boldsymbol{x}^T(n) \\ &= \lambda \hat{\boldsymbol{R}}(n-1) + \boldsymbol{x}(n)\boldsymbol{x}^T(n)\end{aligned}$$

- Cross-correlation vector

$$\begin{aligned}\hat{\boldsymbol{p}}(n) &= \sum_{k=0}^{n} \lambda^{n-k} \boldsymbol{x}(k)d(k) \\ &= \sum_{k=0}^{n-1} \lambda^{n-k} \boldsymbol{x}(k)d(k) + \boldsymbol{x}(n)d(n) \\ &= \lambda \hat{\boldsymbol{p}}(n-1) + \boldsymbol{x}(n)d(n)\end{aligned}$$

- We have recursive updating expressions for $\hat{\boldsymbol{R}}(n)$ and $\hat{\boldsymbol{p}}(n)$
- However, we need a recursive updating for $\hat{\boldsymbol{R}}^{-1}(n)$, as

$$\boldsymbol{w}^o(n) = \hat{\boldsymbol{R}}^{-1}(n)\hat{\boldsymbol{p}}(n)$$

- Applying the matrix inversion lemma to

$$\hat{\boldsymbol{R}}(n) = \lambda\hat{\boldsymbol{R}}(n-1) + \boldsymbol{x}(n)\boldsymbol{x}^T(n)$$

and defining $\boldsymbol{P}(n) = \hat{\boldsymbol{R}}^{-1}(n)$ yields

$$\boldsymbol{P}(n) = \lambda^{-1}\boldsymbol{P}(n-1) - \frac{\lambda^{-2}\boldsymbol{P}(n-1)\boldsymbol{x}(n)\boldsymbol{x}^T(n)\boldsymbol{P}(n-1)}{1+\lambda^{-1}\boldsymbol{x}^T(n)\boldsymbol{P}(n-1)\boldsymbol{x}(n)}$$

The Gain Vector

- Definition

$$\boldsymbol{k}(n) = \frac{\lambda^{-1}\boldsymbol{P}(n-1)\boldsymbol{x}(n)}{1+\lambda^{-1}\boldsymbol{x}^T(n)\boldsymbol{P}(n-1)\boldsymbol{x}(n)}$$

- Using this definition

$$\boldsymbol{P}(n) = \lambda^{-1}\boldsymbol{P}(n-1) - \lambda^{-1}\boldsymbol{k}(n)\boldsymbol{x}^T(n)\boldsymbol{P}(n-1)$$

- $\boldsymbol{k}(n)$ can be written as

$$\begin{aligned}\boldsymbol{k}(n) &= \overbrace{\left[\lambda^{-1}\boldsymbol{P}(n-1) - \lambda^{-1}\boldsymbol{k}(n)\boldsymbol{x}^T(n)\boldsymbol{P}(n-1)\right]}^{\boldsymbol{P}(n)}\boldsymbol{x}(n) \\ &= \hat{\boldsymbol{R}}^{-1}(n)\boldsymbol{x}(n)\end{aligned}$$

➪ $\boldsymbol{k}(n)$ is the repres. of $\boldsymbol{x}(n)$ on the column space of $\hat{\boldsymbol{R}}(n)$

RLS – Recursive Weight Update

- $\boldsymbol{w}(n+1) = \boldsymbol{w}^o(n)$
- Using

$$\begin{aligned}
\boldsymbol{w}(n+1) &= \hat{\boldsymbol{R}}^{-1}(n)\hat{\boldsymbol{p}}(n) = \boldsymbol{P}(n)\hat{\boldsymbol{p}}(n) \\
\hat{\boldsymbol{p}}(n) &= \lambda\hat{\boldsymbol{p}}(n-1) + \boldsymbol{x}(n)d(n) \\
\boldsymbol{P}(n) &= \lambda^{-1}\boldsymbol{P}(n-1) - \lambda^{-1}\boldsymbol{k}(n)\boldsymbol{x}^T(n)\boldsymbol{P}(n-1) \\
\boldsymbol{k}(n) &= \boldsymbol{P}(n)\boldsymbol{x}(n) \\
e(n) &= d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n)
\end{aligned}$$

yields

$$\boxed{\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \boldsymbol{k}(n)e(n)}$$ RLS weight update

RLS – Observations

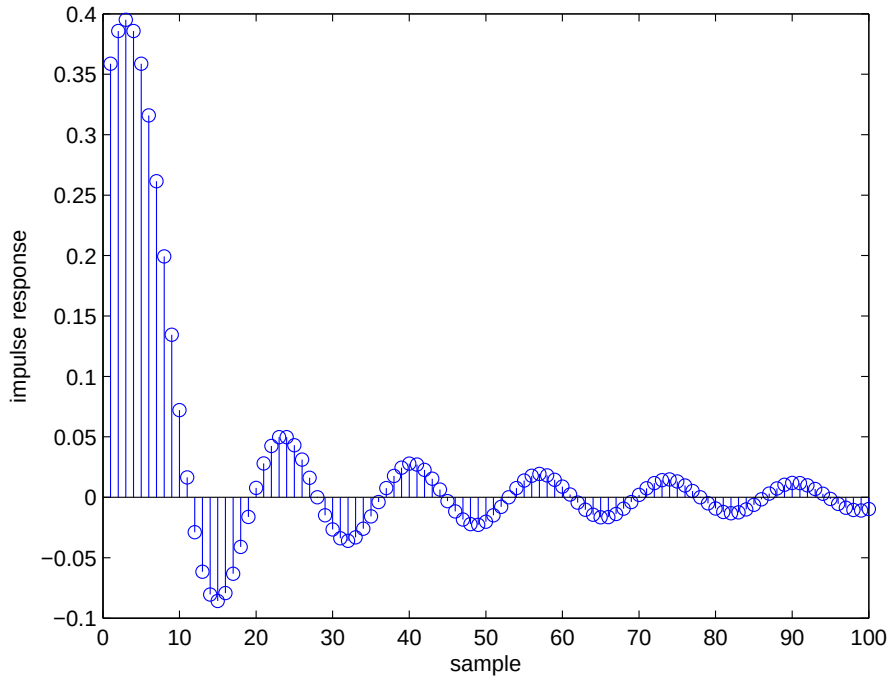
- The RLS convergence speed is not affected by the eigenvalues of $\hat{\boldsymbol{R}}(n)$
- Initialization: $\hat{\boldsymbol{p}}(0) = \boldsymbol{0}$, $\hat{\boldsymbol{R}}(0) = \delta \boldsymbol{I}$, $\delta \simeq (1-\lambda)\sigma_x^2$
- This results in $\hat{\boldsymbol{R}}(n)$ changed to $\lambda^n \delta \boldsymbol{I} + \hat{\boldsymbol{R}}(n)$
  ➪ Biased estimate of $\boldsymbol{w}^o(n)$ for small n
  (No problem for $\lambda < 1$ and large n)
- Numerical problems in finite precision
  - Unstable for $\lambda = 1$
  - Loss of symmetry in $\boldsymbol{P}(n)$
    - ★ Evaluate only upper (lower) part and diagonal of $\boldsymbol{P}(n)$
    - ★ Replace $\boldsymbol{P}(n)$ with $[\boldsymbol{P}(n) + \boldsymbol{P}^T(n)]/2$ after updating from $P(n-1)$
- Numerical problems when $\boldsymbol{x}(n) \to \boldsymbol{0}$ and $\lambda < 1$

Some Research Results

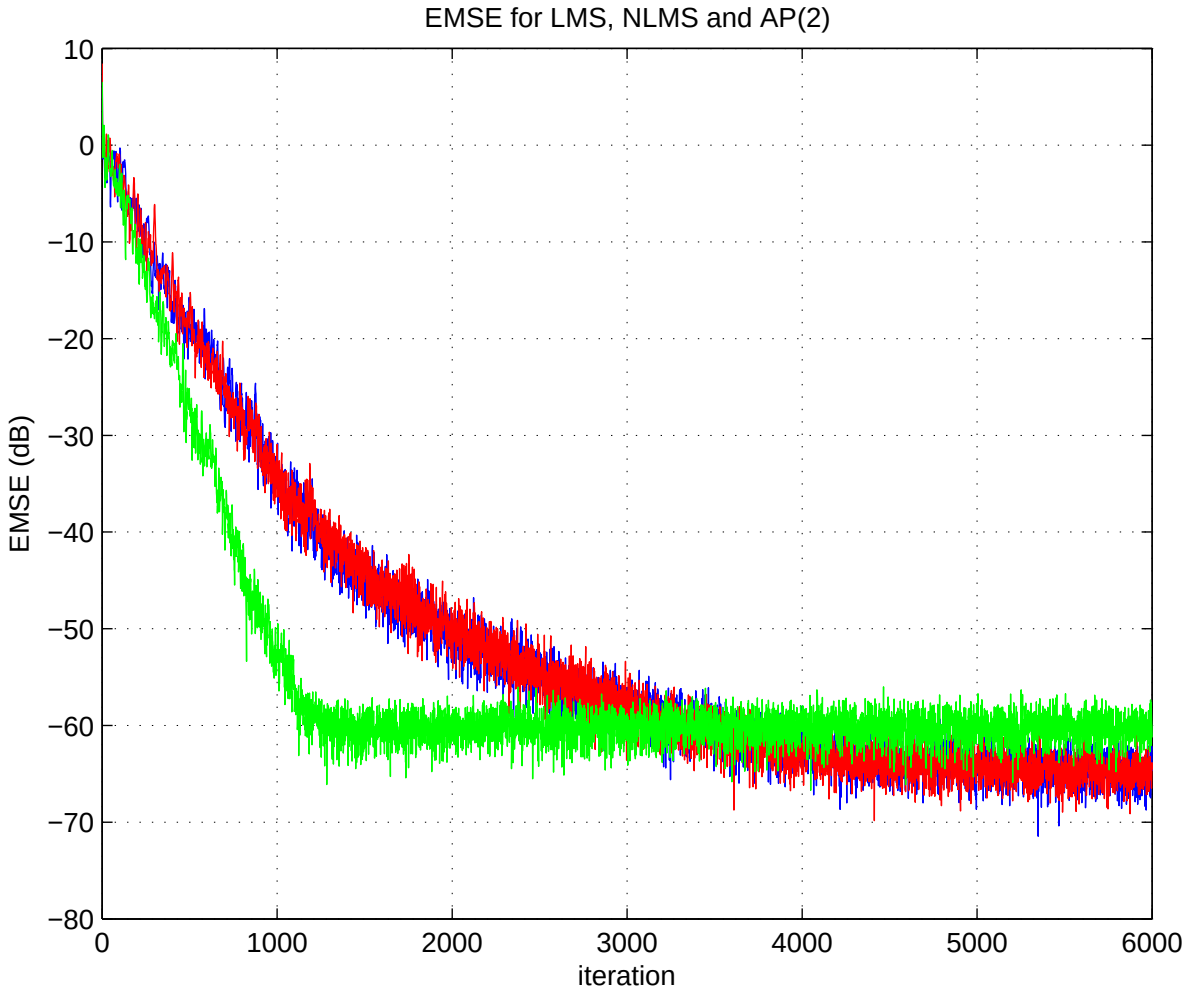
- C. Ludovico and J. C. M. Bermudez, "A recursive least squares algorithm robust to low- power excitation," in. Proc. 2004 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP-2004), Montreal, Canada, vol. II, p. 673-677, 2004.
- C. Ludovico and J. C. M. Bermudez, "An improved recursive least squares algorithm robust to input power variation," in Proc. IEEE Statistical Signal Processing Workshop, Bordeaux, France, 2005.

Performance Comparison

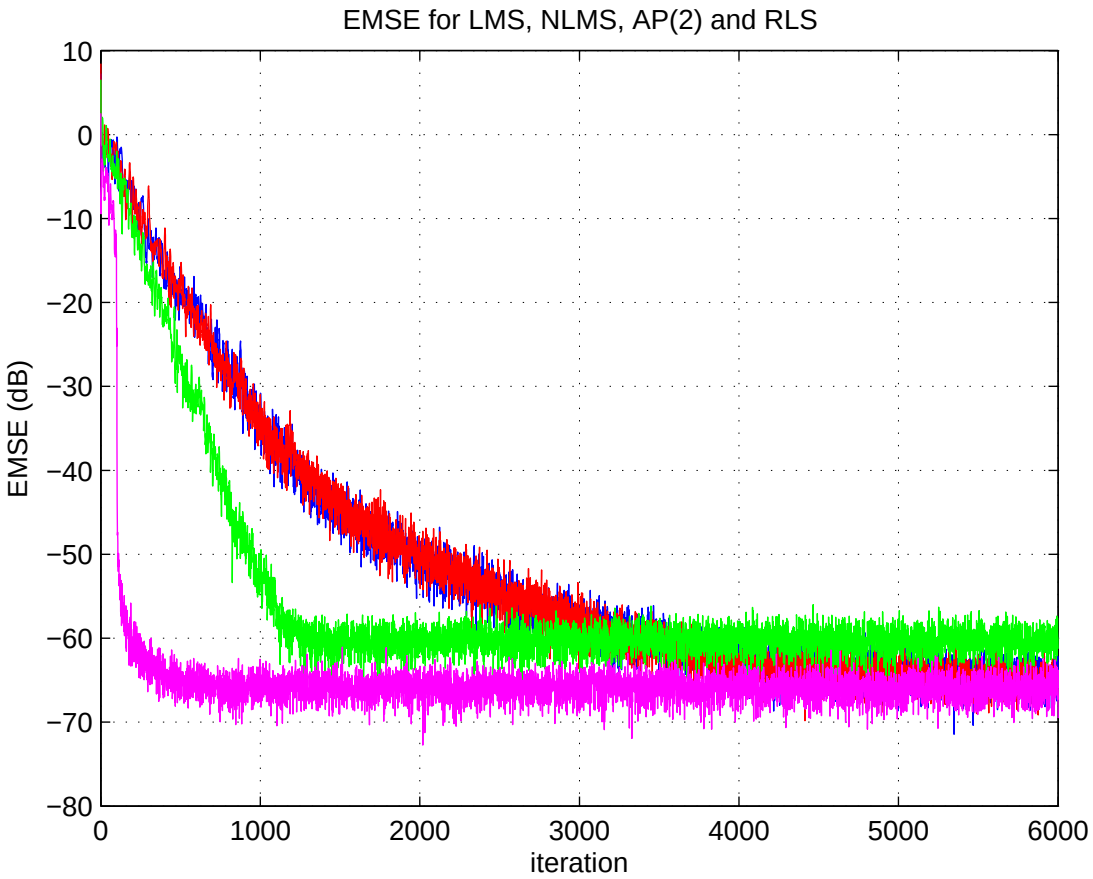
- System identification, N=100
- Input – AR(1) $\quad x(n) = 0.9x(n-1) + v(n)$
- AP algorithm with $P = 2$
- Step sizes designed for equal LMS and NLMS performances
- Impulse response to be estimated



Excess Mean Square Error - LMS, NLMS and AP



Excess Mean Square Error - LMS, NLMS, AP and RLS



Performance Comparisons
Computational Complexity

- Adaptive filter with N real coefficients and real signals
- For the AP algorithm, $K = P + 1$

Algorithm	$\times$	$+$	$/$
LMS	$2N + 1$	$2N$	
NLMS	$3N + 1$	$3N$	1
AP	$(K^2 + 2K)N + K^3 + K$	$(K^2 + 2K)N + K^3 + K^2$	
RLS	$N^2 + 5N + 1$	$N^2 + 3N$	1

For $N = 100$, $P = 2$

Algorithm	$\times$	$+$	$/$	$\simeq$ factor
LMS	201	200		1
NLMS	301	300	1	1.5
AP	1,530	1,536		7.5
RLS	10,501	10,300	1	52.5

Typical values for acoustic echo cancellation ($N = 1024,\ P = 2$)

Algorithm	$\times$	$+$	$/$	$\simeq$ factor
LMS	$2,049$	$2,048$		1
NLMS	$3,073$	$3,072$	1	1.5
AP	$15,390$	$15,396$		7.5
RLS	$1,053,697$	$1,051,648$	1	514

How to Deal with Computational Complexity?

- Not an easy task!!!
- There are "fast" versions for some algorithms (especially RLS)
- What is usually not said is that ... speed can bring
  - Instability
  - Increased need for memory

  ➪ *Most applications rely on simple solutions*

Understanding the Adaptive Filter Behavior

What are Adaptive Filters?

Adaptive filters are, by design, systems that are

- Time-variant ($\boldsymbol{w}(n)$ is time-variant)
- Nonlinear ($y(n)$ is a nonlinear function of $\boldsymbol{x}(n)$)
- Stochastic ($\boldsymbol{w}(n)$ is random)

LMS:

$$\begin{aligned}\boldsymbol{w}(n+1) &= \boldsymbol{w}(n) + \mu e(n)\boldsymbol{x}(n) \\ &= \left[\boldsymbol{I} - \mu\, \boldsymbol{x}(n)\boldsymbol{x}^T(n)\right]\boldsymbol{x}(n) + \mu\, d(n)\boldsymbol{x}(n) \\ y(n) &= \boldsymbol{x}^T(n)\boldsymbol{w}(n)\end{aligned}$$

- They are difficult to understand
- Different applications and signals require different analyses
- Simplifying assumptions are necessary

➪ *Good design requires good analytical models.*

LMS Analysis

Basic equations:

$$
\begin{aligned}
d(n) &= \boldsymbol{x}^T(n)\boldsymbol{w}^o + z(n) \\
e(n) &= d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}(n) \\
\boldsymbol{w}(n+1) &= \boldsymbol{w}(n) + \mu\, e(n)\boldsymbol{x}(n) \\
\boldsymbol{v}(n) &= \boldsymbol{w}(n) - \boldsymbol{w}^o \quad \text{(study about the optimum weight vector)}
\end{aligned}
$$

Mean Weight Behavior:

Neglecting dependence between $\boldsymbol{v}(n)$ *and* $\boldsymbol{x}(n)\boldsymbol{x}^T(n)$,

$$E[\boldsymbol{v}(n+1)] = (\boldsymbol{I} - \mu\boldsymbol{R}_{xx})E[\boldsymbol{v}(n)]$$

- Follows the steepest descent trajectory
- Convergence condition: $0 < \mu < 2/\lambda_{\max}$
- $\lim_{n\to\infty} E[\boldsymbol{w}(n)] = \boldsymbol{w}^o$

However: $\boldsymbol{w}(n)$ is random

➪ *We need to study its fluctuations about* $E[\boldsymbol{w}(n)] \rightarrow$ MSE

The Mean-Square Estimation Error

$$e(n) = e_o(n) - \boldsymbol{x}^T(n)\boldsymbol{v}(n), \quad e_o(n) = d(n) - \boldsymbol{x}^T(n)\boldsymbol{w}^o$$

- Square the error equation
- Take the expected value
- Neglect the statistical dep. between $\boldsymbol{v}(n)$ and $\boldsymbol{x}(n)\boldsymbol{x}^T(n)$

$$J_{ms}(n) = E[e_o^2(n)] + Tr\{\boldsymbol{R}_{xx}\boldsymbol{K}(n)\}, \quad \boldsymbol{K}(n) = E[\boldsymbol{v}(n)\boldsymbol{v}^T(n)]$$

- This expression is independent of the adaptive algorithm
- Effect of the alg. on $J_{ms}(n)$ is determined by $\boldsymbol{K}(n)$
- $J_{ms_{\min}} = E[e_o^2(n)]$
- $J_{ms_{\text{ex}}} = Tr\{\boldsymbol{R}_{xx}\boldsymbol{K}(n)\}$

The Behavior of $\boldsymbol{K}(n)$ for LMS

- Using the basic equations

$$\boldsymbol{v}(n+1) = \left[\boldsymbol{I} + \mu\, \boldsymbol{x}(n)\boldsymbol{x}^T(n)\right] \boldsymbol{v}(n) + \mu\, e_o(n)\boldsymbol{x}(n)$$

- Post-multiply by $\boldsymbol{v}^T(n+1)$
- Take the expected value assuming $\boldsymbol{v}(n)$ and $\boldsymbol{x}(n)\boldsymbol{x}^T(n)$ independent
- Evaluate the expectations
  - Higher order moments of $x(n)$ require input pdf
- Assuming Gaussian inputs

$$\begin{aligned}\boldsymbol{K}(n+1) =& \boldsymbol{K}(n) - \mu\, [\boldsymbol{R}_{xx}\boldsymbol{K}(n) + \boldsymbol{K}(n)\boldsymbol{R}_{xx}] \\ &+ \mu^2\, \{\boldsymbol{R}_{xx}\mathsf{Tr}[\boldsymbol{R}_{xx}\boldsymbol{K}(n)] + 2\boldsymbol{R}_{xx}\boldsymbol{K}(n)\boldsymbol{R}_{xx}\} \\ &+ \mu^2\, \boldsymbol{R}_{xx} J_{ms_{\min}}\end{aligned}$$

Design Guidelines Extrated from the Model

- Stability

$$0 < \mu < \frac{m}{\mathsf{Tr}[\boldsymbol{R}_{xx}]}; \quad m = 2 \text{ or } m = 2/3$$

- $J_{ms}(n)$ converges as a function of

$$\left[(1-\mu\lambda_i)^2 + 2\mu^2\lambda_i^2\right]^n, \quad i = 1, \ldots, N$$

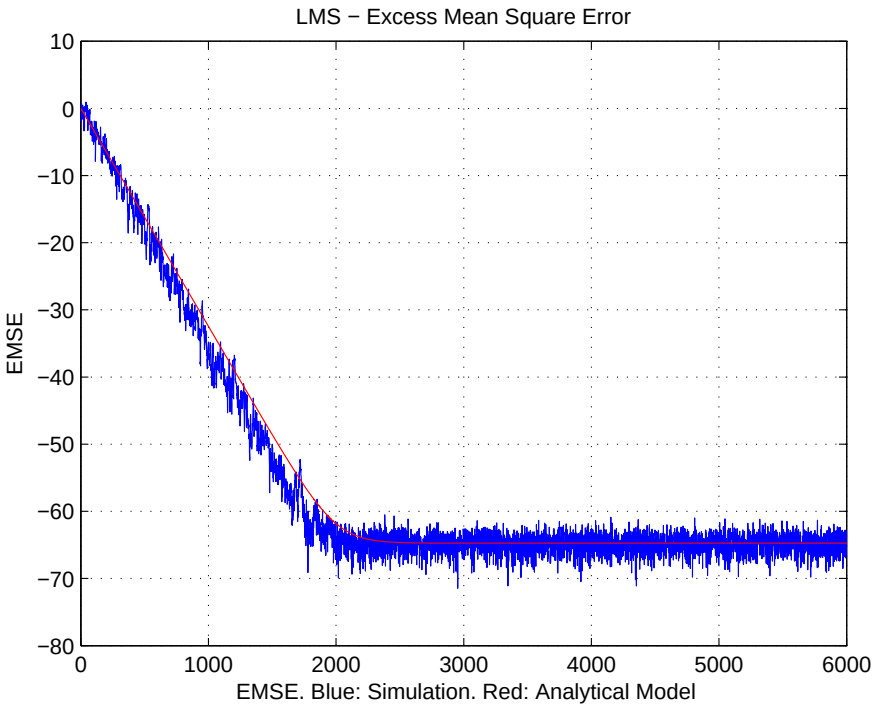
- Steady-State

$$J_{ms_{\text{ex}}}(\infty) \simeq \frac{\mu}{2}\mathsf{Tr}[\boldsymbol{R}_{xx}]J_{ms_{\min}}$$

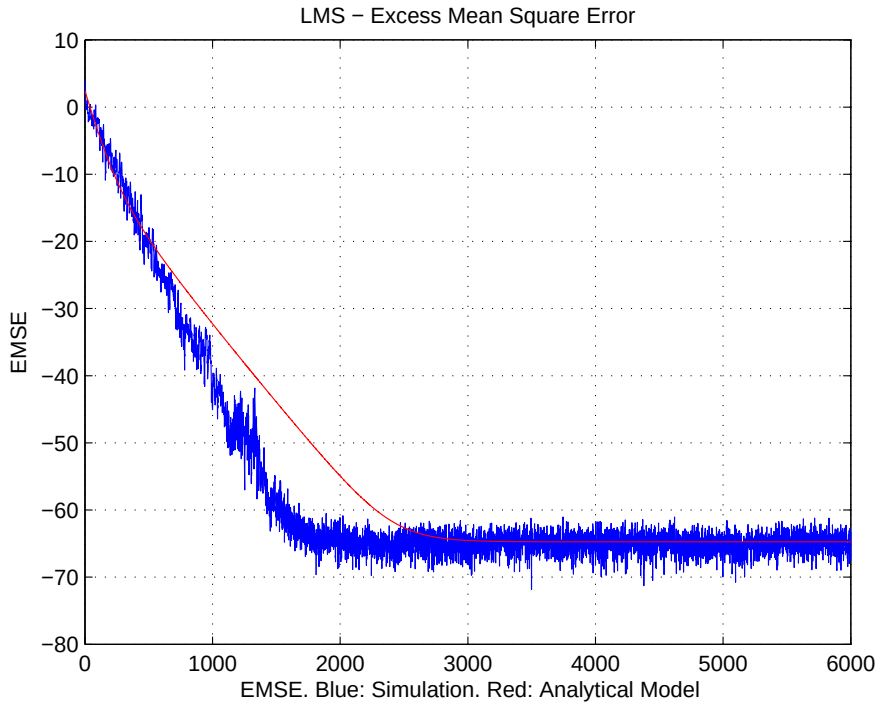
$$\mathcal{M}(\infty) = \frac{J_{ms_{\text{ex}}}(\infty)}{J_{ms_{\min}}} \quad \text{(MSE Misadjustment)}$$

Model Evaluation

$$x(n) = \alpha\, x(n-1) + v(n)$$



Input: White noise



Input: AR(1), $\alpha = 0.8$

- Linear system identification - FIR with 100 coefficients
- Step-size $\mu = 0.05$
- Noise power $\sigma_v^2 = 10^{-6}$

Merci pour votre attention!!!