

A UNIFIED CONVERGENCE ANALYSIS OF BLOCK SUCCESSIVE MINIMIZATION METHODS FOR NONSMOOTH OPTIMIZATION*

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Abstract. The block coordinate descent (BCD) method is widely used for minimizing a continuous function f of several block variables. At each iteration of this method, a single block of variables is optimized, while the remaining variables are held fixed. To ensure the convergence of the BCD method, the subproblem of each block variable needs to be solved to its unique global optimal. Unfortunately, this requirement is often too restrictive for many practical scenarios. In this paper, we study an alternative inexact BCD approach which updates the variable blocks by successively minimizing a sequence of approximations of f which are either locally tight upper bounds of f or strictly convex local approximations of f . The main contributions of this work include the characterizations of the convergence conditions for a fairly wide class of such methods, especially for the cases where the objective functions are either nondifferentiable or nonconvex. Our results unify and extend the existing convergence results for many classical algorithms such as the BCD method, the difference of convex functions (DC) method, the expectation maximization (EM) algorithm, as well as the block forward-backward splitting algorithm, all of which are popular for large scale optimization problems involving big data.

Key words. block coordinate descent, block successive upper-bound minimization, successive convex approximation, successive inner approximation

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1. Introduction. Consider the optimization problem

$$\begin{aligned} \min \quad & f(x_1, \dots, x_n) \\ \text{s.t.} \quad & x_i \in \mathcal{X}_i, \quad i = 1, 2, \dots, n, \end{aligned}$$

where $\mathcal{X}_i \subseteq \mathbb{R}^{m_i}$ is a closed convex set, and $f : \prod_{i=1}^n \mathcal{X}_i \rightarrow \mathbb{R}$ is a continuous function. A popular approach to solving the above optimization problem is the block coordinate descent (BCD) method, which is also known as the Gauss–Seidel method. At each iteration of this method, the function is minimized with respect to a single block of variables while the rest of the blocks are held fixed. More specifically, at iteration r of the algorithm, the block variable x_i is updated by solving the following subproblem:

$$(1.1) \quad x_i^r = \arg \min_{y_i \in \mathcal{X}_i} f(x_1^r, \dots, x_{i-1}^r, y_i, x_{i+1}^{r-1}, \dots, x_n^{r-1}), \quad i = 1, 2, \dots, n.$$

Let us use $\{x^r\}$ to denote the sequence of iterates generated by this algorithm, where $x^r \triangleq (x_1^r, \dots, x_n^r)$. Due to its particular simple implementation, the BCD method has been widely used for solving problems such as power allocation in wireless communication systems [36], clustering [20], image denoising and image reconstruction [9], and dynamic programming [24].

Convergence of the BCD method typically requires the uniqueness of the minimizer at each step or the quasi convexity of the objective function (see [38] and the

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references therein). Without these assumptions, it is possible that the BCD iterates do not get close to any of the stationary points of the problem (see Powell [32] for examples). Unfortunately, these requirements can be quite restrictive in some important practical problems such as the tensor decomposition problem (see [26] and the application section in this work) and the sum rate maximization problem in wireless networks. In fact, for the latter case, even solving the per block subproblem (1.1) is difficult due to the nonconvexity of the objective function.

To overcome such difficulties, one can modify the BCD algorithm by optimizing a well-chosen *approximate* version of the objective function at each iteration. The classical gradient descent method, for example, can be viewed as an implementation of such a strategy. To illustrate, recall that the update rule of the gradient descent method is given by

$$x^{r+1} = x^r - \alpha^{r+1} \nabla f(x^r).$$

This update rule is equivalent to solving the problem

$$x^{r+1} = \arg \min_x g(x, x^r),$$

where

$$g(x, x^r) \triangleq f(x^r) + \nabla f(x^r)(x - x^r) + \frac{1}{2\alpha^{r+1}} \|x - x^r\|^2.$$

Clearly, the function $g(x, x^r)$ is an approximation of $f(\cdot)$ around the point x^r . In fact, as we will see later in this paper, successively optimizing an approximate version of the original objective is the key idea of many important algorithms such as the concave-convex procedure [41], the expectation maximization (EM) algorithm [15], and the proximal minimization algorithm [3]. Furthermore, this idea can be used to simplify the computation and to guarantee the convergence of the original BCD algorithm with the Gauss–Seidel update rule (e.g., [39], [18], [40]). However, despite its wide applicability, there appears to be no general unifying convergence analysis for this class of algorithms.

In this paper, we provide a unified convergence analysis for a general class of inexact BCD methods in which a sequence of approximate versions of the original problem are solved successively. Our focus will be on problems with nonsmooth and nonconvex objective functions. Two types of approximations are considered: one is a locally tight upper bound for the original objective function, and the other is a convex local approximation of the objective function. We provide convergence analyses for both of these successive approximation strategies as well as for various types of updating rules, including the cyclic updating rule and the Gauss–Southwell update rule or the overlapping essentially cyclic update rule. By allowing inexact solution of subproblems, our work unifies and extends several existing algorithms and their convergence analysis, including the difference of convex functions (DC) method, the EM algorithm, and the alternating proximal minimization algorithm. Moreover, our analysis shows that the convergence of these algorithms is guaranteed even when the variables are updated in a block coordinate manner.

2. Technical preliminaries. Throughout the paper, we adopt the following notation. We use \mathbb{R}^m to denote the space of m dimensional real valued vectors, which is also represented as the Cartesian product of n smaller real valued vector spaces, i.e.,

$$\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n},$$

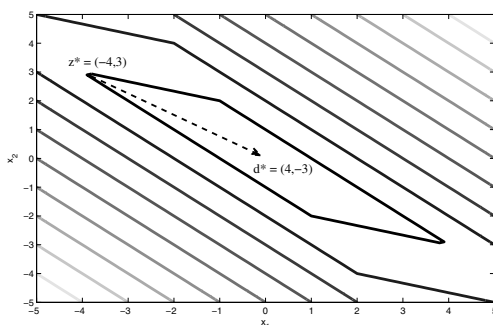


FIG. 2.1. The contour plot of the function $f(z) = \|Az\|_1$, where $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$. The function $f(\cdot)$ is not regular at the point $z^* = (-4, 3)$.

where $\sum_{i=1}^n m_i = m$. We use the notation $(0, \dots, d_k, \dots, 0)$ to denote the vector of all zeros except the k th block, with $d_k \in \mathbb{R}^{m_k}$. The following concepts/definitions are adopted in our paper.

- *Distance of a point from a set.* Let $\mathcal{S} \subseteq \mathbb{R}^m$ be a set and x be a point in \mathbb{R}^m . The distance of the point x from the set \mathcal{S} is defined as

$$d(x, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|x - s\|,$$

where $\|\cdot\|$ denotes the 2-norm in \mathbb{R}^m .

- *Directional derivative.* Let $f : \mathcal{D} \rightarrow \mathbb{R}$ be a function where $\mathcal{D} \subseteq \mathbb{R}^m$ is a convex set. The directional derivative of f at point x in direction d is defined by

$$f'(x; d) \triangleq \liminf_{\lambda \downarrow 0} \frac{f(x + \lambda d) - f(x)}{\lambda}.$$

- *Stationary points of a function.* Let $f : \mathcal{D} \rightarrow \mathbb{R}$ be a function where $\mathcal{D} \subseteq \mathbb{R}^m$ is a convex set. The point x is a stationary point of $f(\cdot)$ if $f'(x; d) \geq 0$ for all d such that $x + d \in \mathcal{D}$. In this paper we use the notation \mathcal{X}^* to denote the set of stationary points of a function.
- *Quasi-convex function.* The function f is quasi-convex if

$$f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\} \quad \forall \theta \in (0, 1), \quad \forall x, y \in \text{dom } f.$$

- *Coordinatewise minimum of a function.* $z \in \text{dom } f \subseteq \mathbb{R}^m$ is the coordinatewise minimum of f with respect to the coordinates in $\mathbb{R}^{m_1}, \mathbb{R}^{m_2}, \dots, \mathbb{R}^{m_n}$, $m_1 + \dots + m_n = m$ if

$$f(z + d_k^0) \geq f(z) \quad \forall d_k \in \mathbb{R}^{m_k} \quad \text{with} \quad z + d_k^0 \in \text{dom } f \quad \forall k = 1, 2, \dots, n,$$

where $d_k = (0, \dots, d_k, \dots, 0)$.

- *Regularity of a function at a point.* The function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is regular at the point $z \in \text{dom } f$ with respect to the coordinates m_1, m_2, \dots, m_n , $m_1 + m_2 + \dots + m_n = m$ if $f'(z; d) \geq 0$ for all $d = (d_1, d_2, \dots, d_n)$ with $f'(z; d_k^0) \geq 0$, where $d_k^0 \triangleq (0, \dots, d_k, \dots, 0)$ and $d_k \in \mathbb{R}^{m_k}$ for all k .

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1 Find a feasible point  $x^0 \in \mathcal{X}$  and set  $r = 0$ 
2 repeat
3    $r = r + 1$ 
4   Let  $\mathcal{X}^r = \arg \min_{x \in \mathcal{X}} u(x, x^{r-1})$ 
5   Set  $x^r$  to be an arbitrary element in  $\mathcal{X}^r$ 
6 until some convergence criterion is met

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FIG. 3.1. Pseudocode of the SUM algorithm.

As an example, consider the function $f(z) = \|Az\|_1$, where $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$. This function is not regular at the point $z^* = (-4, 3)$ with respect to the two standard coordinates since $f'(z^*; d) \geq 0$ for all $d \in \{(d_1, d_2) \in \mathbb{R}^2 | d_1 d_2 = 0\}$; but $f'(z^*; d^*) < 0$ for $d^* = (4, -3)$. This fact can be also observed in the contour plot of the function in Figure 2.1.

For detailed discussion on the regularity of a function, the readers are referred to [38, Lemma 3.1].

3. Successive upper-bound minimization (SUM). To gain some insights to the general inexact BCD method, let us first consider a simple successive upper-bound minimization (SUM) approach in which all the variables are grouped into a *single* block. Although simple in form, the SUM algorithm is the key to many important algorithms such as the DC programming method [41] and the EM algorithm [6].

Consider the optimization problem

$$(3.1) \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathcal{X}, \end{array}$$

where \mathcal{X} is a closed convex set. Without loss of generality, we can assume that $\text{dom } f = \mathcal{X}$. When the objective function $f(\cdot)$ is nonconvex and/or nonsmooth, solving (3.1) directly may not be easy. The SUM algorithm circumvents such difficulty by optimizing a sequence of approximate objective functions instead. More specifically, starting from a feasible point x^0 , the algorithm generates a sequence $\{x^r\}$ according to the update rule

$$(3.2) \quad x^r \in \arg \min_{x \in \mathcal{X}} u(x, x^{r-1}),$$

where x^{r-1} is the point generated by the algorithm at the $(r-1)$ th iteration and $u(x, x^{r-1})$ is an approximation of $f(x)$ at the r th iteration. Typically the approximate function $u(\cdot, \cdot)$ needs to be chosen such that the subproblem (3.2) is easy to solve. Moreover, to ensure the convergence of the SUM algorithm, certain regularity conditions on $u(\cdot, \cdot)$ are required (which will be discussed shortly). Among them is that $u(x, x^{r-1})$ needs to be a global upper bound for $f(x)$ —hence the name of the algorithm. The main steps of the SUM algorithm are presented in Figure 3.1.

We remark that the proposed SUM algorithm is in many ways similar to the inner approximation algorithm (IAA) developed in [28], with the following key differences:

- The IAA approximates *both* the objective functions and the feasible sets. On the contrary, the SUM algorithm approximates only the objective function.
- The IAA is applicable only to problems with smooth objectives, while the SUM algorithm is able to handle nonsmooth objectives as well.

It is worth mentioning that the existing convergence result for the IAA is quite weak. In particular, [28, Theorem 1] states that if the whole sequence converges, then the algorithm should converge to a stationary point. In the following, we show that the SUM algorithm provides stronger convergence guarantees as long as the approximation function $u(\cdot, \cdot)$ satisfies certain mild assumptions¹ which we outline below.

ASSUMPTION 1. *Let the approximation function $u(\cdot, \cdot)$ satisfy the following:*

- (A1) $u(y, y) = f(y) \quad \forall y \in \mathcal{X},$
- (A2) $u(x, y) \geq f(x) \quad \forall x, y \in \mathcal{X},$
- (A3) $u'(x, y; d)|_{x=y} = f'(y; d) \quad \forall d \text{ with } y + d \in \mathcal{X},$
- (A4) $u(x, y)$ is continuous in $(x, y).$

The assumptions (A1) and (A2) imply that the approximate function $u(\cdot, x^{r-1})$ in (3.2) is a tight upper bound of the original function. The assumption (A4) says that the function $u(x, y)$ should be continuous jointly in x and y . The assumption (A3) guarantees that the first order behavior of $u(\cdot, x^{r-1})$ is the same as $f(\cdot)$ locally (note that the directional derivative $u'(x, y; d)$ is only with respect to the variable x). Although directly checking (A3) may not be easy, the following proposition provides a sufficient condition under which (A3) holds true automatically.

PROPOSITION 1. *Assume $f(x) = f_0(x) + f_1(x)$, where $f_0(\cdot)$ is differentiable and the directional derivative of $f_1(\cdot)$ exists at every point $x \in \mathcal{X}$. Consider $u(x, y) = u_0(x, y) + f_1(x)$, where $u_0(x, y)$ is a differentiable function satisfying*

$$(3.3) \quad u_0(y, y) = f_0(y) \quad \forall y \in \mathcal{Y},$$

$$(3.4) \quad u_0(x, y) \geq f_0(x) \quad \forall x, y \in \mathcal{Y},$$

where \mathcal{Y} is an open set containing \mathcal{X} . Then, (A1), (A2), and (A3) hold for $u(\cdot, \cdot)$.

Proof. First, (3.3) and (3.4) imply (A1) and (A2) immediately. Now we prove (A3). For any fixed $y \in \mathcal{X}$, the function $g(x) \triangleq u_0(x, y) - f_0(x)$ achieves its global minimum at the point $x = y$. Hence the first order optimality condition implies

$$u'_0(x, y; d)|_{x=y} - f'_0(y; d) = 0 \quad \forall d,$$

which completes the proof. \square

The following theorem establishes the convergence for the SUM algorithm.

THEOREM 1. *Suppose that Assumption 1 is satisfied. Then every limit point of the iterates generated by the SUM algorithm is a stationary point of the problem (3.1).*

Proof. First, we observe the series of inequalities

$$(3.5) \quad f(x^{r+1}) \stackrel{(i)}{\leq} u(x^{r+1}, x^r) \stackrel{(ii)}{\leq} u(x^r, x^r) = f(x^r) \quad \forall r = 0, 1, 2, \dots,$$

where step (i) is due to (A2), step (ii) follows from the optimality of x^{r+1} (cf. steps 4 and 5 in Figure 3.1), and the last equality is due to (A1). A straightforward consequence of (3.5) is that the sequence of the objective function values are nonincreasing, that is,

$$(3.6) \quad f(x^0) \geq f(x^1) \geq f(x^2) \geq \dots$$

¹These assumptions are weaker than those made to ensure the convergence of the IAA algorithm in [28].

Assume that there exists a subsequence $\{x^{r_j}\}$ converging to a limit point z . Then assumptions (A1), (A2) together with (3.6) imply that

$$u(x^{r_{j+1}}, x^{r_{j+1}}) = f(x^{r_{j+1}}) \leq f(x^{r_j+1}) \leq u(x^{r_j+1}, x^{r_j}) \leq u(x, x^{r_j}) \quad \forall x \in \mathcal{X}.$$

Letting $j \rightarrow \infty$, we obtain

$$u(z, z) \leq u(x, z) \quad \forall x \in \mathcal{X},$$

which implies

$$u'(x, z; d)|_{x=z} \geq 0 \quad \forall d \in \mathbb{R}^m \text{ with } z + d \in \mathcal{X}.$$

Combining this with (A3), we obtain

$$f'(z; d) \geq 0 \quad \forall d \in \mathbb{R}^m \text{ with } z + d \in \mathcal{X},$$

implying that z is a stationary point of $f(\cdot)$. \square

COROLLARY 1. *Assume that the level set $\mathcal{X}^0 = \{x \mid f(x) \leq f(x^0)\}$ is compact and Assumption 1 holds. Then, the sequence of iterates $\{x^r\}$ generated by the SUM algorithm satisfy*

$$\lim_{r \rightarrow \infty} d(x^r, \mathcal{X}^*) = 0,$$

where \mathcal{X}^* is the set of stationary points of (3.1).

Proof. We prove the claim by contradiction. Suppose on the contrary that there exists a subsequence $\{x^{r_j}\}$ such that $d(x^{r_j}, \mathcal{X}^*) \geq \gamma$ for some $\gamma > 0$. Since the sequence $\{x^{r_j}\}$ lies in the compact set \mathcal{X}^0 , it has a limit point z . By further restricting the indices of the subsequence, we obtain

$$d(z, \mathcal{X}^*) = \lim_{j \rightarrow \infty} d(x^{r_j}, \mathcal{X}^*) \geq \gamma,$$

which contradicts the fact that $z \in \mathcal{X}^*$ due to Theorem 1. \square

The above results show that under Assumption 1, the SUM algorithm is globally convergent. In the rest of this work, we derive similar results for a family of more general inexact BCD algorithms.

4. The block successive upper-bound minimization algorithm. In many practical applications, the optimization variables can be decomposed into independent blocks. Such block structure, when judiciously exploited, can lead to low-complexity algorithms that are implementable in a distributed manner. In this section, we introduce the block successive upper-bound minimization (BSUM) algorithm, which effectively takes such block structure into consideration.

Let us assume that the feasible set \mathcal{X} is the Cartesian product of n closed convex sets: $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, with $\mathcal{X}_i \subseteq \mathbb{R}^{m_i}$ and $\sum_i m_i = m$. Accordingly, the optimization variable $x \in \mathbb{R}^m$ can be decomposed as $x = (x_1, x_2, \dots, x_n)$, with $x_i \in \mathcal{X}_i$, $i = 1, \dots, n$. We are interested in solving the problem

$$(4.1) \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathcal{X}. \end{array}$$

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1 Find a feasible point  $x^0 \in \mathcal{X}$  and set  $r = 0$ 
2 repeat
3    $r = r + 1, i = (r \bmod n) + 1$ 
4   Let  $\mathcal{X}^r = \arg \min_{x_i \in \mathcal{X}_i} u_i(x_i, x^{r-1})$ 
5   Set  $x_i^r$  to be an arbitrary element in  $\mathcal{X}^r$ 
6   Set  $x_k^r = x_k^{r-1} \quad \forall k \neq i$ 
7 until some convergence criterion is met

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FIG. 4.1. Pseudocode of the BSUM algorithm.

Different from the SUM algorithm, the BSUM algorithm updates only a single block of variables in each iteration. More precisely, at iteration r , the selected block (say, block i) is computed by solving the subproblem

$$(4.2) \quad \begin{aligned} \min_{x_i} \quad & u_i(x_i, x^{r-1}) \\ \text{s.t.} \quad & x_i \in \mathcal{X}_i, \end{aligned}$$

where $u_i(\cdot, x^{r-1})$ is again an approximation (in fact, a global upper-bound) of the original objective $f(\cdot)$ at the point x^{r-1} . Figure 4.1 summarizes the main steps of the BSUM algorithm. Note that although the blocks are updated following a simple cyclic rule, the algorithm and its convergence results can be easily extended to the (more general) essentially cyclic update rule as well. This point will be further elaborated on in section 7.

Now we are ready to study the convergence behavior of the BSUM algorithm. To this end, the following regularity conditions on the function $u_i(\cdot, \cdot)$ are needed.

ASSUMPTION 2.

- (B1) $u_i(y_i, y) = f(y) \quad \forall y \in \mathcal{X}, \forall i$,
- (B2) $u_i(x_i, y) \geq f(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n) \quad \forall x_i \in \mathcal{X}_i, \forall y \in \mathcal{X}, \forall i$,
- (B3) $u'_i(x_i, y; d_i)|_{x_i=y_i} = f'(y; d) \quad \forall d = (0, \dots, d_i, \dots, 0) \text{ s.t. } y_i + d_i \in \mathcal{X}_i \forall i$,
- (B4) $u_i(x_i, y)$ is continuous in $(x_i, y) \quad \forall i$.

Similar to Proposition 1, we can identify a sufficient condition to ensure (B3).

PROPOSITION 2. Assume $f(x) = f_0(x) + f_1(x)$, where $f_0(\cdot)$ is differentiable and the directional derivative of $f_1(\cdot)$ exists at every point $x \in \mathcal{X}$. Consider $u_i(x_i, y) = u_{0,i}(x_i, y) + f_1(x)$ with $u_{0,i}(x_i, y)$ satisfying

$$\begin{aligned} u_{0,i}(x_i, x) &= f_0(x) \quad \forall x \in \mathcal{Y}, \quad \forall i, \\ u_{0,i}(x_i, y) &\geq f_0(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n) \quad \forall x, y \in \mathcal{Y}, \quad \forall i, \end{aligned}$$

where \mathcal{Y} is an open set containing \mathcal{X} . Then, (B1), (B2), and (B3) hold.

Proof. The proof is exactly the same as the proof in Proposition 1. \square

The convergence results regarding the BSUM algorithm consist of two parts. In the first part, a quasi convexity of the objective function is assumed, which guarantees the existence of the limit points. This is in the same spirit as the classical proof of convergence for the BCD method in [3]. However, if we know that the iterates lie in

a compact set, then a stronger result can be proved. Indeed, in the second part of the theorem, the convergence is obtained by relaxing the quasi-convexity assumption while imposing the compactness assumption of level sets.

THEOREM 2. (a) Suppose that the function $u_i(x_i, y)$ is quasi-convex in x_i for $i = 1, \dots, n$ and Assumption 2 holds. Furthermore, assume that the subproblem (4.2) has a unique solution for any point $x^{r-1} \in \mathcal{X}$. Then, every limit point z of the iterates generated by the BSUM algorithm is a coordinatewise minimum of (4.1). In addition, if $f(\cdot)$ is regular at z , then z is a stationary point of (4.1).

(b) Suppose the level set $\mathcal{X}^0 = \{x \mid f(x) \leq f(x^0)\}$ is compact and Assumption 2 holds. Furthermore, assume that $f(\cdot)$ is regular at any point in \mathcal{X}^0 and the subproblem (4.2) has a unique solution for any point $x^{r-1} \in \mathcal{X}$ for at least $n-1$ blocks. Then, the iterates generated by the BSUM algorithm converge to the set of stationary points, i.e.,

$$\lim_{r \rightarrow \infty} d(x^r, \mathcal{X}^*) = 0.$$

Proof. The proof of part (a) is similar to that in [3] for the BCD approach. First, since a locally tight upper bound of $f(\cdot)$ is minimized at each iteration, we have

$$(4.3) \quad f(x^0) \geq f(x^1) \geq f(x^2) \geq \dots$$

Consider a limit point z . Combining (4.3) with the continuity of $f(\cdot)$ implies

$$(4.4) \quad \lim_{r \rightarrow \infty} f(x^r) = f(z).$$

Let us consider the subsequence $\{x^{r_j}\}$ converging to the limit point z . Since the number of blocks is finite, there exists a block which is updated infinitely often in the subsequence $\{r_j\}$. Without loss of generality, we assume that block n is updated infinitely often. Thus, by further restricting to a subsequence, we can write

$$x_n^{r_j} = \arg \min_{x_n} u_n(x_n, x^{r_j-1}).$$

Now we prove that $x^{r_j+1} \rightarrow z$; in other words, we will show that $x_1^{r_j+1} \rightarrow z_1$. Assume to the contrary that $x_1^{r_j+1}$ does not converge to z_1 . Therefore, by further restricting to a subsequence, there exists $\bar{\gamma} > 0$ such that

$$\bar{\gamma} \leq \gamma^{r_j} = \|x_1^{r_j+1} - x_1^{r_j}\| \quad \forall r_j.$$

Let us normalize the difference between $x_1^{r_j}$ and $x_1^{r_j+1}$, i.e.,

$$s^{r_j} \triangleq \frac{x_1^{r_j+1} - x_1^{r_j}}{\gamma^{r_j}}.$$

Notice that $\|s^{r_j}\| = 1$; thus s^{r_j} belongs to a compact set and has a limit point \bar{s} . By further restricting to a subsequence that converges to \bar{s} , using (B1) and (B2), we obtain

$$(4.5) \quad f(x^{r_j+1}) \leq u_1(x_1^{r_j+1}, x^{r_j})$$

$$(4.6) \quad = u_1(x_1^{r_j} + \gamma^{r_j} s^{r_j}, x^{r_j})$$

$$(4.7) \quad \leq u_1(x_1^{r_j} + \epsilon \bar{\gamma} s^{r_j}, x^{r_j}) \quad \forall \epsilon \in [0, 1]$$

$$(4.8) \quad \leq u_1(x_1^{r_j}, x^{r_j})$$

$$(4.9) \quad = f(x^{r_j}),$$

where (4.5) and (4.9) hold due to (B1) and (B2). The inequalities (4.7) and (4.8) are the result of the quasi convexity of $u(\cdot, x^{r_j})$. Letting $j \rightarrow \infty$ and combining (4.5), (4.7), (4.4), and (4.9) imply

$$f(z) \leq u_1(z_1 + \epsilon \bar{\gamma} \bar{s}, z) \leq f(z) \quad \forall \epsilon \in [0, 1],$$

or, equivalently,

$$(4.10) \quad f(z) = u_1(z_1 + \epsilon \bar{\gamma} \bar{s}, z) \quad \forall \epsilon \in [0, 1].$$

Furthermore,

$$\begin{aligned} u_1(x_1^{r_{j+1}}, x^{r_{j+1}}) &= f(x^{r_{j+1}}) \leq f(x^{r_j+1}) \\ &\leq u_1(x_1^{r_j+1}, x^{r_j}) \leq u_1(x_1, x^{r_j}) \quad \forall x_1 \in \mathcal{X}_1. \end{aligned}$$

Letting $j \rightarrow \infty$, we obtain

$$u_1(z_1, z) \leq u_1(x_1, z) \quad \forall x_1 \in \mathcal{X}_1,$$

which further implies that z_1 is the minimizer of $u_1(\cdot, z)$. On the other hand, we assume that the minimizer is unique, which contradicts (4.10). Therefore, the contrary assumption is not true, i.e., $x^{r_j+1} \rightarrow z$.

Since $x_1^{r_j+1} = \arg \min_{x_1 \in \mathcal{X}_1} u_1(x_1, x^{r_j})$, we get

$$u_1(x_1^{r_j+1}, x^{r_j}) \leq u_1(x_1, x^{r_j}) \quad \forall x_1 \in \mathcal{X}_1.$$

Taking the limit $j \rightarrow \infty$ implies

$$u_1(z_1, z) \leq u_1(x_1, z) \quad \forall x_1 \in \mathcal{X}_1,$$

which further implies

$$u'_1(x_1, z; d_1)|_{x_1=z_1} \geq 0 \quad \forall d_1 \in \mathbb{R}^{m_1} \quad \text{with} \quad z_1 + d_1 \in \mathcal{X}_1.$$

Similarly, by repeating the above argument for the other blocks, we obtain

$$(4.11) \quad u'_k(x_k, z; d_k)|_{x_k=z_k} \geq 0 \quad \forall d_k \in \mathbb{R}^{m_k} \quad \text{with} \quad d_k + z_k \in \mathcal{X}_k \quad \forall k = 1, \dots, n.$$

Combining (B3) and (4.11) implies

$$f'(z; d) \geq 0 \quad \forall d = (0, \dots, d_k, \dots, 0) \quad \text{s.t.} \quad d + z \in \mathcal{X} \quad \forall k;$$

in other words, z is the coordinatewise minimum of $f(\cdot)$.

Now we prove part (b) of the theorem. Without loss of generality, let us assume that (4.2) has a unique solution at every point x^{r-1} for $i = 1, 2, \dots, n-1$. Since the iterates lie in a compact set, we need only show that every limit point of the iterates is a stationary point of $f(\cdot)$. To do so, let us consider a subsequence $\{x^{r_j}\}$ which converges to a limit point $z \in \mathcal{X}^0 \subseteq \mathcal{X}$. Since the number of blocks is finite, there exists a block i which is updated infinitely often in the subsequence $\{x^{r_j}\}$. By further restricting to a subsequence, we can assume that

$$x_i^{r_j} \in \arg \min_{x_i} u_i(x_i, x^{r_j-1}).$$

Since all the iterates lie in a compact set, we can further restrict to a subsequence such that

$$\lim_{j \rightarrow \infty} x^{r_j - i + k} = z^k \quad \forall k = 0, 1, \dots, n,$$

where $z^k \in \mathcal{X}^0 \subseteq \mathcal{X}$ and $z^i = z$. Moreover, due to the update rule in the algorithm, we have

$$u_k(x_k^{r_j - i + k}, x^{r_j - i + k - 1}) \leq u_k(x_k, x^{r_j - i + k - 1}) \quad \forall x_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n.$$

Taking the limit $j \rightarrow \infty$, we obtain

$$(4.12) \quad u_k(z_k^k, z^{k-1}) \leq u_k(x_k, z^{k-1}) \quad \forall x_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n.$$

This, plus (B1) and (B2), implies

$$(4.13) \quad f(z^k) \leq u_k(z_k^k, z^{k-1}) \leq u_k(z_k^{k-1}, z^{k-1}) = f(z^{k-1}), \quad k = 1, \dots, n.$$

On the other hand, the objective function is nonincreasing in the algorithm and it has a limit. Thus, due to the continuity of $f(\cdot)$, we have

$$(4.14) \quad f(z^0) = f(z^1) = \dots = f(z^n).$$

Using (4.13), (4.14), and (4.12), we obtain

$$(4.15) \quad f(z) = u_k(z_k^k, z^{k-1}) \leq u_k(x_k, z^{k-1}) \quad \forall x_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n.$$

Furthermore, $f(z) = f(z^{k-1}) = u_k(z_k^{k-1}, z^{k-1})$, and therefore,

$$(4.16) \quad u_k(z_k^{k-1}, z^{k-1}) \leq u_k(x_k, z^{k-1}) \quad \forall x_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n.$$

The inequalities (4.15) and (4.16) imply that both z_k^{k-1} and z_k^k are the minimizer of $u_k(\cdot, z^{k-1})$. However, according to our assumption, the minimizer is unique for $k = 1, 2, \dots, n-1$, and therefore

$$z^0 = z^1 = z^2 = \dots = z^{n-1} = z.$$

Plugging the above relation into (4.12) implies

$$(4.17) \quad u_k(z_k, z) \leq u_k(x_k, z) \quad \forall x_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n-1.$$

Moreover, by setting $k = n$ in (4.16), we obtain

$$(4.18) \quad u_n(z_n, z) \leq u_n(x_n, z) \quad \forall x_n \in \mathcal{X}_n.$$

The inequalities (4.17) and (4.18) imply that

$$u'_k(x_k, z; d_k) \Big|_{x_k = z_k} \geq 0 \quad \forall d_k \in \mathbb{R}^{m_k} \text{ with } z_k + d_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n.$$

Combining this with (B3) yields

$$f'(z; d) \geq 0 \quad \forall d = (0, \dots, d_k, \dots, 0) \text{ with } z_k + d_k \in \mathcal{X}_k, \quad k = 1, 2, \dots, n,$$

which implies the stationarity of the point z due to the regularity of $f(\cdot)$. \square

```

1 Find a feasible point  $x^0 \in \mathcal{X}$  and set  $r = 0$ 
2 repeat
3    $r = r + 1$ 
4   Let  $k = \arg \min_i \min_{x_i} u_i(x_i, x^{r-1})$ 
5   Let  $\mathcal{X}^r = \arg \min_{x_k \in \mathcal{X}_k} u_k(x_k, x^{r-1})$ 
6   Set  $x_k^r$  to be an arbitrary element in  $\mathcal{X}^r$ 
7   Set  $x_i^r = x_i^{r-1} \quad \forall i \neq k$ 
8 until some convergence criterion is met

```

FIG. 5.1. Pseudocode of the MISUM algorithm.

A key assumption in Theorem 2 is the uniqueness of the minimizer of (4.2), while the classical BCD method requires the uniqueness of the minimizer of (1.1) for convergence. This property is an advantage of BSUM over BCD since the uniqueness of the minimizer of (4.2) depends on the choice of the upper-bound $u(x, y)$, while the uniqueness of the solution of (1.1) depends only on the objective function. Another key assumption in Theorem 2 is the regularity of the objective function. Notice that this assumption is necessary even for the classical BCD method. To see the necessity of this assumption consider the function $f(z) = \|Az\|$ defined in section 2. Clearly, the point $z^* = (-4, 3)$ is a fixed point of the BCD method, while it is not a stationary point of $f(\cdot)$.

Theorem 2 extends the existing result of BCD method [3] and [38] to the BSUM case where only an approximation of the objective function is minimized at each iteration. As we will see in section 8, our result implies the global convergence of several existing algorithms including the EM algorithm or the DC method when the Gauss–Seidel update rule is used.

5. The maximum improvement successive upper-bound minimization algorithm. A key assumption for the BSUM algorithm is the uniqueness of the minimizer of the subproblem. This assumption is necessary even for the simple BCD method [3]. In general, by removing such an assumption, the convergence is not guaranteed (see [32] for examples) unless we assume pseudoconvexity in pairs of the variables [42], [38]. In this section, we explore the possibility of removing such a uniqueness assumption.

Recently, Chen et al. [10] have proposed a related maximum block improvement (MBI) algorithm, which differs from the conventional BCD algorithm only by its update schedule. More specifically, only the block that provides the *maximum improvement* is updated at each step. Remarkably, by utilizing such a modified updating rule (which is similar to the well-known Gauss–Southwell update rule), the per-block subproblems are allowed to have multiple solutions. Inspired by this recent development, we propose modifying the BSUM algorithm similarly by simply updating the block that gives the maximum improvement. We name the resulting algorithm the maximum improvement successive upper-bound minimization (MISUM) algorithm and list its main steps in Figure 5.1.

Clearly the MISUM algorithm is more general than the MBI method proposed in [10], since only an approximate version of the subproblem is solved at each iteration. Theorem 3 states the convergence result for the proposed MISUM algorithm.

THEOREM 3. *Suppose that Assumption 2 is satisfied. Then, every limit point z of*

the iterates generated by the MISUM algorithm is a coordinatewise minimum of (4.1). In addition, if $f(\cdot)$ is regular at z , then z is a stationary point of (4.1).

Proof. Let us define $R_i(y)$ to be the minimum objective value of the i th subproblem at a point y , i.e.,

$$R_i(y) \triangleq \min_{x_i} u_i(x_i, y).$$

Using an argument similar to that in Theorem 2, we can show that the sequence of the objective function values are nonincreasing; that is,

$$f(x^r) = u_i(x_i^r, x^r) \geq R_i(x^r) \geq f(x^{r+1}).$$

Let $\{x^{r_j}\}$ be the subsequence converging to a limit point z . For every fixed block index $i = 1, 2, \dots, n$ and every $x_i \in \mathcal{X}_i$, we have the series of inequalities

$$\begin{aligned} u_i(x_i, x^{r_j}) &\geq R_i(x^{r_j}) \\ &\geq u_k(x_k^{r_j+1}, x^{r_j}) \\ &\geq f(x^{r_j+1}) \\ &\geq f(x^{r_{j+1}}) \\ &= u_i(x_i^{r_{j+1}}, x^{r_{j+1}}), \end{aligned}$$

where we use k to index the block that provides the maximum improvement at iteration $r_j + 1$. The first and second inequalities are due to the definition of the function $R_i(\cdot)$ and the MISUM update rule, respectively. The third inequality is implied by the upper-bound assumption (B2), while the last inequality is due to the nonincreasing property of the objective values.

Letting $j \rightarrow \infty$, we obtain

$$u_i(x_i, z) \geq u_i(z_i, z) \quad \forall x_i \in \mathcal{X}_i, \quad i = 1, 2, \dots, n.$$

The first order optimality condition implies

$$u'_i(x_i, z; d_i) \Big|_{x_i=z_i} \geq 0 \quad \forall d_i \text{ with } z_i + d_i \in \mathcal{X}_i \quad \forall i = 1, 2, \dots, n.$$

Combining this with (B3) yields

$$f'(z; d) \geq 0 \quad \forall d = (0, \dots, d_i, \dots, 0) \text{ with } z_i + d_i \in \mathcal{X}_i, \quad i = 1, 2, \dots, n.$$

In other words, z is the coordinatewise minimum of $f(\cdot)$. \square

The main advantage of the MISUM algorithm over the BSUM algorithm is that its convergence does not rely on the uniqueness of the minimizer for the subproblems. On the other hand, each iteration of the MISUM algorithm is more expensive than the BSUM since the minimization needs to be performed for all the blocks. Nevertheless, the MISUM algorithm is more suitable when parallel processing units are available, since the minimizations with respect to all the blocks can be carried out simultaneously.

6. Successive convex approximation of a smooth function. In the previous sections, we have demonstrated that the stationary solutions of the problems (3.1) and (4.1) can be obtained by successively minimizing a sequence of upper-bounds

```

1 Find a feasible point  $x^0 \in \mathcal{X}$  and set  $r = 0$ 
2 repeat
3    $r = r + 1, i = (r \bmod n) + 1$ 
4   Let  $\mathcal{X}^r = \arg \min_{x_i \in \mathcal{X}_i} h_i(x_i, x^{r-1})$ 
5   Set  $y_i^r$  to be an arbitrary element in  $\mathcal{X}^r$  and set  $y_k^r = x_k^{r-1} \quad \forall k \neq i$ 
6   Set  $d^r = y^r - x^{r-1}$  and choose  $\sigma \in (0, 1)$ 
7   Armijo stepsize rule: Choose  $\alpha^{\text{init}} > 0$  and  $\sigma, \beta \in (0, 1)$ . Let  $\alpha^r$  be the
   largest element in  $\{\alpha^{\text{init}} \beta^j\}_{j=0,1,\dots}$  satisfying
    $f(x^{r-1}) - f(x^{r-1} + \alpha^r d^r) \geq -\sigma \alpha^r f'(x^{r-1}; d^r)$ 
8   Set  $x^r = x^{r-1} + \alpha^r (y^r - x^{r-1})$ 
9 until some convergence criterion is met

```

FIG. 6.1. Pseudocode of the BSCA algorithm.

of $f(\cdot)$. However, in practice, unless the objective $f(\cdot)$ possesses certain convexity/concavity structure, those upper-bounds may not be easily identifiable. In this section, we extend the BSUM algorithm by further relaxing the requirement that the approximation functions $\{u_i(x_i, y)\}$ must be the global upper-bounds of the original objective f .

Throughout this section, we use $h_i(\cdot, \cdot)$ to denote the convex approximation function for the i th block. Suppose that $h_i(x_i, x)$ is no longer a global upper-bound of $f(x)$, but only a first order approximation of $f(x)$ at each point, i.e.,

$$(6.1) \quad h'_i(y_i, x; d_i) \big|_{y_i=x_i} = f'(x; d) \quad \forall d = (0, \dots, d_i, \dots, 0) \quad \text{with } x_i + d_i \in \mathcal{X}_i.$$

In this case, simply optimizing the approximate functions in each step may not even decrease the objective function. Nevertheless, the minimizer obtained in each step can still be used to construct a good search direction, which, when combined with a proper step size selection rule, can yield a sufficient decrease of the objective value.

Suppose that at iteration r , the i th block needs to be updated. Let $y_i^r \in \min_{y_i \in \mathcal{X}_i} h_i(y_i, x^{r-1})$ denote the optimal solution for optimizing the i th approximation function at the point x^{r-1} . We propose using $y_i^r - x_i^{r-1}$ as the search direction and adopt the Armijo rule to guide the step size selection process. We name the resulting algorithm the block successive convex approximation (BSCA) algorithm. Its main steps are given in Figure 6.1.

Note that for $d^r = (0, \dots, d_i^r, \dots, 0)$ with $d_i^r = y_i^r - x_i^{r-1}$, we have

$$(6.2) \quad f'(x^{r-1}; d^r) = h'_i(x_i, x^{r-1}; d_i^r) \big|_{x_i=x_i^r} = \lim_{\lambda \downarrow 0} \frac{h_i(x_i^{r-1} + \lambda d_i^r, x^{r-1}) - h_i(x_i^{r-1}, x^{r-1})}{\lambda} \leq 0,$$

where the inequality is due to the fact that $h_i(\cdot)$ is convex and $y_i^r = x_i^{r-1} + d_i^r$ is the minimizer at iteration r . Moreover, there holds

$$f(x^{r-1}) - f(x^{r-1} + \alpha d^r) = -\alpha f'(x^{r-1}; d^r) + o(\alpha) \quad \forall \alpha > 0.$$

Hence the Armijo step size selection rule in Figure 6.1 is well defined when $f'(x^{r-1}; d^r) \neq 0$, and there exists $j \in \{0, 1, \dots\}$ such that for $\alpha^r = \alpha^{\text{init}} \beta^j$,

$$(6.3) \quad f(x^r) - f(x^r + \alpha^{r+1} d^{r+1}) \geq -\sigma \alpha^{r+1} f'(x^r; d^{r+1}).$$

The following theorem states the convergence result of the proposed algorithm.

THEOREM 4. *Suppose that $f(\cdot)$ is continuously differentiable and that assumption (6.1) holds. Furthermore, assume that $h(x, y)$ is strictly convex in x and continuous in (x, y) . Then every limit point of the iterates generated by the BSCA algorithm is a stationary point of (3.1).*

Proof. First, due to the use of Armijo step size selection rule, we have

$$(6.4) \quad f(x^r) - f(x^{r+1}) \geq -\sigma \alpha^r f'(x^r; d^r) \geq 0.$$

Consider a limit point z and a subsequence $\{x^{r_j}\}_j$ converging to z . Since $\{f(x^r)\}$ is a monotonically decreasing sequence, it follows that

$$\lim_{r \rightarrow \infty} f(x^r) = f(z).$$

Moreover, (6.4) implies

$$(6.5) \quad \lim_{r \rightarrow \infty} \alpha^r f'(x^r; d^r) = 0.$$

By further restricting to a subsequence if necessary, we can assume without loss of generality that in the subsequence $\{x^{r_j}\}_j$ the first block is updated. We first claim that we can restrict to a further subsequence if necessary so that

$$(6.6) \quad \lim_{j \rightarrow \infty} d^{r_j+1} = 0.$$

We prove this by contradiction. Let us assume the contrary so that there exist a δ , $0 < \delta < 1$, and an $\ell \in \{1, 2, \dots\}$ with

$$(6.7) \quad \|d^{r_j+1}\| \geq \delta \quad \forall j \geq \ell.$$

Define $p^{r_j+1} = \frac{d^{r_j+1}}{\|d^{r_j+1}\|}$. Equation (6.5) implies $\alpha^{r_j+1} \|d^{r_j+1}\| f'(x^{r_j}; p^{r_j+1}) \rightarrow 0$. We consider the following two cases.

Case A: $f'(x^{r_j}; p^{r_j+1}) \rightarrow 0$ along a subsequence of $\{r_j\}$. Let us restrict ourselves to that subsequence. Since $\|p^{r_j+1}\| = 1$, there exists a limit point \bar{p} . By further restricting to a subsequence and using the smoothness of $f(\cdot)$, we obtain

$$(6.8) \quad f'(z; \bar{p}) = 0.$$

Furthermore, due to the strict convexity of $h_1(\cdot, z)$,

$$(6.9) \quad h_1(z_1 + \delta \bar{p}_1, z) > h_1(z_1, z) + \delta h'_1(x_1, z; \bar{p}_1)|_{x_1=z_1} \geq h_1(z_1, z),$$

where \bar{p}_1 is the first block of \bar{p} and the last step is due to (6.8) and (6.1). On the other hand, since $x_1^{r_j+1} + \delta p_1^{r_j}$ lies between $x_1^{r_j}$ and $y_1^{r_j}$, we have (from the convexity of $h_1(\cdot, x^{r_j})$)

$$h_1(x_1^{r_j} + \delta p_1^{r_j+1}, x^{r_j}) \leq h_1(x_1^{r_j}, x^{r_j}).$$

Letting $j \rightarrow \infty$ along the subsequence, we obtain

$$(6.10) \quad h_1(z_1 + \delta \bar{p}_1, z) \leq h_1(z_1, z),$$

which contradicts (6.9).

Case B: $\alpha^{r_j+1}\|d^{r_j+1}\| \rightarrow 0$ along a subsequence. Let us restrict ourselves to that subsequence. Due to the hypothesis (6.7),

$$\lim_{j \rightarrow \infty} \alpha^{r_j+1} = 0,$$

which further implies that there exists $j_0 \in \{1, 2, \dots\}$ such that

$$f\left(x^{r_j} + \frac{\alpha^{r_j+1}}{\beta} d^{r_j+1}\right) - f(x^{r_j}) > \sigma \frac{\alpha^{r_j+1}}{\beta} f'(x^{r_j}; d^{r_j+1}) \quad \forall j \geq j_0.$$

Rearranging the terms, we obtain

$$\frac{f\left(x^{r_j} + \frac{\alpha^{r_j+1}}{\beta} \|d^{r_j+1}\| p^{r_j+1}\right) - f(x^{r_j})}{\frac{\alpha^{r_j+1}}{\beta} \|d^{r_j+1}\|} > \sigma f'(x^{r_j}; p^{r_j+1}) \quad \forall j \geq j_0.$$

Letting $j \rightarrow \infty$ along the subsequence that $p^{r_j+1} \rightarrow \bar{p}$, we obtain

$$f'(z; \bar{p}) \geq \sigma f'(z; \bar{p}),$$

which implies $f(z; \bar{p}) \geq 0$ since $\sigma < 1$. Therefore, using an argument similar to the previous case, (6.9) and (6.10) hold, which is a contradiction. Thus, the assumption (6.7) must be false and the condition (6.6) must hold. On the other hand, $y_1^{r_j+1}$ is the minimizer of $h_1(\cdot, x^{r_j})$; thus,

$$(6.11) \quad h_1(y_1^{r_j+1}, x^{r_j}) \leq h_1(x_1, x^{r_j}) \quad \forall x_1 \in \mathcal{X}_1.$$

Note that $y_1^{r_j+1} = x_1^{r_j} + d_1^{r_j+1}$. Combining (6.6) and (6.11) and letting $j \rightarrow \infty$ yield

$$h_1(z_1, z) \leq h_1(x_1, z) \quad \forall x_1 \in \mathcal{X}_1.$$

The first order optimality condition and assumption (6.1) imply

$$f'(z; d) \geq 0 \quad \forall d = (d_1, 0, \dots, 0) \quad \text{with} \quad z_1 + d_1 \in \mathcal{X}_1.$$

On the other hand, since $d^{r_j+1} \rightarrow 0$, it follows that

$$\lim_{j \rightarrow \infty} x^{r_j+1} = z.$$

Therefore, by restricting ourselves to the subsequence that $d^{r_j+1} \rightarrow 0$ and repeating the above argument n times, we obtain

$$f'(z; d) \geq 0 \quad \forall d = (0, \dots, d_k, \dots, 0) \quad \text{with} \quad z_k + d_k \in \mathcal{X}_k, \quad k = 1, \dots, n.$$

Using the regularity of $f(\cdot)$ at point z completes the proof. \square

We remark that the proposed BSCA method is related to the coordinate gradient descent method [39], in which a strictly convex second order approximation of the objective function is minimized at each iteration. It is important to note that the convergence results of these two algorithms do not imply each other. The BSCA algorithm, although more general in the sense that the approximation function can take the form of any strictly convex function satisfying (6.1), covers only the case when the objective function is smooth. Nevertheless, the freedom provided by the BSCA to choose a more general approximation function allows one to better approximate the original function at each iteration. It is also worth noting that the idea of the coordinate line search method has also appeared in [17], where the unconstrained smooth optimization problem is considered. An efficient line search algorithm is proposed so that the subproblems related to certain blocks are solved approximately. Another interesting related work is [5], where the direction d is obtained by projected gradient direction with respect to only one of the coordinates.

7. Overlapping essentially cyclic rule. In both the BSUM and the BSCA algorithms considered in the previous sections, variable blocks are updated in a simple cyclic manner. In this section, we consider a very general block scheduling rule named the overlapping essentially cyclic rule and show that it still ensures the convergence of the BSUM and the BSCA algorithms.

In the so-called overlapping essentially cyclic rule, at each iteration r , a group ϑ^r of the variables is chosen to be updated where

$$\vartheta^r \subseteq \{1, 2, \dots, n\} \quad \text{and} \quad \vartheta^r \neq \emptyset.$$

Furthermore, we assume that the update rule is essentially cyclic with period T , i.e.,

$$\bigcup_{i=1}^T \vartheta^{r+i} = \{1, 2, \dots, n\} \quad \forall r.$$

Notice that in the classical essentially cyclic rule [38], in addition to the above condition, the cardinality of each set ϑ must be one for all r ; while in the overlapping essentially cyclic method, the blocks are allowed to have overlaps. Using the overlapping essentially cyclic update rule, almost all the convergence results presented so far still hold. For example, the following corollary extends the convergence of BSUM to the overlapping essentially cyclic case.

COROLLARY 2. (a) Assume that the function $u_i(x_i, y)$ is quasi-convex in x_i and Assumption 2 is satisfied. Furthermore, assume that the overlapping essentially cyclic update rule with period T is used and the subproblem (4.2) has a unique solution for every block ϑ^r . Then, every limit point z of the iterates generated by the BSUM algorithm is a coordinatewise minimum of (4.1). In addition, if $f(\cdot)$ is regular at z with respect to the updated blocks, then z is a stationary point of (4.1).

(b) Assume the level set $\mathcal{X}^0 = \{x \mid f(x) \leq f(x^0)\}$ is compact and Assumption 2 is satisfied. Furthermore, assume that the overlapping essentially cyclic update rule is used and the subproblem (4.2) has a unique solution for every block ϑ^r . If $f(\cdot)$ is regular (with respect to the updated blocks), then the iterates generated by the BSUM algorithm converge to the set of stationary points, i.e.,

$$\lim_{r \rightarrow \infty} d(x^r, \mathcal{X}^*) = 0.$$

Proof. The proofs of both cases are similar to the proof of the BSUM algorithm with the simple cyclic update rule. Here we present the proof for case (a) only. The proof of part (b) is similar.

Let $\{x^{r_j}\}$ be a convergent subsequence whose limit is denoted by z . Consider every T updating cycle along the subsequence $\{x^{r_j}\}$, namely, $\{(x^{r_j}, x^{r_j+1}, \dots, x^{r_j+T-1})\}$. Since the number of different subblocks ϑ^r is finite, there must exist a (fixed) T tuple of variable blocks, say, $(\vartheta_0, \vartheta_1, \dots, \vartheta_{T-1})$, that has been updated in infinitely many T updating cycles. By restricting our attention to the corresponding subsequence of $\{x^{r_j}\}$, we have

$$x_{\vartheta_i}^{r_j+i+1} = \arg \min_{x_{\vartheta_i}} u_{\vartheta_i}(x_{\vartheta_i}, x^{r_j+i}) \quad \forall i = 0, 1, 2, \dots, T-1.$$

The rest of the proof is the same as the proof of part (a) in Theorem 2. The only difference is that the steps of the proof need to be repeated for the blocks $(\vartheta_0, \vartheta_1, \dots, \vartheta_{T-1})$ instead of $(1, \dots, n)$. \square

In the proof of Corollary 2, we first restrict ourselves to a fixed set of T variable blocks that have been updated in infinitely many consecutive T update cycles. Then, we use the same approach as in the proof of the convergence of cyclic update rule. Using the same technique, we can extend the results in Theorem 4 to the overlapping essentially cyclic update rule. More specifically, we have the following corollary.

COROLLARY 3. *Assume $f(\cdot)$ is smooth and the condition (6.1) is satisfied. Furthermore, assume that $h(x, y)$ is strictly convex in x and the overlapping essentially cyclic update rule is used in the BSCA algorithm. Then every limit point of the iterates generated by the BSCA algorithm is a stationary point of (3.1).*

Notice that the overlapping essentially cyclic rule is not applicable to the MISUM algorithm in which the update order of the variables is given by the amount of improvement. However, one can simply check that the proof of Theorem 3 still applies to the case when the blocks are allowed to have overlaps.

8. Applications. In this section, we provide several applications of the algorithms proposed in the previous sections.

8.1. Linear transceiver design in cellular networks. Consider a K -cell wireless network where each base station k serves a set \mathcal{I}_k of users (see Figure 8.1 for an illustration). Let i_k denote the i th receiver in cell k . For simplicity, suppose that the users and the base stations are all equipped with N antennas. Let us define the set of all users as $\mathcal{I} \triangleq \{i_k \mid 1 \leq k \leq K, i \in \mathcal{I}_k\}$. Let d_{i_k} denote the number of data symbols transmitted simultaneously to user i_k .

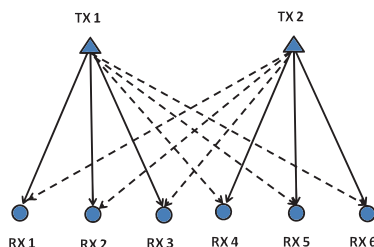


FIG. 8.1. The cellular network model considered in section 8.1. The solid lines represent the direct channels, while the dotted lines represent the interfering channels.

When linear transceivers are used at the base stations and the users, user i_k 's received signal vector, denoted as $\mathbf{y}_{i_k} \in \mathbb{C}^N$, can be written as

$$\mathbf{y}_{i_k} = \underbrace{\mathbf{H}_{i_k k} \mathbf{V}_{i_k} \mathbf{s}_{i_k}}_{\text{desired signal}} + \underbrace{\sum_{\ell \neq i, \ell=1}^{I_k} \mathbf{H}_{i_k k} \mathbf{V}_{\ell k} \mathbf{s}_{\ell k}}_{\text{intracell interference}} + \underbrace{\sum_{j \neq k, j=1}^K \sum_{\ell=1}^{I_j} \mathbf{H}_{i_k j} \mathbf{V}_{\ell j} \mathbf{s}_{\ell j}}_{\text{intercell interference plus noise}} + \mathbf{n}_{i_k} \quad \forall i_k \in \mathcal{I},$$

where $\mathbf{V}_{i_k} \in \mathbb{C}^{M \times d_{i_k}}$ is the linear transmit beamformer used by base station k for user i_k ; $\mathbf{s}_{i_k} \in \mathbb{C}^{d_{i_k} \times 1}$ is user i_k 's data signal. The matrix $\mathbf{H}_{i_k j}$ represents the channel from transmitter j to receiver i_k , and \mathbf{n}_{i_k} denotes the complex additive white Gaussian noise with distribution $\mathcal{CN}(0, \sigma_{i_k}^2 \mathbf{I})$. User i_k estimates the intended message using a linear beamformer $\mathbf{U}_{i_k} \in \mathbb{C}^{M \times d_{i_k}}$: $\hat{\mathbf{s}}_{i_k} = \mathbf{U}_{i_k}^H \mathbf{y}_{i_k}$.

Treating interference as noise, the rate of user i_k is given by

$$R_{i_k}(\mathbf{V}) = \log \det \left(\mathbf{I} + \mathbf{H}_{i_k k} \mathbf{V}_{i_k} (\mathbf{V}_{i_k})^H \mathbf{H}_{i_k k}^H \left(\sigma_{i_k}^2 \mathbf{I} + \sum_{(j,\ell) \neq (k,i)} \mathbf{H}_{i_k j} \mathbf{V}_{\ell j} (\mathbf{V}_{\ell j})^H \mathbf{H}_{i_k j}^H \right)^{-1} \right).$$

We are interested in finding the beamformers \mathbf{V} such that the sum of the users' rates are optimized:

$$(8.1) \quad \begin{aligned} & \max_{\{\mathbf{V}_{i_k}\}_{i_k \in \mathcal{I}}} \sum_{k=1}^K \sum_{i=1}^{I_k} R_{i_k}(\mathbf{V}) \\ & \text{s.t.} \quad \sum_{i=1}^{I_k} \text{Tr}(\mathbf{V}_{i_k} \mathbf{V}_{i_k}^H) \leq \bar{P}_k \quad \forall k \in \mathcal{K}, \end{aligned}$$

where $\mathcal{K} = \{1, 2, \dots, K\}$. Note that we have included a transmit power constraint for each base station. It has been shown in [27] that solving (8.1) is NP-hard. Therefore, we try to obtain the stationary solution for this problem. Furthermore, we can no longer straightforwardly apply the BSUM algorithm that updates \mathbf{V}_{i_k} 's cyclically. This is due to the fact that the users in the set \mathcal{I}_k share a common power constraint. Thus the requirement for the separability of the constraints for different block components in (4.1) is not satisfied.

To devise an efficient and low complexity algorithm for problem (8.1), we will first transform this problem into a more suitable form. We first introduce the function $f_{i_k}(\mathbf{U}_{i_k}, \mathbf{V}) \triangleq \log \det(\mathbf{E}_{i_k}^{-1})$, where \mathbf{E}_{i_k} is the mean square error (MSE) matrix given as

$$\begin{aligned} \mathbf{E}_{i_k} & \triangleq (\mathbf{I} - \mathbf{U}_{i_k}^H \mathbf{H}_{i_k k} \mathbf{V}_{i_k})(\mathbf{I} - \mathbf{U}_{i_k}^H \mathbf{H}_{i_k k} \mathbf{V}_{i_k})^H \\ & + \sum_{(\ell, j) \neq (i, k)} \mathbf{U}_{i_k}^H \mathbf{H}_{i_k j} \mathbf{V}_{\ell j} \mathbf{V}_{\ell j}^H \mathbf{H}_{i_k j}^H \mathbf{U}_{i_k} + \sigma_{i_k}^2 \mathbf{U}_{i_k}^H \mathbf{U}_{i_k}. \end{aligned}$$

In the subsequent presentation we will occasionally use the notation $\mathbf{E}_{i_k}(\mathbf{U}_{i_k}, \mathbf{V})$ to make the dependency of the MSE matrix and the transceivers explicit.

Taking the derivative of $f_{i_k}(\mathbf{U}_{i_k}, \mathbf{V})$ with respect to \mathbf{U}_{i_k} and checking the first order optimality condition, we have

$$\arg \max_{\mathbf{U}_{i_k}} f_{i_k}(\mathbf{U}_{i_k}, \mathbf{V}) = \left(\sigma_{i_k}^2 \mathbf{I} + \sum_{(j,\ell)} \mathbf{H}_{i_k j} \mathbf{V}_{\ell j} \mathbf{V}_{\ell j}^H \mathbf{H}_{i_k j}^H \right)^{-1} \mathbf{H}_{i_k k} \mathbf{V}_{i_k}.$$

Plugging in the optimal value of \mathbf{U}_{i_k} in $f_{i_k}(\cdot)$, we obtain $\max_{\mathbf{U}_{i_k}} f_{i_k}(\mathbf{U}_{i_k}, \mathbf{V}) = R_{i_k}(\mathbf{V})$. Thus, we can rewrite the optimization problem equivalently (8.1) as²

$$(8.2) \quad \begin{aligned} & \min_{\mathbf{V}, \mathbf{U}} \sum_{k=1}^K \sum_{i=1}^{I_k} \log \det(\mathbf{E}_{i_k}) \\ & \text{s.t.} \quad \sum_{i=1}^{I_k} \text{Tr}(\mathbf{V}_{i_k} \mathbf{V}_{i_k}^H) \leq P_k \quad \forall k \in \mathcal{K}. \end{aligned}$$

²Such equivalence is in the sense of one-to-one correspondence of both local and global optimal solutions. See [36] for a detailed argument.

Notice the fact that the function $\log \det(\cdot)$ is a *concave* function on its argument (see, e.g., [7]); then for any feasible \mathbf{E}_{i_k} , $\hat{\mathbf{E}}_{i_k}$, we have

$$(8.3) \quad \begin{aligned} \log \det(\mathbf{E}_{i_k}) &\leq \log \det(\hat{\mathbf{E}}_{i_k}) + \text{Tr}[\nabla_{\mathbf{E}_{i_k}}(\log \det(\hat{\mathbf{E}}_{i_k}))(\mathbf{E}_{i_k} - \hat{\mathbf{E}}_{i_k})] \\ &= \log \det(\hat{\mathbf{E}}_{i_k}) + \text{Tr}[\hat{\mathbf{E}}_{i_k}^{-1}(\mathbf{E}_{i_k} - \hat{\mathbf{E}}_{i_k})] \triangleq u_{i_k}(\mathbf{E}_{i_k}, \hat{\mathbf{E}}_{i_k}). \end{aligned}$$

Utilizing the above transformation and the upper-bound, we can again apply the BSUM algorithm. Let \mathbf{V} and \mathbf{U} be two block variables. Define

$$\begin{aligned} u_{\mathbf{v}}(\mathbf{V}, (\hat{\mathbf{V}}, \hat{\mathbf{U}})) &\triangleq \sum_{k=1}^K \sum_{i=1}^{I_k} u_{i_k}(\mathbf{E}_{i_k}(\mathbf{V}, \hat{\mathbf{U}}_{i_k}), \mathbf{E}_{i_k}(\hat{\mathbf{V}}, \hat{\mathbf{U}}_{i_k})), \\ u_{\mathbf{u}}(\mathbf{U}, (\hat{\mathbf{V}}, \hat{\mathbf{U}})) &\triangleq \sum_{k=1}^K \sum_{i=1}^{I_k} u_{i_k}(\mathbf{E}_{i_k}(\hat{\mathbf{V}}, \mathbf{U}_{i_k}), \mathbf{E}_{i_k}(\hat{\mathbf{V}}, \hat{\mathbf{U}}_{i_k})). \end{aligned}$$

In iteration $2r + 1$, the algorithm solves the following problem:

$$(8.4) \quad \begin{aligned} \min_{\mathbf{V}} \quad & u_{\mathbf{v}}(\mathbf{V}, (\mathbf{V}^{2r}, \mathbf{U}^{2r})) \\ \text{s.t.} \quad & \sum_{i=1}^{I_k} \text{Tr}(\mathbf{V}_{i_k} \mathbf{V}_{i_k}^H) \leq P_k \quad \forall k \in \mathcal{K}. \end{aligned}$$

In iteration $2r + 2$, the algorithm solves the following (unconstrained) problem:

$$(8.5) \quad \min_{\mathbf{U}} \quad u_{\mathbf{u}}(\mathbf{U}, (\mathbf{V}^{2r+1}, \mathbf{U}^{2r})).$$

The above BSUM algorithm for solving (8.1) is called the WMMSE algorithm in [36].

Due to (8.3), we must have that

$$\begin{aligned} u_{\mathbf{v}}(\mathbf{V}, (\mathbf{V}^{2r}, \mathbf{U}^{2r})) &\geq \sum_{k=1}^K \sum_{i=1}^{I_k} \log \det(\mathbf{E}_{i_k}(\mathbf{V}, \mathbf{U}_{i_k}^{2r})) \quad \forall \text{ feasible } \mathbf{V}, \forall i_k, \\ u_{\mathbf{u}}(\mathbf{U}, (\mathbf{V}^{2r+1}, \mathbf{U}^{2r})) &\geq \sum_{k=1}^K \sum_{i=1}^{I_k} \log \det(\mathbf{E}_{i_k}(\mathbf{V}^{2r+1}, \mathbf{U}_{i_k})) \quad \forall \mathbf{U}_{i_k}, \forall i_k. \end{aligned}$$

Moreover, other conditions in Assumption 2 are also satisfied for $u_{\mathbf{v}}(\cdot)$ and $u_{\mathbf{u}}(\cdot)$. Thus the convergence of the WMMSE algorithm to a stationary solution of problem (8.2) follows directly from Theorem 2.

We briefly mention here that the main benefit of using the BSUM approach for solving problem (8.2) is that in each step, the problem (8.4) can be decomposed into K independent convex subproblems, one for each base station $k \in \mathcal{K}$. Moreover, the solutions for these K subproblems can be simply obtained in closed form (subject to an efficient bisection search). For more details on this algorithm, we refer the readers to [36] and [33].

The BSUM approach has been extensively used for resource allocation in wireless networks [25], [11], [35], [30], [23], [34]. However, the convergence of most of the algorithms was not rigorously established.

8.2. Proximal minimization algorithm. The classical proximal minimization algorithm (see, e.g., [4, section 3.4.3]) obtains a solution of the problem $\min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x})$ by solving an equivalent problem

$$(8.6) \quad \min_{\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{X}} f(\mathbf{x}) + \frac{1}{2c} \|\mathbf{x} - \mathbf{y}\|_2^2,$$

where $f(\cdot)$ is a convex function, \mathcal{X} is a closed convex set, and $c > 0$ is a scalar parameter. The equivalent problem (8.6) is attractive in that it is strongly convex in both x and y (but not jointly) so long as $f(x)$ is convex. This problem can be solved by performing the following two steps in an alternating fashion:

$$(8.7) \quad \mathbf{x}^{r+1} = \arg \min_{\mathbf{x} \in \mathcal{X}} \left\{ f(x) + \frac{1}{2c} \|\mathbf{x} - \mathbf{y}^r\|_2^2 \right\},$$

$$(8.8) \quad \mathbf{y}^{r+1} = \mathbf{x}^{r+1}.$$

Equivalently, let $u(\mathbf{x}; \mathbf{x}^r) \triangleq f(\mathbf{x}) + \frac{1}{2c} \|\mathbf{x} - \mathbf{x}^r\|_2^2$; then the iteration (8.7)–(8.8) can be written as

$$(8.9) \quad \mathbf{x}^{r+1} = \arg \min_{\mathbf{x} \in \mathcal{X}} u(\mathbf{x}, \mathbf{x}^r).$$

It can be straightforwardly checked that for all $\mathbf{x}, \mathbf{x}^r \in \mathcal{X}$, the function $u(\mathbf{x}, \mathbf{x}^r)$ serves as an upper bound for the function $f(\mathbf{x})$. Moreover, the conditions listed in Assumption 1 are all satisfied. Clearly, the iteration (8.9) corresponds to the SUM algorithm discussed in section 3. Consequently, the convergence of the proximal minimization procedure can be obtained from Theorem 1.

The proximal minimization algorithm can be generalized in the following way. Consider the problem

$$(8.10) \quad \begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}_1, \dots, \mathbf{x}_n) \\ \text{s.t.} \quad & \mathbf{x}_i \in \mathcal{X}_i, \quad i = 1, \dots, n, \end{aligned}$$

where $\{\mathcal{X}_i\}_{i=1}^n$ are closed convex sets and $f(\cdot)$ is convex in each of its block components but not necessarily strictly convex. A straightforward application of the BCD procedure may fail to find a stationary solution for this problem, as the per-block subproblems may contain multiple solutions. Alternatively, we can consider an *alternating proximal minimization* algorithm [2], [18], in each iteration of which the following subproblem is solved:

$$(8.11) \quad \begin{aligned} \min_{\mathbf{x}_i} \quad & f(\mathbf{x}_1^r, \dots, \mathbf{x}_{i-1}^r, \mathbf{x}_i, \mathbf{x}_{i+1}^r, \dots, \mathbf{x}_n^r) + \frac{1}{2c} \|\mathbf{x}_i - \mathbf{x}_i^r\|_2^2 \\ \text{s.t.} \quad & \mathbf{x}_i \in \mathcal{X}_i. \end{aligned}$$

It is not hard to see that this subproblem always admits a unique solution, as the objective is a strictly convex function of \mathbf{x}_i . Let $u_i(\mathbf{x}_i, \mathbf{x}^r) \triangleq f(\mathbf{x}_1^r, \dots, \mathbf{x}_i, \dots, \mathbf{x}_n^r) + \frac{1}{2c} \|\mathbf{x}_i - \mathbf{x}_i^r\|_2^2$. Again, for each $\mathbf{x}_i \in \mathcal{X}_i$ and $\mathbf{x}^r \in \prod_j \mathcal{X}_j$, the function $u_i(\mathbf{x}_i, \mathbf{x}^r)$ is an upper-bound of the original objective $f(\mathbf{x})$. Moreover, all the conditions in Assumption 2 are satisfied. Utilizing Theorem 2, we conclude that the alternating proximal minimization algorithm must converge to a stationary solution of the problem (8.10). Moreover, our result extends those in [18] to the case of a nonsmooth objective function as well as the case with iteration-dependent coefficient c . The latter case, which

was also studied in the contemporary work [40], will be demonstrated in an example for tensor decomposition shortly. It is also worth noting that the convergence of the alternating proximal minimization algorithm is also studied in [1] for Kurdyka–Lojasiewicz functions.

8.3. Proximal splitting algorithm. The proximal splitting algorithm (see, e.g., [13]) for nonsmooth optimization is also a special case of the BSUM algorithm. Consider the problem

$$(8.12) \quad \min_{\mathbf{x} \in \mathcal{X}} f_1(\mathbf{x}) + f_2(\mathbf{x}),$$

where \mathcal{X} is a closed and convex set. Furthermore, f_1 is convex and lower semicontinuous; f_2 is convex and has Lipschitz continuous gradient, i.e., $\|\nabla f_2(\mathbf{x}) - \nabla f_2(\mathbf{y})\| \leq \beta \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and for some $\beta > 0$.

Define the proximity operator $\text{prox}_{f_i} : \mathcal{X} \rightarrow \mathcal{X}$ as

$$(8.13) \quad \text{prox}_{f_i}(\mathbf{x}) = \arg \min_{\mathbf{y} \in \mathcal{X}} f_i(\mathbf{y}) + \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2.$$

The following forward-backward splitting iteration can be used to obtain a solution for problem (8.12) [13], [14]:

$$(8.14) \quad \mathbf{x}^{r+1} = \underbrace{\text{prox}_{\gamma f_1}}_{\text{backward step}} \left(\underbrace{\mathbf{x}^r - \gamma \nabla f_2(\mathbf{x}^r)}_{\text{forward step}} \right),$$

where $\gamma \in [\epsilon, 2/\beta - \epsilon]$ with $\epsilon \in]0, \min\{1, 1/\beta\}[$. Define

$$(8.15) \quad u(\mathbf{x}, \mathbf{x}^r) \triangleq f_1(\mathbf{x}) + \frac{1}{2\gamma} \|\mathbf{x} - \mathbf{x}^r\|^2 + \langle \mathbf{x} - \mathbf{x}^r, \nabla f_2(\mathbf{x}^r) \rangle + f_2(\mathbf{x}^r).$$

We first show that the iteration (8.14) is equivalent to the following iteration:

$$(8.16) \quad \mathbf{x}^{r+1} = \arg \min_{\mathbf{x} \in \mathcal{X}} u(\mathbf{x}, \mathbf{x}^r).$$

From the definition of the prox operation, we have

$$\begin{aligned} \text{prox}_{\gamma f_1}(\mathbf{x}^r - \gamma \nabla f_2(\mathbf{x}^r)) &= \arg \min_{\mathbf{x} \in \mathcal{X}} \gamma f_1(\mathbf{x}) + \frac{1}{2} \|\mathbf{x} - \mathbf{x}^r + \gamma \nabla f_2(\mathbf{x}^r)\|_2^2 \\ &= \arg \min_{\mathbf{x} \in \mathcal{X}} f_1(\mathbf{x}) + \frac{1}{2\gamma} \|\mathbf{x} - \mathbf{x}^r\|_2^2 + \langle \mathbf{x} - \mathbf{x}^r, \nabla f_2(\mathbf{x}^r) \rangle \\ &= \arg \min_{\mathbf{x} \in \mathcal{X}} u(\mathbf{x}, \mathbf{x}^r). \end{aligned}$$

We then show that $u(\mathbf{x}, \mathbf{x}^r)$ is an upper-bound of the original function $f_1(\mathbf{x}) + f_2(\mathbf{x})$ for all $\mathbf{x}, \mathbf{x}^r \in \mathcal{X}$. Note that from the well-known descent lemma [3, Proposition A.32], we have that

$$\begin{aligned} f_2(\mathbf{x}) &\leq f_2(\mathbf{x}^r) + \frac{\beta}{2} \|\mathbf{x} - \mathbf{x}^r\|^2 + \langle \mathbf{x} - \mathbf{x}^r, \nabla f_2(\mathbf{x}^r) \rangle \\ &\leq f_2(\mathbf{x}^r) + \frac{1}{2\gamma} \|\mathbf{x} - \mathbf{x}^r\|^2 + \langle \mathbf{x} - \mathbf{x}^r, \nabla f_2(\mathbf{x}^r) \rangle, \end{aligned}$$

where the second inequality is from the definition of γ . This result implies that $u(\mathbf{x}, \mathbf{y}) \geq f_1(\mathbf{x}) + f_2(\mathbf{x})$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Moreover, we can again verify that all

the other conditions in Assumption 1 are true. Consequently, we conclude that the forward-backward splitting algorithm is a special case of the SUM algorithm.

Similar to the previous example, we can generalize the forward-backward splitting algorithm to the problem with multiple block components. Consider the problem

$$(8.17) \quad \begin{aligned} \min \quad & \sum_{i=1}^n f_i(\mathbf{x}_i) + f_{n+1}(\mathbf{x}_1, \dots, \mathbf{x}_n) \\ \text{s.t.} \quad & \mathbf{x}_i \in \mathcal{X}_i, i = 1, \dots, n, \end{aligned}$$

where $\{\mathcal{X}_i\}_{i=1}^n$ are closed and convex sets. Each function $f_i(\cdot)$, $i = 1, \dots, n$, is convex and lower semicontinuous with respect to \mathbf{x}_i ; $f_{n+1}(\cdot)$ is convex and has Lipschitz continuous gradient with respect to each of the component \mathbf{x}_i , i.e., $\|\nabla f_{n+1}(\mathbf{x}) - \nabla f_{n+1}(\mathbf{y})\| \leq \beta_i \|\mathbf{x}_i - \mathbf{y}_i\|$ for all $\mathbf{x}_i, \mathbf{y}_i \in \mathcal{X}_i, i = 1, \dots, n$. Then the following block forward-backward splitting algorithm can be shown as a special case of the BSUM algorithm and consequently converges to a stationary solution of the problem (8.17),

$$\mathbf{x}_i^{r+1} = \text{prox}_{\gamma f_i}(\mathbf{x}_i^r - \gamma \nabla_{\mathbf{x}_i} f_{n+1}(\mathbf{x}^r)), \quad i = 1, 2, \dots, n,$$

where $\gamma \in [\epsilon_i, 2/\beta_i - \epsilon_i]$ with $\epsilon_i \in]0, \min\{1, 1/\beta_i\}[$. To the best of our knowledge, the convergence of the block forward-backward splitting method has not been studied before for nonsmooth nonconvex problems.

8.4. CANDECOMP/PARAFAC decomposition of tensors. Another application of the proposed method is in CANDECOMP/PARAFAC (CP) decomposition of tensors. Given a tensor $\mathfrak{X} \in \mathbb{R}^{m_1 \times m_2 \times \dots \times m_n}$ of order n , the idea of CP decomposition is to write the tensor as the sum of rank-one tensors:

$$\mathfrak{X} = \sum_{r=1}^R \mathfrak{X}_r,$$

where $\mathfrak{X}_r = a_{1r} \circ a_{2r} \circ \dots \circ a_{nr}$ and $a_{ir} \in \mathbb{R}^{m_i}$. Here the notation “ \circ ” denotes the outer product.

In general, finding the CP decomposition of a given tensor is NP-hard [22]. In practice, one of the most widely accepted algorithms for computing the CP decomposition of a tensor is the alternating least squares (ALS) algorithm [26], [16], [37]. The ALS algorithm proposed in [8], [19] is in essence a BCD method. For ease of presentation, we will present the ALS algorithm only for tensors of order three.

Let $\mathfrak{X} \in \mathbb{R}^{I \times J \times K}$ be a third order tensor. Let $(A; B; C)$ represent the decomposition

$$(A; B; C) \triangleq \sum_{r=1}^R a_r \circ b_r \circ c_r,$$

where a_r (resp., b_r and c_r) is the r th column of A (resp., B and C). The ALS algorithm minimizes the difference between the original and the reconstructed tensors

$$(8.18) \quad \min_{A, B, C} \|\mathfrak{X} - (A; B; C)\|,$$

where $A \in \mathbb{R}^{I \times R}$, $B \in \mathbb{R}^{J \times R}$, $C \in \mathbb{R}^{K \times R}$, and R is the rank of the tensor.

The ALS approach is a special case of the BCD algorithm in which the three blocks of variables A , B , and C are cyclically updated. In each step of the computation, when

two blocks of variables are held fixed, the subproblem becomes the quadratic least squares problem and admits closed form updates (see [26]).

One of the well-known drawbacks of the ALS algorithm is the *swamp* effect, where the objective value remains almost constant for many iterations before starting to decrease again. Navasca, De Lathauwer, and Kindermann in [29] observed that adding a proximal term in the algorithm could help reduce the swamp effect. More specifically, at each iteration r the algorithm proposed in [29] solves the following problem for updating the variables:

$$(8.19) \quad \|\mathfrak{X} - (A; B; C)\|^2 + \lambda\|A - A^r\|^2 + \lambda\|B - B^r\|^2 + \lambda\|C - C^r\|^2,$$

where $\lambda \in \mathbb{R}$ is a positive constant. As discussed before, this proximal term has been considered in different optimization contexts and its convergence has been shown in [18]. An interesting numerical observation in [29] is that decreasing the value of λ during the algorithm can noticeably improve the convergence of the algorithm. Such an iterative decrease of λ can be accomplished in a number of different ways. Our numerical experiments show that the following simple approach to updating λ can significantly improve the convergence of the ALS algorithm and substantially reduce the swamp effect:

$$(8.20) \quad \lambda^r = \lambda_0 + \lambda_1 \frac{\|\mathfrak{X} - (A^r; B^r; C^r)\|}{\|\mathfrak{X}\|},$$

where λ^r is the proximal coefficient λ at iteration r . Theorem 2 implies the convergence is guaranteed even with this update rule of λ , whereas the convergence result of [18] does not apply in this case since the proximal coefficient is changing during the iterations.

Figure 8.2 shows the performance of different algorithms for the example given in [29], where the tensor \mathfrak{X} is obtained from the decomposition

$$A = \begin{bmatrix} 1 & \cos \theta & 0 \\ 0 & \sin \theta & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & \sqrt{2} \cos \theta & 0 \\ 0 & \sin \theta & 1 \\ 0 & \sin \theta & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To generate the plots, we have used a MATLAB script running on a PC with 4GB RAM memory and a dual core 2.7 GHz CPU. In Figure 8.2, the vertical axis is the value of the objective function where the horizontal axis is the iteration number. In this plot, *ALS* is the classical alternating least squares algorithm. The curve for *Constant Proximal* shows the performance of the BSUM algorithm when we use the objective function in (8.19) with $\lambda = 0.1$. The curve for *Diminishing Proximal* shows the performance of BCD method on (8.19), where the weight λ decreases iteratively according to (8.20) with $\lambda_0 = 10^{-7}$, $\lambda_1 = 0.1$. The other two curves *MBI* and *MISUM* correspond to the MBI algorithm and the MISUM algorithm. In the implementation of the MISUM algorithm, the proximal term is of the form in (8.19) and the weight λ is updated based on (8.20).

Table 8.1 represents the average number of iterations required to get an objective value less than $\epsilon = 10^{-5}$ for different algorithms. The average is taken over 1000 Monte Carlo runs over different initializations. The initial points are generated randomly where the components of the variables A , B , and C are drawn independently from the uniform distribution over the unit interval $[0, 1]$. As can be seen, adding a diminishing proximal term significantly improves the convergence speed of the ALS algorithm.

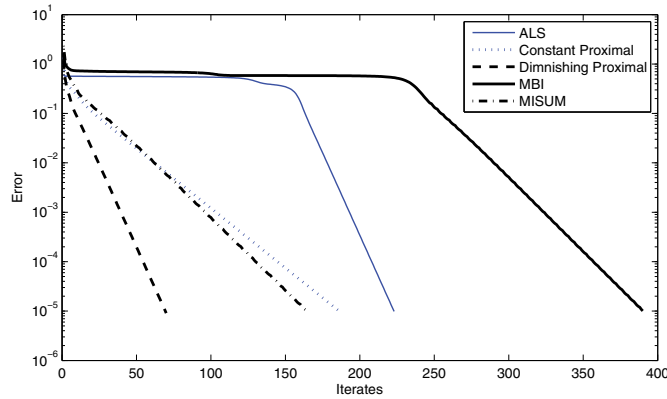


FIG. 8.2. Convergence of different algorithms.

TABLE 8.1
Average number of iterations for convergence.

Algorithm	Average number of iterations for convergence
ALS	277
Constant proximal	140
Diminishing proximal	78
MBI	572
MISUM	175

Figure 8.3 illustrates the performance of different algorithms for the case of $I = J = K = R = 100$. In this experiment, we set $\lambda_0 = 10^{-1}$ and $\lambda_1 = 1$. As can be seen from the figure, adding the proximal terms reduces the swamp effect.

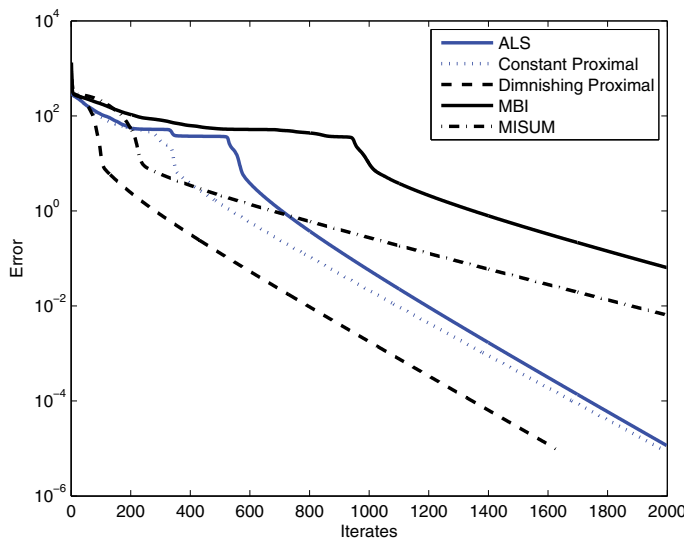


FIG. 8.3. Convergence of different algorithms.

8.5. Expectation maximization algorithm. The EM algorithm in [15] is an iterative procedure for maximum likelihood estimation when some of the random variables are unobserved/hidden. Let w be the observed random vector which is used for estimating the value of θ . The maximum likelihood estimate of θ can be given as

$$(8.21) \quad \hat{\theta}_{\text{ML}} = \arg \max_{\theta} \ln p(w|\theta).$$

Let the random vector z be the hidden/unobserved variable. The EM algorithm starts from an initial estimate θ^0 and generates a sequence $\{\theta^r\}$ by repeating the following steps:

- E-step: Calculate $g(\theta, \theta^r) \triangleq \mathbb{E}_{z|w, \theta^r} \{\ln p(w, z|\theta)\}$.
- M-step: $\theta^{r+1} = \arg \max_{\theta} g(\theta, \theta^r)$.

The EM algorithm can be viewed as a special case of the SUM algorithm [6]. In fact, we are interested in solving the following optimization problem:

$$\min_{\theta} -\ln p(w|\theta).$$

The objective function could be written as

$$\begin{aligned} -\ln p(w|\theta) &= -\ln \mathbb{E}_{z|\theta} p(w|z, \theta) \\ &= -\ln \mathbb{E}_{z|\theta} \left[\frac{p(z|w, \theta^r)p(w|z, \theta)}{p(z|w, \theta^r)} \right] \\ &= -\ln \mathbb{E}_{z|w, \theta^r} \left[\frac{p(z|\theta)p(w|z, \theta)}{p(z|w, \theta^r)} \right] \\ &\leq -\mathbb{E}_{z|w, \theta^r} \ln \left[\frac{p(z|\theta)p(w|z, \theta)}{p(z|w, \theta^r)} \right] \\ &= -\mathbb{E}_{z|w, \theta^r} \ln p(w, z|\theta) + \mathbb{E}_{z|w, \theta^r} \ln p(z|w, \theta^r) \\ &\triangleq u(\theta, \theta^r), \end{aligned}$$

where the inequality is due to Jensen's inequality and the third equality follows from a simple change of the order of integration for the expectation. Since $\mathbb{E}_{z|w, \theta^r} \ln p(z|w, \theta^r)$ is not a function of θ , the M-step in the EM algorithm can be written as

$$\theta^{r+1} = \arg \max_{\theta} u(\theta, \theta^r).$$

Furthermore, it is not hard to see that $u(\theta^r, \theta^r) = -\ln p(w|\theta^r)$. Therefore, under the smoothness assumption, Proposition 1 implies that Assumption 1 is satisfied. As an immediate consequence, the EM algorithm is a special case of the SUM algorithm. Therefore, our result implies not only the convergence of the EM algorithm but also the convergence of the EM algorithm with the Gauss–Seidel/coordinatewise update rule (under the assumptions of Theorem 2). In fact, in the block coordinate EM algorithm (BEM), at each M-step, only one block is updated. More specifically, let $\theta = (\theta_1, \dots, \theta_n)$ be the unknown parameter. Assume w is the observed vector and z is the hidden/unobserved variable as before. The BEM algorithm starts from an initial point $\theta^0 = (\theta_1^0, \dots, \theta_n^0)$ and generates a sequence $\{\theta^r\}$ according to the algorithm in Figure 8.4.

The motivation behind using the BEM algorithm instead of the EM algorithm could be the difficulties in solving the M-step of EM for the entire set of variables, while solving the same problem per block of variables is easy. As an example, consider

```

1 Initialize with  $\theta^0$  and set  $r = 0$ 
2 repeat
3    $r = r + 1, i = r \bmod n + 1$ 
4   E-step:  $g_i(\theta_i, \theta^r) = \mathbb{E}_{z|w, \theta^r} \{\ln p(w, z | \theta_1^r, \dots, \theta_{i-1}^r, \theta_i, \theta_{i+1}^r, \dots, \theta_n^r)\}$ 
5   M-step:  $\theta_i^{r+1} = \arg \max_{\theta_i} g_i(\theta_i, \theta^r)$ 
6 until some convergence criterion is met

```

FIG. 8.4. Pseudocode of the BEM algorithm.

the mixture of the Gaussian model [21] where different Gaussian distributions have the same mean but different variances. It can be checked that the update rule of the EM algorithm [15] cannot be done in closed form. However, fixing the variance, the update rule of the mean could be done in closed form. Furthermore, by fixing the mean, the variance could be updated in closed form, and hence the BEM algorithm can be applied. To the best of our knowledge, the BEM algorithm and its convergence behavior have not been analyzed before.

8.6. Concave-convex procedure/difference of convex functions. A popular algorithm for solving unconstrained problems, which also belongs to the class of successive upper-bound minimization, is the concave-convex procedure (CCCP) introduced in [41]. In CCCP, also known as the difference of convex functions (DC) programming, we consider the unconstrained problem

$$\min_{x \in \mathbb{R}^m} f(x),$$

where $f(x) = f_{cve}(x) + f_{cvx}(x) \forall x \in \mathbb{R}^m$, where $f_{cve}(\cdot)$ is a concave function and $f_{cvx}(\cdot)$ is convex. The CCCP generates a sequence $\{x^r\}$ by solving the equation

$$\nabla f_{cvx}(x^{r+1}) = -\nabla f_{cve}(x^r),$$

which is equivalent to

$$(8.22) \quad x^{r+1} = \arg \min_x g(x, x^r),$$

where $g(x, x^r) \triangleq f_{cvx}(x) + (x - x^r)^T \nabla f_{cve}(x^r) + f_{cve}(x^r)$. Clearly, $g(x, x^r)$ is a tight convex upper-bound of $f(x)$ and hence CCCP is a special case of the SUM algorithm and its convergence is guaranteed by Theorem 1 under certain assumptions. Furthermore, if the updates are done in a block coordinate manner, the algorithm becomes a special case of BSUM whose convergence is guaranteed by Theorem 2. To the best of our knowledge, the general block coordinate version of the CCCP algorithm and its convergence have not been studied before. For an application of the block coordinate version of the constrained CCCP method in optimizing cognitive radio systems, the readers are referred to [25].

In summary, in this section we have demonstrated the generality of the BSUM/SUM framework by presenting a wide range of its applications. In particular, we have derived the block coordinate versions of several popular algorithms and shown their

convergence using the general convergence framework established in this paper. Additional applications of the related methods can be found in recent works such as [35], [12], [31], [43].

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