Compressive time delay estimation off the grid

Yongsung ParkWoojae SeongYoungmin Choo

Citation: The Journal of the Acoustical Society of America **141**, EL585 (2017); doi: 10.1121/1.4985612
View online: http://dx.doi.org/10.1121/1.4985612
View Table of Contents: http://asa.scitation.org/toc/jas/141/6
Published by the Acoustical Society of America

Compressive time delay estimation off the grid

Yongsung Park
Department of Naval Architecture and Ocean Engineering, Seoul National University, Seoul, 151-744, Korea
ysparkwin@snu.ac.kr

Woojae Seong
Department of Naval Architecture and Ocean Engineering and Research Institute of Marine System Engineering, Seoul National University, Seoul, 151-744, Korea
wseong@snu.ac.kr

Youngmin Choo[a)]
Department of Defense System Engineering, Sejong University, Seoul, 143-747, Korea
ychoo@sejong.ac.kr

Abstract: This paper describes a time delay estimation (TDE) technique using compressive sensing (CS) off the grid, which estimates the channel impulse response in a continuous time domain. The TDE can be formulated into a sparse signal reconstruction problem where the CS technique can be applied. Previous works have used standard finite dimensional CS with evenly discretized grids. However, the actual time delays will not always lie on the discrete grid, and this mismatch between the actual and discretized time delays results in reconstruction degradation. To overcome the basis mismatch, a TDE technique using an off the grid CS framework is proposed by modifying the scheme in the off the grid direction of arrival (DOA) estimation [Xenaki and Gerstoft, J. Acoust. Soc. Am. **137**(4), 1923–1935 (2015)]. The effectiveness of the suggested method is demonstrated on real data from a water tank experiment. The off the grid CS TDE is shown to have super-resolution, which enables close arrivals to be distinguished.

[DRD]
Date Received: April 13, 2017 **Date Accepted:** May 27, 2017

1. Introduction

The time delay estimation (TDE) of signals transmitted from a source is the first step that feeds into succeeding processes for identifying, localizing, and tracking the source in many different areas of science and engineering.[1–3] In ocean acoustics, acoustic signals emitted from the source travel via multiple propagation paths and thus arrive at different times corresponding to different paths. TDE for these arrivals is required to estimate the channel impulse response.

Since TDE estimates the time delays of a few (K sparse) arrivals in the limited time interval of received data, this TDE problem can be considered as a sparse signal reconstruction problem. The received time-domain signal $\mathbf{y} \in \mathbb{R}^M$ can be represented as a sum of different time-delayed source waveforms. The estimation problem is to find a sparse vector $\mathbf{x} \in \mathbb{R}^N$, given $\mathbf{y}$, which is related by the underdetermined linear system $\mathbf{y} = \mathbf{A}_{M\times N}\mathbf{x}$ with $M < N$. The sensing matrix $\mathbf{A}$ is made up of N columns (*a priori* bases), which are time-delayed versions of source waveform in a discrete time domain. The TDE problem, accordingly, is to find a linear combination of K sparse bases in *a priori* N bases of the sensing matrix $\mathbf{A}$.

Compressive sensing (CS)[4] has been proposed as an efficient way to solve the sparse signal reconstruction problem by using sparsity of the vector $\mathbf{x}$, where sparsity corresponds to the number of non-zero elements of $\mathbf{x}$. There have been various works which employed standard finite dimensional CS in TDE problems,[5,6] and CS based TDE algorithms outperformed existing TDE algorithms in terms of resolution.[6]

Continuous time delay estimation problems are inevitably digitized into finite discrete grids and one of the major drawbacks of CS in estimation problems is the basis mismatch.[7] This arises when the exact K support does not fall on the discretized grid due to inadequate N *a priori* bases of the sensing matrix. This basis mismatch can severely affect the accuracy of TDE scheme resulting in incorrect determination of the

---

[a)]Author to whom correspondence should be addressed.

number and delays of arrivals. Finer grids can be employed to alleviate the basis mismatch, however, leading to numerical instability issues.[7]

The idea to cope with basis mismatch is to work directly in the continuous parameter domain, which is referred to as off the grid CS. In the direction-of-arrival (DOA) estimation problem, Xenaki and Gerstoft[8] adopted off the grid CS to mitigate errors from the basis mismatch arising in grid-based CS beamforming. They formulate the estimation problem in a continuous angular spectrum and employ atomic norm minimization, the continuous analog of l_1-norm minimization, for enforcing sparsity. Our off the grid TDE method utilizes the scheme of Xenaki and Gerstoft[8] which is the foundational work on off the grid CS used for DOA estimation, except that we implement the off the grid CS to TDE problem. To do so, we reformulate the TDE problem into a form of atomic norm minimization problem in a continuous domain. The atomic norm minimization problem defined in an infinite dimensional space (which comes from the continuous domain) can be solved in a finite dimensional space using its dual problem by introducing a Lagrange multiplier.[10] Our approach distinctly differs from existing TDE approaches based on CS where the estimation is performed over a discretized domain; in our approach, TDE is performed over a continuous domain. The proposed off the grid TDE technique results in high resolution and accuracy even in noisy environment.

The present off the grid TDE technique is applied to acoustic signal (70–100 kHz) transmission and reception experiments for various source-receiver ranges. The channel impulse responses of the measured data obtained from our technique are compared with matched filter results. Section 2 specifies the TDE problem and describes our off the grid TDE methodology. In Sec. 3, analyses of our technique are given. A discussion of effects of noise parameter choice on the off the grid TDE technique are described. Also, water tank experiments are presented and the results of our approach are discussed with experimental data. Section 4 provides a summary of the present work.

2. Off the grid TDE for channel impulse response estimate

The channel impulse response $h(t)$ of the source-receiver geometry for a given ocean environment is assumed to be a weighted superposition of K spikes in the time domain, given by $h(t) = \Sigma_{j=1}^{K} a_j \delta(t-\tau_j)$, where a_j is the amplitude and $\delta(t-\tau_j)$ is a Dirac measure at time delay τ_j. Note that the TDE for channel impulse response estimate is comparable to reconstructing the sparse solution $h(t)$, which has sparse non-zero components at the time delays of multiple arrivals. For simplicity, we limit our problem to where there are no frequency-dependent propagation effects (such as absorption, diffraction, and scattering). The measured signal $y(t)$ consists of K multiple arrivals of the source waveform $s(t)$ with different time delays τ_j and amplitudes a_j, i.e., $y(t) = \sum_{j=1}^{K} a_j s(t-\tau_j)$. In the frequency domain, the spectrum of the received signal $\mathbf{Y} \in \mathbb{C}^M$ is band-limited when the source spectrum is band-limited and is in the form of the M Fourier series coefficients,

$$Y(\omega_m) = S(\omega_m)\int e^{-i\omega_m t} h(t) dt = S(\omega_m)\sum_{j=1}^{K} a_j e^{-i\omega_m \tau_j}, \quad m = 0, 1, \ldots, M-1, \tag{1}$$

where $S(\omega_m)$ is the source spectrum, the Fourier transform of the source-signal, $s(t)$, and ω_0 and ω_{M-1} are the lower and the upper band limits of the signal spectrum band, respectively. The goal of our off the grid TDE technique is to compute estimates $\tilde{h}(t)$ of the actual channel impulse response $h(t)$ from the spectrum of the received signal $\mathbf{Y}$.

For the sparse signal reconstruction problem where CS is applicable, most algorithms have been implemented by using l_1-minimization techniques in a discretized parameter space.[5,6] The sensing matrix is made up of finite discrete bases, where the bases are time-delayed versions of the source waveform in a discrete time domain. However, signals exist in a continuous basis. A new framework to handle sparsity-enforcing optimization over a continuous basis has been receiving a lot of attention.[8–11] To impose sparsity on a continuous optimization variable, the atomic norm is introduced and this sparsity-enforcing measure can be interpreted as being the continuous analog to the l_1-norm of the amplitudes, i.e., $\|h(t)\|_{\mathrm{A}} = \|h(t)\|_1 = \sum_{j=1}^{K} |a_j|$.[8]

Our off the grid TDE technique utilizes the atomic norm minimization approach, as a computational method, to achieve sparsity-enforcement over a continuous domain. To do so, we modify the construction of the TDE problem to a simple convex optimization problem, which can essentially be recast as semi-definite programming (SDP).[10] Employing the atomic norm, the off the grid TDE is solved with the atomic norm minimization problem,

$$\underset{\tilde{\mathbf{h}}}{\text{minimize}}\ \|\tilde{\mathbf{h}}\|_{\mathrm{A}}\ \text{subject to}\ \hat{\mathbf{Y}} = \mathbf{F}_M\tilde{\mathbf{h}}, \tag{2}$$

where $\hat{Y}(\omega_m) = Y(\omega_m)/S(\omega_m)$, $m = 0, 1, \ldots, M-1$, $\tilde{\mathbf{h}}$ is the estimated impulse response, and $\mathbf{F}_M$ is the linear mapping matrix for inverse Fourier transform which maps the continuous variable $\tilde{\mathbf{h}}$ to the M Fourier series coefficients in the frequency band of the source spectrum, i.e., $[\omega_0, \omega_{M-1}]$. Note that the spectrum $\mathbf{Y}$ is normalized by the source spectrum, $\mathbf{S}$, so that we can achieve a mathematically equivalent structure to Xenaki and Gerstoft[8] by filtering out the source spectrum effect on the received signal.[11]

In practical applications, the observed signal is unavoidably contaminated with noise. The off the grid TDE problem with additive Gaussian white noise $\mathbf{N}(\omega_m) \in \mathbb{C}^M$ can be solved by the following convex program:

$$\underset{\tilde{\mathbf{h}}}{\text{minimize}}\ \|\tilde{\mathbf{h}}\|_{\mathrm{A}}\ \text{subject to} \begin{cases} \hat{\mathbf{Y}} = \mathbf{F}_M\tilde{\mathbf{h}} + \hat{\mathbf{N}}, \\ \|\hat{\mathbf{N}}\|_2 \le \delta. \end{cases} \tag{3}$$

Here, $\hat{N}(\omega_m) = N(\omega_m)/S(\omega_m)$, $m = 0, 1, \ldots, M-1$, represents source spectrum normalized perturbation due to additive noise $\mathbf{N}$, such that $\|\hat{\mathbf{N}}\|_2 \le \delta$.

The atomic norm minimization [Eq. (3)] finds the position of non-zero components on the continuous parameter with infinite precision, i.e., $\tilde{\mathbf{h}}$ is an infinite dimensional variable. Having finite dimensional dual variables, the Lagrange dual problem of the primal problem [Eq. (3)] can be used to solve Eq. (3) in a finite manner. The dual function $g(\mathbf{c},\xi)$ is the infimum of the Lagrangian for Eq. (3), and has the form[12]

$$g(\mathbf{c}, \xi) = \inf_{\tilde{\mathbf{h}}} \|\tilde{\mathbf{h}}\|_{\mathrm{A}} + \mathrm{Re}\left[\mathbf{c}^H(\hat{\mathbf{Y}} - \mathbf{F}_M\tilde{\mathbf{h}} - \hat{\mathbf{N}})\right] + \xi(\hat{\mathbf{N}}^H\hat{\mathbf{N}} - \delta^2), \tag{4}$$

where $\mathbf{c}(\omega_m) \in \mathbb{C}^M$ is the vector of the Lagrange dual variables related to the equality constraints $\hat{\mathbf{Y}} = \mathbf{F}_M\tilde{\mathbf{h}} + \hat{\mathbf{N}}$, and $\xi \in \mathbb{R}^+$ is a Lagrange multiplier related to the inequality constraint $\|\hat{\mathbf{N}}\|_2 \le \delta$. The Lagrange dual problem of Eq. (3) is formulated by maximizing the dual function $g(\mathbf{c},\xi)$,[12]

$$\underset{\mathbf{c}}{\text{maximize}}\ \mathrm{Re}\left[\mathbf{c}^H\hat{\mathbf{Y}}\right] - \delta\|\mathbf{c}\|_2\ \text{subject to}\ \|\mathbf{F}_M^H\mathbf{c}\|_\infty \le 1. \tag{5}$$

The dual problem can be recast as SDP,[8,11]

$$\begin{aligned} &\underset{\mathbf{c}}{\text{maximize}}\ \mathrm{Re}\left[\mathbf{c}^H\hat{\mathbf{Y}}\right] - \delta\|\mathbf{c}\|_2, \quad \text{subject to} \begin{bmatrix} \mathbf{Q} & \mathbf{c} \\ \mathbf{c}^H & 1 \end{bmatrix} \succcurlyeq 0, \\ &\sum_{i=1}^{M-j} Q_{i,i+j} = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, M-1. \end{cases} \end{aligned} \tag{6}$$

Herein, we use the CVX program[13] (available in MATLAB) for solving SDP [Eq. (6)] and it can obtain the vector of the Lagrange dual variables $\mathbf{c}$. From Lemma 3.1 of Ref. 11, the support $\hat{T}$, which represents the position of non-zero components, of the primal solution for Eq. (3) is related to the maximum modulus of the trigonometric polynomial $(\mathbf{F}_M^H\mathbf{c})(t) = \sum_{m=0}^{M-1} c(\omega_m)e^{-i\omega_m t}$, and can be determined by finding points where $(\mathbf{F}_M^H\mathbf{c})(t)$ becomes one, i.e., $\hat{T} = \{t_j | |(\mathbf{F}_M^H\mathbf{c})(t_j)| = 1\}$. With this estimated support $\hat{T}$, we can obtain the amplitudes of the waveforms by solving the system of equations $S(\omega_m)\sum_{t\in\hat{T}} a_t e^{-i\omega_m t} = Y(\omega_m)$, $m = 0, 1, \ldots, M-1$, using the method of least squares. The mathematical theorems and proofs underlying our off the grid TDE technique are guaranteed by the prior work of Xenaki and Gerstoft.[8]

3. Analysis

3.1 Effects of noise parameter choice on the off the grid TDE method

In this section, we analyze the effect of the noise parameter on the off the grid TDE technique. The choice of the noise floor δ controls the balance between sparsity of the solution and best data fit with respect to the l_2-norm, i.e., $\|\hat{\mathbf{Y}} - \mathbf{F}_M\tilde{\mathbf{h}}\|_2$. Similar results, based on l_1-norm minimization, have been presented that deal with the regularization parameter in the evolution of the least absolute shrinkage and selection operator (LASSO) solution.[14] The noise floor δ plays the same role as the regularization parameter μ in Gerstoft *et al.*[14] Large δ results in a very sparse solution, where data fit is poor and allows a considerable amount of noise. However, if δ becomes small, the sparsity of the solution is relieved and the solution becomes better fit to the data.

Figure 1 shows simulation results of the off the grid TDE technique in a noisy environment with four different δ. The additive noise is Gaussian white noise and the SNR is 6 dB. The source signal is a 70–100 kHz linear frequency modulated chirp signal of 0.1 ms duration. Seven arrivals with different time delays exist in the time-domain received data having a sampling period of 0.001 ms. In this case, because the noise is simulated, the noise level $\|\mathbf{N}\|_2$ is known, which corresponds to the noise parameter $\delta = 0.36$. With proper choice of the noise parameter, the off the grid TDE technique gives high resolution with acceptable accuracy, as shown in Fig. 1(c). Note that the difference between the estimated and true solutions should become smaller as SNR increases.[11] For a large value of δ (e.g., $\delta = 0.78$), the most sparse solution with only one arrival [Fig. 1(a)] is obtained by using the proposed method. As δ becomes smaller, other non-zero components appear. Note that activated components' amplitudes remain constant, while more components are activated as δ becomes smaller. For the smallest value of $\delta = 0.08$ [Fig. 1(d)], the number of solutions exceeds seven true solutions to facilitate the fit to noisy data. These additional miscellaneous components have to do with the potential for overfitting, and they follow the noise in the measurements. Thus, selecting the noise floor δ, depending on the purpose, is an important part of our TDE framework, as it balances the fit of the solution to the data versus the sparsity (number of non-zeros) of the solution.

3.2 Experimental results

The proposed method is applied to controlled water tank experimental data. The source and receiver arrangement as well as the water tank geometry are shown in Fig. 2. The omni-directional hydrophone (B&K 8104) and transducer (Reson TC4038) were used as the source and receiver, respectively. The source signal was a 70–100 kHz linear frequency modulated chirp signal of 0.4 ms duration with a sampling period of 0.001 ms. Several experiments were conducted by varying the distances L from wall B, ranging from 5 to 6 m [Fig. 2(a)], whereas the depths of the source and receiver were fixed at 2.3 and 1 m, respectively [Fig. 2(b)]. A stainless steel cylinder (SUS304, d = 267 mm, h = 650 mm) was located at depth 2 m and 5.60 m from wall B, referenced to the center of the cylinder. The sound speed in the water tank was maintained at an isovelocity of 1460 m/s.

TDE analysis in our work focuses on single-reflected paths and double-reflected paths, as shown in Fig. 2(c). Single-reflected paths [see Fig. 2(c), solid line] contain wall-reflected and cylinder-reflected paths. For the cases of double-reflected paths [see Fig. 2(c), dotted line], wall-to-water surface reflections are considered. Among the various distances, $L = 5.48$, 5.59, and 5.85 m are considered.

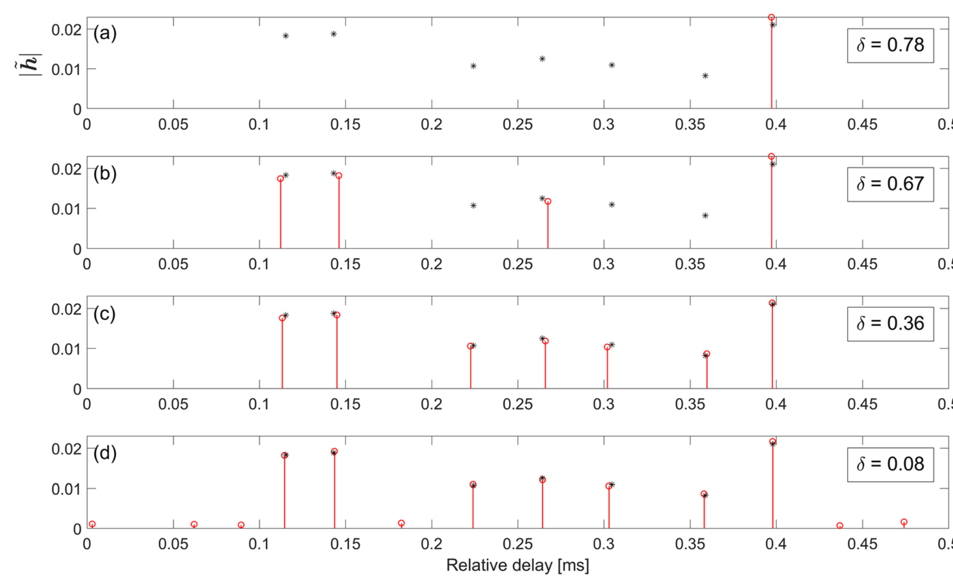

Fig. 1. (Color online) Compressive time delay estimation off the grid for the noisy scenario. The estimated impulse response (○) and true impulse response (∗) for (a) $\delta = 0.78$, (b) $\delta = 0.67$, (c) $\delta = 0.36$, and (d) $\delta = 0.08$. Seven arrivals (∗) are at [0.115, 0.143, 0.224, 0.264, 0.305, 0.359, 0.398] ms. SNR is 6 dB.

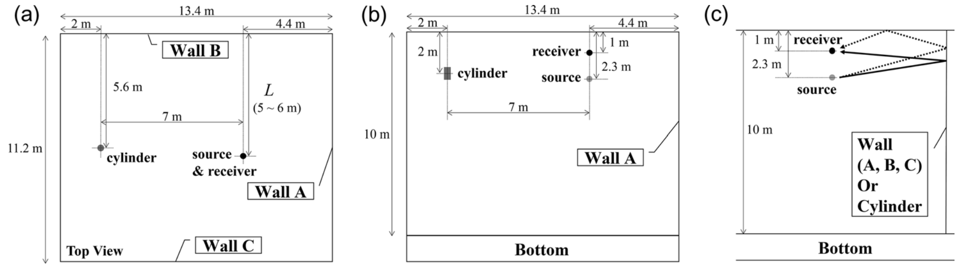

Fig. 2. Schematic plan of water tank experiment: (a) Top view of the water tank, (b) horizontal cross-section view of the water tank showing arrangement of source, receiver, and cylinder, and (c) schematic paths of signals showing single-reflected (wall-reflected or cylinder-reflected) path (solid line) and double-reflected (wall-to-water surface-reflected) path (dotted line).

Figure 3 shows the TDE results, computed by our off the grid TDE scheme when $L = 5.85$ m. Note that the support-detection of the solution of the atomic norm minimization, which is the estimated time delays of the existing signals, is obtained from the magnitude of polynomials $(\mathbf{F}_M^H \mathbf{c})(t)$. The magnitude of these polynomials is equal to 1, i.e., $|(\mathbf{F}_M^H \mathbf{c})(t)| = 1$, at the support of the solution as shown in Fig. 3(a). Figure 3(b) shows the estimated relative time delays of our method compared to the matched filter result. The relative delay represents elapsed time after the direct arrival from the source to the receiver. We selected the noise parameter $\delta = 0.39$ that shows interpretable obvious peaks. Both methods provide high-resolution time delay estimation with noticeable peaks. Single-reflected paths [see Fig. 2(c), solid line] from three walls A, B, C, and the cylinder are clearly visible. The source and receiver are located at distances of 4.40, 5.35, 5.85 m from three walls A, C, B, and 7.06 m from the cylinder, respectively. The distinct peaks are water surface-reflected (2.260 ms), wall A-reflected (6.092 ms), wall C-reflected (7.382 ms), wall B-reflected (8.063 ms), and steel cylinder-reflected (9.712 ms) paths, in the order of arrival.

Figure 4(a) shows the estimated time delays when $L = 5.59$ m where two time delays from walls B and C are very close to each other. We focus on a zoomed region of the relative delay (6–8.5 ms) to distinguish between different arrivals. We selected the noise parameter $\delta = 0.40$ that shows only single-reflected paths. Three points of support elements are detected, which correspond to wall A-reflected (6.201 ms), wall B-reflected (7.709 ms), and wall C-reflected (7.736 ms) paths; the distances from walls A, B, and C are 4.48, 5.59, and 5.61 m, respectively. While the present technique offers very accurate resolution, the matched filter method, having resolution limit 0.033 ms [1/(frequency band-width)], merges the two peaks from walls B and C. Additionally, we observe from Fig. 4(b), ($L = 5.48$), that our method can resolve the weak arrivals, e.g., double-reflected (wall-to-water surface-reflected) arrivals [see Fig. 2(c), dotted line]. In this case, we selected the noise parameter $\delta = 0.37$. The distances of the source and receiver from the two walls B and C are 5.48 and 5.72 m, respectively. Five points of support elements are detected, which correspond to, from the first component, wall A-reflected (6.092 ms), wall B-reflected (7.559 ms), wall B-to-water surface-reflected

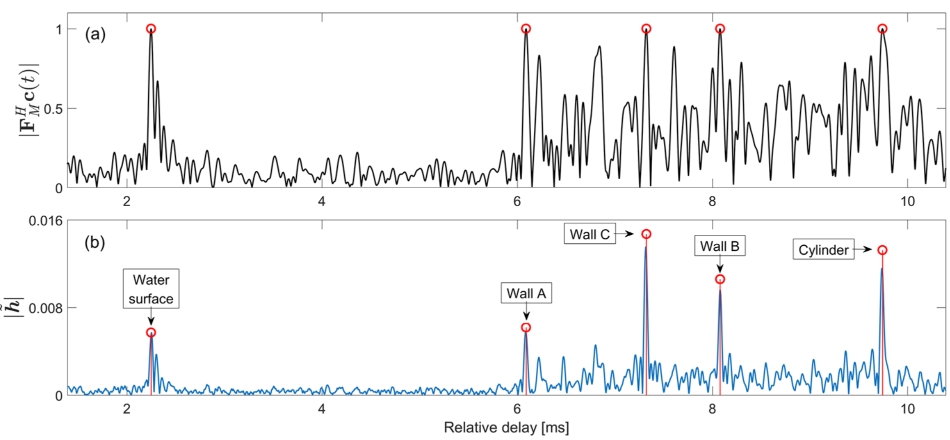

Fig. 3. (Color online) Compressive time delay estimation off the grid. (a) Support-locating polynomial and the points at which its magnitude equals one (circle) and (b) impulse response reconstruction with off the grid method (circle) and matched filter method (solid line). Source and receiver are located at a distance of $L = 5.85$ m from wall B.

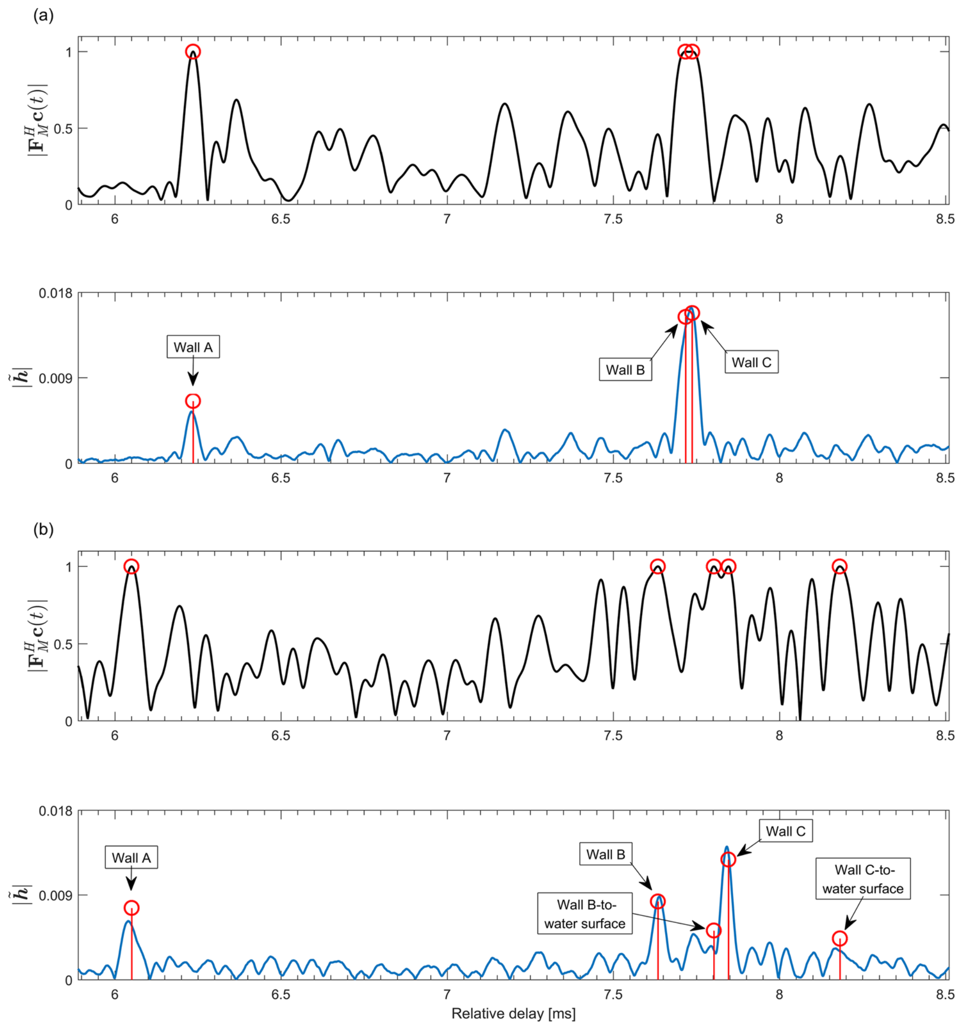

Fig. 4. (Color online) Same as Fig. 3 except for the source and receiver distance of (a) $L = 5.59$ m and (b) $L = 5.48$ m from wall B, respectively.

(7.839 ms), wall C-reflected (7.885 ms), and wall C-to-water surface-reflected (8.155 ms) paths. On the other hand, the matched filter method fails to detect the signals coming from double-reflected (wall-to-water surface-reflected) paths. This is because the difference of the time delays between wall B-to-water surface-reflected path and wall C-reflected path exceeds the resolution limit of the matched filter method. Moreover, noticeable biased peaks occur from 7 to 8.5 ms, which probably reflect the effect of the interference of noise or side lobes, and these peaks make it hard to identify the time delays of the actual signals. By contrast, the present off the grid method resolves the time delays with high resolution in all cases. With our off the grid scheme, the resolvable minimum separation condition of time delays and the minimum required signal strength are related to the frequency spectrum of the source signal and the signal-to-noise ratio. Further theoretical investigations are needed to derive an explicit expression for the resolution limit.

4. Summary

Estimating time delays of signals is important for estimating channel impulse response, especially in a multipath environment. In this paper, we proposed a feasible CS based TDE technique working directly in the continuous parameter domain for exactly estimating the continuous time delay. The atomic norm minimization was implemented to impose a penalty on the sparsity of the solution in a continuous domain. By using the continuous model, the very sensitive issue of basis mismatch in the conventional CS method can be avoided. We examined various cases using our method that provide a variation of support-detection of the solution with respect to the noise parameter. Finally, from our experimental data, it was observed that not only does the present method offer super-resolution, it can also be applied to detect weak signals. Some interesting areas of research for future work include finding a resolvable minimum separation condition of the time delays.

Acknowledgments

This work was supported by the Agency for Defense Development in Korea under Contract No. UD160015DD and the Ministry of Trade, Industry & Energy (MOTIE) (project code: 10045337).

References and links

[1]K. Gedalyahu and Y. C. Eldar, "Time-delay estimation from low-rate samples: A union of subspaces approach," IEEE Trans. Sign. Process. **58**(6), 3017–3031 (2010).
[2]A. H. Quazi, "An overview on the time delay estimate in active and passive systems for target localization," IEEE Trans. Acoust. Speech Sign. Process. **29**(3), 527–533 (1981).
[3]J. Chen, Y. Huang, and J. Benesty, *Audio Signal Processing for Next-Generation Multimedia Communication Systems* (Springer, New York, 2004), pp. 1–370.
[4]E. J. Candès and M. B. Wakin, "An introduction to compressive sampling," IEEE Sign. Process. Mag. **25**(2), 21–30 (2008).
[5]C. Ekanadham, D. Tranchina, and E. P. Simoncelli, "Recovery of sparse translation-invariant signals with continuous basis pursuit," IEEE Trans. Sign. Process. **59**(10), 4735–4744 (2011).
[6]K. Fyhn, F. Marco, and S. Holdt, "Compressive parameter estimation for sparse translation-invariant signals using polar interpolation," IEEE Trans. Sign. Process. **63**(4), 870–881 (2015).
[7]Y. Chi, L. L. Scharf, A. Pezeshki, and A. R. Calderbank, "Sensitivity to basis mismatch in compressed sensing," IEEE Trans. Sign. Process. **59**(5), 2182–2195 (2011).
[8]A. Xenaki and P. Gerstoft, "Grid-free compressive beamforming," J. Acoust. Soc. Am. **137**(4), 1923–1935 (2015).
[9]G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, "Compressed sensing off the grid," IEEE Trans. Inf. Theory **59**(11), 7465–7490 (2013).
[10]E. J. Candès and C. Fernandez-Granda, "Towards a mathematical theory of super-resolution," Comm. Pure Appl. Math. **67**(6), 906–956 (2014).
[11]E. J. Candès and C. Fernandez-Granda, "Super-resolution from noisy data," J. Fourier Anal. Appl. **19**(6), 1229–1254 (2013).
[12]S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, New York, 2004), pp. 1–684.
[13]M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx (Last viewed April 7, 2017).
[14]P. Gerstoft, A. Xenaki, and C. F. Mecklenbräuker, "Multiple and single snapshot compressive beamforming," J. Acoust. Soc. Am. **138**(4), 2003–2014 (2015).