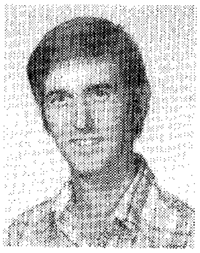
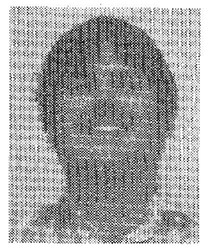


John R. Treichler (S'69-M'78) was born in Velasco, TX, on September 22, 1947. He received the B.A. and M.A. degrees in electrical engineering from Rice University, Houston, TX, in 1970, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, CA, in 1977.

From 1970 to 1974 he was in the U.S. Navy. He spent the next three years at Stanford University, performing doctoral research in the area of adaptive signal processing. Since 1977 he has been a member of the Technical Staff, ARGOSystems, Inc., Sunnyvale, CA. In addition, he serves as a lecturer at Stanford University. His current interests include estimation theory, communication theory, and adaptive signal processing, particularly as applied to communications.



Brian G. Agee (S'76-M'80) received the B.S. degree in electrical engineering and in mathematics in 1977, and the M.S. degree in electrical engineering in 1982, all from the University of California, Davis.

From 1978 to 1980, he was a Regents Fellow and Research Assistant in the Department of Electrical Engineering at the University of California, Davis, where he worked in areas of optimal and adaptive signal extraction. Since 1980 he has been employed as a member of the Technical Staff at ARGOSystems, Inc., in Sunnyvale, CA, where he has worked on problems in adaptive signal extraction, direction finding, and spread spectrum communications.

Mr. Agee is a member of Tau Beta Pi and Pi Mu Epsilon.

Fundamental Limitations in Passive Time Delay Estimation—Part I: Narrow-Band Systems

ANTHONY J. WEISS AND EHUD WEINSTEIN, MEMBER, IEEE

***Abstract*—Time delay estimation of a noise-like random signal observed at two or more spatially separated receivers is a problem of considerable practical interest in passive radar/sonar applications. A new method is presented to analyze the mean-square error performance of delay estimation schemes based on a modified (improved) version of the Ziv-Zakai lower bound (ZZLB). This technique is shown to yield the tightest results on the attainable system performance for a wide range of signal-to-noise ratio (SNR) conditions.**

For delay estimation using narrow-band (ambiguity-prone) signals, the fundamental result of this study is illustrated in Fig. 3. The entire domain of SNR is divided into several disjoint segments indicating several distinct modes of operation. If the available SNR does not exceed SNR_1, signal observations from the receiver outputs are completely dominated by noise thus essentially useless for the delay estimation. As a result, the attainable mean-square error $\bar{\epsilon}^2$ is bounded only by the *a priori* parameter domain. If $\mathrm{SNR}_1 < \mathrm{SNR} < \mathrm{SNR}_2$, the modified ZZLB coincides with the Barankin bound. In this regime differential delay observations are subject to ambiguities. If $\mathrm{SNR} > \mathrm{SNR}_3$ the modified ZZLB coincides with the Cramer-Rao lower bound indicating that the ambiguity in the differential delay estimation can essentially be resolved. The transition from the ambiguity-dominated mode of operation to the ambiguity-free mode of operation starts at SNR_2 and ends at SNR_3. This is the threshold phenomenon in time delay estimation.

The various deflection points SNR_i and the various segments of the bound (Fig. 3) are given as functions of such important system parameters as time-bandwidth product (WT), signal bandwidth to center frequency ratio (W/ω_0) and the number of half wavelengths of the signal center frequency contained in the spacing between receivers. With this information the composite bound illustrated in Fig. 3 provides the most complete characterization of the attainable system performance under any prespecified SNR conditions.

Manuscript received September 28, 1981; revised October 15, 1982. This work was supported in part by the Office of Naval Research under Contract N00014-82-C-0152. It is also a Woods Hole Oceanographic Institution Contribution 5326.

A. J. Weiss is with the Department of Electronic Systems, School of Engineering, Tel Aviv University, Tel Aviv, Israel 69978.

E. Weinstein is with the Department of Ocean Engineering, Woods Hole Oceanographic Institution, Woods Hole, MA 02543 and with the Department of Electronic Systems, School of Engineering, Tel Aviv University, Ramat-Aviv, Tel Aviv, Israel.

I. Introduction

The location of a radiating source can be determined by passive observation of its signal at three or more spatially separated receivers. In the absence of detailed knowledge concerning the signal waveshape, all information about source location (i.e., bearing and range) is contained in the relative (differential) delay of the signal wavefront to the various receiver pairs. Differential delay estimation has therefore attracted a great deal of interest in recent literature (e.g., [1]-[7]).

In most of the systems which have been analyzed the Cramer-Rao inequality has been used to set bounds on the attainable mean-square error (MSE). Its use was justified by invoking an asymptotic theorem asserting that the maximum likelihood (ML) estimator approaches the Cramer-Rao lower bound (CRLB) arbitrarily closely for sufficiently long observation times. However, in many situations of practical interest "sufficiently long" may, in fact, be unreasonably long: to come close to the CRLB one must be able to select from many possi-

0096-3518/83/0400-0472$01.00 © 1983 IEEE

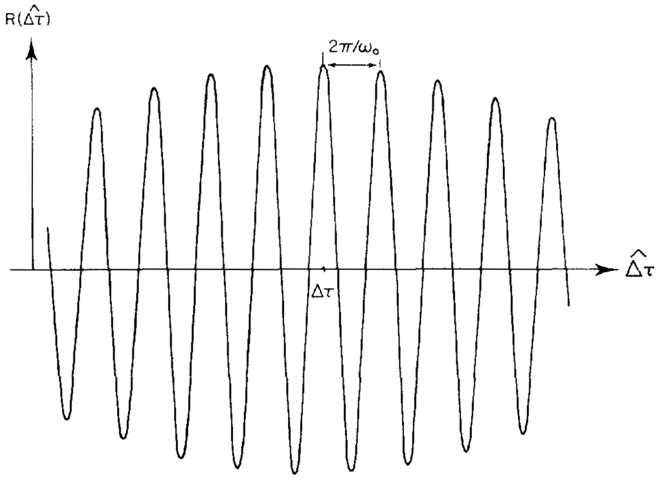


Fig. 1. Typical narrow-band signal cross correlation.

ble solutions of the likelihood equation that one which corresponds to the absolute minimum variance. To appreciate the difficulty consider a narrow-band source and a pair of widely separated receivers. The problem is to estimate the differential delay parameter.

The ML estimation for the two receivers delay estimation problem simply forms the cross correlation of the two receiver outputs ([2] and [7]). For narrow-band source signals the cross correlator output (Fig. 1) peaks at $\Delta\tau$ (the true differential delay) but it is quasi-periodic with a period of $2\pi/\omega_0$, where ω_0 is the signal center frequency. To identify the correct differential delay one must be able to distinguish unambiguously between adjacent peaks of the correlation function. If the signal bandwidth is a small fraction of the center frequency (i.e., $W/\omega_0 << 1$) adjacent peaks have very nearly equal height and identification of the largest one will require either very large signal-to-noise (SNR) or exceedingly long observation times. In many practical important situations, therefore, the performance of the ML method or, in fact, any other estimation scheme may be very drastically inferior to that predicted by the CRLB.

A method which has been used with some success to address the ambiguity problem is based on a simplified version of the Barankin lower bound. The basic result derived in [8] indicates a distinct threshold phenomenon. Above a critical SNR the lower bound on the MSE is characterized by the Cramer-Rao inequality. Below the threshold the Barankin bound exceeds the CRLB by a large factor as suggested by Fig. 2. This approach, however, suffers from two fundamental limitations. 1) Above the threshold the Barankin bound diverges, i.e., one obtains the trivial equation $\bar{\epsilon}^2 \geqslant 0$ where $\epsilon = \hat{\Delta\tau} - \Delta\tau$ is the estimation error. The exact transition to the CRLB, therefore, cannot be clearly specified. 2) In the formulation of the Barankin bound, one completely ignores *a priori* information about the parameter under investigation. In particular, the receiver-to-receiver delay cannot exceed certain limits depending on the spacing between the receivers. Such prior information concerning $\Delta\tau$ is rather critical since the spacing between receivers limits the number of local peaks of the narrow-band cross-correlation function and thus determines the order of ambiguity. As a result of 1) and 2), the approach

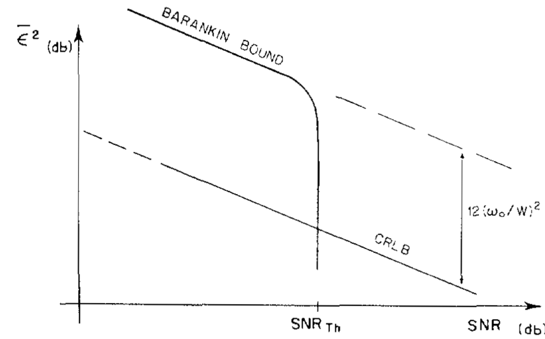


Fig. 2. The Barankin lower bound.

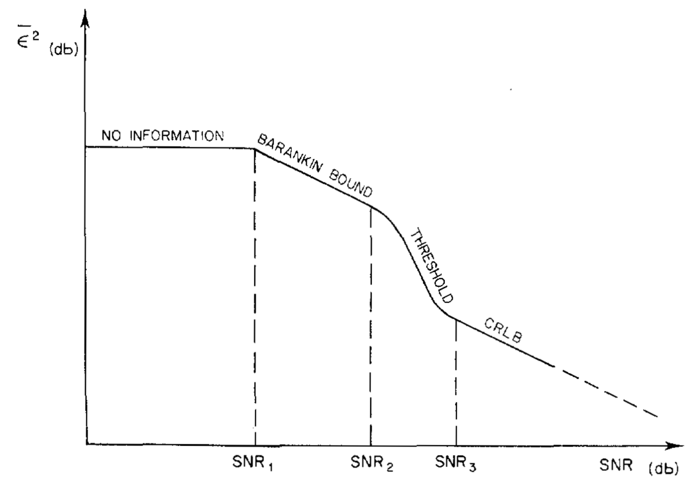


Fig. 3. The modified Ziv-Zakai lower bound.

based on the Barankin theory appears to give useful numerical information only for a limited range of SNR conditions.

In this paper a new method is presented to investigate the performance of delay estimation schemes based on a modified version of the Ziv-Zakai lower bound (ZZLB). This method does not suffer from limitations 1) and 2) and thus can be applied to a wide range of SNR's. Consequently, it yields the tightest known lower bound on the attainable MSE as suggested by Fig. 3.

One immediately observes that the entire domain of SNR is divided into several disjoint segments indicating several distinct modes of operation. If the available SNR does not exceed SNR_1, signal observations from the receiver outputs are completely dominated by noise and are thus essentially useless in the differential delay estimation. As a result, the attainable MSE is bounded only by the *a priori* parameter domain.

If $SNR > SNR_1$ but $SNR < SNR_2$, the modified ZZLB coincides with the Barankin bound. In this regime differential delay observations are subject to ambiguities. As already indicated in [8], all the essential information is contained in the received signal envelope, not in the signal phase. A near optimal estimation scheme is therefore obtained by forming the cross correlation between the received signal envelopes.

If $SNR > SNR_3$ the ambiguity in the differential delay estimation can essentially be resolved. Cross correlating the received signal waveforms yields a $\Delta\tau$ estimate whose performance is closely characterized by the CRLB. The transition from the ambiguity-dominated mode of operation to the ambiguity-free mode of operation starts at SNR_2 and ends at SNR_3. This is the threshold phenomenon in the differential delay estimation.

The ZZLB on the MSE of signal parameter estimates is derived in [9]. In Section II we shall rederive this basic result for two particular reasons. First, since this bound is not as well known as the CRLB or even as the Barankin lower bound, it might be the place to introduce the reader to the rather clever philosophy behind the ZZLB. The second and by far the more important reason is that the ZZLB closely characterizes estimator performance only in ambiguity-free problems such as estimation of the arrival time of a rectangular radar pulse contaminated by additive noise (see [9]). Here we are concerned with the estimation of time delay between spatially separated receivers using narrow-band (ambiguity-prone) random signals. To set a tight bound on the performance of such a system, one of the steps in the derivation must be modified. This rather crucial point becomes clear only by going through a step-by-step derivation of the bound.

In Section III the Ziv-Zakai philosophy is adopted to the delay estimation problem and the basic bound is derived. Our ultimate interest in this study is to investigate the threshold effect in delay estimation using ambiguity-prone (narrow-band) signals. The remainder of the paper is therefore devoted to the analysis of narrow-band systems. In that context analytical considerations leading to the composite result plotted in Fig. 3 are presented and the various deflection points SNR_i are calculated as functions of such important system parameters as time-bandwidth product (WT), signal bandwidth to center frequency (W/ω_0) and the number of half-wavelengths contained in the spacing between receivers. With this information the composite bound illustrated in Fig. 3 provides the most complete characterization of the attainable system performance under any prespecified SNR conditions.

We finally note that Section III can be read independently of Section II. Section II which contains the mathematical derivation of the basic bound can therefore be bypassed for the convenience of the reader.

II. The Ziv-Zakai Lower Bound (ZZLB) and Its Improved Version

The Ziv-Zakai formulation of the lower bound is based on the probability of deciding correctly between two possible values (a) and ($a + x$) of the parameter under investigation. For our purposes it is the receiver-to-receiver delay $\Delta\tau$.

A detection scheme which minimizes the probability of error simply forms the likelihood ratio test (LRT) between the two hypothesized delays. Consider now the following suboptimal detection procedure based on some arbitrary estimate $\widehat{\Delta\tau}$ of $\Delta\tau$

$$\left|\widehat{\Delta\tau} - a\right| \underset{\Delta\tau = a}{\overset{\Delta\tau = a + x}{\gtrless}} \left|\widehat{\Delta\tau} - a - x\right|. \tag{1}$$

The decision is therefore made to the favor of (a) if $|\widehat{\Delta\tau} - a| < |\widehat{\Delta\tau} - a - x|$ and ($a + x$) if $|\widehat{\Delta\tau} - a| \geq |\widehat{\Delta\tau} - a - x|$. If the two hypothesized delays are equally likely to occur, the probability of error for this suboptimal detection scheme is given by

$$\tfrac{1}{2}P\{\widehat{\Delta\tau} - a > x/2 | a\} + \tfrac{1}{2}P\{\widehat{\Delta\tau} - a - x \leq -x/2 | a + x\} \tag{2}$$

where according to (1) $\{\widehat{\Delta\tau} - a > x/2|a\}$ is the event of deciding on ($a + x$) when (a) is correct, $\{\widehat{\Delta\tau} - a - x \leq -x/2|a + x\}$ is the event of deciding on (a) when ($a + x$) is correct.

Let $P_e(a, a + x)$ denote the minimum attainable probability of error (associated with the LRT). It immediately follows that

$$P_e(a, a + x) \leq \tfrac{1}{2}P\{\widehat{\Delta\tau} - a > x/2|a\} + \tfrac{1}{2}P\{\widehat{\Delta\tau} - a - x \leq -x/2|a + x\}. \tag{3}$$

The last inequality holds for arbitrary estimate $\widehat{\Delta\tau}$ of $\Delta\tau$. If $\epsilon = \widehat{\Delta\tau} - \Delta\tau$ denotes the estimation error, the inequality in (3) then reads

$$P_e(a, a + x) \leq \tfrac{1}{2}P\{\epsilon > x/2|a\} + \tfrac{1}{2}P\{\epsilon \leq -x/2|a + x\}. \tag{4}$$

Let the *a priori* domain of $\Delta\tau$ be given by $[-D/2, D/2]$. Since (4) holds for any preselected a and x, it certainly holds for all combinations of a and x such that $a, (a + x) \in [-D/2, D/2]$, or

$$-D/2 \leq a \leq D/2 - x \qquad 0 \leq x \leq D. \tag{5}$$

Integrating (4) with respect to a over $[-D/2, D/2 - x]$ one obtains

$$\begin{aligned}\int_{-D/2}^{D/2-x} & P_e(a, a + x)\, da \\ &\leq \tfrac{1}{2}\int_{-D/2}^{D/2-x} [P\{\epsilon > x/2|a\} + P\{\epsilon \leq -x/2|a + x\}]\, da \\ &= \tfrac{1}{2}\int_{-D/2}^{D/2-x} P\{\epsilon > x/2|a\}\, da \\ &\quad + \tfrac{1}{2}\int_{-D/2+x}^{D/2} P\{\epsilon \leq -x/2|a\}\, da \\ &\leq \tfrac{1}{2}\int_{-D/2}^{D/2} P\{|\epsilon| \geq x/2|a\}\, da.\end{aligned} \tag{6}$$

Now define

$$F(x) \triangleq \frac{1}{D}\int_{-D/2}^{D/2} P\{|\epsilon| \geq x|a\}\, da. \tag{7}$$

$F(x)$ can therefore be regarded as the average of $P\{|\epsilon| \geq x|a\}$ where a is uniformly distributed in $[-D/2, D/2]$. In terms of $F(x)$ the inequality in (6) reads

$$\int_{-D/2}^{D/2-x} P_e(a, a + x)\, da \leq \frac{D}{2}F(x/2). \tag{8}$$

Note that (8) is useful only for $x \leq D$, since for $x > D$ the integral is negative and therefore zero is a better bound. Multiplying both sides of (8) by $2x/D$ and integrating with respect to x over $[0, D]$ one obtains

$$\frac{2}{D}\int_0^D x\,dx \int_{-D/2}^{D/2-x} P_e(a, a+x)\,da \leq \int_0^D xF(x/2)\,dx$$
$$= 4\int_0^{D/2} xF(x)\,dx \leq 4\int_0^D xF(x)\,dx$$
$$= 2x^2F(x)\big|_0^{D^+} - 2\int_0^D x^2\,dF(x). \tag{9}$$

One can always assume that $F(D^+) = 0$ (otherwise the $\Delta\tau$ estimate can be improved by an obvious modification). One further observes that

$$\overline{\epsilon^2} = -\int_0^D x^2\,dF(x) \tag{10}$$

where $\overline{\epsilon^2}$ is, by definition, the MSE when $\Delta\tau$ is uniformly distributed in $[-D/2, D/2]$. Substituting (10) into (9) one immediately obtains

$$\overline{\epsilon^2} \geq \frac{1}{D}\int_0^D x\,dx \int_{-D/2}^{D/2-x} P_e(a, a+x)\,da. \tag{11}$$

This is the basic bound derived in [9]. Bellini and Tartara [10] observed that $F(x)$ is a nonincreasing function of x. The right-hand side of (8) can therefore be bounded more tightly by

$$G\left[\int_{-D/2}^{D/2-x} P_e(a, a+x)\,da\right] \leq \frac{D}{2}F(x/2) \tag{12}$$

where $G[\cdot]$ is a nonincreasing function of x obtained by filling the valleys (if there are any) in the bracketed function. By definition

$$G\left[\int_{-D/2}^{D/2-x} P_e(a, a+x)\,da\right] \geq \int_{-D/2}^{D/2-x} P_e(a, a+x)\,da. \tag{13}$$

Following the derivation leading from (8) to (11), the improved version of ZZLB is therefore given by

$$\overline{\epsilon^2} \geq \frac{1}{D}\int_0^D xG\left[\int_{-D/2}^{D/2-x} P_e(a, a+x)\,da\right]dx. \tag{14}$$

If $P_e(a, a+x) = P_e(x)$ independent of a, the ZZLB and its improved version assume the simplified forms

$$\overline{\epsilon^2} \geq \frac{1}{D}\int_0^D x(D-x)P_e(x)\,dx \tag{15}$$

and

$$\overline{\epsilon^2} \geq \frac{1}{D}\int_0^D xG[(D-x)P_e(x)]\,dx. \tag{16}$$

We shall find (16) to be the relevant bound for the delay estimation problem.

III. Fundamental Limitations in Passive Time Delay Estimation

The basic system of interest here consists of a stationary source radiating a narrow-band signal towards two spatially separated receivers. Each receiver is contaminated by an additive noise component so that the actual waveforms observed at the receiver outputs are

$$\begin{aligned} r_1(t) &= s(t) + n_1(t) \\ r_2(t) &= s(t-\Delta\tau) + n_2(t) \end{aligned} \qquad -T/2 \leq t \leq T/2. \tag{17}$$

$s(t)$, $n_1(t)$ and $n_2(t)$ are assumed to be sample functions from uncorrelated zero-mean stationary Gaussian random processes with spectral densities $S(\omega)$, $N_1(\omega)$ and $N_2(\omega)$, respectively.[1] The *a priori* domain for the receiver-to-receiver delay is given by

$$-D/2 \leq \Delta\tau \leq D/2 \tag{18}$$

where $\Delta\tau = \pm D/2$ is the resulting differential delay when the signal source is endfire to the receiving array.

The fundamental bound on the mean-square estimation error is given by (16), rewritten here for reference

$$\overline{\epsilon^2} \geq \frac{1}{D}\int_0^D xG[(D-x)P_e(x)]\,dx. \tag{19}$$

In general, a closed analytical form for $P_e(x)$ cannot be found. However, for observation times long compared with the correlation time (inverse bandwidth) of signal and noise (i.e., $WT/2\pi >> 1$), $P_e(x)$ can be closely approximated by

$$P_e(x) = e^{a(x)+b(x)}\phi(\sqrt{2b(x)}) \tag{20}$$

where

$$a(x) = -\frac{T}{2\pi}\int_0^\infty \ln\,[1+\gamma(\omega, x)]\,d\omega \tag{21}$$

$$b(x) = \frac{T}{2\pi}\int_0^\infty \frac{\gamma(\omega, x)}{1+\gamma(\omega, x)}\,d\omega \tag{22}$$

$$\gamma(\omega, x) = \frac{S^2(\omega)\cdot\sin^2\omega x/2}{[N_1(\omega)+N_2(\omega)]\,S(\omega) + N_1(\omega)N_2(\omega)} \tag{23}$$

and

$$\phi(y) = \frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-t^2/2}\,dt. \tag{24}$$

Analytical considerations leading to (20) are contained in Appendix A. Moderately large WT product is usually required for the ML (cross correlator) estimator performance to ap-

[1] We are interested in the ambiguity problem in time-delay estimation, implying the use of widely separated receivers. The assumption of noise incoherence from receiver-to-receiver is therefore not very restrictive.

proach the CRLB in ambiguity-free situations (i.e., when only local (small) estimation errors are allowed). In the approximation of (20) one requires large WT only in that sense. This condition is very generally satisfied in problems of practical interest here.

The formulation of the ZZLB and its improved version is rather different from the formulation of the well-known CRLB (Appendix B). In the analysis to come the relation between the various bounds in the limit of large observation interval will be demonstrated. One first observes that

$$0 \leqslant \gamma/(1+\gamma) \leqslant 1 \tag{25}$$

where $\gamma = \gamma(\omega, x)$. Hence,

$$\begin{aligned} 0 &\leqslant \ln(1+\gamma) - \gamma/(1+\gamma) = -\ln\left[1 - \frac{\gamma}{1+\gamma}\right] - \frac{\gamma}{1+\gamma} \\ &= \sum_{n=2}^{\infty} \frac{1}{n} \cdot \left(\frac{\gamma}{1+\gamma}\right)^n = \frac{1}{2}\left(\frac{\gamma}{1+\gamma}\right)^2 \sum_{n=0}^{\infty} \frac{2}{n+2}\left(\frac{\gamma}{1+\gamma}\right)^n \\ &\leqslant \frac{1}{2}\left(\frac{\gamma}{1+\gamma}\right)^2 \sum_{n=0}^{\infty}\left(\frac{\gamma}{1+\gamma}\right)^n = \frac{1}{2}\frac{\gamma^2}{1+\gamma} \leqslant \frac{1}{2}\gamma^2. \end{aligned} \tag{26}$$

It immediately follows that

$$\begin{aligned} 0 &\geqslant [a(x) + b(x)] \\ &= -\frac{T}{2\pi}\int_0^\infty \left\{\ln[1+\gamma(\omega,x)] - \frac{\gamma(\omega,x)}{1+\gamma(\omega,x)}\right\} d\omega \\ &\geqslant -\frac{T}{4\pi}\int_0^\infty \gamma^2(\omega,x)\, d\omega. \end{aligned} \tag{27}$$

Furthermore, since $\gamma/(1+\gamma) \leqslant \gamma$ then

$$0 \leqslant b(x) \leqslant \frac{T}{2\pi}\int_0^\infty \gamma(\omega,x)\, d\omega. \tag{28}$$

It follows that

$$\begin{aligned} P_e(x) &\geqslant \exp\left\{-\frac{T}{4\pi}\int_0^\infty \gamma^2(\omega,x)\, d\omega\right\} \\ &\quad \cdot \phi\left(\sqrt{\frac{T}{\pi}\int_0^\infty \gamma(\omega,x)\, d\omega}\right). \end{aligned} \tag{29}$$

The integrals appearing in (29) can further be bounded as follows

$$\begin{aligned} &\frac{T}{\pi}\int_0^\infty \gamma(\omega,x)\, d\omega \\ &\quad = \frac{T}{\pi}\int_0^\infty \frac{S^2(\omega)\sin^2 \omega x/2}{[N_1(\omega)+N_2(\omega)]\, S(\omega) + N_1(\omega) N_2(\omega)}\, d\omega \\ &\quad \leqslant \frac{T}{\pi}\int_0^\infty \frac{S^2(\omega)(\omega x/2)^2}{[N_1(\omega)+N_2(\omega)]\, S(\omega) + N_1(\omega) N_2(\omega)}\, d\omega \\ &\quad \triangleq \beta^2 x^2 \end{aligned} \tag{30}$$

where

$$\beta^2 = \frac{T}{4\pi}\int_0^\infty \frac{\omega^2 S^2(\omega)\, d\omega}{[N_1(\omega)+N_2(\omega)]\, S(\omega) + N_1(\omega) N_2(\omega)} \tag{31}$$

in the second version of (30) we used the relation $0 \leqslant (\sin x/x)^2 \leqslant 1$. Following the same considerations

$$\frac{T}{4\pi}\int_0^\infty \gamma^2(\omega,x)\, d\omega \leqslant \delta^4 x^4 \tag{32}$$

where

$$\begin{aligned} \delta^4 &= \frac{T}{64\pi}\int_0^\infty \\ &\quad \cdot \left(\frac{\omega^2 S^2(\omega)}{[N_1(\omega)+N_2(\omega)]\, S(\omega) + N_1(\omega) N_2(\omega)}\right)^2 d\omega. \end{aligned} \tag{33}$$

Hence,

$$P_e(x) \geqslant e^{-\delta^4 x^4}\phi(\beta x). \tag{34}$$

Substituting (34) into (19), one obtains

$$\begin{aligned} \overline{\epsilon^2} &\geqslant \frac{1}{D}\int_0^D xG[(D-x)P_e(x)]\, dx \\ &\geqslant \frac{1}{D}\int_0^D x(D-x)P_e(x) \\ &\geqslant \frac{1}{D}\int_0^D x(D-x)\, e^{-\delta^4 x^4}\phi(\beta x)\, dx \\ &= \frac{1}{\beta^2}\cdot\frac{1}{\beta D}\int_0^{\beta D} y(\beta D - y)\, e^{-[y^4/(\beta/\delta)^4]}\phi(y)\, dy \end{aligned} \tag{35}$$

where

$$\left(\frac{\beta}{\delta}\right)^4 = \frac{4T}{\pi}\frac{\left[\int_0^\infty \frac{\omega^2 S^2(\omega)\, d\omega}{[N_1(\omega)+N_2(\omega)]\, S(\omega)+N_1(\omega)N_2(\omega)}\right]^2}{\int_0^\infty \left(\frac{\omega^2 S^2(\omega)}{[N_1(\omega)+N_2(\omega)]\, S(\omega)+N_1(\omega)N_2(\omega)}\right)^2 d\omega}. \tag{36}$$

As $T \to \infty$ $(\beta/\delta)^4 \to \infty$. Since the significant contribution to the integral (35) results from small values of y, it can be approximated without incurring any significant error by

$$\begin{aligned} \overline{\epsilon^2} &\geqslant \frac{1}{\beta^2}\cdot\frac{1}{\beta D}\int_0^{\beta D} y(\beta D - y)\phi(y)\, dy \\ &= \frac{1}{\beta^2}\left\{\left[\frac{1}{4} - \frac{1}{2}\phi(\beta D)\right] + \frac{(\beta D)^2}{6}\right. \\ &\quad \cdot \left[\phi(\beta D) - \frac{1}{\sqrt{2\pi}\,\beta D} e^{-(\beta D)^2/2}\right] \\ &\quad - \frac{2}{3\sqrt{2\pi}\,\beta D}\left[1 - e^{(\beta D)^2/2}\right]. \end{aligned} \tag{37}$$

For $\beta D >> 1$ (37) approaches the limit

$$\overline{\epsilon^2} \geqslant 1/4\beta^2. \tag{38}$$

The right-hand side of (38) is the CRLB derived in Appendix B. Thus if T increases without a limit the ZZLB and its improved version converge to the CRLB. This result is consistent with the well known theorem in statistics asserting that the CRLB can be approached arbitrarily closely if the observation time is long enough. We further note that since (23) uses arbitrary spectral functions the above conclusion is equally applicable to wide-band as well as narrow-band systems. In the next section we shall quantify the actual amount of WT product (observation interval) required to achieve this limit for time delay estimation using narrow-band signals.

A. Narrow-Band Signals

We shall now concentrate our attention to signals whose power is narrowly distributed about the center frequency. For analytical convenience consider the following illustration:

$$\frac{S(\omega)}{N_i(\omega)} = \begin{cases} S/N_i \text{ (constant)} & |\omega \pm \omega_0| \leqslant W/2 \\ 0 & |\omega \pm \omega_0| > W/2. \end{cases} \tag{39}$$

Hence,

$$\gamma(\omega, x) = \begin{cases} \text{SNR} \sin^2 \omega x/2 & |\omega \pm \omega_0| \leqslant W/2 \\ 0 & |\omega \pm \omega_0| > W/2 \end{cases} \tag{40}$$

where

$$\text{SNR} \triangleq \frac{(S/N_1)(S/N_2)}{1 + S/N_1 + S/N_2}. \tag{41}$$

Since our ultimate interest is to set lower bound on the mean-square estimation error, we shall find (29) to be very useful. With the above convention, the integrals appearing in (29) assume the simplified form

$$\frac{T}{\pi} \int_0^\infty \gamma(\omega, x)\, d\omega = \frac{T}{\pi} \text{SNR} \int_{\omega_0 - W/2}^{\omega_0 + W/2} \sin^2 \omega x/2\, d\omega = \frac{WT}{2\pi} \text{SNR} \left(1 - \frac{\sin Wx/2}{Wx/2} \cos \omega_0 x\right). \tag{42}$$

Similarly

$$\frac{T}{4\pi} \int_0^\infty \gamma^2(\omega, x)\, d\omega = \frac{T}{4\pi} \text{SNR}^2 \int_{\omega_0 - W/2}^{\omega_0 + W/2} \sin^4 \omega x/2\, d\omega = \frac{WT}{16\pi} \text{SNR}^2 \left(3 - 4 \frac{\sin Wx/2}{Wx/2} \cos \omega_0 x + \frac{\sin Wx}{Wx} \cos 2\omega_0 x\right). \tag{43}$$

Intuitively, one may expect $P_e(x)$ to be a monotonically decreasing function of x indicating that better detection performance can be achieved if the separation between the two hypothesized delays (a) and $(a + x)$ increases. Perhaps the most striking feature of (42) and (43), however, is that they are quasi-periodic functions of x with a period of $2\pi/\omega_0$ and therefore (29) is also quasi-periodic with the same period. Similar observation can be made with respect to (20). This phenomenon reflects the ambiguity problem in the narrow-band case; the observed source signal looks much like a sinusoid with random amplitude and phase. In that case the receiver-to-receiver delay is very nearly translated into differential phase which, in turn, can be identified only modulu 2π. Further note that the ambiguity problem affects the lower bound (19) significantly only when the *a priori* parameter domain is at least several periods of $P_e(x)$ i.e., $D >> 2\pi/\omega_0$. As $D = 2L/c$ and $\omega_0/2\pi = c/\lambda$, where L is the spacing between receivers and λ is the wavelength of the signal center frequency, the latter inequality can easily be converted to $L >> \lambda/2$. Since we are primarily interested in the ambiguity phenomenon (the large estimation errors mode of operation) we shall therefore assume that this condition holds.

With the above convention, $(D - x) P_e(x)$ and $G[(D - x) \cdot P_e(x)]$ assume the general form illustrated in Fig. 4. The difference between the two functions is rather substantial, indicating that the improved version of the ZZLB is likely to yield significantly tighter results. Fig. 4 further suggests that $G[(D - x) P_e(x)]$ can be closely approximated from knowledge of the bracketed function at the discrete set $x_n = (2\pi/\omega_0)\, n$. Substituting $x = x_n$ into (42) and (43) one obtains

$$\frac{T}{\pi} \int_0^\infty \gamma(\omega, x_n)\, d\omega = \frac{WT}{2\pi} \text{SNR} \left(1 - \frac{\sin Wx_n/2}{Wx_n/2}\right) \leqslant \frac{WT}{2\pi} \text{SNR} \frac{(Wx_n/2)^2}{6} = \eta^2 x_n^2 \tag{44}$$

where

$$\eta^2 = \frac{W^3 T}{48\pi} \text{SNR}. \tag{45}$$

Similarly

$$\frac{T}{4\pi} \int_0^\infty \gamma^2(\omega, x_n)\, d\omega = \frac{WT}{16\pi} \text{SNR}^2 \left(3 - 4 \frac{\sin Wx_n/2}{Wx_n/2} + \frac{\sin Wx_n}{Wx_n}\right) \leqslant \frac{WT}{16\pi} \text{SNR}^2 \frac{(Wx_n/2)^4}{20} \triangleq \mu^4 x_n^4 \tag{46}$$

where

$$\mu^4 = \frac{W^5 T}{5 \cdot 2^{10} \pi} \text{SNR}^2. \tag{47}$$

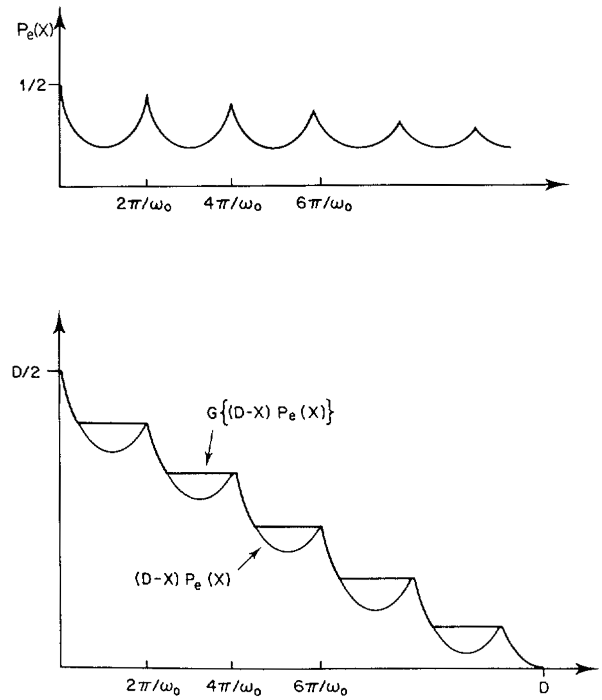


Fig. 4. The relation between $(D - x) P_e(x)$ and $G[(D - x) P_e(x)]$.

Substituting (44) and (46) into (29) one immediately obtains

$$P_e(x_n) \geqslant e^{-\mu^4 x_n^4} \phi(\eta x_n). \tag{48}$$

One further observes that

$$\begin{aligned} G&[(D-x)P_e(x)] \\ &\geqslant (D - x_n) P_e(x_n)\, \Pi\,(x; x_n) \\ &\geqslant (D - x_n)\, e^{-\mu^4 x_n^4} \phi(\eta x_n)\, \Pi\,(x; x_n) \\ &\geqslant (D - 2\pi/\omega_0 - x)\, e^{-\mu^4 (x + 2\pi/\omega_0)^4} \phi[\eta(x + 2\pi/\omega_0)] \end{aligned} \tag{49}$$

where

$$\Pi\,(x; x_n) = \begin{cases} 1 & x_{n-1} \leqslant x \leqslant x_n \\ 0 & \text{elsewhere.} \end{cases} \tag{50}$$

$(D - x_n) P_e(x_n)\, \Pi\,(x;\, x_n)$ is therefore a monotonically decreasing step function with a step size of $2\pi/\omega_0$. The first inequality in (49) results from the definition of $G[\cdot]$ (see also Fig. 4). The second inequality is obtained by connecting the lower edges of the indicated step function. The approximation in (49) is rather tight except for values of x in the vicinity of $x = 0$ where $P_e(x)$ drops quite sharply. In that region (34) (which is based on the inequality $\sin x/x \leqslant 1$ and thus appears to be rather tight for small x) can be used to yield

$$G[(D - x) P_e(x)] \geqslant (D - x) P_e(x) \geqslant (D - x)\, e^{-\delta^4 x^4} \phi(\beta x). \tag{51}$$

For constant in-band SNR β and δ are given by substituting (39) into (31) and (33), respectively. The resulting terms are

$$\begin{aligned} \beta^2 &= \frac{T}{4\pi} \int_{\omega_0 - W/2}^{\omega_0 + W/2} \text{SNR} \cdot \omega^2 d\omega \\ &= \frac{WT}{4\pi}\, \omega_0^2\, \text{SNR} \left(1 + \frac{W^2}{12\omega_0^2}\right) \\ &\approx \frac{WT}{4\pi}\, \omega_0^2\, \text{SNR}. \end{aligned} \tag{52}$$

Similarly

$$\begin{aligned} \delta^4 &= \frac{T}{64\pi} \int_{\omega_0 - W/2}^{\omega_0 + W/2} [\text{SNR} \cdot \omega^2]^2\, d\omega \\ &\approx \frac{WT}{64\pi}\, \omega_0^4\, \text{SNR}^2. \end{aligned} \tag{53}$$

Combining (49) and (51), $G[(D - x) P_e(x)]$ is tightly bounded by

$$G[(D - x) P_e(x)] \geqslant \begin{cases} (D - x)\, e^{-\delta^4 x^4} \phi(\beta x) & 0 \leqslant x < z \\ \left(D - \dfrac{2\pi}{\omega_0} - x\right) e^{-\mu^4 [x + (2\pi/\omega_0)]^4} \phi\left[\eta\left(x + \dfrac{2\pi}{\omega_0}\right)\right] & z \leqslant x < D - \dfrac{2\pi}{\omega_0} \\ 0 & D - \dfrac{2\pi}{\omega_0} \leqslant x \leqslant D \end{cases} \tag{54}$$

where z is defined by the equation

$$\beta \cdot z = \eta(z + 2\pi/\omega_0). \tag{55}$$

The lower bound on the MSE is therefore given by

$$\begin{aligned} \overline{e^2} &\geqslant \frac{1}{D} \int_0^D x G[(D - x) P_e(x)]\, dx \\ &\geqslant \frac{1}{D} \int_0^z x(D - x)\, e^{-\delta^4 x^4} \phi(\beta x)\, dx \\ &\quad + \frac{1}{D} \int_z^{D - 2\pi/\omega_0} x(D - 2\pi/\omega_0 - x) \\ &\quad \cdot e^{-\mu^4 (x + 2\pi/\omega_0)^4} \phi[\eta(x + 2\pi/\omega_0)]\, dx. \end{aligned} \tag{56}$$

Making the change of variables $y = \beta x$ in the first integral and $y = \eta(x + 2\pi/\omega_0)$ in the second integral and using the relation in (55) for the upper limit of the first integral one immediately obtains

$$\overline{\epsilon^2} \geqslant \frac{1}{\beta^2} \cdot \frac{1}{\beta D} \int_0^{\eta(z+2\pi/\omega_0)} y(\beta D - y) \cdot e^{-y^4/8(WT/2\pi)} \phi(y)\, dy$$

$$+ \frac{1}{\eta^2} \cdot \frac{1}{\eta D} \int_{\eta(z+2\pi/\omega_0)}^{\eta D} (y - \eta \cdot 2\pi/\omega_0)(\eta D - y) \cdot e^{-9y^4/40(WT/2\pi)} \phi(y)\, dy \tag{57}$$

where in (57) we substitute

$$(\beta/\delta)^4 = 8(WT/2\pi), \qquad (\eta/\mu)^4 = \frac{40}{9}(WT/2\pi). \tag{58}$$

The inequality in (57) provides a closed form, however, still rather complicated solution for the lower bound. This result can further be simplified following the considerations outlined below. First note from (55) that

$$z = \frac{1}{\beta/\eta - 1} \cdot \frac{2\pi}{\omega_0} = \frac{1}{\sqrt{12}\,\omega_0/W - 1} \cdot \frac{2\pi}{\omega_0}. \tag{59}$$

Further note that

$$\beta D = \frac{\beta}{\eta} \cdot \frac{D}{2\pi/\omega_0} \cdot \eta \cdot \frac{2\pi}{\omega_0} = \frac{\sqrt{12}\,\omega_0}{W} \cdot \frac{L}{\lambda/2} \cdot \eta \cdot \frac{2\pi}{\omega_0}. \tag{60}$$

Since $\omega_0/W >> 1$ and $L >> \lambda/2$ it follows that $z << 2\pi/\omega_0$ and $\beta D >> \eta/f_0$ where $f_0 = \omega_0/2\pi$. The lower bound can therefore be closely approximated by

$$\overline{\epsilon^2} \geqslant \frac{1}{\beta^2} \int_0^{\eta/f_0} y e^{-y^4/8(WT/2\pi)} \phi(y)\, dy$$

$$+ \frac{1}{\eta^2} \frac{1}{\eta D} \int_{\eta/f_0}^{\eta D} (y - \eta/f_0)(\eta D - y) \cdot e^{-9y^4/40(WT/2\pi)} \phi(y)\, dy. \tag{61}$$

The significant contribution to the integrals in (61) results from small values of y. For $WT/2\pi >> 1$ one can therefore approximate $e^{-y^4/8(WT/2\pi)} \approx 1$ in the first integral without incurring any significant error. Following the same consideration in the second integral one obtains

$$\overline{\epsilon^2} \geqslant \frac{1}{\beta^2} \int_0^{\eta/f_0} y\phi(y)\, dy + \frac{1}{\eta^2} e^{-[9(y/f_0)^4/40(WT/2\pi)]}$$

$$\cdot \frac{1}{\eta D} \int_{\eta/f_0}^{\eta D} (y - \eta/f_0)(\eta D - y)\, \phi(y)\, dy. \tag{62}$$

Performing the integrations required in (62) and observing that $\beta^2/\eta^2 = 12\omega_0^2/W^2 >> 1$ and that $f_0 D = L/(\lambda/2) >> 1$, thus ignoring small order terms, one finds

$$\overline{\epsilon^2} \geqslant \frac{1}{2\beta^2} \left[\frac{1}{2} - \phi(\eta/f_0)\right] + \frac{1}{2\eta^2} e^{-[9(\eta/f_0)^4/40(WT/2\pi)]}$$

$$\cdot \Big\{[1 + (\eta/f_0)^2]\, \phi(\eta/f_0) - [1 - (\eta D)^2/3]\, \phi(\eta D)$$

$$- [\eta/f_0 + 4/3(\eta D)]\, e^{-(\eta/f_0)^2/2}/\sqrt{2\pi}$$

$$- \frac{1}{3}(\eta D - 4/\eta D)\, e^{-(\eta D)^2/2}/\sqrt{2\pi}\Big\}. \tag{63}$$

Numerical integration indicates that (63) is an excellent approximation to (19). The former equation can therefore be used to closely characterize the lower bound. Let us now consider few limiting cases.

$\underline{\eta/f_0 > 1}$: The dominant term in (63) becomes

$$\overline{\epsilon^2} \geqslant \frac{1}{2\beta^2}\left[\frac{1}{2} - \phi(\eta/f_0)\right] \approx \frac{1}{4\beta^2} = \frac{\pi}{WT\omega_0^2\, \text{SNR}} \tag{64}$$

the calculated bound approaches the CRLB. This is the ambiguity-free mode of operation.

$\underline{\eta/f_0 < 1 \textit{ but } \eta D > 1}$: The dominant term in (63) becomes

$$\overline{\epsilon^2} \geqslant \frac{1}{2\eta^2} \phi(\eta/f_0) \approx \frac{1}{4\eta^2} = \frac{12\pi}{W^3 T\, \text{SNR}} \tag{65}$$

the calculated bound exceeds the CRLB by a factor of $\eta^2/\beta^2 = 12(\omega_0/W)^2$. This is the ambiguity-dominated mode of operation.

$\underline{\eta D < 1}$: It immediately follows that $\eta/f_0 = \eta D/f_0 D << 1$. One can therefore make the following approximations:

$$\phi(\eta/f_0) \approx 1/2 \qquad e^{-(\eta/f_0)^2/2} \approx 1$$

$$\phi(\eta D) \approx 1/2 - \eta D/\sqrt{2\pi} \qquad e^{-(\eta D)^2/2} \approx 1 - (\eta D)^2/2. \tag{66}$$

Substituting (66) into (63) and ignoring terms on the order of $(\eta D)^3$ one obtains

$$\overline{\epsilon^2} \geqslant D^2/12 \tag{67}$$

The bound in (67) is completely independent of SNR. We further observe that if we set $\hat{\Delta\tau} \equiv 0$, the estimation error is uniformly distributed in $[-D/2, D/2]$ and its MSE equals $D^2/12$. This limit, therefore, is always attainable regardless of signal observations. This is the noise-dominated region where information from the receiver outputs is essentially useless for the delay estimation.

We have therefore identified three different asymptotes. The transition (3 dB point) from the $D^2/12$ asymptote to the second asymptote is simply obtained by intersecting (67) with (65). The result is

$$\text{SNR}_1 = \frac{18}{\pi^2 (WT/2\pi)} \left(\frac{\omega_0}{W}\right)^2 \frac{1}{f_0 D}. \tag{68}$$

Both (65) and (64) vary like the first inverse power of SNR thus never intersect, indicating a transition region. To define

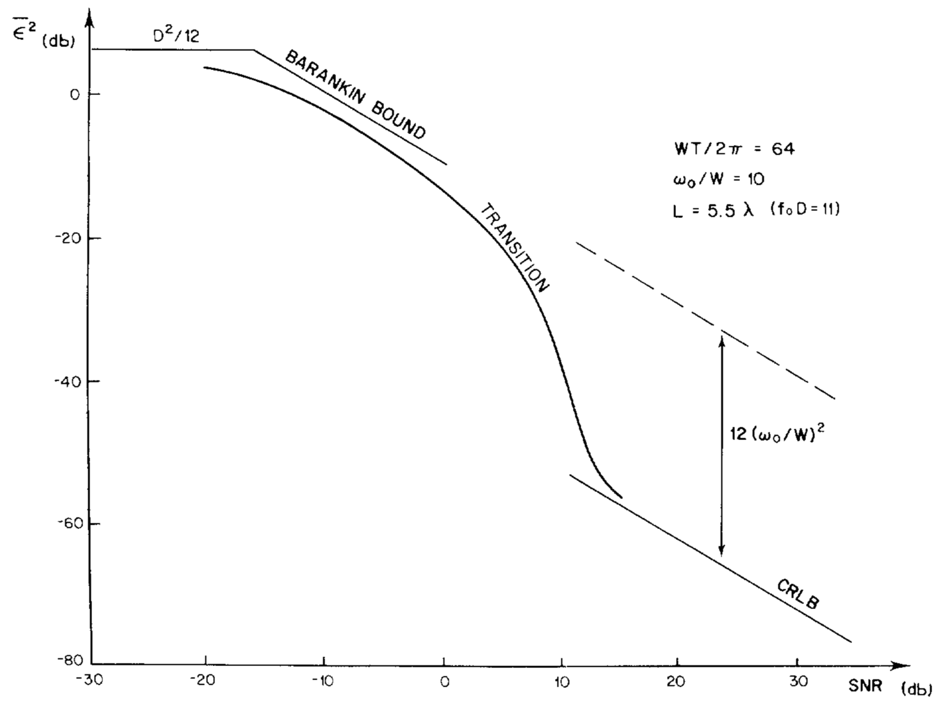


Fig. 5. Composite bound on the attainable mean-square estimation error.

its limits, let us express the bound as the sum of terms appearing in (64) and (65)

$$\bar{\epsilon}^2 \geqslant \frac{1}{2\eta^2}\,\phi(\eta/f_0) + \frac{1}{2\beta^2}\left[\frac{1}{2} - \phi(\eta/f_0)\right]. \tag{69}$$

The transition from the first term to the second term starts at

$$\phi(\eta/f_0) = 1/4 \tag{70}$$

and ends at

$$\frac{1}{2\eta^2}\,\phi(\eta/f_0) = \frac{1}{2\beta^2}\left[\frac{1}{2} - \phi(\eta/f_0)\right]. \tag{71}$$

In terms of SNR (70) and (71) read

$$\text{SNR}_2 = \frac{6 \cdot 0.46}{\pi^2 (WT/2\pi)}\left(\frac{\omega_0}{W}\right)^2 \tag{72}$$

$$\text{SNR}_3 \approx \frac{6}{\pi^2 (WT/2\pi)}\left(\frac{\omega_0}{W}\right)^2 \left[\phi^{-1}\left(\frac{W^2}{24\omega_0^2}\right)\right]^2. \tag{73}$$

Exact numerical integration of (19) indicates that the asymptotic characterization of the lower bound illustrated in Fig. 3 can be used with some confidence to describe the modified ZZLB (e.g., Fig. 5).

The form of (68), (72) and (73) furnishes some interesting insights:

1) The various SNR_i are proportional to the first inverse power of the WT product. Thus, for any prespecified SNR conditions one can find WT large enough to yield an ambiguity-free estimate characterized by the CRLB. This observation is consistent with the asymptotic theorem asserting that the CRLB is attainable for sufficiently long observation times. Making use of (73), however, one can actually specify the minimum amount of WT required to achieve this limit.

2) Taking the ratio of (72) to (71) and (73) to (72), respectively, one obtains

$$\text{SNR}_2/\text{SNR}_1 = 0.153\,(f_0 D)^2 \tag{74}$$

$$\text{SNR}_3/\text{SNR}_2 = 2.17\,[\phi^{-1}(W^2/24\omega_0^2)]^2. \tag{75}$$

Thus the SNR band (in dB scale) associated with the ambiguity-dominated mode of operation depends on $f_0 D = L/\lambda/2)$, the number of half-wavelengths of the signal center-frequency contained in the spacing between the receivers, or the order of ambiguity. One further observes that (74) is completely independent of the system WT product. A similar observation can be made with respect to (75), the SNR band over which the threshold phenomenon occurs. The CRLB, therefore, can never be achieved uniformly for all SNR. As WT increases (i.e., large observation periods) SNR_3 is shifted to the left however SNR_2 and SNR_1 are shifted by the same amount.

Comparison with the CRLB: The lower bound on the attainable MSE predicted by the Cramer-Rao inequality is given by (64), rewritten here for reference

$$\bar{\epsilon}^2 \geqslant \frac{\pi}{WT \cdot \omega_0^2\ \text{SNR}}. \tag{76}$$

One first observes that (76) depends on W and T only through their product. Smaller signal bandwidth can therefore be accommodated by longer observation interval. One must remember, however, that in the analysis based on the CRLB it is assumed that only local estimation errors are likely to occur, thus completely ignore the ambiguity effect. The above statement is therefore correct only for SNR in access of the threshold point. For constant WT product, as W decreases ω_0/W increases indicating a more serious ambiguity problem. This is illustrated in Fig. 6 where the various performance characteristics were generated by exact numerical integration of (19).

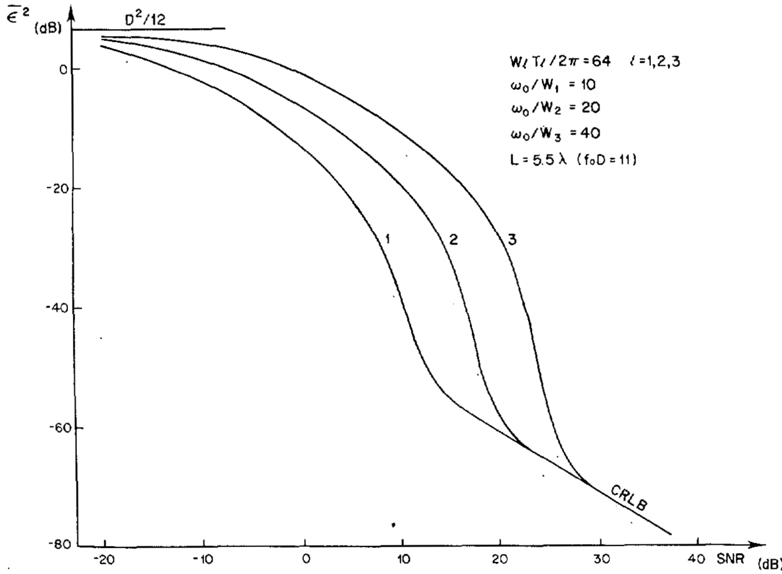


Fig. 6. Performance characteristics for WT = constant.

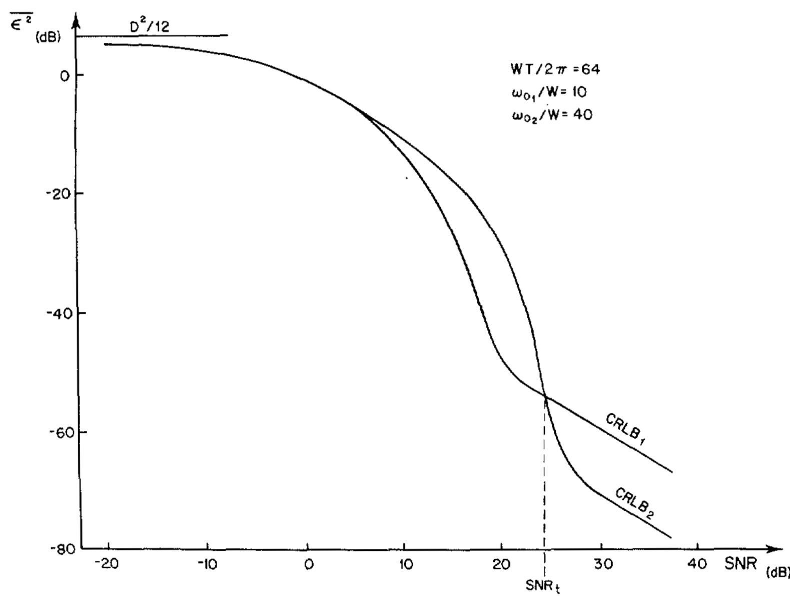


Fig. 7. Performance characteristics as a function of ω_0.

One further observes that (76) varies with the second inverse power of ω_0, indicating that system MSE performance can be improved simply by up-shifting the received signals in the frequency domain. However, this statement holds only in the small error regime as illustrated in Fig. 7. One immediately observes that the two performance characteristics [generated by exact numerical integration of (19)] intersect. Above SNR_t the advantage rests very clearly with the delay estimation system centered about ω_{0_2}, below SNR_t the advantage rests with the delay estimation system centered about ω_{0_1}. This rather interesting phenomenon should not be surprising; consider a typical signal cross correlation illustrated in Fig. 1. In the analysis based on the CRLB it is assumed that the peak of the cross correlator output associated with the true differential delay can be identified. This, in turn, requires prior knowledge of the receiver-to-receiver delay to within $2\pi/\omega_0$. As ω_0 increases, indicating that stronger *a priori* information is assumed, the local MSE characterized by the CRLB (or the width of the true cross correlation peak) improves, however the cross correlation peaks adjacent to the true peak have very nearly equal height indicating a more critical ambiguity problem.

One therefore concludes that MSE predictions based on the Cramer–Rao theory must be considered very carefully. Perhaps the most useful information is provided by the combination of SNR and WT required to achieve the CRLB or the small error regime. From (73) one immediately obtains

$$(WT/2\pi)\,\mathrm{SNR} \geqslant \frac{6}{\pi^2}\left(\frac{\omega_0}{W}\right)^2 [\phi^{-1}(W^2/24\omega_0^2)]^2. \tag{77}$$

$(WT/2\pi)$ SNR can be interpreted as the postintegration SNR. Thus for 10 percent signal bandwidth (i.e., $\omega_0/W = 10$) the product in (77) must exceed 28.3 dB. For 1 percent signal bandwidth, the postintegration SNR must exceed 50.8 dB.

Comparison with the Barankin Lower Bound: Analysis of the attainable MSE based on a simplified (analytically tractable) version of the Barankin bound is carried out in [8] and the basic result is illustrated in Fig. 2. Direct comparison between Figs. 2 and 3 indicates that the improved ZZLB and the Barankin bound coincide for $\mathrm{SNR}_1 \leqslant \mathrm{SNR} \leqslant \mathrm{Min}\ \{\mathrm{SNR}_{\mathrm{Th}}, \mathrm{SNR}_2\}$. Below SNR_1 the Barankin lower bound exceeds the $D^2/12$ asymptote. As already indicated before, however, the $D^2/12$ performance level can always be achieved regardless of signal observations. It is not difficult to find an explanation to the apparent paradox; the number of test points used in the derivation of the Barankin inequality is unconstrained. This is equivalent to the assumption that the spacing between receivers is arbitrarily large. This assumption can most easily be taken into consideration in the present analysis by letting $f_0 D \to \infty$. In that limit $\mathrm{SNR}_1 \to -\infty$ and the calculated bound converge to the Barankin bound whenever SNR is below the threshold point. As a practical matter, however, (68) provides important additional information concerning the fundamental limitations on system performance.

From the analysis based on the Barankin inequality the threshold SNR is found to be

$$\mathrm{SNR}_{\mathrm{Th}} = \omega_0^2 / W^3 T. \tag{78}$$

The critical SNR predicted in this analysis is given by (72) rewritten here for reference

$$\mathrm{SNR}_2 = \frac{6 \cdot 0.46}{\pi^2 (WT/2\pi)} \left(\frac{\omega_0}{W}\right)^2 \approx 1.8 \omega_0^3 / W^3 T. \tag{79}$$

The result in (79) exceeds the result stated in (78) by approximately 2.5 dB. One must remember, however, that SNR_2 is defined as the point at which the modified ZZLB is 3 dB below the Barankin bound. $\mathrm{SNR}_{\mathrm{Th}}$ is the critical point at which the Barankin inequality still characterizes the lower bound.

Above the threshold, however, error analysis based on the Barankin inequality diverges thus completely fails to provide any useful information. The analysis based on the modified ZZLB clearly traces the transition from the ambiguity dominated mode of operation to the ambiguity free mode of operation. The SNR band over which the threshold phenomenon occurs is given by (75) rewritten here for reference

$$\mathrm{SNR}_3 / \mathrm{SNR}_2 = 2.17\ [\phi^{-1}(W^2/24\omega_0^2)]). \tag{80}$$

It is suggested in [8] that the ambiguity error is strictly additive. Once the SNR exceeds the critical point the CRLB is immediately attainable. With regard to (80), however, this observation is rather optimistic. The actual point at which the CRLB is attained within 3 dB is specified only by SNR_3. For $\omega_0/W = 10$ $\mathrm{SNR}_3/\mathrm{SNR}_2 = 13.8$ dB. For $\omega_0/W = 100$ $\mathrm{SNR}_3/\mathrm{SNR}_2 = 16.3$ dB. SNR_3 therefore exceeds the threshold point given by (78) or (79) by a rather significant amount.

One therefore concludes that while a first attempt to analyze the ambiguity and threshold phenomena in passive time delay estimation is reported in [8], the results of that study are clearly dominated by the results reported in this paper.

B. Generalization to M Receivers

In the previous section we analyzed the two receivers delay estimation as a completely isolated problem. In passive radar/sonar applications, however, source localization is generally accomplished by utilizing arrays of large number of receivers. The problem we are addressing in this section is, therefore, how the quality of the differential delay estimate for a particular receiver pair can be improved by the availability of information from other receivers.

The basic system of interest here consists of M-receiver array and a narrow-band Gaussian source. Suppose the range from the source to the center of the array is large compared with the spacing between receivers so that the signal wavefront is essentially planar across the array baseline. For analytical convenience we shall confine ourselves to linear arrays of evenly spaced receivers. In this setting, therefore, the signal waveform observed at the output of the ith receiver is given by

$$r_i(t) = s[t - (i-1)\Delta\tau] + n_i(t)$$
$$-T/2 \leqslant t \leqslant T/2, \quad i = 1, 2, \cdots, M. \tag{81}$$

$\Delta\tau$ is the differential time delay between adjacent receivers. For $M = 2$ (81) reduces to (17). In complete analogy with the two receiver case we shall assume that the noise components at the various receiver outputs are statistically uncorrelated zero-mean Gaussian random processes. We further assume, for analytical convenience, that the spectral density function of the various noise components are identical, i.e., $N_i(\omega) = N(\omega)$, $i = 1, 2, \cdots, M$.

The basic bound on the attainable MSE is given by (19) where $P_e(x)$ must now be calculated for the M receivers case. Following some rather extensive algebra manipulations one finds that $P_e(x)$ is still characterized by (20) where $a(x)$ and $b(x)$ are given by (21) and (22), respectively, and $\gamma(\omega, x)$ is given by

$$\gamma(\omega, x) = \frac{S^2(\omega)/N^2(\omega)}{1 + MS(\omega)/N(\omega)} \sum_{i=1}^{M-1} i \sin^2 [(M-i)\,\omega x/2]$$
$$= \frac{1}{4} \frac{S^2(\omega)/N^2(\omega)}{1 + MS(\omega)/N(\omega)} \left(M^2 - \frac{\sin^2 M\omega x/2}{\sin^2 \omega x/2}\right). \tag{82}$$

For $M = 2$ (and $N_i(\omega) = N(\omega)$, $i = 1, 2$) (82) reduces to (23). For constant in-band signal-to-noise ratio, as assumed in (39)

$$\gamma(\omega, x) = \begin{cases} \dfrac{S^2/N^2}{1 + MS/N} \displaystyle\sum_{i=1}^{M-1} i \sin^2 [(M-i)\,\omega x/2] \\ \qquad |\omega \pm \omega_0| \leqslant W/2 \\ 0 \qquad |\omega \pm \omega_0| > W/2. \end{cases} \tag{83}$$

Since

$$\sum_{i=1}^{M-1} i \sin^2 [(M-i)\,\omega x/2]$$
$$\leqslant (\omega x/2)^2 \sum i (M-i)^2 = \frac{M^2(M^2-1)}{48} \omega^2 x^2. \tag{84}$$

It immediately follows that

$$\frac{T}{\pi}\int_0^\infty \gamma(\omega, x)\, d\omega \leqslant \beta_M^2 x^2 \tag{85}$$

$$\frac{T}{4\pi}\int_0^\infty \gamma^2(\omega, x)\, d\omega \leqslant \delta_M^4 x^4 \tag{86}$$

where

$$\beta_M^2 = \frac{WT}{4\pi}\,\omega_0^2\,\mathrm{SNR}_M \tag{87}$$

$$\delta_M^4 = \frac{WT}{64\pi}\,\omega_0^4\,\mathrm{SNR}_M^2 \tag{88}$$

and

$$\mathrm{SNR}_M = \rho_M \cdot \mathrm{SNR}. \tag{89}$$

SNR is given by substituting $S/N_i = S/N$ $i = 1, 2$ into (41). ρ_M is given by

$$\rho_M = \frac{M^2(M^2 - 1)}{12} \cdot \frac{1 + 2S/N}{1 + MS/N}. \tag{90}$$

Substituting (85) and (86) into (29) one immediately obtains

$$P_e(x) \geqslant e^{-\delta_M^4 x^4}\,\phi(\beta_M x). \tag{91}$$

Equation (91) is rather tight for small x. Tighter bound on $P_e(x)$ can further be found for $x_n = (2\pi/\omega_0)\,n$ following the same set of considerations as in the two receivers case. The result is

$$P_e(x_n) \geqslant e^{-\mu_M^4 x_n^4}\phi(\eta_M x_n) \tag{92}$$

where

$$\eta_M^2 = \frac{W^3 T}{48\pi}\,\mathrm{SNR}_M \tag{93}$$

$$\mu_M^4 = \frac{W^5 T}{5 \cdot 2^{10}\pi}\,\mathrm{SNR}_M^2. \tag{94}$$

$G[(D - x)\,P_e(x)]$ can now be tightly bounded following the exact same considerations leading to (54). The resulting bound on the MSE, therefore, assumes the exact same form as in the two receiver cases where SNR is replaced by SNR_M. To put this observation into perspective we note that

$$\rho_M = \frac{M(M^2 - 1)}{6} \cdot \frac{M(1 + 2S/N)}{2(1 + MS/N)} \geqslant \frac{M(M^2 - 1)}{6}. \tag{95}$$

Equation (95) is tight whenever $2S/N > 1$.

The lower bound for the M receivers case, therefore, assumes the general form illustrated in Fig. 3 where the linear segments of the bound characterizing the ambiguity-dominated regime and the ambiguity-free regime are shifted down by (at least) $10 \log_{10} [M(M^2 - 1)/6]$ dB while the various deflection points are shifted to the left by the exact same amount. The same effect on the composite bound results from increasing T. Essentially, therefore, array size can be traded by observation interval. However, since $\rho_M \propto M^3$, then to accommodate smaller arrays, one therefore needs exceedingly long observation periods. Finally, observe that in the ambiguity-dominated mode of operation $\overline{\epsilon^2} \propto 1/W^3$. In that regime, therefore, array size can be traded by signal bandwidth in an essentially linear fashion.

The above discussion is applied to linear arrays of evenly spaced receivers. The realizable improvement in the differential delay information obtained by using an unevenly spaced array of receivers may be more significant than one would have interpreted from array size alone. Investigating the effect of different array configurations on system performance is therefore proposed as a further possible research direction.

IV. Conclusion

In this paper a new method is presented to investigate the mean-square error (MSE) performance of time delay estimation schemes based on a modified (improved) version of the Ziv-Zakai lower bound (ZZLB). This method is found to be particularly useful for analyzing the threshold and ambiguity phenomena (i.e., the large estimation errors mode of operation).

For delay estimation using narrow-band (ambiguity-prone) signals the fundamental results of this study is illustrated in Fig. 3, where the various segments of the composite bound are given by

$$\overline{\epsilon^2} \geqslant \begin{cases} D^2/12 & \mathrm{SNR} \leqslant \mathrm{SNR}_1 \\ \dfrac{12\pi}{W^3 T\,\mathrm{SNR}} & \mathrm{SNR}_1 < \mathrm{SNR} \leqslant \mathrm{SNR}_2 \\ \text{THRESHOLD} & \mathrm{SNR}_2 < \mathrm{SNR} \leqslant \mathrm{SNR}_3 \\ \dfrac{\pi}{WT\omega_0^2\,\mathrm{SNR}} & \mathrm{SNR} > \mathrm{SNR}_3. \end{cases}$$

$[-D/2, D/2]$ is the *a priori* parameter domain, T is the observation interval, W is the signal bandwidth and ω_0 is the signal center frequency.

Hence, if the available SNR does not exceed SNR_1 signal observations from the receiver outputs are completely dominated by noise thus essentially useless for the delay estimation. As a result, the attainable MSE is bounded only by the *a priori* parameter domain. If $\mathrm{SNR}_1 < \mathrm{SNR} \leqslant \mathrm{SNR}_2$ the modified ZZLB coincides with the Barankin bound. In this regime differential delay observations are subject to ambiguities. If $\mathrm{SNR} > \mathrm{SNR}_3$ the modified ZZLB coincides with the Cramer-Rao lower bound (CRLB) indicating that the ambiguity in the differential delay estimation can essentially be resolved. The transition from the ambiguity-dominated mode of operation to the ambiguity-free mode of operation starts at SNR_2 and ends at SNR_3. This is the threshold effect in time delay estimation.

Note that both the second and fourth lines in the above equation vary with the first inverse power of SNR. However, below threshold the minimum attainable MSE exceeds the CRLB by a factor of $12(\omega_0/W)^2$.

The various deflection points are given by

$$\mathrm{SNR}_1 = \frac{72}{WT/2\pi}\left(\frac{\omega_0}{W}\right)^2 \cdot \frac{1}{(\omega_0 D)^2}$$

$$\mathrm{SNR}_2 = \frac{2.17}{\pi^2 (WT/2\pi)}\left(\frac{\omega_0}{W}\right)^2$$

$$\mathrm{SNR}_3 = \frac{6}{\pi^2 (WT/2\pi)}\left(\frac{\omega_0}{W}\right)^2 \left[\phi^{-1}\left(\frac{W^2}{24\omega_0^2}\right)\right]^2$$

where

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-t^2/2}\, dt.$$

With the above information the composite result illustrated in Fig. 3 yields the tightest known bound on the attainable MSE for any prespecified SNR conditions and should therefore be used as the standard of comparison for delay estimation systems.

Appendix A

Calculation of $P_e(a, a+x)$

The binary detection problem under consideration here is given by

$$H_0:\ \Delta\tau = \Delta\tau_0 = a$$

$$H_1:\ \Delta\tau = \Delta\tau_1 = a + x. \tag{A1}$$

A binary detection scheme which minimizes the probability of error simply forms the likelihood ratio test between the two hypothesized delays. Since the signal carrying the differential delay information is assumed to be a sample function from a Gaussian random process, a closed form analytical expression for $P_e(a, a+x)$, the minimum attainable probability of error, cannot be found. For $WT/2\pi >> 1$, however, $P_e(a, a+x)$ can be closely approximated using Chernoff formula [11, p. 125, eq. (484)]

$$P_e(a, a+x) \approx \frac{1}{2}\exp\left\{\mu(s_m) + \frac{s_m^2}{2}\mu''(s_m)\right\}\phi(s_m\sqrt{\mu''(s_m)}$$
$$+ \frac{1}{2}\exp\left\{\mu(s_m) + \frac{(1-s_m)^2}{2}\mu''(s_m)\right\}$$
$$\cdot\, \phi[(1-s_m)\sqrt{\mu''(s_m)} \tag{A2}$$

where s_m is the point for which $\mu'(s_m) = 0$, and $\mu(s)$ is given by

$$\mu(s) = \ln \iint_{-\infty}^{+\infty} \cdots \int [P(r/H_1)]^s\, [P(r/H_0)]^{1-s}\, dr. \tag{A3}$$

$P(\boldsymbol{r}/H_i)$ is the conditional probability density of $\boldsymbol{r}$ under H_i hypothesis (i.e., $\Delta\tau = \Delta\tau_i$) and $\boldsymbol{r}$ is the data vector generated by Fourier analyzing $r_i(t)$ in (17). Since signal and noise are assumed to be zero-mean Gaussian processes and the components of $\boldsymbol{r}$ are generated by linear operations on these time functions $\boldsymbol{r}$ has a multivariate Gaussian distribution

$$P(\boldsymbol{r}/H_i) = \frac{1}{\det(\pi K_i)}\, e^{-\boldsymbol{r}^* K_i^{-1}\boldsymbol{r}} \tag{A4}$$

where

$$K_i = E\{\boldsymbol{r}\boldsymbol{r}^*/H_i\} \qquad i = 1, 2. \tag{A5}$$

$\boldsymbol{r}^*$ denotes the conjugate transpose of $\boldsymbol{r}$ and $E\{\cdot\}$ denotes the statistical expectation of the bracketed quantity. Substituting (A4) into (A3) and carrying out the indicated integration one obtains

$$\mu(s) = -\{s \ln[\det K_1] + (1-s)\ln[\det K_0]$$
$$+ \ln[\det(sK_1^{-1} + (1-s)K_0^{-1})]\} \tag{A6}$$

the data vector is given by

$$\boldsymbol{r}^T = (R_1(\omega_1), R_2(\omega_1), R_1(\omega_2), R_2(\omega_2),$$
$$\cdots, R_1(\omega_N), R_2(\omega_N)) \tag{A7}$$

where

$$R_i(\omega_l) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} r_i(t)\, e^{-j\omega_l t}\, dt, \quad \omega_l = \frac{2\pi}{T} l. \tag{A8}$$

For $WT/2\pi >> 1$ the Fourier coefficients associated with different frequencies are statistically uncorrelated [12] so that the data covariance matrix assumes the form

$$K_i = \begin{pmatrix} K_i(\omega_1) & & (0) \\ & K_i(\omega_2) & \\ & & \ddots \\ (0) & & K_i(\omega_N) \end{pmatrix} \tag{A9}$$

where $K_i(\omega_l)$ is the 2×2 matrix

$$K_i(\omega_l) = \begin{pmatrix} s(\omega_l) + N_1(\omega_l) & s(\omega_l)\, e^{-j\omega_l \Delta\tau_i} \\ s(\omega_l)\, e^{j\omega_l \Delta\tau_i} & s(\omega_l) + N_2(\omega_l) \end{pmatrix}. \tag{A10}$$

Substituting (A9) into (A6) one immediately obtains

$$\mu(s) = -\sum_l \{s \ln[\det K_1(\omega_l)] + (1-s)\ln[\det K_0(\omega_l)]$$
$$+ \ln[\det(sK_1^{-1}(\omega_l) + (1-s)K_0^{-1}(\omega_l))]\}$$
$$= -\sum_l \ln\{\det[K(\omega_l)] \cdot [\det(sK_1^{-1}(\omega_l)$$
$$+ (1-s)K_0^{-1}(\omega_l)]\}. \tag{A11}$$

In the second version (A11) we used the relation $\det K_0(\omega_l) = \det K_1(\omega_l) \triangleq \det K(\omega_l)$. Substituting (A10) into (A11) and carrying the indicated matrix algebra manipulations one finds

$$\mu(s) = -\sum_l \ln[1 + 4s(1-s)\,\gamma(\omega_l, x)] \tag{A12}$$

where $\gamma(\omega, x)$ is defined in (23) and $x = \Delta\tau_1 - \Delta\tau_0$. Now

$$\mu'(s) = -\sum_l \frac{4(1-2s)\,\gamma(\omega_l, x)}{1 + 4s(1-s)\,\gamma(\omega_l, x)} = 0 \Rightarrow s = 1/2. \tag{A13}$$

Thus

$$\mu(1/2) = -\sum_l \ln\,[1+\gamma(\omega_l, x)] \xrightarrow[WT/2\pi \gg 1]{} -\frac{T}{2\pi}\int_0^\infty \ln\,[1+\gamma(\omega, x)]\,d\omega. \tag{A14}$$

Similarly one finds

$$\mu''(1/2) = \sum_l \frac{8\gamma(\omega_l, x)}{1+\gamma(\omega_l, x)} \xrightarrow[WT/2\pi \gg 1]{} \frac{4T}{\pi}\int_0^\infty \frac{\gamma(\omega, x)}{1+\gamma(\omega, x)}\,d\omega. \tag{A15}$$

Substituting (A14), (A15), and $s_m = \frac{1}{2}$ into (A2) immediately yields (20).

Appendix B

Derivation of the CRLB

The Cramer-Rao inequality asserts that the mean-square error of any unbiased estimate $\widehat{\Delta\tau}$ of $\Delta\tau$ is bounded by

$$\overline{\epsilon^2} \geqslant -1/E\{d^2 \ln P(\boldsymbol{r}/\Delta\tau)/d\Delta\tau^2\} \tag{B-1}$$

where $E\{\cdot\}$ stands for the statistical expectation of the bracketed term. $P(\boldsymbol{r}/\Delta\tau)$ is the conditional probability density of $\boldsymbol{r}$ when $\Delta\tau$ is the true differential delay and $\boldsymbol{r}$ is the data vector generated by Fourier analyzing the $r_i(t)$ in (17). Since signal and noise are assumed to be zero-mean Gaussian processes and the components of $\boldsymbol{r}$ are generated by linear operations on these time functions, $\boldsymbol{r}$ has a multivariate Gaussian distribution

$$P(\boldsymbol{r}/\Delta\tau) = \frac{1}{\det(\pi K)}\, e^{-\boldsymbol{r}^* K^{-1} \boldsymbol{r}} \tag{B2}$$

where $K = E\{\boldsymbol{r}\boldsymbol{r}^*/\Delta\tau\}$. Substituting (B2) into (B1) and carrying out the indicated operations one finds [15]

$$\overline{\epsilon^2} \geqslant 1/\mathrm{Tr}\left[\left(K^{-1}\frac{dK}{d\Delta\tau}\right)^2\right] \tag{B3}$$

where Tr $[\cdot]$ denotes the trace of the bracketed matrix. For $WT/2\pi \gg 1$ the Fourier coefficients associated with different frequencies are statistically uncorrelated [12] so that K assumes the block diagonal form

$$K = \begin{pmatrix} K(\omega_1) & & & \\ & K(\omega_2) & (0) & \\ & & \ddots & \\ & (0) & & K(\omega_N) \end{pmatrix} \tag{B4}$$

where $K(\omega_l)$ is the 2×2 matrix

$$K(\omega_l) = \begin{pmatrix} s(\omega_l) + N_1(\omega_l) & s(\omega_l)\, e^{-j\omega_l \Delta\tau} \\ s(\omega_l)\, e^{j\omega_l\Delta\tau} & s(\omega_l) + N_2(\omega_l) \end{pmatrix}. \tag{B5}$$

It follows that

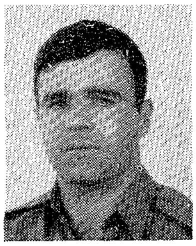
$$\overline{\epsilon^2} \geqslant 1\Big/\sum_{l=1}^{N} \mathrm{Tr}\left[\left(K^{-1}_{(\omega_l)}\frac{dK(\omega_l)}{d\Delta\tau}\right)^2\right] = 1/2\sum_{l=1}^{N} \cdot \frac{\omega_l^2 S^2(\omega_l)}{[N_1(\omega_l)+N_2(\omega_l)]\,S(\omega_l) + N_1(\omega_l)N_2(\omega_l)} \xrightarrow[WT/2\pi \gg 1]{} 1\Big/\frac{T}{\pi}\int_0^\infty \frac{\omega^2 S^2(\omega)\,d\omega}{[N_1(\omega)+N_2(\omega)]\,S(\omega)+N_1(\omega)N_2(\omega)} \triangleq \frac{1}{4\beta^2}. \tag{B6}$$

Acknowledgment

The authors wish to thank Prof. P. M. Schultheiss from Yale University and Dr. J. P. Ianniello from the Naval Underwater Systems Center for their helpful comments, and the Acoustic Group at the Woods Hole Oceanographic Institution for their continuous support and encouragement throughout this study, in particular to B. Pratt for her graphical assistance and to C. Muzzey for her secretarial assistance.

References

[1] V. H. MacDonald and P. M. Schultheiss, "Optimum passive bearing estimation in a spatially incoherent noise environment," *J. Acoust. Soc. Amer.*, vol. 46, pp. 37–43, 1969.
[2] W. R. Hahn, "Optimum signal processing for passive sonar range and bearing estimation," *J. Acoust. Soc. Amer.*, vol. 58, pp. 201–207, 1981.
[3] G. C. Carter, "Time delay estimation for passive sonar signal processing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 463–470, June 1981.
[4] Y. T. Chan, R. V. Hattin, and J. B. Plant, "The least squares estimation of time delay and its use in signal detection," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-26, pp. 217–222, 1978.
[5] W. R. Hahn and S. A. Tretter, "Optimum processing for delay-vector estimation in passive signal arrays," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 608–614, 1973.
[6] B. V. Hamon and E. J. Hannan, "Spectral estimation of time delay for dispersive and non-dispersive systems," *J. Appl. Stat.*, vol. 23, pp. 134–142, 1974.
[7] C. H. Knapp and G. C. Carter, "The generalized correlation method for estimation of time delay," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 320–327, 1976.
[8] S. K. Chow and P. M. Schultheiss, "Delay estimation using narrowband processes," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 478–484, June 1981.
[9] D. Chazan, M. Zakai, and J. Ziv, "Improved lower bounds on signal parameter estimation," *IEEE Trans. Inform. Theory*, vol. IT-21, pp. 90–93, January 1975.
[10] S. Bellini and G. Tartara, "Bounds on error in signal parameter estimation," *IEEE Trans. Commun.*, vol. COM-22, pp. 340–342, Mar. 1974.
[11] H. L. Van Trees, *Detection, Estimation and Modulation Theory, Part I.* New York: Wiley, 1968.
[12] A. Papoulis, *Probability Random Variables and Stochastic Processes.* New York: McGraw-Hill, 1965, ch. 13.
[13] C. R. Rao, *Linear Statistical Inference and Its Applications.* New York: Wiley, 1965.
[14] E. Weinstein, "Decentralization of the Gaussian maximum likelihood estimator and its applications to passive array processing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, Oct. 1981.
[15] W. J. Bangs, "Array processing with generalized beamformers," Ph.D. dissertation, Yale Univ., New Haven, CT, 1971.
[16] E. Weinstein and A. Weiss, "Fundamented limitations in passive time-delay estimation, Part II: Baseband and bandpass systems," *IEEE Trans. Acoust., Speech, Signal Processing*, to be published.



Anthony J. Weiss was born in London, England, in March 1951. He received the B.Sc. degree in electrical engineering from the Technion–Israel Institute of Technology, Haifa, in 1973, and the M.S.E.E. degree from Tel-Aviv University, in 1982. He is currently working towards his Ph.D.

From 1973 to 1982 he was involved in research and development of various communication systems. He is currently investigating the problem of time-delay estimation at Tel-Aviv University.

Ehud Weinstein (M'82) was born in Tel-Aviv, Israel, in May 1950. He received the B.Sc. degree in electrical engineering from the Technion–Israel Institute of Technology, Haifa, in 1975, and the M.Sc. and Ph.D. degrees from Yale University, New Haven, CT, in 1976 and 1978, respectively.

In 1978 he joined the Ocean Engineering Department at Woods Hole Oceanographic Institute, where he is currently an Associate Scientist. In 1980 he also joined the Department of Electronic Systems, School of Engineering, Tel-Aviv University, where he is currently a Senior Lecturer. His research activities are in the area of statistical estimation theory and its applications to signal processing problems in radar/sonar systems.

An Explicit Solution and Data Extension in the Maximum Entropy Method

NAI-LONG WU

Abstract—**The Lagrange undetermined multiplier method and the cepstrum analysis method are used to determine an explicit solution and a data extension model in the maximum entropy method of power spectrum analysis. The 1-dimensional case for a real, causal, stable, and all-pole signal is treated. Only the definition of entropy as $H_2 = -\int S(f) \log S(f)\, df$ is treated although the same treatment has been applied to $H_1 = \int \log S(f)\, df$. Relevant results of cepstrum analysis are provided. This paper concludes with a numerical example.**

I. Introduction

The maximum entropy method (MEM) is a nonlinear technique for estimating power spectra or for image reconstruction with improved resolution. This technique was originally employed by Burg [1]. Among all the spectra consistent with the available data he chose the one that maximized the entropy of a stationary random process as a solution. For a stationary, Gaussian second-order process the entropy is (see [2])

$$H_1 = \int \log S(f)\, df$$

where $\log = \log_e$. Later Frieden [3], Gull and Daniell [4] suggested using the quantity

$$H_2 = -\int S(f) \log S(f)\, df$$

Manuscript received February 12, 1982; revised July 8, 1982, and October 7, 1982. This work was supported by a University of Sydney Postgraduate Studentship.

The author is with the School of Electrical Engineering, University of Sydney, Sydney, Australia.

as an entropy measure, called configurational entropy, and the application to radio astronomy was successful [4].

Thus, two different definitions of entropy are in use: H_1 and H_2. Much work has been done for MEM1 (the MEM with H_1). An explicit solution (see [1]), a data extension model (see [5], [6]) and the statistical properties (see [7]) were determined in the 1-dimensional case. Iterative algorithms in the 2-dimensional case were also established (see [8], [9], [10]).

In contrast, little work has been done for MEM2 (the MEM with H_2). Neither an explicit solution nor an equivalent extrapolating model has been worked out even in the 1-dimensional case. So it is almost impossible to understand its statistical properties. Iteration is the only algorithm we can use (see [4]). Its merits are determined on the basis of enhanced resolution, positiveness of the solution, etc.

In this paper, the Lagrange undetermined multiplier (LUM) method and the cepstrum analysis method are first used to work out an explicit solution and an extrapolating model for the MEM2 in the 1-dimensional case. This will strengthen the rationale of the MEM2, can be used for computing purposes, and makes statistical analysis of the MEM2 possible. A discussion of the conditions imposed on the signal is then presented and finally a numerical example is demonstrated.

Familiarity with the cepstrum analysis method is assumed. In this respect a book by Oppenheim and Schafer [11] is an excellent reference. However, for convenience a diagram (Fig. 1) and the relations between the input $x(n)$ and output $\hat{x}(n)$ for cepstrum analysis of a real, causal and minimum-phase sequence are shown. $x(n)$ is assumed to have a rational

0096-3518/83/0400-0486$01.00 © 1983 IEEE