Chamblandes 2013 — Problème 3

1. Les sommets A et B correspondent au minimum et au maximum de la fonction f sur l'intervalle $[0\,;\pi]$. Il s'agit donc d'étudier la croissance de la fonction f.

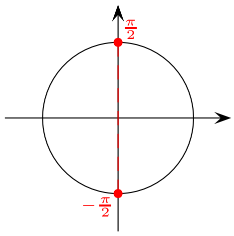
$$f'(x) = \big(3 - 2\,\sin(2\,x)\big)' = 0 - 2\,\cos(2\,x)\cdot\underbrace{(2\,x)'}_{2} = -4\,\cos(2\,x)$$

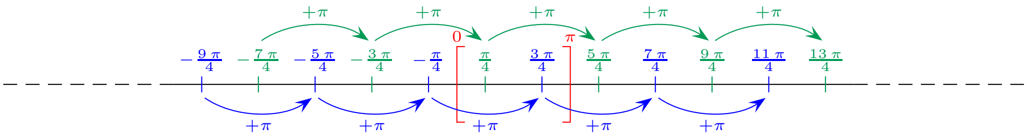
$-4\,\cos(2\,x) = 0$

$\cos(2\,x) = 0$

$$\begin{cases} 2\,x_1 = \frac{\pi}{2} + 2\,k\,\pi & \text{où } k \in \mathbb{Z} \\ 2\,x_2 = -\frac{\pi}{2} + 2\,k\,\pi & \text{où } k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} x_1 = \frac{\pi}{4} + k\,\pi & \text{où } k \in \mathbb{Z} \\ x_2 = -\frac{\pi}{4} + k\,\pi & \text{où } k \in \mathbb{Z} \end{cases}$$





Les seules solutions comprises dans l'intervalle $[0\,;\pi]$ sont $\frac{\pi}{4}$ et $\frac{3\pi}{4}$.

	0		$\frac{\pi}{4}$		$\frac{3\pi}{4}$		π
f'		$-$		$+$		$-$	
f		$\searrow$	min	$\nearrow$	max	$\searrow$	

$f\left(\frac{\pi}{4}\right) = 3 - 2\,\sin\left(2\cdot\frac{\pi}{4}\right) = 3 - 2\,\sin\left(\frac{\pi}{2}\right) = 3 - 2\cdot 1 = 1$ $\qquad \boxed{A\left(\frac{\pi}{4}\,;1\right)}$

$f\left(\frac{3\pi}{4}\right) = 3 - 2\,\sin\left(2\cdot\frac{3\pi}{4}\right) = 3 - 2\,\sin\left(\frac{3\pi}{2}\right) = 3 - 2\cdot(-1) = 5$ $\qquad \boxed{B\left(\frac{3\pi}{4}\,;5\right)}$

2. Le sommet C correspond au zéro de la fonction g sur l'intervalle $[0\,;\pi]$.

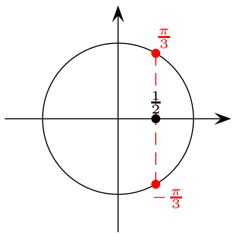
$2\,\cos(\frac{x}{2}) - 1 = 0$

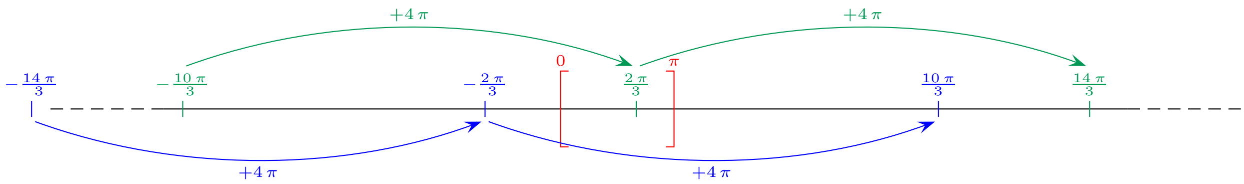
$2\,\cos(\frac{x}{2}) = 1$

$\cos(\frac{x}{2}) = \frac{1}{2}$

$$\begin{cases} \frac{x_1}{2} = \frac{\pi}{3} + 2\,k\,\pi & \text{où } k \in \mathbb{Z} \\ \frac{x_2}{2} = -\frac{\pi}{3} + 2\,k\,\pi & \text{où } k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} x_1 = \frac{2\pi}{3} + 4\,k\,\pi & \text{où } k \in \mathbb{Z} \\ x_2 = -\frac{2\pi}{3} + 4\,k\,\pi & \text{où } k \in \mathbb{Z} \end{cases}$$





L'unique solution comprise dans l'intervalle $[0\,;\pi]$ est $\frac{2\pi}{3}$.

Donc $\boxed{C(\frac{2\pi}{3}\,;0)}$

3. $$\begin{aligned}\int_0^{\frac{\pi}{2}} \Big(f(x)-g(x)\Big)\,dx &= \int_0^{\frac{\pi}{2}} \Big(\big(3-2\,\sin(2\,x)\big)-\big(2\,\cos(\tfrac{x}{2})-1\big)\Big)\,dx \\
&= \int_0^{\frac{\pi}{2}} \Big(4-2\,\sin(2\,x)-2\,\cos(\tfrac{x}{2})\Big)\,dx \\
&= \int_0^{\frac{\pi}{2}} \Big(4-\sin(\underbrace{2\,x}_{g})\cdot\underbrace{2}_{g'}-4\,\cos(\underbrace{\tfrac{x}{2}}_{g})\cdot\underbrace{\tfrac{1}{2}}_{g'}\Big)\,dx \\
&= \Big[4\,x-\big(-\cos(2\,x)\big)+4\,\sin(\tfrac{x}{2})\Big]_0^{\frac{\pi}{2}} = \Big[4\,x+\cos(2\,x)+4\,\sin(\tfrac{x}{2})\Big]_0^{\frac{\pi}{2}} \\
&= \Big(4\cdot\tfrac{\pi}{2}+\cos(2\cdot\tfrac{\pi}{2})+4\,\sin\Big(\frac{\frac{\pi}{2}}{2}\Big)\Big)-\Big(4\cdot 0+\cos(2\cdot 0)+4\,\sin\Big(\frac{0}{2}\Big)\Big) \\
&= \Big(2\,\pi+\cos(\pi)+4\,\sin(\tfrac{\pi}{4})\Big)-\Big(0+\cos(0)-4\,\sin(0)\Big) \\
&= \Big(2\,\pi+(-1)-4\cdot\tfrac{\sqrt{2}}{2}\Big)-\Big(1-4\cdot 0\Big) \\
&= \boxed{2\,\pi-2\,\sqrt{2}-2}\end{aligned}$$