2.1 1) $\begin{cases} 2x + y - z = 1 \\ x + 2y + z = 8 \\ 3x - y + 2z = 7 \end{cases} \overset{L_1 \leftrightarrow L_2}{\Longrightarrow} \begin{cases} x + 2y + z = 8 \\ 2x + y - z = 1 \\ 3x - y + 2z = 7 \end{cases} \overset{\substack{L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - 3L_1}}{\Longrightarrow}$

$$\begin{cases} x + 2y + z = 8 \\ -3y - 3z = -15 \\ -7y - z = -17 \end{cases} \overset{\substack{L_2 \to -\frac{1}{3}L_2 \\ L_3 \to -L_3}}{\Longrightarrow} \begin{cases} x + 2y + z = 8 \\ y + z = 5 \\ 7y + z = 17 \end{cases} \overset{L_3 \to L_3 - 7L_2}{\Longrightarrow}$$

$$\begin{cases} x + 2y + z = 8 \\ y + z = 5 \\ -6z = -18 \end{cases} \overset{L_3 \to -\frac{1}{6}L_3}{\Longrightarrow} \begin{cases} x + 2y + z = 8 \\ y + z = 5 \\ z = 3 \end{cases} \overset{\substack{L_1 \to L_1 - L_3 \\ L_2 \to L_2 - L_3}}{\Longrightarrow}$$

$$\begin{cases} x + 2y = 5 \\ y = 2 \\ z = 3 \end{cases} \overset{L_1 \to L_1 - 2L_2}{\Longrightarrow} \begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases}$$

En résumé, $S = \{(1\,;2\,;3)\}$.

2) $\begin{cases} 2x - y + 3z = 4 \\ 3x + 4y - z = -5 \\ x + 5y - 4z = -9 \end{cases} \overset{L_1 \leftrightarrow L_3}{\Longrightarrow} \begin{cases} x + 5y - 4z = -9 \\ 3x + 4y - z = -5 \\ 2x - y + 3z = 4 \end{cases} \overset{\substack{L_2 \to L_2 - 3L_1 \\ L_3 \to L_3 - 2L_1}}{\Longrightarrow}$

$$\begin{cases} x + 5y - 4z = -9 \\ -11y + 11z = 22 \\ -11y + 11z = 22 \end{cases} \overset{\substack{L_2 \to -\frac{1}{11}L_2 \\ L_3 \to -\frac{1}{11}L_3}}{\Longrightarrow} \begin{cases} x + 5y - 4z = -9 \\ y - z = -2 \\ y - z = -2 \end{cases} \overset{L_3 \to L_3 - L_2}{\Longrightarrow}$$

$$\begin{cases} x + 5y - 4z = -9 \\ y - z = -2 \\ 0 = 0 \end{cases} \overset{L_1 \to L_1 - 5L_2}{\Longrightarrow} \begin{cases} x + z = 1 \\ y - z = -2 \end{cases}$$

La variable z est libre : on pose $z = \alpha$.
Il en résulte $x = 1 - \alpha$ et $y = -2 + \alpha$.
Finalement, $S = \{(1 - \alpha\,; -2 + \alpha\,; \alpha) : \alpha \in \mathbb{R}\}$.

3) $\begin{cases} 2x + y + 3z = 3 \\ 3x - y + 4z = 2 \\ 4x + y - z = 5 \\ x + y + z = 4 \end{cases} \overset{L_1 \leftrightarrow L_4}{\Longrightarrow} \begin{cases} x + y + z = 4 \\ 3x - y + 4z = 2 \\ 4x + y - z = 5 \\ 2x + y + 3z = 3 \end{cases} \overset{\substack{L_2 \to L_2 - 3L_1 \\ L_3 \to L_3 - 4L_1 \\ L_4 \to L_4 - 2L_1}}{\Longrightarrow}$

$$\begin{cases} x + y + z = 4 \\ -4y + z = -10 \\ -3y - 5z = -11 \\ -y + z = -5 \end{cases} \overset{\substack{L_2 \to -L_2 \\ L_3 \to -L_3 \\ L_4 \to -L_4}}{\Longrightarrow} \begin{cases} x + y + z = 4 \\ 4y - z = 10 \\ 3y + 5z = 11 \\ y - z = 5 \end{cases} \overset{L_2 \leftrightarrow L_4}{\Longrightarrow}$$

$$\begin{cases} x + y + z = 4 \\ y - z = 5 \\ 3y + 5z = 11 \\ 4y - z = 10 \end{cases} \overset{\substack{L_3 \to L_3 - 3L_2 \\ L_4 \to L_4 - 4L_2}}{\Longrightarrow} \begin{cases} x + y + z = 4 \\ y - z = 5 \\ 8z = -4 \\ 3z = -10 \end{cases} \overset{\substack{L_3 \to \frac{1}{8}L_3 \\ L_4 \to \frac{1}{3}L_4}}{\Longrightarrow}$$

$$\begin{cases} x+y+z= & 4 \\ y-z= & 5 \\ z= & -\frac{1}{2} \\ z= & -\frac{10}{3} \end{cases} \overset{L_4 \to L_4 - L_3}{\Longrightarrow} \begin{cases} x+y+z= & 4 \\ y-z= & 5 \\ z= & -\frac{1}{2} \\ 0= & -\frac{17}{6} \end{cases}$$

La quatrième ligne montre que ce système est impossible : $\mathrm{S} = \varnothing$.

4) $$\begin{cases} x-3y+z-t=0 \\ 2x+y-z+2t=0 \end{cases} \overset{L_2 \to L_2 - 2L_1}{\Longrightarrow} \begin{cases} x-3y+z-t=0 \\ 7y-3z+4t=0 \end{cases}$$

$$\overset{L_1 \to 7L_1 + 3L_2}{\Longrightarrow} \begin{cases} 7x-2z+5t=0 \\ 7y-3z+4t=0 \end{cases} \overset{\substack{L_1 \to \frac{1}{7}L_1 \\ L_2 \to \frac{1}{7}L_2}}{\Longrightarrow}$$

$$\begin{cases} x-\frac{2}{7}z+\frac{5}{7}t=0 \\ y-\frac{3}{7}z+\frac{4}{7}t=0 \end{cases}$$

Il y a donc deux variables libres z et t ; on pose $z=\alpha$ et $t=\beta$.
On obtient alors $x = \frac{2}{7}\alpha - \frac{5}{7}\beta$ et $y = \frac{3}{7}\alpha - \frac{4}{7}\beta$.
Ainsi $\mathrm{S} = \{(\frac{2}{7}\alpha - \frac{5}{7}\beta ; \frac{3}{7}\alpha - \frac{4}{7}\beta ; \alpha ; \beta) : \alpha, \beta \in \mathbb{R}\}$.

On obtient une solution plus simple en posant plutôt $z = 7\alpha$ et $t = 7\beta$.
Alors $x = 2\alpha - 5\beta$ et $y = 3\alpha - 4\beta$.
Donc $\mathrm{S} = \{(2\alpha - 5\beta ; 3\alpha - 4\beta ; 7\alpha ; 7\beta) : \alpha, \beta \in \mathbb{R}\}$.

5) $$\begin{cases} x+2y+3z= & 9 \\ x-y+4z= & 15 \\ -x+7y-6z= & -27 \end{cases} \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 + L_1}}{\Longrightarrow} \begin{cases} x+2y+3z= & 9 \\ -3y+z= & 6 \\ +9y-3z= & -18 \end{cases} \overset{L_3 \to L_3 + 3L_2}{\Longrightarrow}$$

$$\begin{cases} x+2y+3z=9 \\ -3y+z=6 \\ 0=0 \end{cases} \overset{L_1 \to 3L_1 + 2L_2}{\Longrightarrow} \begin{cases} 3x+11z=39 \\ -3y+z=6 \end{cases}$$

$$\overset{\substack{L_1 \to \frac{1}{3}L_1 \\ L_2 \to -\frac{1}{3}L_2}}{\Longrightarrow} \begin{cases} x+\frac{11}{3}z=13 \\ y-\frac{1}{3}z=-2 \end{cases}$$

Il y a une variable libre : z ; on pose donc $z = \alpha$.
Alors $x = 13 - \frac{11}{3}\alpha$ et $y = -2 + \frac{1}{3}\alpha$.
En résumé, $\mathrm{S} = \{(13 - \frac{11}{3}\alpha ; -2 + \frac{1}{3}\alpha ; \alpha) : \alpha \in \mathbb{R}\}$.

Là encore, on obtient une solution plus simple en posant plutôt $z = 3\alpha$.
Alors $x = 13 - 11\alpha$ et $y = -2 + \alpha$.
D'où $\mathrm{S} = \{(13 - 11\alpha ; -2 + \alpha ; 3\alpha) : \alpha \in \mathbb{R}\}$.

6) $$\begin{cases} x+2y-5z+4t= & 1 \\ 2x-3y+2z+3t= & 18 \\ 4x-7y+z-6t= & -5 \\ x+y-z+t= & 1 \end{cases} \overset{\substack{L_2 \to L_2 - 2L_1 \\ L_3 \to L_3 - 4L_1 \\ L_4 \to L_4 - L_1}}{\Longrightarrow} \begin{cases} x+2y-5z+4t= & 1 \\ -7y+12z-5t= & 16 \\ -15y+21z-22t= & -9 \\ -y+4z-3t= & 0 \end{cases}$$

$$\overset{L_4 \leftrightarrow L_2}{\Longrightarrow} \quad \begin{cases} x + 2y - 5z + 4t = 1 \\ - y + 4z - 3t = 0 \\ -15y + 21z - 22t = -9 \\ -7y + 12z - 5t = 16 \end{cases} \quad \overset{\substack{L_3 \to L_3 - 15\,L_2 \\ L_4 \to L_4 - 7\,L_2}}{\Longrightarrow}$$

$$\begin{cases} x + 2y - 5z + 4t = 1 \\ - y + 4z - 3t = 0 \\ -39z + 23t = -9 \\ -16z + 16t = 16 \end{cases} \quad \overset{L_4 \leftrightarrow L_3}{\Longrightarrow} \quad \begin{cases} x + 2y - 5z + 4t = 1 \\ - y + 4z - 3t = 0 \\ -16z + 16t = 16 \\ -39z + 23t = -9 \end{cases}$$

$$\overset{L_3 \to -\frac{1}{16}L_3}{\Longrightarrow} \quad \begin{cases} x + 2y - 5z + 4t = 1 \\ - y + 4z - 3t = 0 \\ z - t = -1 \\ -39z + 23t = -9 \end{cases} \quad \overset{L_4 \to L_4 + 39\,L_3}{\Longrightarrow}$$

$$\begin{cases} x + 2y - 5z + 4t = 1 \\ - y + 4z - 3t = 0 \\ z - t = -1 \\ -16t = -48 \end{cases} \quad \overset{\substack{L_2 \to -L_2 \\ L_4 \to -\frac{1}{16}L_4}}{\Longrightarrow} \quad \begin{cases} x + 2y - 5z + 4t = 1 \\ y - 4z + 3t = 0 \\ z - t = -1 \\ t = 3 \end{cases}$$

$$\overset{\substack{L_1 \to L_1 - 4\,L_4 \\ L_2 \to L_2 - 3\,L_4 \\ L_3 \to L_3 + L_4}}{\Longrightarrow} \quad \begin{cases} x + 2y - 5z = -11 \\ y - 4z = -9 \\ z = 2 \\ t = 3 \end{cases} \quad \overset{\substack{L_1 \to L_1 + 5\,L_3 \\ L_2 \to L_2 + 4\,L_3}}{\Longrightarrow}$$

$$\begin{cases} x + 2y = -1 \\ y = -1 \\ z = 2 \\ t = 3 \end{cases} \quad \overset{L_1 \to L_1 - 2\,L_2}{\Longrightarrow} \quad \begin{cases} x = 1 \\ y = -1 \\ z = 2 \\ t = 3 \end{cases}$$

On conclut que $S = \{(1\,;-1\,;2\,;3)\}$.