8.5

1) $a_{12}\,\mathrm{C}_{12} + a_{22}\,\mathrm{C}_{22} = a_{12}\,(-1)^{1+2}\,a_{21} + a_{22}\,(-1)^{2+2}\,a_{11}$
$= -a_{12}\,a_{21} + a_{22}\,a_{11}$
$= a_{11}\,a_{22} - a_{21}\,a_{12}$
$= \det(\mathrm{A})$

2) (a) $\begin{vmatrix} \lambda\,a_{11} & a_{12} \\ \lambda\,a_{21} & a_{22} \end{vmatrix} = \lambda\,a_{11}\,a_{22} - \lambda\,a_{21}\,a_{12} = \lambda\,(a_{11}\,a_{22} - a_{21}\,a_{12}) = \lambda\,\det(\mathrm{A})$

(b) $\begin{vmatrix} a_{11} & \lambda\,a_{12} \\ a_{21} & \lambda\,a_{22} \end{vmatrix} = a_{11}\,\lambda\,a_{22} - a_{21}\,\lambda\,a_{12} = \lambda\,(a_{11}\,a_{22} - a_{21}\,a_{12}) = \lambda\,\det(\mathrm{A})$

3) (a) $\begin{vmatrix} a_{11} + \lambda\,a_{12} & a_{12} \\ a_{21} + \lambda\,a_{22} & a_{22} \end{vmatrix} = (a_{11} + \lambda\,a_{12})\,a_{22} - (a_{21} + \lambda\,a_{22})\,a_{12}$
$= a_{11}\,a_{22} + \lambda\,a_{12}\,a_{22} - a_{21}\,a_{12} - \lambda\,a_{22}\,a_{12}$
$= a_{11}\,a_{22} - a_{21}\,a_{12}$
$= \det(\mathrm{A})$

(b) $\begin{vmatrix} a_{11} & a_{12} + \lambda\,a_{11} \\ a_{21} & a_{22} + \lambda\,a_{21} \end{vmatrix} = a_{11}\,(a_{22} + \lambda\,a_{21}) - a_{21}\,(a_{12} + \lambda\,a_{11})$
$= a_{11}\,a_{22} + a_{11}\,\lambda\,a_{21} - a_{21}\,a_{12} - a_{21}\,\lambda\,a_{11}$
$= a_{11}\,a_{22} - a_{21}\,a_{12}$
$= \det(\mathrm{A})$

4) $\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{12}\,a_{21} - a_{22}\,a_{11} = -(a_{11}\,a_{22} - a_{21}\,a_{12}) = -\det(\mathrm{A})$

5) $\det({}^t\mathrm{A}) = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21} = a_{11}\,a_{22} - a_{21}\,a_{12} = \det(\mathrm{A})$

6) $\det(\mathrm{AB}) = \det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)$
$= \begin{vmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} & a_{11}\,b_{12} + a_{12}\,b_{22} \\ a_{21}\,b_{11} + a_{22}\,b_{21} & a_{21}\,b_{12} + a_{22}\,b_{22} \end{vmatrix}$
$= (a_{11}\,b_{11} + a_{12}\,b_{21})\,(a_{21}\,b_{12} + a_{22}\,b_{22})$
$\quad - (a_{21}\,b_{11} + a_{22}\,b_{21})\,(a_{11}\,b_{12} + a_{12}\,b_{22})$
$= a_{11}\,a_{21}\,b_{11}\,b_{12} + a_{11}\,a_{22}\,b_{11}\,b_{22} + a_{12}\,a_{21}\,b_{21}\,b_{12} + a_{12}\,a_{22}\,b_{21}\,b_{22}$
$\quad - a_{21}\,a_{11}\,b_{11}\,b_{12} - a_{21}\,a_{12}\,b_{11}\,b_{22} - a_{22}\,a_{11}\,b_{21}\,b_{12} - a_{22}\,a_{12}\,b_{21}\,b_{22}$
$= a_{11}\,a_{22}\,b_{11}\,b_{22} - a_{11}\,a_{22}\,b_{21}\,b_{12} - a_{21}\,a_{12}\,b_{11}\,b_{22} + a_{12}\,a_{21}\,b_{21}\,b_{12}$
$= a_{11}\,a_{22}\,(b_{11}\,b_{22} - b_{21}\,b_{12}) - a_{21}\,a_{12}\,(b_{11}\,b_{22} - b_{21}\,b_{12})$
$= (a_{11}\,a_{22} - a_{21}\,a_{12})\,(b_{11}\,b_{22} - b_{21}\,b_{12})$
$= \det(\mathrm{A})\ \det(\mathrm{B})$