8.9 1) $$\begin{vmatrix} 1 & b+c & a \\ 1 & c+a & b \\ 1 & a+b & c \end{vmatrix} \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - L_1}}{=} \begin{vmatrix} 1 & b+c & a \\ 0 & a-b & b-a \\ 0 & a-c & c-a \end{vmatrix} = 1 \begin{vmatrix} a-b & b-a \\ a-c & c-a \end{vmatrix}$$

$$= (a-b)\,(a-c) \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \overset{L_2 \to L_2 - L_1}{=} (a-b)\,(a-c) \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= (a-b)\,(a-c) \cdot 0 = 0$$

2) $$\begin{vmatrix} x & 1 & a \\ 1 & 1 & a \\ 1 & 1 & x \end{vmatrix} \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - L_1}}{=} \begin{vmatrix} x & 1 & a \\ 1-x & 0 & 0 \\ 1-x & 0 & x-a \end{vmatrix} = (-1) \begin{vmatrix} 1-x & 0 \\ 1-x & x-a \end{vmatrix}$$

$$= (x-1) \begin{vmatrix} 1 & 0 \\ 1 & x-a \end{vmatrix} = (x-1)\,(x-a)$$

3) $$\begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix} \overset{C_1 \to C_1 + C_2 + C_3}{=} \begin{vmatrix} x+a+b & a & b \\ x+a+b & x & b \\ x+a+b & b & x \end{vmatrix}$$

$$= (x+a+b) \begin{vmatrix} 1 & a & b \\ 1 & x & b \\ 1 & b & x \end{vmatrix} \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - L_1}}{=} (x+a+b) \begin{vmatrix} 1 & a & b \\ 0 & x-a & 0 \\ 0 & b-a & x-b \end{vmatrix}$$

$$= (x+a+b) \begin{vmatrix} x-a & 0 \\ b-a & x-b \end{vmatrix} = (x+a+b)\,(x-a)\,(x-b)$$

4) $$\begin{vmatrix} 3-t & -1 & 1 \\ 5 & -3-t & 1 \\ 6 & -6 & 4-t \end{vmatrix} \overset{\substack{C_1 \to C_1 - (3-t)\,C_3 \\ C_2 \to C_2 + C_3}}{=} \begin{vmatrix} 0 & 0 & 1 \\ t+2 & -2-t & 1 \\ -t^2+7\,t-6 & -2-t & 4-t \end{vmatrix}$$

$$= 1 \begin{vmatrix} t+2 & -2-t \\ -t^2+7\,t-6 & -2-t \end{vmatrix} = -(t+2) \begin{vmatrix} t+2 & 1 \\ -t^2+7\,t-6 & 1 \end{vmatrix} \overset{L_1 \to L_1 - L_2}{=}$$

$$-(t+2) \begin{vmatrix} t^2-6\,t+8 & 0 \\ -t^2+7\,t-6 & 1 \end{vmatrix} = -(t+2)\,(t^2-6\,t+8) = -(t+2)\,(t-2)\,(t-4)$$

5) $$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y\,z & z\,x & x\,y \end{vmatrix} \overset{\substack{C_2 \to C_2 - C_3 \\ C_3 \to C_3 - C_1}}{=} \begin{vmatrix} x & y-z & z-x \\ x^2 & y^2-z^2 & z^2-x^2 \\ y\,z & z\,x-x\,y & x\,y-y\,z \end{vmatrix}$$

$$= \begin{vmatrix} x & y-z & z-x \\ x^2 & (y-z)\,(y+z) & (z-x)\,(z+x) \\ y\,z & -x\,(y-z) & -y\,(z-x) \end{vmatrix}$$

$$= (y-z)\,(z-x) \begin{vmatrix} x & 1 & 1 \\ x^2 & y+z & z+x \\ y\,z & -x & -y \end{vmatrix} \overset{\substack{C_1 \to C_1 - x\,C_2 \\ C_3 \to C_3 - C_2}}{=}$$

$$(y-z)\,(z-x) \begin{vmatrix} 0 & 1 & 0 \\ x^2-x\,y-x\,z & y+z & x-y \\ x^2+y\,z & -x & x-y \end{vmatrix}$$

$$(y-z)\,(z-x)\,(-1)\begin{vmatrix} x^2-x\,y-x\,z & x-y \\ x^2+y\,z & x-y \end{vmatrix}$$

$$=-(x-y)\,(y-z)\,(z-x)\begin{vmatrix} x^2-x\,y-x\,z & 1 \\ x^2+y\,z & 1 \end{vmatrix} \overset{L_1 \to L_1 - L_2}{=}$$

$$-(x-y)\,(y-z)\,(z-x)\begin{vmatrix} -x\,y-x\,z-y\,z & 0 \\ x^2+y\,z & 1 \end{vmatrix} =$$

$$(x-y)\,(y-z)\,(z-x)\begin{vmatrix} x\,y+x\,z+y\,z & 0 \\ x^2+y\,z & 1 \end{vmatrix} =$$

$$(x-y)\,(y-z)\,(z-x)\,(x\,y+y\,z+z\,x)$$