3.6 Soient $u(x) = \sum_{k=0}^{+\infty} a_k\, x^k$, $v(x) = \sum_{k=0}^{+\infty} b_k\, x^k$, $w(x) = \sum_{k=0}^{+\infty} c_k\, x^k$ (avec un nombre fini de $a_k \neq 0$, $b_k \neq 0$ et $c_k \neq 0$) des polynômes à coefficients réels et α, β deux scalaires.

1) (a) $\big(u(x) + v(x)\big) + w(x) = \left(\sum_{k=0}^{+\infty} a_k\, x^k + \sum_{k=0}^{+\infty} b_k\, x^k\right) + \sum_{k=0}^{+\infty} c_k\, x^k =$
$\sum_{k=0}^{+\infty} (a_k + b_k)\, x^k + \sum_{k=0}^{+\infty} c_k\, x^k = \sum_{k=0}^{+\infty} \big((a_k + b_k) + c_k\big)\, x^k =$
$\sum_{k=0}^{+\infty} \big(a_k + (b_k + c_k)\big)\, x^k = \sum_{k=0}^{+\infty} a_k\, x^k + \sum_{k=0}^{+\infty} (b_k + c_k)\, x^k =$
$\sum_{k=0}^{+\infty} a_k\, x^k + \left(\sum_{k=0}^{+\infty} b_k\, x^k + \sum_{k=0}^{+\infty} c_k\, x^k\right) = u(x) + \big(v(x) + w(x)\big)$

(b) Posons $o(x) = 0$.
Alors $u(x) + o(x) = u(x) + 0 = u(x)$ et $o(x) + u(x) = 0 + u(x) = u(x)$.

(c) Posons $-u(x) = \sum_{k=0}^{+\infty} (-a_k)\, x^k$.
$u(x) + \big(-u(x)\big) = \sum_{k=0}^{+\infty} a_k\, x^k + \sum_{k=0}^{+\infty} (-a_k)\, x^k = \sum_{k=0}^{+\infty} \big(a_k + (-a_k)\big)\, x^k =$
$\sum_{k=0}^{+\infty} 0 \cdot x^k = 0 = o(x)$

(d) $u(x) + v(x) = \sum_{k=0}^{+\infty} a_k\, x^k + \sum_{k=0}^{+\infty} b_k\, x^k = \sum_{k=0}^{+\infty} (a_k + b_k)\, x^k =$
$\sum_{k=0}^{+\infty} (b_k + a_k)\, x^k = \sum_{k=0}^{+\infty} b_k\, x^k + \sum_{k=0}^{+\infty} a_k\, x^k = v(x) + u(x)$

2) (a) $\alpha \cdot \big(\beta \cdot u(x)\big) = \alpha \cdot \left(\beta \cdot \sum_{k=0}^{+\infty} a_k\, x^k\right) = \alpha \cdot \sum_{k=0}^{+\infty} (\beta\, a_k)\, x^k = \sum_{k=0}^{+\infty} \big(\alpha\, (\beta\, a_k)\big)\, x^k =$
$\sum_{k=0}^{+\infty} \big((\alpha\, \beta)\, a_k\big)\, x^k = (\alpha\, \beta) \cdot \sum_{k=0}^{+\infty} a_k\, x^k = (\alpha\, \beta) \cdot u(x)$

(b) $(\alpha + \beta) \cdot u(x) = (\alpha + \beta) \cdot \sum_{k=0}^{+\infty} a_k\, x^k = \sum_{k=0}^{+\infty} \big((\alpha + \beta)\, a_k\big)\, x^k =$
$\sum_{k=0}^{+\infty} (\alpha\, a_k + \beta\, a_k)\, x^k = \sum_{k=0}^{+\infty} (\alpha\, a_k)\, x^k + \sum_{k=0}^{+\infty} (\beta\, a_k)\, x^k =$

$$\alpha \cdot \sum_{k=0}^{+\infty} a_k\, x^k + \beta \cdot \sum_{k=0}^{+\infty} a_k\, x^k = \alpha \cdot u(x) + \beta \cdot u(x)$$

(c) $\alpha \cdot \big(u(x) + v(x)\big) = \alpha \cdot \left(\sum_{k=0}^{+\infty} a_k\, x^k + \sum_{k=0}^{+\infty} b_k\, x^k\right) = \alpha \cdot \sum_{k=0}^{+\infty} (a_k + b_k)\, x^k =$

$$\sum_{k=0}^{+\infty} \big(\alpha\,(a_k+b_k)\big)\, x^k = \sum_{k=0}^{+\infty} \big(\alpha\, a_k + \alpha\, b_k\big)\, x^k = \sum_{k=0}^{+\infty} (\alpha\, a_k)\, x^k + \sum_{k=0}^{+\infty} (\alpha\, b_k)\, x^k =$$

$$\alpha \cdot \sum_{k=0}^{+\infty} a_k\, x^k + \alpha \cdot \sum_{k=0}^{+\infty} b_k\, x^k = \alpha \cdot u(x) + \alpha \cdot v(x)$$

(d) $1 \cdot u(x) = 1 \cdot \sum_{k=0}^{+\infty} a_k\, x^k = \sum_{k=0}^{+\infty} (1 \cdot a_k)\, x^k = \sum_{k=0}^{+\infty} a_k\, x^k = u(x)$