2.4 1) $\left\{\begin{array}{rcrcrcr} x & - & y & - & z & = & 6 \\ x & - & 2y & - & 3z & = & 10 \\ 5x & + & 6y & + & z & = & 2 \end{array}\right. \Longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 6 \\ 1 & -2 & -3 & 10 \\ 5 & 6 & 1 & 2 \end{array}\right) \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - 5L_1}}{\Longrightarrow}$

$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 6 \\ 0 & -1 & -2 & 4 \\ 0 & 11 & 6 & -28 \end{array}\right) \overset{L_3 \to L_3 + 11L_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 6 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & -16 & 16 \end{array}\right)$

$\overset{\substack{L_2 \to -L_2 \\ L_3 \to -\frac{1}{16} L3}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 6 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right) \overset{\substack{L_1 \to L_1 + L_3 \\ L_2 \to L_2 - 2L3}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$

$\overset{L_1 \to L_1 + L_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \Longrightarrow \left\{\begin{array}{l} x = 3 \\ y = -2 \\ z = -1 \end{array}\right.$

On a donc obtenu $S = \{(3\,;-2\,;-1)\}$.

2) $\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 9 \\ x & + & 2y & + & 3z & = & 14 \\ 3x & + & 2y & + & z & = & 22 \end{array}\right. \Longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 14 \\ 3 & 2 & 1 & 22 \end{array}\right) \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - 3L_1}}{\Longrightarrow}$

$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -2 & -5 \end{array}\right) \overset{L_3 \to L_3 + L_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \to L_1 - L_2}{\Longrightarrow}$

$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \left\{\begin{array}{rcrcr} x & & - & z = & 4 \\ & y & + & 2z = & 5 \end{array}\right.$

Comme z est une variable libre, on pose $z = \alpha$.
Par suite, $x = 4 + \alpha$ et $y = 5 - 2\alpha$.
En résumé, $S = \{(4 + \alpha\,; 5 - 2\alpha\,; \alpha) : \alpha \in \mathbb{R}\}$.

3) $\left\{\begin{array}{rcrcrcrcrcr} x & + & y & - & z & + & v & - & w & = & 8 \\ -2x & + & y & - & 2z & + & 3v & & & = & 6 \\ 3x & - & 2y & - & z & + & v & + & 2w & = & 8 \\ x & + & 3y & & & - & 2v & - & 5w & = & 1 \\ x & + & 2y & + & 3z & + & v & - & 3w & = & -1 \end{array}\right. \Longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ -2 & 1 & -2 & 3 & 0 & 6 \\ 3 & -2 & -1 & 1 & 2 & 8 \\ 1 & 3 & 0 & -2 & -5 & 1 \\ 1 & 2 & 3 & 1 & -3 & -1 \end{array}\right)$

$\overset{\substack{L_2 \to L_2 + 2L_1 \\ L_3 \to L_3 - 3L_1 \\ L_4 \to L_4 - L_1 \\ L_5 \to L_5 - L_1}}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 3 & -4 & 5 & -2 & 22 \\ 0 & -5 & 2 & -2 & 5 & -16 \\ 0 & 2 & 1 & -3 & -4 & -7 \\ 0 & 1 & 4 & 0 & -2 & -9 \end{array}\right) \overset{L_5 \leftrightarrow L_2}{\Longrightarrow}$

$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & -5 & 2 & -2 & 5 & -16 \\ 0 & 2 & 1 & -3 & -4 & -7 \\ 0 & 3 & -4 & 5 & -2 & 22 \end{array}\right) \overset{\substack{L_3 \to L_3 + 5L_2 \\ L_4 \to L_4 - 2L_2 \\ L_5 \to L_5 - 3L_2}}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & 0 & 22 & -2 & -5 & -61 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & -16 & 5 & 4 & 49 \end{array}\right)$

$$\overset{L_3 \leftrightarrow L_4}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 22 & -2 & -5 & -61 \\ 0 & 0 & -16 & 5 & 4 & 49 \end{array}\right) \overset{\substack{L_4 \to 7\,L_4 + 22\,L_3 \\ L_5 \to 7\,L_5 - 16\,L_3}}{\Longrightarrow}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 0 & -80 & -35 & -185 \\ 0 & 0 & 0 & 83 & 28 & 167 \end{array}\right) \overset{L_4 \to -\frac{1}{5}\,L4}{\Longrightarrow}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 0 & 16 & 7 & 37 \\ 0 & 0 & 0 & 83 & 28 & 167 \end{array}\right) \overset{L_5 \to 16\,L_5 - 83\,L_4}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 0 & 16 & 7 & 37 \\ 0 & 0 & 0 & 0 & -133 & -399 \end{array}\right)$$

$$\overset{L_5 \to -\frac{1}{133}\,L_5}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & -1 & 8 \\ 0 & 1 & 4 & 0 & -2 & -9 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 0 & 16 & 7 & 37 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \overset{\substack{L_1 \to L_1 + L_5 \\ L_2 \to L_2 + 2\,L_5 \\ L_4 \to L_4 - 7\,L_5}}{\Longrightarrow}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 0 & 11 \\ 0 & 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 0 & 16 & 0 & 16 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \overset{L_4 \to \frac{1}{16}\,L_4}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 0 & 11 \\ 0 & 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & -7 & -3 & 0 & 11 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\overset{\substack{L_1 \to L_1 - L_4 \\ L_3 \to L_3 + 3\,L_4}}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & -7 & 0 & 0 & 14 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \overset{L_3 \to -\frac{1}{7}\,L_3}{\Longrightarrow}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \overset{\substack{L_1 \to L_1 + L_3 \\ L_2 \to L_2 - 4\,L_3}}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\overset{L_1 \to L_1 - L_2}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \Longrightarrow \left\{\begin{array}{rr} x = & 3 \\ y = & 5 \\ z = & -2 \\ v = & 1 \\ w = & 3 \end{array}\right.$$

On a donc trouvé $S = \{(3\,;5\,;-2\,;1\,;3)\}$.

4) $\left\{\begin{array}{rcrcrcrcrcr} 3x & - & 3y & - & z & + & 2v & - & 9w & = & 13 \\ x & - & y & + & 2z & - & v & - & 6w & = & -6 \\ x & - & y & + & z & + & v & - & 6w & = & 1 \\ -x & + & y & - & z & - & 2v & + & 7w & = & -3 \end{array}\right. \Longrightarrow \left(\begin{array}{ccccc|c} 3 & -3 & -1 & 2 & -9 & 13 \\ 1 & -1 & 2 & -1 & -6 & -6 \\ 1 & -1 & 1 & 1 & -6 & 1 \\ -1 & 1 & -1 & -2 & 7 & -3 \end{array}\right)$

$\overset{L_1 \leftrightarrow L_2}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & -6 & -6 \\ 3 & -3 & -1 & 2 & -9 & 13 \\ 1 & -1 & 1 & 1 & -6 & 1 \\ -1 & 1 & -1 & -2 & 7 & -3 \end{array}\right) \overset{\substack{L_2 \to L_2 - 3L_1 \\ L_3 \to L_3 - L_1 \\ L_4 \to L_4 + L_1}}{\Longrightarrow}$

$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & -6 & -6 \\ 0 & 0 & -7 & 5 & 9 & 31 \\ 0 & 0 & -1 & 2 & 0 & 7 \\ 0 & 0 & 1 & -3 & 1 & -9 \end{array}\right) \overset{L_2 \leftrightarrow L_4}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & -6 & -6 \\ 0 & 0 & 1 & -3 & 1 & -9 \\ 0 & 0 & -1 & 2 & 0 & 7 \\ 0 & 0 & -7 & 5 & 9 & 31 \end{array}\right)$

$\overset{\substack{L_3 \to L_3 + L_2 \\ L_4 \to L_4 + 7L2}}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & -6 & -6 \\ 0 & 0 & 1 & -3 & 1 & -9 \\ 0 & 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & -16 & 16 & -32 \end{array}\right) \overset{\substack{L_3 \to -L_3 \\ L_4 \to L_4 - 16L3}}{\Longrightarrow}$

$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -1 & -6 & -6 \\ 0 & 0 & 1 & -3 & 1 & -9 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{\substack{L_1 \to L_1 + L_3 \\ L_2 \to L_2 + 3L3}}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -7 & -4 \\ 0 & 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\overset{L_1 \to L_1 - 2L_2}{\Longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow$

$\left\{\begin{array}{rcrcrcr} x - y & & & & - 3w & = & 2 \\ & + & z & & - 2w & = & -3 \\ & & & v & - \; w & = & 2 \end{array}\right.$

Ce système échelonné réduit comporte deux variables libres y et w ; on pose $y = \alpha$ et $w = \beta$.

Dès lors, $x = 2 + \alpha + 3\beta$, $z = -3 + 2\beta$ et $v = 2 + \beta$.

En résumé, $\mathrm{S} = \{(2 + \alpha + 3\beta\,; \alpha\,; -3 + 2\beta\,; 2 + \beta\,; \beta) : \alpha, \beta \in \mathbb{R}\}$.