11.11 Le plan vectoriel ayant $\begin{pmatrix}1\\1\\1\end{pmatrix}$ pour vecteur normal admet pour équation :

$$x+y+z=0 \Longrightarrow \begin{cases} x=-\beta-\gamma \\ y=\beta \\ z=\gamma \end{cases} = -\beta\begin{pmatrix}1\\-1\\0\end{pmatrix} - \gamma\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Posons $e_1' = \frac{1}{\sqrt{1^2+(-1)^2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{pmatrix}$ et $e_3' = \frac{1}{\sqrt{1^2+1^2+1^2}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$

$$e_2' = e_3' \times e_1' = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix} \times \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\-\frac{2}{\sqrt{6}}\end{pmatrix}$$

On considère encore la base orthonormée $\mathcal{B}' = (e_1'\,;e_2'\,;e_3')$ et la matrice de passage P de la base canonique à la base $\mathcal{B}'$:

$$\mathrm{P} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

De $\cos(\alpha) = \frac{3}{5}$ et $\cos^2(\alpha)+\sin^2(\alpha)=1$, on déduit que $\sin(\alpha) = \pm\frac{4}{5}$.
Dans la base $\mathcal{B}'$, il y a deux matrices possibles :

$$\mathrm{R}_1' = \begin{pmatrix}\frac{3}{5} & -\frac{4}{5} & 0\\ \frac{4}{5} & \frac{3}{5} & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{ou} \quad \mathrm{R}_2' = \begin{pmatrix}\frac{3}{5} & \frac{4}{5} & 0\\ -\frac{4}{5} & \frac{3}{5} & 0\\ 0 & 0 & 1\end{pmatrix}$$

Il s'agit de déterminer les matrices correspondantes dans la base canonique.

1) $$\mathrm{R}_1 = \mathrm{P}\mathrm{R}_1'\mathrm{P}^{-1} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\frac{3}{5} & -\frac{4}{5} & 0\\ \frac{4}{5} & \frac{3}{5} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{4}{5\sqrt{6}}+\frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{6}}-\frac{4}{5\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{4}{5\sqrt{6}}-\frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{6}}+\frac{4}{5\sqrt{2}} & \frac{1}{\sqrt{3}}\\ -\frac{8}{5\sqrt{6}} & -\frac{6}{5\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

$$= \frac{1}{5\sqrt{6}}\begin{pmatrix}4+3\sqrt{3} & 3-4\sqrt{3} & 5\sqrt{2}\\ 4-3\sqrt{3} & 3+4\sqrt{3} & 5\sqrt{2}\\ -8 & -6 & 5\sqrt{2}\end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3} & -\sqrt{3} & 0\\ 1 & 1 & -2\\ \sqrt{2} & \sqrt{2} & \sqrt{2}\end{pmatrix}$$

$$= \frac{1}{5\sqrt{6}\sqrt{6}}\begin{pmatrix}22 & 4-8\sqrt{3} & 4+8\sqrt{3}\\ 4+8\sqrt{3} & 22 & 4-8\sqrt{3}\\ 4-8\sqrt{3} & 4+8\sqrt{3} & 22\end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix} 11 & 2-4\sqrt{3} & 2+4\sqrt{3} \\ 2+4\sqrt{3} & 11 & 2-4\sqrt{3} \\ 2-4\sqrt{3} & 2+4\sqrt{3} & 11 \end{pmatrix}$$

$$2)\ \mathrm{R}_2 = \mathrm{P}\mathrm{R}_2'\mathrm{P}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5\sqrt{6}} + \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{6}} + \frac{4}{5\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{4}{5\sqrt{6}} - \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{6}} - \frac{4}{5\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{8}{5\sqrt{6}} & -\frac{6}{5\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \frac{1}{5\sqrt{6}}\begin{pmatrix} -4+3\sqrt{3} & 3+4\sqrt{3} & 5\sqrt{2} \\ -4-3\sqrt{3} & 3-4\sqrt{3} & 5\sqrt{2} \\ 8 & -6 & 5\sqrt{2} \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{5\sqrt{6}\sqrt{6}}\begin{pmatrix} 22 & 4+8\sqrt{3} & 4-8\sqrt{3} \\ 4-8\sqrt{3} & 22 & 4+8\sqrt{3} \\ 4+8\sqrt{3} & 4-8\sqrt{3} & 22 \end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix} 11 & 2+4\sqrt{3} & 2-4\sqrt{3} \\ 2-4\sqrt{3} & 11 & 2+4\sqrt{3} \\ 2+4\sqrt{3} & 2-4\sqrt{3} & 11 \end{pmatrix}$$