4.18 Attendu que $\dim\big(\mathrm{M}_2(\mathbb{R})\big) = 4$, il suffit de montrer que la famille $\left(\begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} ; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} ; \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\right)$ est libre, car elle comporte 4 éléments.

$$\alpha_1 \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

équivaut au système suivant :

$$\left\{\begin{array}{rcrcrcrcl} 3\,\alpha_1 & & & & & + & \alpha_4 & = & 0 \\ 6\,\alpha_1 & + & \alpha_2 & - & 8\,\alpha_3 & & & = & 0 \\ 3\,\alpha_1 & - & \alpha_2 & - & 12\,\alpha_3 & - & \alpha_4 & = & 0 \\ -6\,\alpha_1 & & & - & 4\,\alpha_3 & + & 2\,\alpha_4 & = & 0 \end{array}\right. \overset{\substack{\mathrm{L}_2 \to \mathrm{L}_2 - 2\,\mathrm{L}_1 \\ \mathrm{L}_3 \to \mathrm{L}_3 - \mathrm{L}_1 \\ \mathrm{L}_4 \to \mathrm{L}_4 + 2\,\mathrm{L}_1}}{\Longrightarrow} \left\{\begin{array}{rcrcrcrcl} 3\,\alpha_1 & & & & & + & \alpha_4 & = & 0 \\ & & \alpha_2 & - & 8\,\alpha_3 & - & 2\,\alpha_4 & = & 0 \\ & - & \alpha_2 & - & 12\,\alpha_3 & - & 2\,\alpha_4 & = & 0 \\ & & & - & 4\,\alpha_3 & + & 4\,\alpha_4 & = & 0 \end{array}\right.$$

$$\overset{\mathrm{L}_3 \to \mathrm{L}_3 + \mathrm{L}_2}{\Longrightarrow} \left\{\begin{array}{rcrcrcrcl} 3\,\alpha_1 & & & & & + & \alpha_4 & = & 0 \\ & & \alpha_2 & - & 8\,\alpha_3 & - & 2\,\alpha_4 & = & 0 \\ & & & - & 20\,\alpha_3 & - & 4\,\alpha_4 & = & 0 \\ & & & - & 4\,\alpha_3 & + & 4\,\alpha_4 & = & 0 \end{array}\right. \overset{\mathrm{L}_4 \to 5\,\mathrm{L}_4 - \mathrm{L}_3}{\Longrightarrow}$$

$$\left\{\begin{array}{rcrcrcrcl} 3\,\alpha_1 & & & & & + & \alpha_4 & = & 0 \\ & & \alpha_2 & - & 8\,\alpha_3 & - & 2\,\alpha_4 & = & 0 \\ & & & - & 20\,\alpha_3 & - & 4\,\alpha_4 & = & 0 \\ & & & & & & 24\,\alpha_4 & = & 0 \end{array}\right. \overset{\substack{\mathrm{L}_1 \to 24\,\mathrm{L}_1 - \mathrm{L}_4 \\ \mathrm{L}_2 \to 12\,\mathrm{L}_2 + \mathrm{L}_4 \\ \mathrm{L}_3 \to 6\,\mathrm{L}_3 + \mathrm{L}_4}}{\Longrightarrow} \left\{\begin{array}{rcrcrcl} 72\,\alpha_1 & & & & & & = 0 \\ & & 12\,\alpha_2 & - & 96\,\alpha_3 & & = 0 \\ & & & - & 120\,\alpha_3 & & = 0 \\ & & & & & 24\,\alpha_4 & = 0 \end{array}\right.$$

$$\overset{\mathrm{L}_2 \to 5\,\mathrm{L}_2 - 4\,\mathrm{L}_3}{\Longrightarrow} \left\{\begin{array}{rrrrl} 72\,\alpha_1 & & & & = 0 \\ & 60\,\alpha_2 & & & = 0 \\ & & -\ 120\,\alpha_3 & & = 0 \\ & & & 24\,\alpha_4 & = 0 \end{array}\right. \overset{\substack{\mathrm{L}_1 \to \frac{1}{72}\,\mathrm{L}_1 \\ \mathrm{L}_2 \to \frac{1}{60}\,\mathrm{L}_2 \\ \mathrm{L}_3 \to -\frac{1}{120}\,\mathrm{L}_3 \\ \mathrm{L}_4 \to \frac{1}{24}\,\mathrm{L}_4}}{\Longrightarrow} \left\{\begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \\ \alpha_4 = 0 \end{array}\right.$$

On a ainsi montré que la famille $\left(\begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix} ; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix} ; \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\right)$ est libre.