4.19 Comme $\dim(\mathbb{R}_2[x]) = 3$, il suffit de montrer que la famille $(2-x\,;1+2\,x\,;1-x^2)$ engendre $\mathbb{R}_2[x]$.

Soit $p = a\,x^2 + b\,x + c$ un élément quelconque de $\mathbb{R}_2[x]$.

Résoudre

$$\alpha_1\,(2-x) + \alpha_2\,(1+2\,x) + \alpha_3\,(1-x^2) = p = a\,x^2 + b\,x + c$$

revient à résoudre le système

$$\begin{cases} 2\,\alpha_1 + \alpha_2 + \alpha_3 = c \\ -\alpha_1 + 2\,\alpha_2 = b \\ -\alpha_3 = a \end{cases} \overset{\mathrm{L}_2 \to 2\,\mathrm{L}_2 + \mathrm{L}_1}{\Longrightarrow} \begin{cases} 2\,\alpha_1 + \alpha_2 + \alpha_3 = c \\ 5\,\alpha_2 + \alpha_3 = 2\,b + c \\ -\alpha_3 = a \end{cases}$$

$$\overset{\substack{\mathrm{L}_1 \to \mathrm{L}_1 + \mathrm{L}_3 \\ \mathrm{L}_2 \to \mathrm{L}_2 + \mathrm{L}_3}}{\Longrightarrow} \begin{cases} 2\,\alpha_1 + \alpha_2 = a + c \\ 5\,\alpha_2 = a + 2\,b + c \\ -\alpha_3 = a \end{cases} \overset{\mathrm{L}_1 \to 5\,\mathrm{L}_1 - \mathrm{L}_2}{\Longrightarrow}$$

$$\begin{cases} 10\,\alpha_1 = 4\,a - 2\,b + 4\,c \\ 5\,\alpha_2 = a + 2\,b + c \\ -\alpha_3 = a \end{cases} \overset{\substack{\mathrm{L}_1 \to \frac{1}{10}\,\mathrm{L}_1 \\ \mathrm{L}_2 \to \frac{1}{5}\,\mathrm{L}_2 \\ \mathrm{L}_3 \to -\mathrm{L}_3}}{\Longrightarrow} \begin{cases} \alpha_1 = \frac{2}{5}\,a - \frac{1}{5}\,b + \frac{2}{5}\,c \\ \alpha_2 = \frac{1}{5}\,a + \frac{2}{5}\,b + \frac{1}{5}\,c \\ \alpha_3 = -a \end{cases}$$

Il en résulte que la famille $(2-x\,;1+2\,x\,;1-x^2)$ engendre $\mathbb{R}_2[x]$.

1) Si $p = x^2$, alors $a = 1$, $b = c = 0$.

$$\begin{cases} \alpha_1 = \frac{2}{5}\cdot 1 - \frac{1}{5}\cdot 0 + \frac{2}{5}\cdot 0 = \frac{2}{5} \\ \alpha_2 = \frac{1}{5}\cdot 1 + \frac{2}{5}\cdot 0 + \frac{1}{5}\cdot 0 = \frac{1}{5} \\ \alpha_3 = -1 \end{cases}$$

Dans la base $(2-x\,;1+2\,x\,;1-x^2)$, on a $x^2 = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \\ -1 \end{pmatrix}$.

2) Si $p = (2\,x-1)^2 = 4\,x^2 - 4\,x + 1$, alors $a = 4$, $b = -4$, $c = 1$.

$$\begin{cases} \alpha_1 = \frac{2}{5}\cdot 4 - \frac{1}{5}\cdot(-4) + \frac{2}{5}\cdot 1 = \frac{14}{5} \\ \alpha_2 = \frac{1}{5}\cdot 4 + \frac{2}{5}\cdot(-4) + \frac{1}{5}\cdot 1 = -\frac{3}{5} \\ \alpha_3 = -4 \end{cases}$$

Dans la base $(2-x\,;1+2\,x\,;1-x^2)$, on a $(2\,x-1)^2 = \begin{pmatrix} \frac{14}{5} \\ -\frac{3}{5} \\ -4 \end{pmatrix}$.

3) Si $p = 2\,x^2 - 4\,x + 3$, alors $a = 2$, $b = -4$, $c = 3$.

$$\begin{cases} \alpha_1 = \frac{2}{5}\cdot 2 - \frac{1}{5}\cdot(-4) + \frac{2}{5}\cdot 3 = \frac{14}{5} \\ \alpha_2 = \frac{1}{5}\cdot 2 + \frac{2}{5}\cdot(-4) + \frac{1}{5}\cdot 3 = -\frac{3}{5} \\ \alpha_3 = -2 \end{cases}$$

Dans la base $(2-x\,;1+2\,x\,;1-x^2)$, on a $(2\,x-1)^2 = \begin{pmatrix} \frac{14}{5} \\ -\frac{3}{5} \\ -2 \end{pmatrix}$.