7.6 Soit $\mathcal{B} = (e_1\,;e_2)$ la base canonique de $\mathbb{R}^2$.

$u = h(a) = h(1 \cdot e_1 + 2 \cdot e_2) = 1 \cdot h(e_1) + 2 \cdot h(e_2)$

$v = h(b) = h(3 \cdot e_1 - 1 \cdot e_2) = 3 \cdot h(e_1) - 1 \cdot h(e_2)$

Déterminons $h(e_1)$ et $h(e_2)$ en fonction de u et v :

$$\left(\begin{array}{cc|c} 1 & 2 & u \\ 3 & -1 & v \end{array}\right) \xrightarrow{\mathrm{L}_2 \to \mathrm{L}_2 - 3\,\mathrm{L}_1} \left(\begin{array}{cc|c} 1 & 2 & u \\ 0 & -7 & -3\,u + v \end{array}\right) \xrightarrow{\mathrm{L}_2 \to -\frac{1}{7}\,\mathrm{L}_2}$$

$$\left(\begin{array}{cc|c} 1 & 2 & u \\ 0 & 1 & \frac{3}{7}\,u - \frac{1}{7}\,v \end{array}\right) \xrightarrow{\mathrm{L}_1 \to \mathrm{L}_1 - 2\,\mathrm{L}_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{7}\,u + \frac{2}{7}\,v \\ 0 & 1 & \frac{3}{7}\,u - \frac{1}{7}\,v \end{array}\right)$$

$$h(e_1) = \tfrac{1}{7}\,u + \tfrac{2}{7}\,v = \tfrac{1}{7}\begin{pmatrix}1\\0\\1\end{pmatrix} + \tfrac{2}{7}\begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}\frac{1}{7}+0\\0+\frac{2}{7}\\\frac{1}{7}-\frac{4}{7}\end{pmatrix} = \begin{pmatrix}\frac{1}{7}\\\frac{2}{7}\\-\frac{3}{7}\end{pmatrix}$$

$$h(e_2) = \tfrac{3}{7}\,u - \tfrac{1}{7}\,v = \tfrac{3}{7}\begin{pmatrix}1\\0\\1\end{pmatrix} - \tfrac{1}{7}\begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}\frac{3}{7}-0\\0-\frac{1}{7}\\\frac{3}{7}-(-\frac{2}{7})\end{pmatrix} = \begin{pmatrix}\frac{3}{7}\\-\frac{1}{7}\\\frac{5}{7}\end{pmatrix}$$

$$\mathrm{A} = \begin{pmatrix}\frac{1}{7} & \frac{3}{7}\\ \frac{2}{7} & -\frac{1}{7}\\ -\frac{3}{7} & \frac{5}{7}\end{pmatrix} = \frac{1}{7}\begin{pmatrix}1 & 3\\2 & -1\\-3 & 5\end{pmatrix}$$

$$h\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{7} & \frac{3}{7}\\ \frac{2}{7} & -\frac{1}{7}\\ -\frac{3}{7} & \frac{5}{7}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{1}{7}\,x + \frac{3}{7}\,y\\ \frac{2}{7}\,x - \frac{1}{7}\,y\\ -\frac{3}{7}\,x + \frac{5}{7}\,y\end{pmatrix}$$

Pour déterminer $\mathrm{Ker}(h)$, il faut résoudre le système $\begin{cases} \frac{1}{7}\,x + \frac{3}{7}\,y = 0\\ \frac{2}{7}\,x - \frac{1}{7}\,y = 0\\ -\frac{3}{7}\,x + \frac{5}{7}\,y = 0\end{cases}$:

$$\left(\begin{array}{cc|c} \frac{1}{7} & \frac{3}{7} & 0\\ \frac{2}{7} & -\frac{1}{7} & 0\\ -\frac{3}{7} & \frac{5}{7} & 0\end{array}\right) \xrightarrow[\mathrm{L}_3 \to 7\,\mathrm{L}_3]{\mathrm{L}_1 \to 7\,\mathrm{L}_1,\ \mathrm{L}_2 \to 7\,\mathrm{L}_2} \left(\begin{array}{cc|c} 1 & 3 & 0\\ 2 & -1 & 0\\ -3 & 5 & 0\end{array}\right) \xrightarrow[\mathrm{L}_3 \to \mathrm{L}_3 + 3\,\mathrm{L}_1]{\mathrm{L}_2 \to \mathrm{L}_2 - 2\,\mathrm{L}_1}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 0\\ 0 & -7 & 0\\ 0 & 14 & 0\end{array}\right) \xrightarrow[\mathrm{L}_3 \to \frac{1}{14}\,\mathrm{L}_3]{\mathrm{L}_2 \to -\frac{1}{7}\,\mathrm{L}_2} \left(\begin{array}{cc|c} 1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{array}\right) \xrightarrow[\mathrm{L}_3 \to \mathrm{L}_3 - \mathrm{L}_2]{\mathrm{L}_1 \to \mathrm{L}_1 - 3\,\mathrm{L}_2} \left(\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{array}\right)$$

Puisque l'unique solution est $\begin{cases} x = 0\\ y = 0\end{cases}$, on conclut que $\mathrm{Ker}(h) = \{0\}$.

Déterminons $\mathrm{Im}(h)$ en échelonnant la matrice suivante :

$$\begin{pmatrix} \frac{1}{7} & \frac{2}{7} & -\frac{3}{7} \\ \frac{3}{7} & -\frac{1}{7} & \frac{5}{7} \\ x & y & z \end{pmatrix} \overset{\substack{\mathrm{L}_1 \to 7\,\mathrm{L}_1 \\ \mathrm{L}_2 \to 7\,\mathrm{L}_2}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ x & y & z \end{pmatrix} \overset{\substack{\mathrm{L}_2 \to \mathrm{L}_2 - 3\,\mathrm{L}_1 \\ \mathrm{L}_3 \to \mathrm{L}_3 - x\,\mathrm{L}_1}}{\Longrightarrow}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 14 \\ 0 & -2\,x + y & 3\,x + z \end{pmatrix} \overset{\mathrm{L}_2 \to -\frac{1}{7}\,\mathrm{L}_2}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & -2\,x + y & 3\,x + z \end{pmatrix}$$

$$\overset{\substack{\mathrm{L}_1 \to \mathrm{L}_1 - 2\,\mathrm{L}_2 \\ \mathrm{L}_3 \to \mathrm{L}_3 - (-2\,x + y)\,\mathrm{L}_2}}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -x + 2\,y + z \end{pmatrix}$$

Ainsi $\mathrm{Im}(h) = \Pi\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\right) = \left\{(x\,;y\,;z) \in \mathbb{R}^3 : -x + 2\,y + z = 0\right\}.$