8.17 1) $\det(A) = \begin{vmatrix} -2 & 4 & 2 \\ -4 & 8 & 4 \\ 5 & 10 & 5 \end{vmatrix} \overset{L_2 \to L_2 - 2\,L_1}{=} \begin{vmatrix} -2 & 4 & 2 \\ 0 & 0 & 0 \\ 5 & 10 & 5 \end{vmatrix} = 0$

La matrice A n'est donc pas inversible.

2) $\det(A) = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{vmatrix} \overset{C_1 \to C_1 + 3\,C_2}{=} \begin{vmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 4 \end{vmatrix}$

$$= -(-1) \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix} = 1 \cdot 4 - (-1) \cdot 1 = 5 \neq 0$$

$$A^{-1} = \frac{1}{5}\, {}^t\!\begin{pmatrix} \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{5}\, {}^t\!\begin{pmatrix} 5 & 10 & 0 \\ 4 & 12 & 1 \\ -1 & -3 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

3) $\det(A) = \begin{vmatrix} 5 & -8 & -4 \\ 8 & -15 & -8 \\ -10 & 20 & 11 \end{vmatrix} \overset{C_1 \to C_1 + C_3}{=} \begin{vmatrix} 1 & -8 & -4 \\ 0 & -15 & -8 \\ 1 & 20 & 11 \end{vmatrix} \overset{L_3 \to L_3 - L_1}{=}$

$$= \begin{vmatrix} 1 & -8 & -4 \\ 0 & -15 & -8 \\ 0 & 28 & 15 \end{vmatrix} = \begin{vmatrix} -15 & -8 \\ 28 & 15 \end{vmatrix} = (-15) \cdot 15 - 28 \cdot (-8) = -1$$

$$A^{-1} = \frac{1}{-1}\, {}^t\!\begin{pmatrix} \begin{vmatrix} -15 & -8 \\ 20 & 11 \end{vmatrix} & -\begin{vmatrix} 8 & -8 \\ -10 & 11 \end{vmatrix} & \begin{vmatrix} 8 & -15 \\ -10 & 20 \end{vmatrix} \\ -\begin{vmatrix} -8 & -4 \\ 20 & 11 \end{vmatrix} & \begin{vmatrix} 5 & -4 \\ -10 & 11 \end{vmatrix} & -\begin{vmatrix} 5 & -8 \\ -10 & 20 \end{vmatrix} \\ \begin{vmatrix} -8 & -4 \\ -15 & -8 \end{vmatrix} & -\begin{vmatrix} 5 & -4 \\ 8 & -8 \end{vmatrix} & \begin{vmatrix} 5 & -8 \\ 8 & -15 \end{vmatrix} \end{pmatrix}$$

$$= -\, {}^t\!\begin{pmatrix} -5 & -8 & 10 \\ 8 & 15 & -20 \\ 4 & 8 & -11 \end{pmatrix} = \begin{pmatrix} 5 & -8 & -4 \\ 8 & -15 & -8 \\ -10 & 20 & 11 \end{pmatrix} = A$$

4) $\det(A) = \det\left(\frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}\right) = \left(\frac{1}{3}\right)^3 \begin{vmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{vmatrix} \overset{\substack{C_1 \to C_1 + 2\,C_2 \\ C_3 \to C_3 + 2\,C_2}}{=}$

$$= \frac{1}{27} \begin{vmatrix} 0 & -1 & 0 \\ 6 & 2 & 3 \\ 3 & 2 & 6 \end{vmatrix} = -\frac{-1}{27} \begin{vmatrix} 6 & 3 \\ 3 & 6 \end{vmatrix} = \frac{1}{27} \cdot 3^2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \tfrac{1}{3}\,(2\cdot 2 - 1\cdot 1) = \tfrac{1}{3}\cdot 3 = 1 \neq 0$$

$$\mathrm{A}^{-1} = \tfrac{1}{1}\,{}^t\!\begin{pmatrix} \left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} & -\left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} & \left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} \\[2ex] -\left(\frac{1}{3}\right)^2 \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} & \left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} & -\left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\[2ex] \left(\frac{1}{3}\right)^2 \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} & -\left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} & \left(\frac{1}{3}\right)^2 \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \end{pmatrix}$$

$$= {}^t\!\begin{pmatrix} \frac{1}{9}\cdot 6 & -\frac{1}{9}\cdot 3 & \frac{1}{9}\cdot 6 \\ -\frac{1}{9}\cdot(-6) & \frac{1}{9}\cdot 6 & -\frac{1}{9}\cdot 3 \\ \frac{1}{9}\cdot(-3) & -\frac{1}{9}\cdot(-6) & \frac{1}{9}\cdot 6 \end{pmatrix} = \tfrac{1}{3}\,{}^t\!\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

$$= \tfrac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$$

5) $$\det(\mathrm{A}) = \begin{vmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{vmatrix} \overset{\substack{\mathrm{C}_1 \to \mathrm{C}_1 - m\,\mathrm{C}_3 \\ \mathrm{C}_2 \to \mathrm{C}_2 - \mathrm{C}_3}}{=} \begin{vmatrix} 0 & 0 & 1 \\ 1-m & m-1 & 1 \\ 1-m^2 & 1-m & m \end{vmatrix}$$

$$= \begin{vmatrix} 1-m & m-1 \\ 1-m^2 & 1-m \end{vmatrix} = \begin{vmatrix} 1-m & -(1-m) \\ (1-m)\,(1+m) & 1-m \end{vmatrix}$$

$$= (1-m)^2 \begin{vmatrix} 1 & -1 \\ 1+m & 1 \end{vmatrix} = (1-m)^2\,\big(1\cdot 1 - (1+m)\cdot(-1)\big)$$

$$= (1-m)^2\,(2+m) = (m-1)^2\,(m+2)$$

Ainsi $\det(\mathrm{A}) = 0$ si $m = 1$ ou si $m = -2$: A n'est alors pas inversible.

Si $m \neq 1$ et $m \neq 2$, alors A est inversible et son inverse vaut :

$$\mathrm{A}^{-1} = \tfrac{1}{(m-1)^2\,(m+2)}\,{}^t\!\begin{pmatrix} \begin{vmatrix} m & 1 \\ 1 & m \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & m \end{vmatrix} & \begin{vmatrix} 1 & m \\ 1 & 1 \end{vmatrix} \\[2ex] -\begin{vmatrix} 1 & 1 \\ 1 & m \end{vmatrix} & \begin{vmatrix} m & 1 \\ 1 & m \end{vmatrix} & -\begin{vmatrix} m & 1 \\ 1 & 1 \end{vmatrix} \\[2ex] \begin{vmatrix} 1 & 1 \\ m & 1 \end{vmatrix} & -\begin{vmatrix} m & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} m & 1 \\ 1 & m \end{vmatrix} \end{pmatrix}$$

$$= \tfrac{1}{(m-1)^2\,(m+2)}\,{}^t\!\begin{pmatrix} m^2-1 & -(m-1) & 1-m \\ -(m-1) & m^2-1 & -(m-1) \\ 1-m & -(m-1) & m^2-1 \end{pmatrix}$$

$$= \tfrac{1}{(m-1)^2\,(m+2)}\,(m-1)\,{}^t\!\begin{pmatrix} m+1 & -1 & -1 \\ -1 & m+1 & -1 \\ -1 & -1 & m+1 \end{pmatrix}$$

$$= \frac{1}{(m-1)\,(m+2)} \begin{pmatrix} m+1 & -1 & -1 \\ -1 & m+1 & -1 \\ -1 & -1 & m+1 \end{pmatrix}$$

6) $\det(\mathrm{A}) = \begin{vmatrix} 1 & -a & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \quad$ (matrice triangulaire supérieure)

$$\mathrm{A}^{-1} = \frac{1}{1} \; {}^{t}\!\begin{pmatrix} \begin{vmatrix} 1 & -a \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -a \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -a & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -a \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -a & 0 \\ 1 & -a \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -a \end{vmatrix} & \begin{vmatrix} 1 & -a \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= {}^{t}\!\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$