10.6 1) $$\begin{aligned} x \cdot y &= (\alpha_1\, e_1 + \alpha_2\, e_2) \cdot (\beta_1\, e_1 + \beta_2\, e_2) \\ &= (\alpha_1\, e_1 + \alpha_2\, e_2) \cdot (\beta_1\, e_1) + (\alpha_1\, e_1 + \alpha_2\, e_2) \cdot (\beta_2\, e_2) \\ &= (\alpha_1\, e_1) \cdot (\beta_1\, e_1) + (\alpha_2\, e_2) \cdot (\beta_1\, e_1) + (\alpha_1\, e_1) \cdot (\beta_2\, e_2) + (\alpha_2\, e_2) \cdot (\beta_2\, e_2) \\ &= (\alpha_1\, \beta_1)\, (e_1 \cdot e_1) + (\alpha_2\, \beta_1)\, (e_2 \cdot e_1) + (\alpha_1\, \beta_2)\, (e_1 \cdot e_2) + (\alpha_2\, \beta_2)\, (e_2 \cdot e_2) \\ &= (\alpha_1\, \beta_1)\, \|e_1\|^2 + (\alpha_1\, \beta_2 + \alpha_2\, \beta_1)\, (e_1 \cdot e_2) + (\alpha_2\, \beta_2)\, \|e_2\|^2 \end{aligned}$$

2) $$x \cdot y = (\alpha_1\, \beta_1)\, \underbrace{\|e_1\|^2}_{1} + (\alpha_1\, \beta_2 + \alpha_2\, \beta_1)\, \underbrace{(e_1 \cdot e_2)}_{0} + (\alpha_2\, \beta_2)\, \underbrace{\|e_2\|^2}_{1} = \alpha_1\, \beta_1 + \alpha_2\, \beta_2$$

3) $$x \cdot y = \left(\sum_{i=1}^{n} \alpha_i\, e_i\right) \cdot \left(\sum_{j=1}^{n} \beta_j\, e_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i\, \beta_j)\, (e_i \cdot e_j)$$

Comme $(e_1\,;\dots\,;e_n)$ est une base orthonormée, on a $e_i \cdot e_j = \begin{cases} 1 & \text{si} \quad i = j \\ 0 & \text{si} \quad i \neq j \end{cases}$.

Donc $x \cdot y = \displaystyle\sum_{i=1}^{n} \alpha_i\, \beta_i = \alpha_1\, \beta_1 + \ldots + \alpha_n\, \beta_n$.