Chamblandes 2012 — Problème 4

a) **Signe**

$f(x) = e^{2\,x} - 2\,e^x = \left(e^x\right)^2 - 2\,e^x = e^x\,(e^x - 2)$

On rappelle que $e^x > 0$ pour tout $x \in \mathbb{R}$.

$$\begin{cases} e^x < 2 & \text{si } x < \ln(2) \\ e^x = 2 & \text{si } x = \ln(2) \\ e^x > 2 & \text{si } x > \ln(2) \end{cases} \iff \begin{cases} e^x - 2 < 0 & \text{si } x < \ln(2) \\ e^x - 2 = 0 & \text{si } x = \ln(2) \\ e^x - 2 > 0 & \text{si } x > \ln(2) \end{cases}$$

		$\ln(2)$	
e^x	$+$		$+$
$e^x - 2$	$-$	0	$+$
f	$-$	0	$+$

Croissance

$f'(x) = \left(e^{2\,x} - 2\,e^x\right)' = \left(e^{2\,x}\right)' - 2\left(e^x\right)' = e^{2\,x}\cdot(2\,x)' - 2\,e^x\cdot(x)' = 2\,e^{2\,x} - 2\,e^x$

$\quad = 2\left(e^x\right)^2 - 2\,e^x = 2\,e^x\,(e^x - 1)$

On sait déjà que $2 > 0$ et que $e^x > 0$ pour tout $x \in \mathbb{R}$.

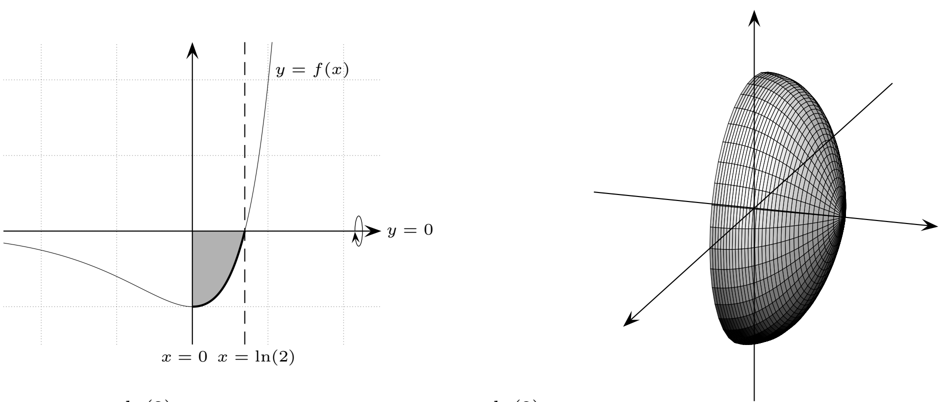
$$\begin{cases} e^x < 1 & \text{si } x < 0 \\ e^x = 1 & \text{si } x = 0 \\ e^x > 1 & \text{si } x > 0 \end{cases} \iff \begin{cases} e^x - 1 < 0 & \text{si } x < 0 \\ e^x - 1 = 0 & \text{si } x = 0 \\ e^x - 1 > 0 & \text{si } x > 0 \end{cases}$$

		0	
$2\,e^x$	$+$		$+$
$e^x - 1$	$-$	0	$+$
f'	$-$	0	$+$
f	$\searrow$	min	$\nearrow$

$f(0) = e^{2\cdot 0} - 2\,e^0 = 1 - 2\cdot 1 = -1$

Le point $(0\,;-1)$ est un minimum global.

b)



$$\mathrm{V} = \pi\int_0^{\ln(2)}\left(e^{2\,x} - 2\,e^x\right)^2 dx = \pi\int_0^{\ln(2)}\left(\left(e^{2\,x}\right)^2 - 2\cdot e^{2\,x}\cdot 2\,e^x + \left(2\,e^x\right)^2\right) dx$$

$$= \pi \int_0^{\ln(2)} \left(e^{4\,x} - 4\,e^{3\,x} + 4\,e^{2\,x}\right) dx = \pi \int_0^{\ln(2)} \left(\tfrac{1}{4}\,e^{4\,x} \cdot 4 - \tfrac{4}{3}\,e^{3\,x} \cdot 3 + \tfrac{4}{2}\,e^{2\,x} \cdot 2\right) dx$$

$$= \pi \left[\tfrac{1}{4}\,e^{4\,x} - \tfrac{4}{3}\,e^{3\,x} + 2\,e^{2\,x}\right]_0^{\ln(2)} = \pi \left[\tfrac{1}{4}\left(e^{x}\right)^4 - \tfrac{4}{3}\left(e^{x}\right)^3 + 2\left(e^{x}\right)^2\right]_0^{\ln(2)}$$

$$= \pi \left(\left(\tfrac{1}{4}\left(e^{\ln(2)}\right)^4 - \tfrac{4}{3}\left(e^{\ln(2)}\right)^3 + 2\left(e^{\ln(2)}\right)^2\right) - \left(\tfrac{1}{4}\left(e^{0}\right)^4 - \tfrac{4}{3}\left(e^{0}\right)^3 + 2\left(e^{0}\right)^2\right)\right)$$

$$= \pi \left(\left(\tfrac{1}{4} \cdot 2^4 - \tfrac{4}{3} \cdot 2^3 + 2 \cdot 2^2\right) - \left(\tfrac{1}{4} \cdot 1^4 - \tfrac{4}{3} \cdot 1^3 + 2 \cdot 1^2\right)\right)$$

$$= \pi \left(\left(4 - \tfrac{32}{3} + 8\right) - \left(\tfrac{1}{4} - \tfrac{4}{3} + 2\right)\right) = \pi \left(\tfrac{4}{3} - \tfrac{11}{12}\right) = \boxed{\tfrac{5\,\pi}{12}}$$