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Department Computer Science and Engineering

&

Department Information Science and Engineering

Course Name : DATA STRUCTURES AND APPLICATIONS

Course Code : BCS304

III Semester

2022 Scheme

MODULE 4:
TREES

4.1 DEFINITION

A *tree* is a finite set of one or more nodes such that

- There is a specially designated node called *root*.
- The remaining nodes are partitioned into $n >= 0$ disjoint set $T_1 \ldots, Tn$, where each of these sets is a tree. $T_1, \ldots, T_n$ are called the *subtrees* of the root. Refer Fig 4.1

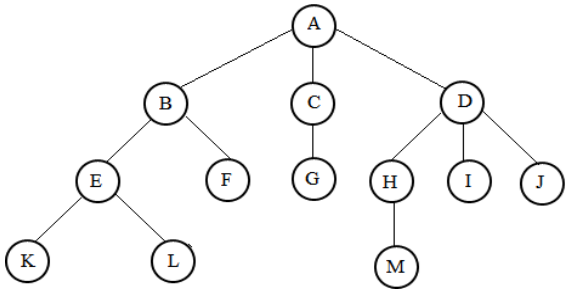


Fig 4.1 Tree Structure

Every node in the tree is the root of some subtree

4.1.1 TERMINOLOGY

- ***Node:*** The item of information plus the branches to other nodes
- ***Degree:*** The number of subtrees of a node
- ***Degree of a tree:*** The maximum of the degree of the nodes in the tree.
- ***Terminal nodes (or leaf):*** nodes that have degree zero or node with no successor
- ***Nonterminal nodes***: nodes that don't belong to terminal nodes.
- ***Parent and Children:*** Suppose N is a node in T with left successor S1 and right successor S2, then N is called the Parent (or father) of S1 and S2. Here, S1 is called left child (or Son) and S2 is called right child (or Son) of N.
- ***Siblings:*** Children of the same parent are said to be siblings.
- ***Edge:*** A line drawn from node N of a T to a successor is called an edge
- ***Path:*** A sequence of consecutive edges from node N to a node M is called a path.
- ***Ancestors of a node:*** All the nodes along the path from the root to that node.
- ***The level of a node:*** defined by letting the root be at level zero. If a node is at level *l*, then it children are at level *l+1*.
- ***Height (or depth):*** The maximum level of any node in the tree

Example

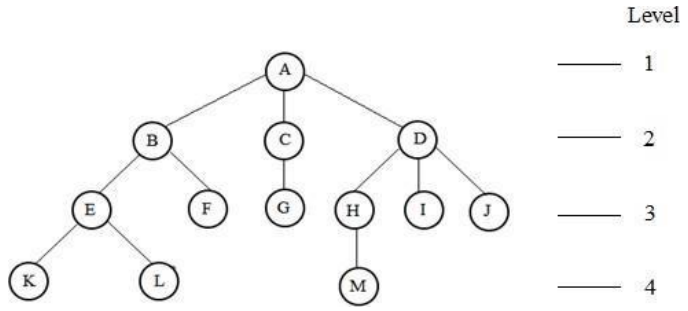
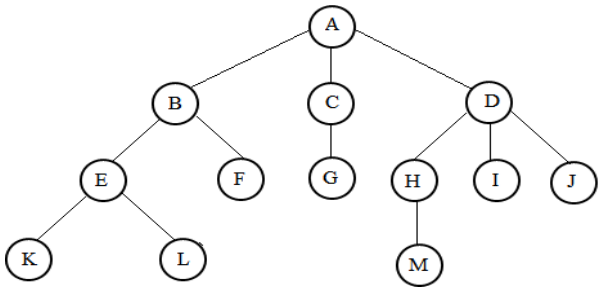


Fig 4.2 Levels in a tree

A is the root node
B is the parent of E and F
C and D are the sibling of B E and F are the children of B
K, L, F, G, M, I, J are external nodes, or leaves A, B, C, D, E,
H are internal nodes (Refer Fig 4.2)
The level of E is 3
The height (depth) of the tree is 4 The degree of node B is 2
The degree of the tree is 3
The ancestors of node M is A, D, H
The descendants of node D is H, I, J,

Representation of Trees

There are several ways to represent a given tree such as:



4.3 Figure (A) Representation of trees

1. List Representation
2. Left Child- Right Sibling Representation
3. Representation as a Degree-Two tree

List Representation:

The tree can be represented as a List. The tree of **figure 4.3** could be written as the list.

(A (B (E (K, L), F), C (G), D (H (M), I, J)))

- The information in the root node comes first.

- The root node is followed by a list of the subtrees of that node.

Tree node is represented by a memory node that has fields for the data and pointers to the tree node's children

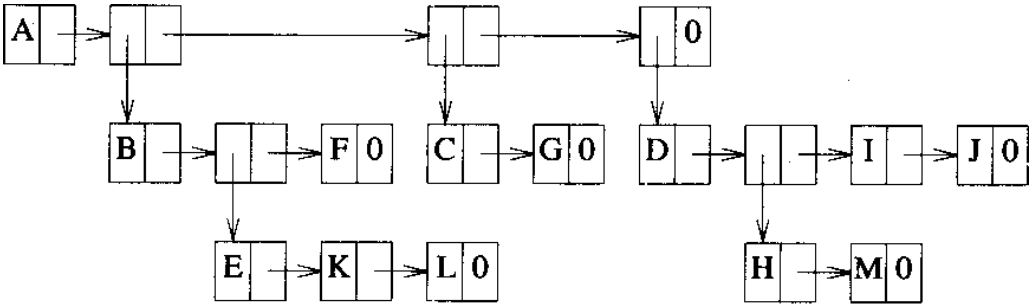


Figure (B): List representation of the tree of figure (A)

4.3 Figure (B) Representation of trees

Since the degree of each tree node may be different, so memory nodes with a varying number of pointer fields are used. (Refer Fig 4.3)

For a tree of degree k, the node structure can be represented as below figure. Each child field is used to point to a subtree.

4.4 Node Representation

Left Child-Right Sibling Representation

The below figure show the node structure used in the left child-right sibling representation

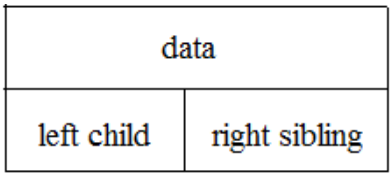


Figure (c): Left child right sibling node structure

To convert the tree of Figure 4.4 into this representation:

1. First note that every node has at most one leftmost child
2. At most one closest right sibling.

Ex:

- In Figure (4.3 A), the leftmost child of A is B, and the leftmost child of Dis H.
- The closest right sibling of B is C, and the closest right sibling of H is I.
- Choose the nodes based on how the tree is drawn. The left child field of each node points to its leftmost child (if any), and the right sibling field points to its closest right sibling (if any).

Figure (D) shows the tree of Figure (4.3 A) redrawn using the left child-right sibling representation.

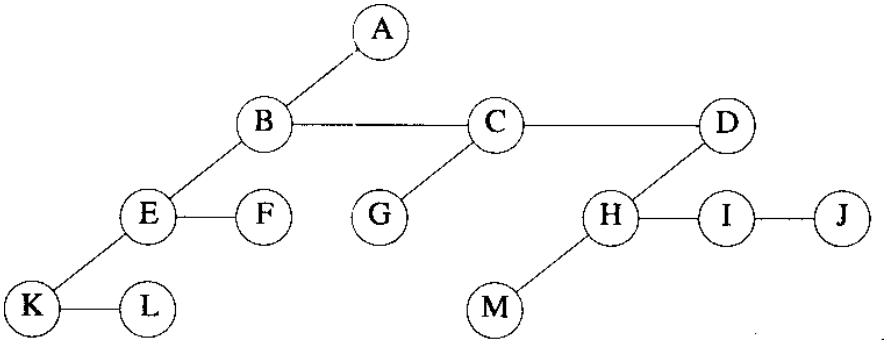


Figure (4.4D): Left child-right sibling of tree (4.A)

Representation as a Degree-Two Tree

To obtain the degree-two tree representation of a tree, simply rotate the right-sibling pointers in a left child-right sibling tree clockwise by 45 degrees. This gives us the degree-two tree displayed in Figure 4.4(E).

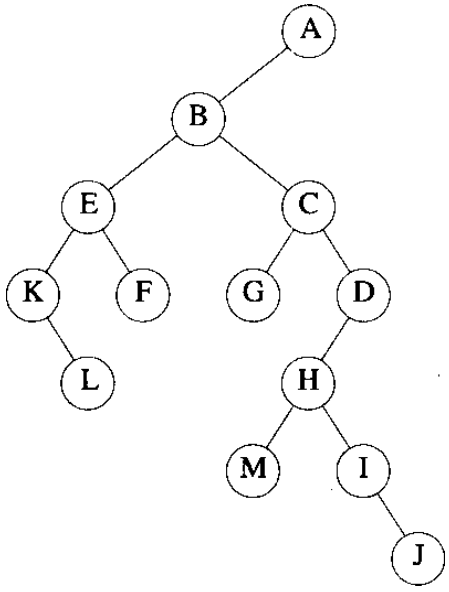


Figure (4.4E): degree-two representation

In the degree-two representation, a node has two children as the left and right children.

4.2 BINARY TREES

Definition: A binary tree T is defined as a finite set of nodes such that,

- T is empty or
- T consists of a root and two disjoint binary trees called the left subtree and the right subtree.

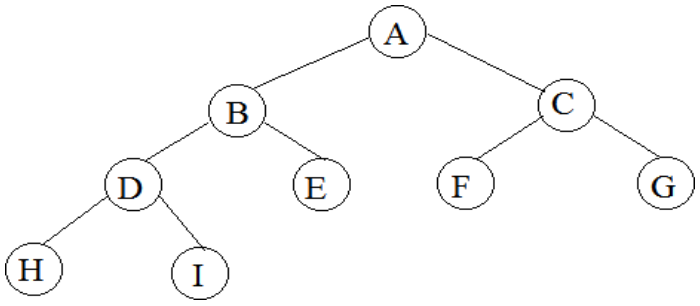


Figure 4.6: Binary Tree

Different kinds of Binary Tree

Skewed Tree

A skewed tree is a tree, skewed to the left or skews to the right.

or

It is a tree consisting of only left subtree or only right subtree.

- A tree with only left subtrees is called Left Skewed Binary Tree.
- A tree with only right subtrees is called Right Skewed Binary Tree.

Complete Binary Tree

A binary tree T is said to complete if all its levels, except possibly the last level, have the maximum number node 2^i, $i \geq 0$ and if all the nodes at the last level appears as far left as possible.

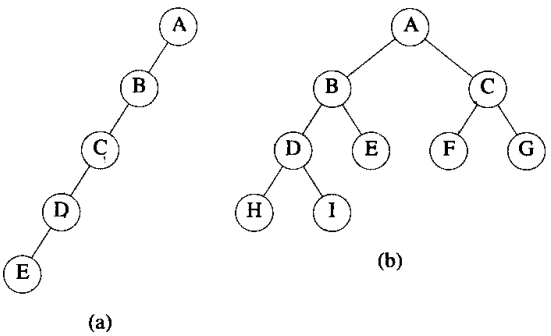


Figure (a): Skewed binary tree

Figure (b): Complete binary tree

Full Binary Tree

A full binary tree of depth 'k' is a binary tree of depth k having $2^k - 1$ node, $k \geq 1$. (Refer Fig 4.7)

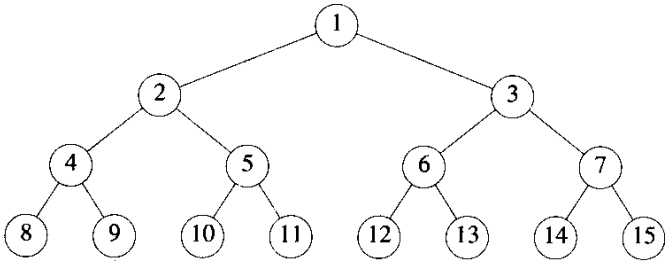


Figure 4.7: Full binary tree of level 4 with sequential node number

Extended Binary Trees or 2-trees

An *extended binary tree* is a transformation of any binary tree into a complete binary tree. This transformation consists of replacing every null subtree of the original tree with "special nodes." The nodes from the original tree are then *internal nodes*, while the special nodes are *external nodes*.

For instance, consider the following binary tree.

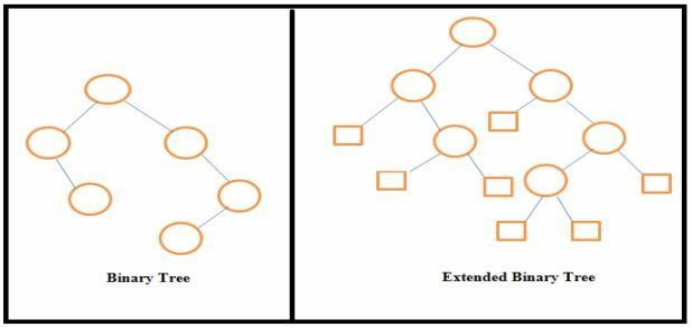


Fig 4.8 Extended Binary Trees

The following tree is its extended binary tree. The circles represent internal nodes, and square represent external nodes. (Refer Fig 4.8)

Every internal node in the extended tree has exactly two children, and every external node is a leaf. The result is a complete binary tree.

4.2.1 PROPERTIES OF BINARY TREES

Lemma 1: [Maximum number of nodes]:

(1) The maximum number of nodes on level i of a binary tree is 2^{i-1}, $i \geq 1$.

(2) The maximum number of nodes in a binary tree of depth k is $2^k - 1$, $k \geq 1$.

Proof:

(1) The proof is by induction on i.

Induction Base: The root is the only node on level i = 1. Hence, the maximum number of nodes on level i =1 is $2^{i-1} = 2^0 = 1$.

Induction Hypothesis: Let i be an arbitrary positive integer greater than 1. Assume that the maximum number of nodes on level i -1is 2^{i-2}

Induction Step: The maximum number of nodes on level i -1 is 2^{i-2} by the induction hypothesis. Since each node in a binary tree has a maximum degree of 2, the maximum number of nodes on level i is two times the maximum number of nodes on level i-1, or 2^{i-1}

3. The maximum number of nodes in a binary tree of depth k is k
4. Lemma 2: [Relation between number of leaf nodes and degree-2 nodes]:

For any nonempty binary tree, T, if n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2, then $n_0 = n_2 + 1$.

Proof: Let n_1 be the number of nodes of degree one and n the total number of nodes.

Since all nodes in T are at most of degree two, we have

$n = n_0 + n_1 + n_2$ (1)

Count the number of branches in a binary tree. If B is the number of branches, then n =B + 1.

All branches stem from a node of degree one or two. Thus,

$B = n_1 + 2n_2$.

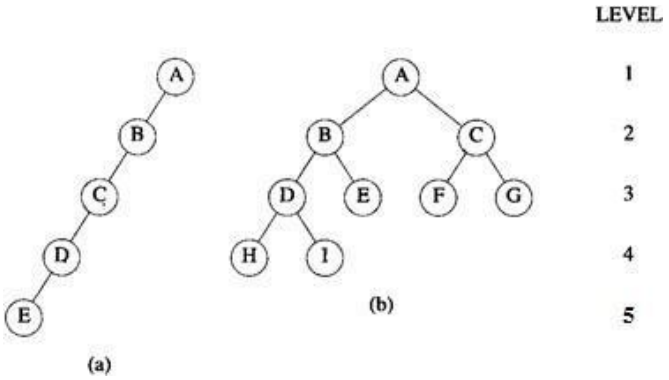
Hence, we obtain

$n = B + 1 = n_1 + 2n_2 + 1$ (2)

Subtracting Eq. (2) from Eq. (1) and rearranging terms, we get

$\mathbf{n_0 = n_2 + 1}$

Consider the figure:



Here, For Figure (b) $n_2=4$, $n_0= n_2+1=$ 4+1=5 Therefore, the total number of leaf node=5

4.2.2 BINARY TREE REPRESENTATION

The storage representation of binary trees can be classified as

1. Array representation
2. Linked representation.

4.4.1 Array representation:

- A tree can be represented using an array, which is called sequential representation.
- The nodes are numbered from 1 to n, and one dimensional array can be used to store the nodes.
- Position 0 of this array is left empty and the node numbered *i* is mapped to position *i* of the array.

Below figure shows the array representation for both the trees of figure 1 (a).

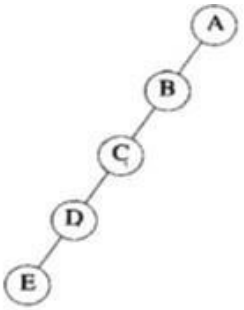


Figure 1(a) Skewed binary tree

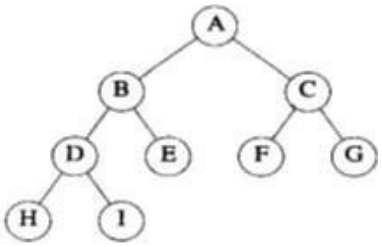


Figure 1(b) Complete binary tree

	tree
[0]	–
[1]	A
[2]	B
[3]	–
[4]	C
[5]	–
[6]	–
[7]	–
[8]	D
[9]	–
.	.
.	.
.	.
[16]	E

(a). Tree of figure 1(a)

	tree
[0]	–
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I
.	.
.	.
.	.
[16]	

(b). Tree of figure 1(b)

- For complete binary tree the array representation is ideal, as no space is wasted.
- For the skewed tree less than half the array is utilized.

Linked representation:

The problems in array representation are:

- It is good for complete binary trees, but more memory is wasted for skewed and many other binary trees.
- The insertion and deletion of nodes from the middle of a tree require the movement of many nodes to reflect the change in level number of these nodes.

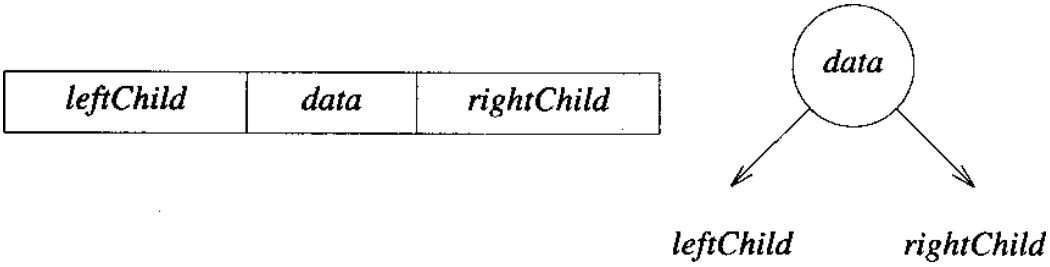
These problems can be easily overcome by linked

representation Each node has three fields,

- LeftChild - which contains the address of left subtree
- RightChild - which contains the address of right subtree.
- Data - which contains the actual information

C Code for node:

```
typedef struct node *treepointer; typedef struct {
int data;
treepointer leftChild, rightChild;
} node;
```



Node representation

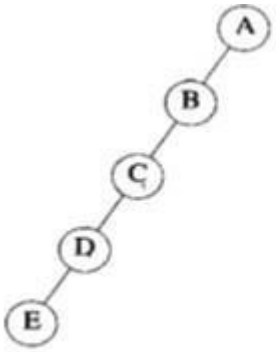


Figure 1(a) Skewed binary tree

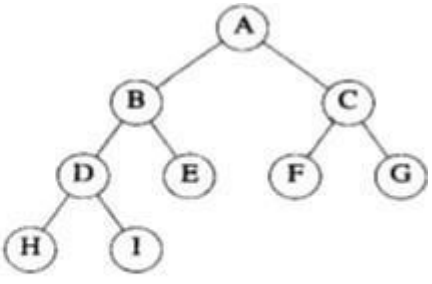
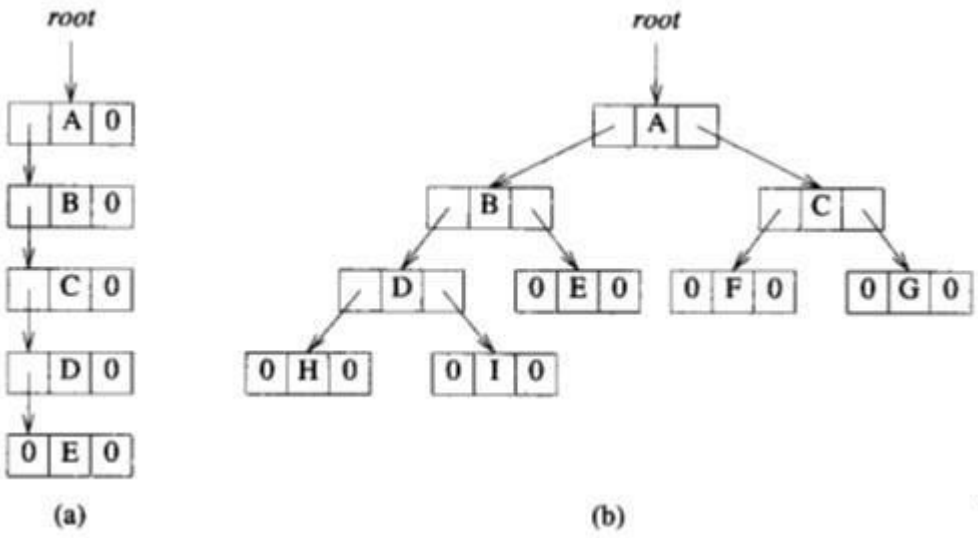


Figure 1(b) Complete binary tree



4.10 Linked representation of the binary tree

4.3 BINARY TREE TRAVERSALS

Visiting each node in a tree exactly once is called tree traversal
The different methods of traversing a binary tree are:

1. Preorder
2. Inorder
3. Postorder
4. Iterative Inorder Traversal
5. Level-Order traversal

Preorder:

Preorder is the procedure of visiting a node, traverse left and continue. When you cannot continue, move right and begin again or move back until you can move right and resume.

Recursion function:

The Preorder traversal of a binary tree can be recursively defined as

- Visit the root
- Traverse the left subtree in preorder.
- Traverse the right subtree in preorder

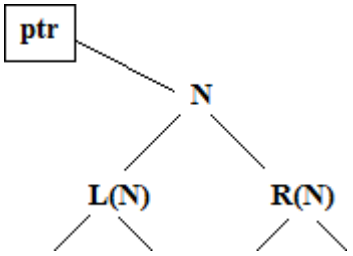
void preorder (treepointerptr)

```
{
if (ptr)
{
                printf ("%d", ptr→data)
                preorder (ptr→leftchild);
                preorder
}               (ptr→rightchild);
}
```

Inorder:

Inorder traversal calls for moving down the tree toward the left until you cannot go further. Then visit the node, move one node to the right and continue. If no move can be done, then go back one more node.

Let ptr is the pointer which contains the location of the node N currently being scanned. L(N) denotes the leftchild of node N and R(N) is the right child of node N



4.11 Linked representation of the binary tree

Recursion function:

The Inorder traversal of a binary tree can be recursively defined as

- Traverse the left subtree in Inorder.
- Visit the root.
- Traverse the right subtree in Inorder.

```
void inorder(treepointer ptr)
{
if (ptr)
{
                inorder(ptr→leftchild);
                printf("%d",ptr→data);
                inorder (ptr→rightchild);
  }
}
```

Postorder:

Postorder traversal calls for moving down the tree towards the left until you can go no further. Then move to the right node and then visit the node and continue.

Recursion function:

The Postorder traversal of a binary tree can be recursively defined as

- Traverse the left subtree in postorder.
- Traverse the right subtree in postorder.
- Visit the root

```
void   postorder (treepointer ptr)
{
if (ptr)
{
                postorder
                (ptr→leftchild);
                postorder
}               (ptr→rightchild); printf
}               ("%d", ptr→data); |
```

Iterative inorder Traversal:

Iterative inorder traversal explicitly make use of stack function.

The left nodes are pushed into stack until a null node is reached, the node is then removed from the stack and displayed, and the node's right child is stacked until a null node is reached. The traversal then continues with the left child. The traversal is complete when the stack is empty.

```
void iterInorder(treePointer node)
{
   int top = -1; /* initialize stack */
   treePointer stack[MAX_STACK_SIZE];
   for (;;) {
      for(; node; node = node→leftChild)
         push(node); /* add to stack */
      node = pop(); /* delete from stack */
      if (!node) break; /* empty stack */
      printf("%d", node→data);
      node = node→rightChild;
   }
}
```

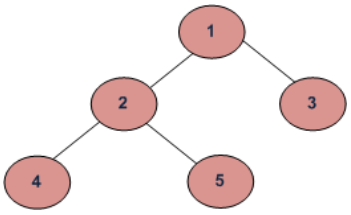
Program : Iterative inorder traversal

Level-Order traversal:

Visiting the nodes using the ordering suggested by the node numbering is called level ordering traversing.

The nodes in a tree are numbered starting with the root on level 1 and so on.

Firstly, visit the root, then the root's left child, followed by the root's right child. Thus continuing in this manner, visiting the nodes at each new level from the leftmost node to the rightmost node.



4.12 Level order traversal

Level order traversal: 1 2 3 4 5

Initially in the code for level order add the root to the queue. The function operates by deleting the node at the front of the queue, printing the nodes data field and adding the nodes left and right children to the queue.

Function for level order traversal of a binary tree:

```
void levelOrder(treePointer ptr)
{/* level order tree traversal */
   int front = rear = 0;
   treePointer queue[MAX_QUEUE_SIZE];
   if (!ptr) return; /* empty tree */
   addq(ptr);
   for (;;) {
      ptr = deleteq();
      if (ptr) {
         printf("%d",ptr→data);
         if(ptr→leftChild)
            addq(ptr→leftChild);
         if (ptr→rightChild)
            addq(ptr→rightChild);
      }
      else break;
   }
}
```

Program : Level-order traversal of a binary tree

4.4 ADDITIONAL BINARY TREE OPERATIONS

Copying a Binary tree

This operation will perform a copying of one binary tree to

another. C function to copy a binary tree:

```
treepointer copy (treepointer original)
{if(original)
{ MALLOC (temp, sizeof(*temp));
temp→leftchild=copy(original→leftchild);
temp→rightchild=copy(original→rightchild
); temp→data=original→data;
return temp;
}
return NULL;
}
```

Testing Equality

This operation will determine the equivalence of two binary tree. Equivalence binary tree have the same structure and the same information in the corresponding nodes.

C function for testing equality of a binary tree:

```
int equal (treepointer first, treepointer
second)
{
return ((! first &&! second) || (first && second && (first→ data== second →data)
&& equal (first→ leftchild, second→ leftchild) && equal (first→ rightchild,
second →rightchild))
}
```

This function will return TRUE if two trees are equivalent and FALSE if they are not.

The Satisfiability problem

- Consider the formula that is constructed by set of variables: $x_1, x_2, \ldots, x_n$ and operators $\wedge$(*and*), $\vee$(*or*), $\neg$ (*not*).
- The variables can hold only of two possible values, *true* or *false*.
- The expression can form using these variables and operators is defined by the following rules.
- A variable is an expression
- If x and y are expressions, then $\neg x$, $x \wedge y$, $x \vee y$ are expressions
- Parentheses can be used to alter the normal order of evaluation ($\neg > \wedge > \vee$)

Example: $x_1 \vee (x_2 \wedge \neg x_3)$ If x_1 and x_3 are *false* and x_2 is *true*

$= false \vee (true \wedge \neg false)$

$= false \vee true$

$= true$

The satisfiability problem for formulas of the propositional calculus asks if there is

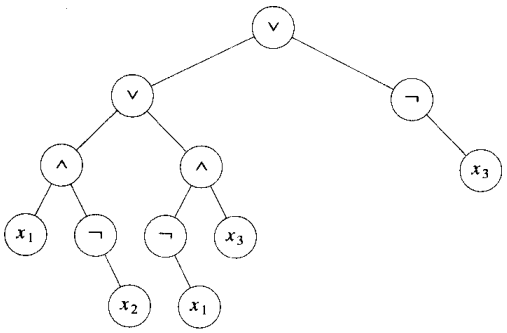


Figure : Propositional formula in a binary tree

Figure 4.13 Propositional formula in a binary tree

an assignment of values to the variable that causes the value of the expression to be *true*. Let's assume the formula in a binary tree

$(x1 \wedge \neg x2) \vee (\neg x1 \wedge x3) \vee \neg x3$

The in order traversal of this tree is

$$x1 \wedge \neg x2 \vee \neg x1 \wedge x3 \vee \neg x3$$

The algorithm to determine satisfiability is to let (x1, x2, x3) takes on all the possible combination of true and false values to check the formula for each combination.

For *n* value of an expression, there are 2^n possible combinations of *true* and *false*

For example n=3, the eight combinations are (t, t, t), (t, t, f), (t, f, t), (t,f,f), (f,t,t), (f,t,f), (f,f,t), (f,f,f).

The algorithm will take O (g 2^n), where g is the time to substitute values for $x1, x2, \ldots xn$ and evaluate the expression.

Node structure:

For the purpose of evaluation algorithm, assume each node has four fields:

left_child	*data*	*value*	*right_child*

Figure : Node structure for the satisfiability problem

Define this node structure in C as:

```
typedef enum {not,and,or,true,false} logical;
typedef struct node *tree_pointer;
typedef struct node {
        tree_pointer left_child;
        logical      data;
        short int    value;
        tree_pointer right_child;
        } ;
```

Satisfiability function: The first version of Satisfiability algorithm

```
for (all 2^n possible combinations) {
   generate the next combination;
   replace the variables by their values;
   evaluate root by traversing it in postorder;
   if (root→value) {
      printf(<combination>);
      return;
   }
}
printf("No satisfiable combination\n");
```

4.5 THREADED BINARY TREE

The limitations of binary tree are:

- In binary tree, there are n+1 null links out of 2n total links.
- Traversing a tree with binary tree is time consuming. These limitations can be overcome by threaded binary tree.

In the linked representation of any binary tree, there are more null links than actual pointers. These null links are replaced by the pointers, called **threads**, which points to other nodes in the tree.

To construct the threads, use the following rules:

1. Assume that **ptr** represents a node. If ptr →left Child is null, then replace the null link with a pointer to the inorder predecessor of ptr.
2. If ptr →right Child is null, replace the null link with a pointer to the inorder successor of ptr.

Ex: Consider the binary tree as shown in below figure 4.15:

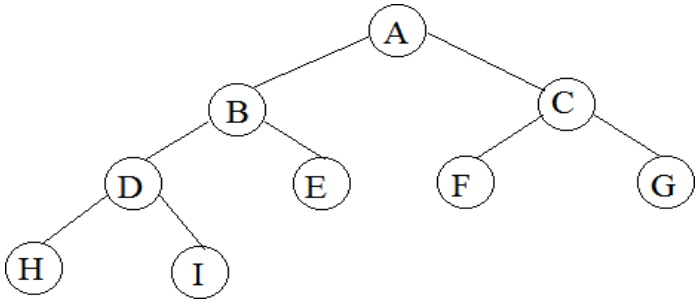


Figure 4.15: Binary Tree

There should be no loose threads in threaded binary tree. But in **Figure** two threads have been left dangling: one in the left child of *H*, the other in the right child of G.

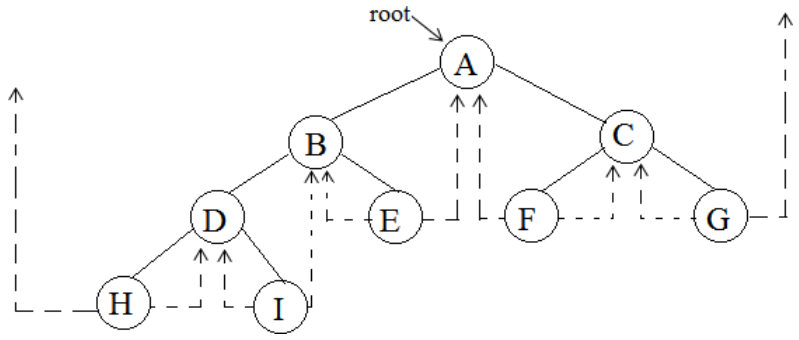


Figure 4.16: Threaded tree corresponding to Figure A

In above figure 4.16 the new threads are drawn in broken lines. This tree has 9 nodes and 10 0 -links which has been replaced by threads.

When trees are represented in memory, it should be able to distinguish between threads and pointers. This can be done by adding two additional fields to node structure, ie., left Thread and *right Thread*

- If `ptr→ leftThread = TRUE,` then ptr→ `leftChild` contains a thread, otherwise it contains a pointer to the left child.
- If `ptr→ rightThread = TRUE, then ptr→ rightChild` contains a thread, otherwise it contains a pointer to the right child.

Node Structure:

The node structure is given in C declaration

```
typedef struct threadTree
*threadPointer typedef struct {
short int leftThread; threadPointer
leftChild; char data;
threadPointer rightChild; short int
rightThread;
} threadTree;
```

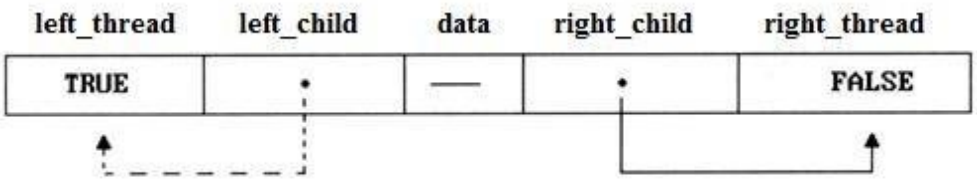


Figure An empty threaded tree

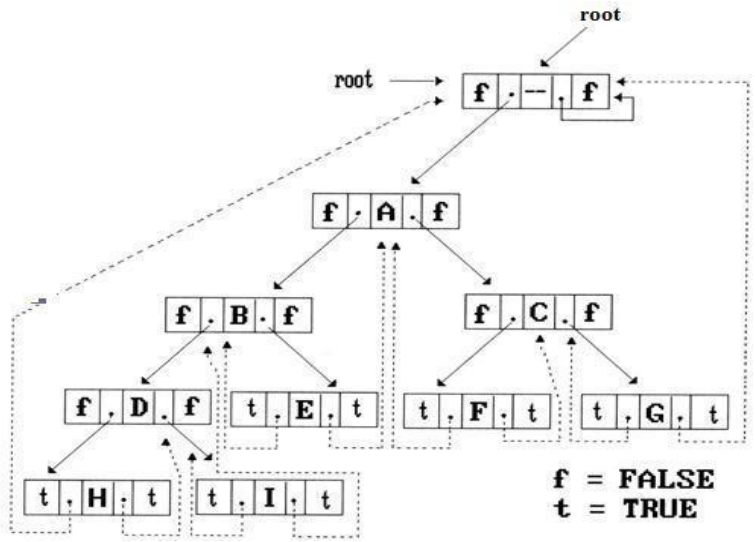


Fig 4.16: Memory representation for the tree

The complete memory representation for the tree of figure is shown in Figure 4.16

The variable ***root*** points to the header node of the tree, while root →**leftChild** points to the start of the first node of the actual tree. This is true for all threaded trees. Here the problem of the loose threads is handled by pointing to the head node called ***root***.

Inorder Traversal of a Threaded Binary Tree

- By using the threads, an inorder traversal can be performed without making use of a stack.
- For any node, **ptr**, in a threaded binary tree, if **ptr→rightThread =TRUE,** the inorder successor of **ptr** is **ptr →rightChild** by definition of the threads. Otherwise we obtain the inorder successor of **ptr** by following a path of **left-child links** from the **right-child** of **ptr** until we reach a node with **leftThread = TRUE.**
- The function in succ **()** finds the inorder successor of any node in a threadedtree without using a stack.

```
threadedpointer insucc(threadedPointer tree)
{  /* find the inorder successor of tree in a threaded binary
tree */ threadedpointer  temp;
temp = tree→rightChild;
if (!tree→rightThread)
while (!temp→leftThread)
temp = temp→leftChild;
 return temp;
```

Program: Finding inorder successor of a node

```
function void inorder (threadedpointer tree)
{
Threadedpointer temp = tree;
for(; ;){
temp = insucc(temp);
if (temp  == tree)

break;
printf("%3c",  temp→data);
}
}
```

Program: Inorder traversal of a threaded binary tree

Inserting a Node into a Threaded Binary Tree

In this case, the insertion of **r** as the right child of a node **s** is studied.

The cases for insertion are:

- If ***s*** has an **empty** right subtree, then the insertion is simple and diagrammed in Figure
- If the right subtree of s is not **empty**, then this right subtree is made the right subtree of ***r*** after insertion. When this is done, r becomes the inorder predecessor of a node that has a **leftThread == true** field, and consequently there is a thread which has to be updated to point to r. The node containing this thread was previously the inorder successor of s. (Refer Fig 4.17)

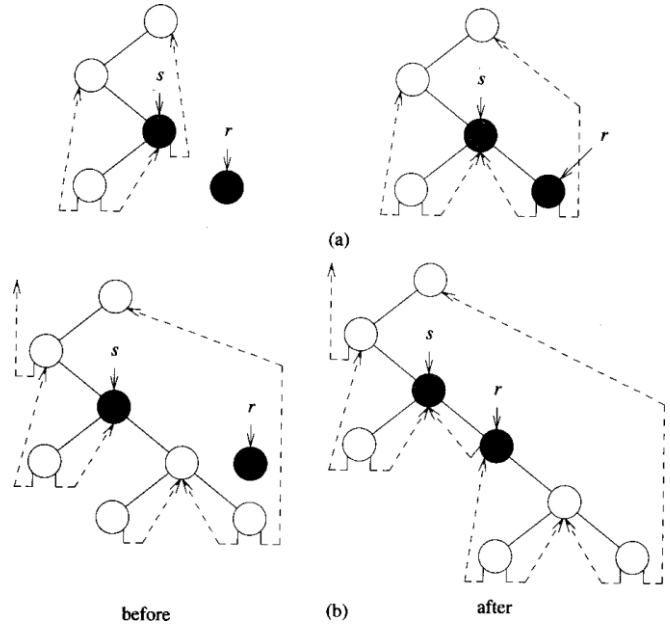


Fig 4.17 Threaded Binary Tree.

```
void insertRight(threadedPointer Sf threadedPointer r)
{ /* insert r as the right child of s */ threadedpointer
```

```
temp; r→rightChild = parent→rightChild;
r→rightThread = parent→rightThread;
r→leftChild = parent;
r→leftThread = TRUE;
s→rightChild = child;
s→rightThread = FALSE;
if (!r→rightThread)
{
temp = insucc(r);
 temp→leftChild = r;
}
}
```

4.6 Binary Search Tree (BST)

A Binary Search Tree (BST) is a tree in which all the nodes follow the below-mentioned properties –

- The left sub-tree of a node has a key less than or equal to its parent node's key.
- The right sub-tree of a node has a key greater than to its parent node's key.

Thus, BST divides all its sub-trees into two segments; the left sub-tree and the right sub-tree and can be defined as

left_subtree (keys) ≤ node (key) ≤ right_subtree (keys)

Representation

BST is a collection of nodes arranged in a way where they maintain BST properties. Each node has a key and an associated value. While searching, the desired key is compared to the keys in BST and if found, the associated value is retrieved. Following is a pictorial representation of BST –

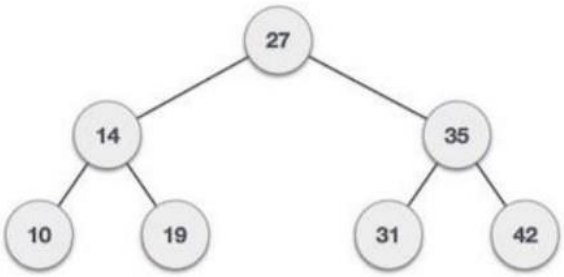


Fig 4.18 BST

We observe that the root node key (27) has all less-valued keys on the left sub-tree and the higher valued keys on the right sub-tree

Basic Operations

- Following are the basic operations of a tree –
- Search − Searches an element in a tree.

- Insert − Inserts an element in a tree.
- Pre-order Traversal − Traverses a tree in a pre-order manner.
- In-order Traversal − Traverses a tree in an in-order manner.
- Post-order Traversal − Traverses a tree in a post-order manner.

Node

Define a node having some data, references to its left and right child nodes.

```
struct node {
  int data;
  struct node *leftChild;
  struct node *rightChild;
};
```

Search Operation

Whenever an element is to be searched, start searching from the root node. Then if the data is less than the key value, search for the element in the left subtree. Otherwise, search for the element in the right subtree. Follow the same algorithm for each node.

Algorithm

```
struct node* search(int data){
  struct node *current = root;
  printf("Visiting elements: ");

  while(current->data != data){

    if(current != NULL) {
      printf("%d ",current->data);

      //go to left tree
      if(current->data > data){
        current = current->leftChild;
      } //else go to right tree
      else {
        current = current->rightChild;
      }

      //not found
```

```
      if(current == NULL){
        return NULL;
      }
    }
  }

  return current;
}
```

Insert Operation

Whenever an element is to be inserted, first locate its proper location. Start searching from the root node, then if the data is less than the key value, search for the empty location in the left subtree and insert the data. Otherwise, search for the empty location in the right subtree and insert the data.

Algorithm

```
void insert(int data) {
  struct node *tempNode = (struct node*) malloc(sizeof(struct node));
  struct node *current;
  struct node *parent;

  tempNode->data = data;
  tempNode->leftChild = NULL;
  tempNode->rightChild = NULL;

  //if tree is empty
  if(root == NULL) {
    root = tempNode;
  } else {
    current = root;
    parent = NULL;
```

```
    while(1) {
      parent = current;

      //go to left of the tree
      if(data < parent->data) {
        current = current->leftChild;
        //insert to the left

        if(current == NULL) {
          parent->leftChild = tempNode;
          return;
        }
      } //go to right of the tree
      else {
        current = current->rightChild;

        //insert to the right
        if(current == NULL) {
          parent->rightChild = tempNode;
          return;
        }
      }
    }
  }
}
```