MODULE-II

JOINT PROBABILITY DISTRIBUTION AND MARKOV CHAIN

Topic Learning Objectives:

Upon Completion of this module, student will be able to:

- Construct joint probability distributions and and measuring the expectation, Covariance and Correlation.
- Investigating a sequence of repeated trials of an experiment in which the outcome an any step in the sequence depends, at most, on the outcome of the preceding step and not on any other previous outcome will be investigated (Markov Process).
- Deal with Stochastic matrices, higher transition probabilities, regular stochastic matrices, probability vectors.

Introduction

In module-I, the focus was on probability distributions for a single random variable. For example, in module-I, the number of successes in a Binomial experiment was explored and several popular distributions for both continuous and discrete random variable were considered. In this chapter, examples of the general situation will be described where several random variables, e.g. X and Y, are observed. The joint probability mass function (discrete case) or the joint density (continuous case) are used to compute probabilities involving X and Y.

Joint probability distribution:

Consider a random experiment and let S denotes its sample space. Let X and Y be two discrete random variables defined on S. Let the image set of these be

$$X: \; x_1 \;\; x_2 \;\; x_3 \; . \; . \; . \; x_m$$

$$Y: \; y_1 \;\; y_2 \;\; y_3 \; . \; . \; . \; y_n$$

Suppose that there exists a correlation between the random variables X and Y. Then X and Y are jointly related/distributed variables Also, note that X and Y together assumes values. The same can be shown by means of a matrix or a table.

$X \backslash Y$	y_1	y_2	y_3	..	..	y_n
x_1	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	..	..	(x_1, y_n)
x_2	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	..	..	(x_2, y_n)
x_3	(x_3, y_1)	(x_3, y_2)	(x_3, y_3)	..	..	(x_3, y_n)
..	..	..	..	..	..	..
..	..	..	..	..	..	..

x_m	(x_m, y_1)	(x_m, y_2)	(x_m, y_3)	..	..	(x_m, y_n)

Joint Probability Function/Joint probability mass function:

The probability of the event $[X = x_i,\ Y = y_j]$ is called joint probability function, denoted by $f(x_i, y_j) = P[X = x_i,\ Y = y_j]$. Here, i take the values from 1 to m and j assumes the values right from 1 to n. The function $f(x_i, y_j)$ has the following properties:

1. $f(x_i, y_j) \geq 0$

2. $0 \leq f(x_i, y_j) \leq 1$
3. $\sum_i \sum_j f(x_i, y_j) = 1$

Note: One caution with discrete random variables is that probabilities of events must be calculated individually. From the preceding sections, it is clear that in the current problem, there totally $m \cdot n$ events. Thus, it is necessary to compute the probability of each and every event. This can also be shown by means of table.

X \ Y	y_1	y_2	y_3	..	..	y_n
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	$f(x_1, y_3)$	..	..	$f(x_1, y_n)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	$f(x_2, y_3)$	..	..	$f(x_2, y_n)$
x_3	$f(x_3, y_1)$	$f(x_3, y_2)$	$f(x_3, y_3)$	..	..	$f(x_3, y_n)$
..	..	..	..	..	..	..
..	..	..	..	..	..	..
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	$f(x_m, y_3)$	..	..	$f(x_m, y_n)$

Note:
1. Exceptions(means)

$$E(X) = \sum_i x_i f(x_i)$$

$$E(Y) = \sum_j y_j f(y_j)$$

$$E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j)$$

2. **Covariance:** Covariance is a measure of the linear relationship between two random variables

$\text{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$

3. Variance:

$\sigma^2{}_x = E\left(X^2\right) - \{E(X)\}^2$

$\sigma^2{}_y = E\left(Y^2\right) - \{E(Y)\}^2$

4. Correlation: A common measure of the relationship between two random variables

$\rho(x,y) = \dfrac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$.

Problems:

1. Consider the joint distribution of X and Y

X \ Y	-4	2	7
1	1/8	¼	1/8
5	2/8	1/8	1/8

Compute E(X), E(Y), E (XY), Covar (X, Y), σ_x, σ_y and $\rho(x,y)$.

Solution: First, we obtain the distribution functions of X and Y. To get the distribution of X, it is sufficient to add the probabilities row wise. Thus,

X	1	5
$f(x_i)$	1/2	½

$E(X) = 1\cdot(1/2) + 5\cdot(1/2) = 3$

$E\left(X^2\right) = 1\cdot(1/2) + 25\cdot(1/2) = 13$

$\text{Var}(X) = E\left(X^2\right) - \{E(X)\}^2 = 4$ and $\sigma_x = 2$.

Similarly,

Y	-4	2	7
$g(y_j)$	3/8	3/8	2/8

$E(Y) =$ -4×(3/8)+2×(3/8)+7×(2/8)=1.0

$E\left(Y^2\right) =$ 16×(3/8)+4×(3/8)+49×(2/8)=39.5

$\mathrm{Var}(\boldsymbol{Y}) = \boldsymbol{E}\left(\boldsymbol{Y}^2\right) - \{\boldsymbol{E}(\boldsymbol{Y})\}^2$ = 38.5

$\sigma_y = \sqrt{\mathbf{38.5}}$ = 6.2048.

Consider $\boldsymbol{E}(\boldsymbol{XY}) = \sum_i \sum_j x_i y_j \boldsymbol{f}(x_i, y_j)$

= (2)(1)(0.2) + (3)(2)(0.4) + (4)(2)(0.1) + (4)(3)(0.2) + (5)(3)(0.2) = 8.8.

With E(X) = 3, and E(Y) = 1, it follows that

$\mathrm{Cov}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{E}(\boldsymbol{XY}) - \boldsymbol{E}(\boldsymbol{X}) \cdot \boldsymbol{E}(\boldsymbol{Y})$ = 5.8 and

$\rho(\boldsymbol{x}, \boldsymbol{y}) = \dfrac{\mathrm{Cov}(\boldsymbol{X},\boldsymbol{Y})}{\sigma_{\boldsymbol{X}} \cdot \sigma_{\boldsymbol{y}}} = \dfrac{5.8}{(2.0)\times(6.2048)}$ =0.4673 .

2. A fair coin is tossed 3 times. Let X denote the random variable equals to 0 or 1 accordingly as a head or a tail occurs on the first toss, and let Y be the random variable representing the total number of heads that occurs. Find the joint distribution function of (X, Y) and marginal distribution.

Solution: S = {HHT, HHT, HTH, THH, THT, TTH, HTT, TTT}. Thus, |S| = 8. Here, X takes the values 0 or 1 accordingly as a H appears on the I toss or a Tail appears on the I toss, while Y takes the values 0, 1, 2, 3 where these numbers represent the number of heads appearing in the experiment. Observe that joint variables(X, Y) assume eight values. Thus, there are (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3) – totally 8 events. Thus, we need to find the probabilities of all these 8 events. First we shall list the respective events corresponding to X and Y.

- [X = 0] = {HHH, HHT, HTH, HTT},
- [X = 1] = {THH, TTH, THT, TTT}
- [Y = 0] = {TTT},
- [Y = 1] = {HTT, HTH, TTH],
- [Y = 2] = {HHT, HTH, THH}
- [Y = 3] = {HHH}

Therefore, [X = 0, Y = 0] = { }, a null event, hence $\boldsymbol{P}[\boldsymbol{X} = \boldsymbol{0}, \boldsymbol{Y} = \boldsymbol{0}] = \boldsymbol{f}(\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0}$. Similarly, $[\boldsymbol{X} = \boldsymbol{0}, \boldsymbol{Y} = \boldsymbol{1}]$ = {HTT}, so $\boldsymbol{P}[\boldsymbol{X} = \boldsymbol{0}, \boldsymbol{Y} = \boldsymbol{1}] = \boldsymbol{f}(\boldsymbol{0},\boldsymbol{1}) = \boldsymbol{1}/\boldsymbol{8}$. Note that $[\boldsymbol{X} = \boldsymbol{0}, \boldsymbol{Y} = \boldsymbol{2}]$ = {HHT, HTH}, thus $P[X=0,Y=2]=f(0,2)=\frac{2}{8}$. Another example, consider the event [X = 1, Y = 2] = {THH} implying that $P[X=1,Y=2]=f(1,2)=\frac{1}{8}$.

Continuing like this, one can compute the probabilities of all the events. The results are shown in the form of the following table;

X \ Y	0	1	2	3
0	$f(0,0)=0$	$f(0,1)=\frac{1}{8}$	$f(0,2)=\frac{2}{8}$	$f(0,3)=\frac{1}{8}$
1	$f(1,0)=\frac{1}{8}$	$f(1,1)=\frac{2}{8}$	$f(1,2)=\frac{1}{8}$	$f(1,3)=0$

The distribution function of X is got by adding the probabilities row wise. Thus,

x_i	:	0	1
$f(x_i)$	:	$\frac{1}{2}$	$\frac{1}{2}$

and the marginal distribution function of Y is to be obtained by the adding the probabilities column wise. Therefore,

y_i	0	1	2	3
$g(y_j)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

A discussion on Independent Random Variables

Let X and Y be two discrete random variables. One says that X and Y are independent whenever the joint distribution function is the product of the respective marginal distribution functions. Equivalently, suppose that $f(x_i)$, $g(y_j)$ and $f(x_i, y_j)$ denote respectively, the distribution function of X, Y and that of X and Y, and $f(x_i, y_j)=f(x_i)\cdot g(y_j)$, then we say that X and Y are independent random variables.

3. If X and Y are independent random variables with marginal distribution

X	0	1	2	3
$f(x_i)$	0.2	0.2	0.5	0.1

y	3	5	6
$g(y_j)$	0.2	0.5	0.3

Compute joint distribution function of X and Y.

Solution: joint distribution function of X and Y may be obtained as follows:

$f(x_{i,}y_j)=f(x_i)\cdot g(y_j)$	3	5	6
0	0.04	0.10	0.06
1	0.04	0.10	0.06
2	0.10	0.25	0.15
3	0.02	0.05	0.03

4. Consider the joint distribution of X and Y

X \ Y	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Find (i) marginal distribution of *x* and *y* and also verify X and Y are independent.

Solution: First, we obtain the marginal distribution functions of X and Y.

marginal distribution function in X

X	1	5
$f(x_i)$	0.3	0.7

marginal distribution function in Y

Y	2	3	4
$g(y_j)$	0.2	0.5	0.3

From table we note that

0.06=0.3x0.2 0.15=0.3X0.5 0.09=0.3X0.3	0.14=0.7x0.2 0.35=0.7X0.5 0.21=0.7X0.3

We observe that $f(x_{i,}y_j)=f(x_i)\cdot g(y_j)$

Therefore, X and Y are independent

5. the joint distribution of two random variables are as follows

X \ Y	0	1	2	3
0	0	$1/8$	$2/8$	$1/8$
1	$1/8$	$2/8$	$1/8$	0

Compute marginal distribution of X and Y.

Solution: Marginal distribution function in X

X	0	1
$f(x_i)$	0.5	0.5

marginal distribution function in Y

Y	0	1	2	3
$g(y_j)$	$1/8$	$3/8$	$3/8$	$1/8$

6. The joint probability distribution of two random variables x and y is given by $f(x,y) = k(2x+y)$ where x and y are integer such that $0 \le x \le 2$, $0 \le y \le 3$

Find (i) find the value of k (ii) marginal distribution of x and y

Solution:

Let $x = \{0,1,2\}$ *and* $y = \{0,1,2,3\}$

joint probability distribution table formed as follows

x \ y	0	1	2	3	sum
0	0	k	2k	3k	6k
1	2k	3k	4k	5k	14k
2	4k	5k	6k	7k	22k
sum	6k	9k	12k	15k	42k

(i) We must have 42k =1

K=1/42

(ii) Marginal distribution function in X

X	0	1	2
$f(x_i)$	$1/7$	$1/3$	$11/21$

marginal distribution function in Y

Y	0	1	2	3
$g(y_j)$	$1/7$	$3/14$	$2/7$	$5/14$

Exercises:

1. The joint probability distribution table for two random variables X and Y is as follows

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal probability distributions of X and Y. Also compute (a) Expectations of X and Y (b) Standard deviation of X and Y (c) Covariance of X and Y (d) Correlation of X and Y.

2. The joint probability distribution of two random variables *x* and *y* is given below

x \ *y*	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Find (i) marginal distribution of *x* and *y* (ii) Cov(*xy*).

3. The joint probability distribution of two random variables *x* and *y* is given below

x \ *y*	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Find (i) marginal distribution of *x* and *y* (ii) Expectation of x and y

4. The joint probability distribution of two discrete random variables x and y is given by the table. Determine the marginal distributions of x and y. Also find whether x and y are independent

x \ *y*	1	3	6
1	1/9	1/6	1/18
3	1/6	1/4	1/12
6	1/18	1/12	1/36

5. The joint distribution of two random variables X and Y is as follows

X \ Y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find (i) marginal distribution of *x* and *y* (ii) Covariance of x and y

Introduction to Stochastic Process

Random variable: Let S be the sample space of a random experiment and 'R' be the set of all real numbers. A random variable 'X' is a function 'f' from S to R i.e. X = f(s), $s \in S$.

Stochastic Process: It is a family of random variables, $\{X(t)/t \in T\}$, defined on a given probability space, indexed by the parameter t, where t varies over an index set $T \subset R$. Usually 't' is considered as the parameter such as time.

If $T = \{t_1, t_2, t_3, ...\}$, then one can have a family of random variables like $X(t_1), X(t_2), X(t_3).........$. for a particular value of parameter t, the values taken by X(t) are called **states.** The set of all possible states is referred to as State space denoted by I.

Thus one can define a Stochastic process as a function X(t, s) where $t \in T$ and $s \in I$

Here we can have many options:

- One can fix t, and vary s.
- 't' may be varied, but 's' could be fixed.
- Both can be fixed to some specific values.
- Both t & s may be allowed to vary.

When both s and t are varied, we generate a family of random variables constituting a Stochastic Process.

Type of stochastic Process:-

1. If the state space I and index set T of a Stochastic Process is both discrete, then it is called discrete – state – discrete parameter (time) process (Markov Process). Here I and T takes the values 0, 1, 2, 3, ……..
2. If State space is continuous, and index set T is discrete, then we have a continuous – state – discrete parameter process.
3. Similarly, we have discrete – state – continuous parameter process and
4. Continuous - state – continuous parameter process.

Thus there are four Stochastic processes. Among the various process, Markov process is more useful. Eg: No. of jobs coming to a computer centre & it is done.

Markov Process (Memory less process):-

It is a discrete – state – discrete – parameter Stochastic process (I and T are discrete)

A Stochastic process $\{X(t)/t \in T\}$ is called a Markov process if for any $t_0 < t_1 < t_2 < ...... < t_n < t$, the conditional distribution of X(t) for given values, $X(t_0)$, $X(t_1)$, $X(t_2)$, …….. , $X(t_n)$ depends only on $X(t_n)$. i.e. Probability of occurrence of an event in future is completely dependent on the chance of occurrence of the event in the present state but not on the previous or past records.

Eg: Cricketers playing cricket.

Rahul dravid, Sachin, Kumble all have good record. But this will not help them to win a match. Winning depends on how they play today.

Mathematically this can be explained as:

$$P[X(t) \le X / X(t_n) = X_n, X(t_{n-1}) = X_{n-1}, \cdots\cdots\cdots, X(t_0) = X_0] = P\{X(t) \le X / X(t_n) = X_n]$$

In a Markov chain (a Markov process where state space I takes discrete value) , the past history is completely summarized in the current state, and future is independent of its past but depends only on present state.

Basic concepts pertaining to Markov chains:

Probability Vector:- A vector $\vec{V} = (V_1, V_2, ..., V_n)$ is called a probability vector if all the components are non – negative and their sum is one.

i.e. $V_i \ge 0$ & $V_1 + V_2 + V_3 + \cdots\cdots + V_n = 1$

eg: $u = (0, 1)$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, $w = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ are all probability vectors.

Stochastic Matrix: A square matrix $P = [p_{ij}]$ is called a stochastic matrix if each row of 'P' is a probability vector.

Example: (i) Identity matrix of any order $I_{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

Note: If A & B are Stochastic Matrices, then their product AB is also a Stochastic matrix. In fact, any power of A is a stochastic matrix.

Regular Stochastic Matrix: A Stochastic matrix P is said to be a regular Stochastic matrix if all the entries in some power of P (i.e. P^n) are positive. (>0)

Note: 'n' is the order of regular Stochastic matrix.

Example: $A=\begin{bmatrix}0 & 1\\ 1/2 & 1/2\end{bmatrix}$, then $A^2=\begin{bmatrix}0 & 1\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1/2 & 1/2\\ 1/4 & 3/4\end{bmatrix}$

All the entries in A^2 are positive (>0). Therefore 'A' is a regular Stochastic matrix of order 2(n =2).

<u>**Properties of a regular Stochastic Matrix**:</u> Let 'P' be a regular Stochastic Matrix of order n. then

1. 'P' has a unique fixed probability vector v = (v_1, v_2, ……., v_n) such that v.P = v
2. The sequence P, P^2, P^3, ………. Of powers of P approaches the matrix V whose rows are each the fixed probability vector 'v'
3. If 'q' is any vector, then the sequence of vectors q, qP, qP^2, qP^3, …… approaches the unique fixed probability vector v.

Problems:

1. Determine which of the following are Stochastic matrix?

(a) $\begin{pmatrix}1/3 & 1/3 & 1/3\\ 1/2 & 0 & 1/2\end{pmatrix}$ $\rightarrow$ This is not a square matrix. Hence it is not a stochastic matrix.

(b) $\begin{pmatrix}3/4 & 1/4\\ 2/3 & 2/3\end{pmatrix}$ $\rightarrow$ It is a square matrix & non-negative. But 2^{nd} row sum $\neq 1$. $\therefore$ It is not a Stochastic matrix.

(c) $\begin{pmatrix}3/4 & -1/4\\ 2/3 & 1/3\end{pmatrix}$ $\rightarrow$ This is a square matrix, but contains negative value. $\therefore$ its is not a Stochastic Matrix.

(d) $A=\begin{pmatrix}2/5 & 1/5 & 2/5\\ 1/2 & 0 & 1/2\\ 1/3 & 1/3 & 1/3\end{pmatrix}$, A is a Stochastic Matrix.

2. Which of the following are regular Stochastic Matrix?

(i) $B=\begin{pmatrix}1/2 & 1/4 & 1/4\\ 0 & 1 & 0\\ 1/2 & 0 & 1/2\end{pmatrix}$ (ii) $D=\begin{pmatrix}0 & 0 & 1\\ 1/2 & 1/4 & 1/4\\ 0 & 1 & 0\end{pmatrix}$

Solution: i) $B^2=\begin{pmatrix}1/2 & 1/4 & 1/4\\ 0 & 1 & 0\\ 1/2 & 0 & 1/2\end{pmatrix}\begin{pmatrix}1/2 & 1/4 & 1/4\\ 0 & 1 & 0\\ 1/2 & 0 & 1/2\end{pmatrix}=\begin{pmatrix}3/8 & 3/8 & 1/4\\ 0 & 1 & 0\\ 1/2 & 1/8 & 3/8\end{pmatrix}$

$B^3=\begin{pmatrix}3/8 & 3/8 & 1/4\\ 0 & 1 & 0\\ 1/2 & 1/8 & 3/8\end{pmatrix}\begin{pmatrix}1/2 & 1/4 & 1/4\\ 0 & 1 & 0\\ 1/2 & 0 & 1/2\end{pmatrix}=\begin{pmatrix}5/16 & 15/32 & 7/32\\ 0 & 1 & 0\\ 7/16 & 1/4 & 5/16\end{pmatrix}$ & so on.

From above we observe that 2nd row always contains 0. Hence it is not regular Stochastic matrix.

(ii) $D^2=\begin{pmatrix}0 & 0 & 1\\ 1/2 & 1/4 & 1/4\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1/2 & 1/4 & 1/4\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 1/8 & 5/16 & 9/16\\ 1/2 & 1/4 & 1/4\end{pmatrix}$

$D^3=\begin{pmatrix}0 & 1 & 0\\ 1/8 & 5/16 & 9/16\\ 1/2 & 1/4 & 1/4\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1/2 & 1/4 & 1/4\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}1/2 & 1/4 & 1/4\\ 5/32 & 41/64 & 13/64\\ 1/8 & 5/16 & 9/16\end{pmatrix}$

In D^3 all the entries are positive (>0). Hence D is regular stochastic Matrix.

3. Show that $P=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1/2 & 1/2 & 0\end{bmatrix}$ is a regular Stochastic Matrix.

Answer: $P^5=\begin{bmatrix}1/4 & 1/4 & 1/2\\ 1/4 & 1/2 & 1/4\\ 1/8 & 3/8 & 1/2\end{bmatrix}$ In P^5 all the entries are positive. Hence P is a regular Stochastic Matrix.

4. Find the unique fixed probability vector of the following Stochastic matrices.

Note: If A is a Stochastic matrix, one can find a unique fixed probability vector such that $v=vA$

(i) $A=\begin{pmatrix}1/3 & 2/3\\ 1 & 0\end{pmatrix}$

Solution: We have to find a unique fixed probability vector, $v=[x, y]$, where $x+y=1 \Rightarrow y=1-x$

Such that $v = vA$, where $v = [x, 1-x]$

Consider $v = vA$

i.e. $[x, 1-x] = [x, 1-x]\begin{pmatrix} 1/3 & 2/3 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow [x, 1-x] = \left[\frac{x}{3} + 1 - x, \frac{2x}{3}\right] \Rightarrow x = \frac{x}{3} + 1 - x$ and $1 - x = \frac{2x}{3}$

$\Rightarrow 3x = x + 3 - 3x \Rightarrow 5x = 3 \Rightarrow x = \frac{3}{5}$ and $y = 1 - x = 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore$ unique fixed probability vector is $v = \left(\frac{3}{5}, \frac{2}{5}\right)$

(ii) $A = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Solution: Since the given matrix A of order 3, the required fixed probability vector ‘v’ must also be of order 3.

Let $v = (x, y, z)$,where $x + y + z = 1 \Rightarrow z = 1 - x - y$, Such that $v = vA$, where $v = (x, y, 1-x-y)$

Consider $v = vA$

$(x, y, 1-x-y) = (x, y, 1-x-y)\begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow (x, y, 1-x-y) = \left(\frac{y}{2}, \frac{3x}{4} + \frac{y}{2} + 1 - x - y, \frac{x}{4}\right)$

$\Rightarrow x = \frac{y}{2} \Rightarrow y = 2x$ ……… (1)

$y = \frac{3x}{4} + \frac{y}{2} + 1 - x - y \Rightarrow 4y = 3x + 2y + 4 - 4x - 4y \Rightarrow x + 6y = 4$ ……… (2)

Solving (1) and (2) $x + 6(2x) = 4 \Rightarrow 13x = 4 \Rightarrow x = 4/13$

$y = 2x = 8/13$ & $z = 1 - x - y = 1 - 4/13 - 8/13 = \frac{13-4-8}{13} = 1/13$

Hence $v = \left(\frac{4}{13}, \frac{8}{13}, \frac{1}{13}\right)$

5. Find the unique fixed probability vector for the regular stochastic matrix.

(i) $A=\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$ Answer: $v=\left(\frac{1}{10},\frac{6}{10},\frac{3}{10}\right)$

(ii) $A=\begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$ Answer: $\left(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right)$

Solution: Since the given matrix A of order 4, the required fixed probability vector 'v' must also be of order 4.

$$A=\begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

We have to find fixed probability vector v = {a, b, c, d} such that vP = v & a + b + c + d = 1

i.e. $[a, b, c, d]\begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}=[a, b, c, d]$

i.e. $[a, b, c, d]\frac{1}{4}\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}=[a, b, c, d]$

i.e. $\frac{1}{4}[2b+2c+2d, 2a+2c+2d, a+b, a+b]=[a, b, c, d]$

i.e. 2b + 2c + 2d = 4a, 2a + 2c + 2d = 4b, a + b = 4c & a + b = 4d, also a + b + c + d = 1

we get b + c + d = 2a ---- (1)

a + c + d = 2b ------ (2)

a + b = 4c --------- (3)

a + b = 4d --------- (4)

Now a + b +c + d = 1 $\Rightarrow$ b + c + d = 1-a & a + c + d = 1 – b

Hence from (1) & (2) we get 1- a = 2a $\Rightarrow$ a = 1/3 & 1-b = 2b $\Rightarrow$ b = 1/3

From (3) (4) we get 2/3 = 4c $\Rightarrow$ c=1/6 & 2/3 = 4d $\Rightarrow$ d = 1/6

Thus v = (1/3, 1/3, 1/6, 1/6) is the required unique fixed probability vector.

Transition Matrix of a Markov Chain:

Consider a Markov chain, a finite stochastic process consisting of a finite sequence of trials whose outcomes, say $X_1, \quad X_2, \quad X_3, \cdots\cdots\cdots$ satisfy the following properties.

i) Each outcome belongs to the state space $I = \{a_1, a_2, a_3, \cdots\cdots\cdots, a_m\}$

ii) The outcome of any trial depends at most on the outcome of the previous trial and not on any other previous outcomes.

If the outcome in n^{th} trial is a_i, then we say that the system is in state a_i at the n^{th} stage. Thus with each pair of states (a_i, a_j), we associate a probability value p_{ij}, which indicate probability of system reaching the state a_j from the state a_i in one step. These probabilities form a matrix called transition probability matrix or Transition Matrix. This transition probability matrix (t.p.m.) is a square matrix of order m, denoted by P

$$i.e.\ \ P = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & \cdots\cdots & p_{1m} \\ p_{21} & p_{22} & \cdots\cdots & p_{2m} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ p_{m1} & p_{m2} & \cdots\cdots & p_{mm} \end{bmatrix}$$

i^{th} row represents the probability that the system will change from a_i to $a_1, a_2, \cdots\cdots, a_n$ in a single step. $p_{ij} = P[X_n = j / X_{n-1} = i]$

The element of $'P'$ have the following properties (i) $0 \le p_{ij} \le 1$ (ii) $\sum_{j=1}^{m} p_{ij} = 1$

Hence transition matrix of a Markov chain is a stochastic matrix.

Examples for writing t.p.m. of a Markov chain:

(1) Three boys A, B, C are throwing a ball to each other. 'A' always throws the ball to B, and B always throws the ball to C. But 'C' is just as likely to throw the ball to B as to A.

Solution: State space = {A, B, C} and the t.p.m. 'P' is as follows:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

(2) A man either takes a bus or drives his car to work each day. Suppose he never takes the bus 2 days in a row; but if he drives to work, then the next day he is just as likely to drive again as he is to take the bus.

Solution: The state space is {Bus (B), Car (C)}

This stochastic process is a Markov chain since the outcome on any day depends only on the happening of the previous day.

The t.p.m is

$$P = \begin{array}{c} \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} B & C \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

The first row of the matrix P correspond to the fact that the man never takes the bus 2 days in a row and so he definitely will drives. The second row of P corresponds to the fact that the day after he drives the car, he will drive or take the bus with equal probability.

Higher Transition Probability:- (n – step transition probabilities)

- The entry p_{ij} in the transition probability matrix 'P' of the Markov chain is the probability that the system changes from state a_i to a_j in a single step. i.e. $a_i \to a_j \Rightarrow p_{ij} = p_{ij}^{(1)}$
- The probability that the system changes from the state a_i to the state a_j in exactly n – steps is denoted by $p_{ij}^{(n)}$
- The matrix formed by the probability $p_{ij}^{(n)}$ is called n – step transition matrix denoted by $P^{(n)}$, which is a stochastic matrix.

Evaluation of n – step Transition Probability Matrix:

It can be proved that the n – step transition matrix is equal to thy n^th power of P. i.e. $P^{(n)} = P^n$

Suppose that a system at time t = 0 is in the state $a = (a_1, a_2, a_3, \cdots\cdots\cdots, a_m)$, where the process begins, then corresponding probability vector $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, a_3^{(0)}, \cdots\cdots\cdots, a_m^{(0)})$, denotes the initial probability distribution.

Similarly the n^th step probability distribution at the end of n steps is given by $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \cdots\cdots\cdots, a_m^{(n)})$, Thus we have

$a^{(1)} = a^{(0)}.P$

$a^{(2)} = a^{(1)}.P = a^{(0)}.P.P = a^{(0)}P^2$

$a^{(3)} = a^{(2)}.P = a^{(0)}.P^2.P = a^{(0)}P^3$

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$a^{(n)} = a^{(0)}.P^n$

Thus the probability distribution of a Markov chain is completely determined from the one step transition probability matrix 'P' & the initial probability distribution $a^{(0)}$. i.e. $a^{(n)} = a^{(0)}.P^n$

Problems:

1. Three boys A, B, C are throwing a ball to each other, A always throws the ball to B and B always throws the ball to C. But C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball, find the probability that A has the ball for the fifth throw.

Solution: State space = {A, B, C} and the t.p.m. is as follows:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Then $P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$$P^4 = P^2 \times P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P^4 \times P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

Suppose that C was the person having the ball first then $a^{(0)} = (0, 0, 1)$

Consider $a^{(5)} = a^{(0)}P^5 = [0, 0, 1]\begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix} = [1/8 \quad 3/8 \quad 1/2] = \left[p_A^{(5)} \quad p_B^{(5)} \quad p_C^{(5)}\right]$

This implies that after 5 throws the probability that the ball is with A is $1/8$, the ball with B is $3/8$, the ball with C is $1/2$.

$\therefore$ The probability that A has the ball for the fifth throw is $= 1/8$

<u>Stationary Distribution of regular Markov Chain</u>:

A Markov chain is ***regular*** if the associated t.p.m. P is regular.

If P is a regular Stochastic Matrix of the Markov chain, then the sequence of n step transition matrices $P^2, P^3, \cdots\cdots\cdots, P^n$ approaches the matrix V, whose rows are each the unique fixed probability vector v of P.

We have $a^{(n)} = a^{(0)} P^n$, where $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \cdots\cdots\cdots, a_m^{(n)})$,

As $n \to \infty$, $a_i^{(n)} = v_i$ where i = 1, 2, 3, …….,m

This is called the Stationary distribution of the Markov chain and $v = (v_1, v_2, v_3, \cdots\cdots, v_m)$ is called the stationary (fixed) probability vector of the Markov chain.

Note: A Markov chain is said to be irreducible if every state can be reached from every other state in a finite number of steps.

i.e. $p_{ij}^{(n)} > 0$ for some $n \geq 1$.

This is equivalent to saying that a Markov chain is irreducible if the associated transition probability matrix is regular.

Absorbing State of a Markov Chain: (Absorbing means remains in the same state)

In a Markov chain the process reaches to a certain state after which it continues to remain in the same state. Such a state is called an absorbing state of the Markov chain. In an absorbing state the transition probabilities p_{ij} are such that $p_{ij} = 1$ for $i = j$ & $p_{ij=0}$ otherwise.

Thus a state a_i of the Markov chain is absorbing if the i[th] row of the t.p.m. has 1 on the principle diagram & zeros elsewhere.

Example: 1. $P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \end{array}$ The state a_2 is absorbing.

2. $P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$ The states a_3 & a_4 are absorbing

Transient state: A state 'i' is said to be a transient state if the system is in this state at some step and there is a chance that it will not return to that state in a subsequent step.

Recurrent state: A state 'i' is said to be recurrent state if starting from state i , the system does eventually return to the same state. For example, consider the two state chain for which the transition matrix is

$$P=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ P^2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ P^3=\begin{bmatrix}0&1\\1&0\end{bmatrix}=P$$

Since $P^3=P$, the system returns to a state after two steps. Therefore both states of the chain are recurrent.

Problems:

1. Prove that the Markov chain with transition matrix $P=\begin{bmatrix}0&2/3&1/3\\1/2&0&1/2\\1/2&1/2&0\end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector.

Solution: Now we shall find the fixed probability vector of P. We shall find B = (x, y, z) such that BP = B when x + y + z = 1

i.e. [x, y, z] $\begin{bmatrix}0&2/3&1/3\\1/2&0&1/2\\1/2&1/2&0\end{bmatrix}=[x,y,z]$ i.e. . [x, y, z] . $\frac{1}{6}\begin{bmatrix}0&4&2\\3&0&3\\3&3&0\end{bmatrix}=[x,y,z]$

i.e. $\frac{1}{6}[3y+3z,\ 4x+3z,\ 2x+3y]=[x,y,z]$

i.e. 6x - 3y - 3z = 0

4x - 6y +3z = 0

2x + 3y – 6z = 0 & x + y + z = 1

Solving we get $x=1/3,\ \ y=10/27\ \&\ z=8/27$

$\therefore$ the required probability vector is $(1/3, 10/27, 8/27)$

Note that the given matrix is a stochastic matrix, as each element is ≥ 0 and sum of each row is 1.

Now by matrix multiplication,

$$P^2=P.P=\begin{bmatrix}0&2/3&1/3\\1/2&0&1/2\\1/2&1/2&0\end{bmatrix}\begin{bmatrix}0&2/3&1/3\\1/2&0&1/2\\1/2&1/2&0\end{bmatrix}=\begin{bmatrix}1/2&1/6&1/3\\1/4&7/12&1/6\\1/4&1/3&5/12\end{bmatrix}$$

The entries in this matrix are $p_{ij}^{(2)}$ which are all positive.

Thus there is a $k \geq 1$, namely k = 2 for which $p_{ij}^{(k)} > 0$ for all i & j. Therefore the given transition matrix P is regular. Hence the given Markov chain is irreducible.

2. A students study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night he is 60% sure not to study the next night. In the long run how often does he study?

Solution: The state of the system is {A, B}, where A: studying, B: not studying.

The associated transition matrix P is as follows;

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

In order to find the happening in the long run we have to find the unique fixed probability vector V of P. i.e we have to find V = (x, y) such that VP = P and x + y = 1

Consider $[x, y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.4 \end{bmatrix} = [x, y] \Rightarrow 0.3x + 0.4y = x, \quad 0.7x + 0.4y = y$

Solving these equation with x + y = 1 we get $x = 4/11$ & $y = 7/11$

Therefore $V = \left(\frac{4}{11}, \frac{7}{11}\right) = (P_A, P_B) \Rightarrow$ in the long run the student will study 4/11 of the time or 36.36% of the time.

3. A habitual gambler is a member of two clubs A and B. He visits either of the clubs everyday for playing cards. He never visits club A on two consecutive days. But , if he visits club B on a particular day, then the next day he is as likely to visit club B or club A. Find the transition matrix of this Markov chain. Also

(a) Show that the matrix is a regular stochastic matrix and find the unique fixed probability vector.

(b) If the person had visited club on Monday, find the probability that he visits club A on Thursday.

Solution: The transition matrix P of the Markov chain is

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

(a) Now consider $P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

Since all the entries of P^2 are positive P is a regular stochastic matrix.

To find the unique fixed probability vector. i.e. to find v = (x, y) such that vP = v where x + y =1

i.e. $[x, y]\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$

i.e. $[x, y].\frac{1}{2}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = [x, y] \Rightarrow \frac{1}{2}[y, 2x + y] = [x, y]$

i.e. $y = 2x, \quad 2x - y = 0$, on solving we get $x = 1/3$ & $y = 2/3$

Thus $v = (1/3, 2/3)$

(b) Let us suppose Monday as day 1, then Thursday will be 3 days after Monday. Given that the person had visited club B on Monday the probability that he visits club A after 3 days is equivalent to finding $p_{21}^{(3)}$ from P^3

Now $P^3 = P^2.P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$

$\therefore p_{21}^{(3)} = 3/8.$ Thus the required probability is $3/8$

4. Two boys B_1, B_2 and two girls G_1, G_2 are throwing ball from one to the other. Each boy throws the ball to the other boy with probability ½ and to each girl with probability ¼. On the otherhand each girl throws the ball to each boy with probability ½ and never to the other girl. In the long run how often does each receive the ball.

Solution: State space = $\{B_1, B_2, C_1, C_2\}$and the associated t. p. m. P is

$$P = \begin{array}{c|cccc} & B_1 & B_2 & C_1 & C_2 \\ \hline B_1 & 0 & 1/2 & 1/4 & 1/4 \\ B_2 & 1/2 & 0 & 1/4 & 1/4 \\ C_1 & 1/2 & 1/2 & 0 & 0 \\ C_2 & 1/2 & 1/2 & 0 & 0 \end{array}$$

We have to find fixed probability vector v = {a, b, c, d} such that vP = v & a + b + c + d = 1

i.e. $[a, b, c, d]\begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a, b, c, d]$

$$\text{i.e. } [a, b, c, d]\,\frac{1}{4}\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix} = [a, b, c, d]$$

$$\text{i.e. } \frac{1}{4}[2b+2c+2d,\ 2a+2c+2d,\ a+b,\ a+b] = [a, b, c, d]$$

i.e. 2b + 2c + 2d = 4a, 2a + 2c + 2d = 4b, a + b = 4c & a + b = 4d, also a + b + c + d = 1

we get b + c + d = 2a ---- (1)

a + c + d = 2b ------ (2)

a + b = 4c --------- (3)

a + b = 4d --------- (4)

Now a + b +c + d = 1 ⇒ b + c + d = 1-a & a + c + d = 1 – b

Hence from (1) & (2) we get 1- a = 2a ⇒ a = 1/3 & 1-b = 2b ⇒ b = 1/3

From (3) (4) we get 2/3 = 4c ⇒ c=1/6 & 2/3 = 4d ⇒ d = 1/6

Thus v = (1/3, 1/3, 1/6, 1/6) is the required unique fixed probability vector.

Thus, in the long run each boy receives the ball 1/3 of the time and each girl 1/6 of the time.

5. Each year a man trades his car for a new car in 3 bands of the popular company Maruti Udyog limited. If he has a 'Standard' he trades it for 'Zen'. If he has a 'Zen' he trades it for a 'Esteem'. If he has a 'Esteem' he is just as likely to trade it for a new Esteem or for a Zen or a Standard one. In 1996 he brought his first car which was Esteem.

(i) Find the probability that he has a) 1998 Esteem b) 1998 Standard c) 1999 Zen d)1999 Esteem

(ii) In the long run, how often will he have a Esteem?

Solution: State space = {A, B, C} where A: Standard B: Zen C: Esteem

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

(i) With 1996 as the first year, 1998 is to be regarded as 2 years after and 1999 as 3 years after

$$\text{Then } P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$P^3 = P^2 \times P = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 4/27 & 4/27 \end{bmatrix}$$

a) 1998 Esteem=$\frac{4}{9}$ b) 1998 Standard=$\frac{1}{9}$ c) 1999 Zen =$\frac{7}{27}$ d)1999 Esteem=$\frac{16}{27}$

$$[x, y, z]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 0 \end{bmatrix} = [x, y, z] \qquad \text{i.e. . } [x, y, z] \cdot \frac{1}{3}\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} = [x, y, z]$$

i.e. $[z,\ 3x + z,\ 3y + z] = [3x,\ 3y, 3z]$

Solving we get $x = 1/6, y = 1/3 \& z = 1/2$

$\therefore$ the required probability vector is $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$.

In the long run, Probability of having Esteem is $p^c = \frac{1}{2}$

Thus in the long run in 50% of the time he will have Esteem.

Exercise:

1. A student's study habits are as follows: If he studies one night, he is 60% sure not to study the next night. On the other hand if he does not study the night, he is 80% since to study the next. In the long run how often does he study?

2. Find the fixed probability vector for the following regular stochastic matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

3. A software engineer goes to his office everyday by motorbike or by car. He never goes by bike on two consecutive days, but if he goes by car on a day then he is equally likely to go by car or by bike the next day. Find the t. p. m. of the Markov chain. If car is used on the first day of the week find the probability that after 4 days (a) bike is used (b) car is used.

4. Assume that a computer system is in one of the three states; busy, idle or undergoing repair denoted by states 0, 1, 2. Observing its state at a certain specified time on each day, it is found that the system approximately behaves like a Markov chain with the transition matrix

$$\begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix}$$

Prove that the chain is irreducible and determine the steady – state probability.

VIDEO LINKS:

https://www.youtube.com/watch?v=eYthpvmqcf0
https://www.youtube.com/watch?v=o-jdJxXL_W4
https://www.youtube.com/watch?v=afIhgiHVnj0