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GENERAL THEORY OF THE ABERRATION IN THE FOCAL PLANE OF A PARABOLIC REFLECTOR,

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In the present article, which is to be regarded as a continuation of my paper in *A.J.* 413, I have derived the general rigorous expressions for the focal plane coördinates of *any* ray of starlight reflected from any point of a parabolic surface.

The method of derivation of the various formulas can be more readily followed with the aid of the accompanying diagram.

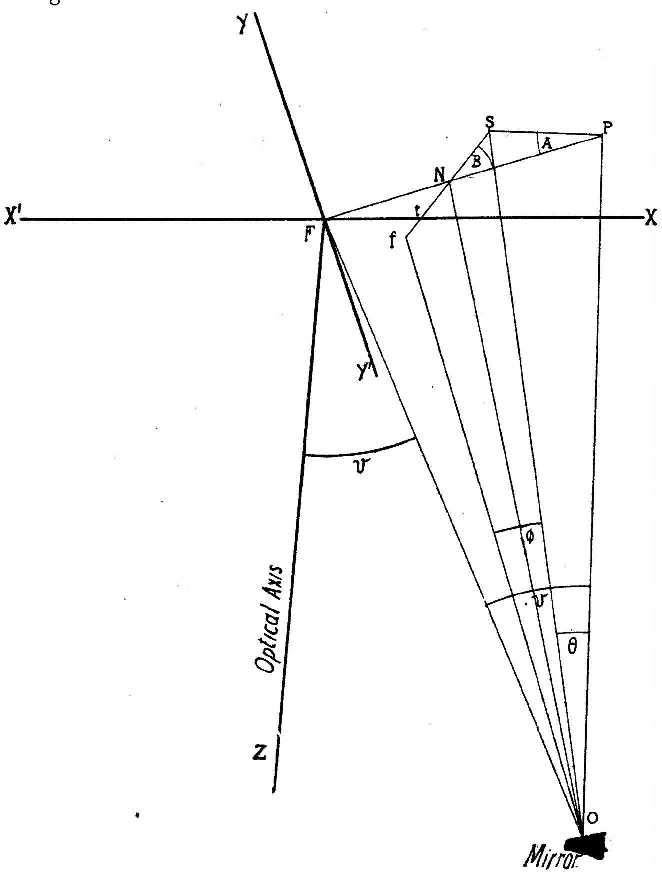

FIGURE I.

Let O be any point of the mirror's surface at which the reflection takes place.

" $OF = \rho$ denote the distance of this point from the principal focus.

" $FOP = OPZ = v$ denote the angle, at the focus, between the optical axis and a line drawn to O.

" $SOP = \theta$ denote the angle which the direct ray makes with the line OP drawn parallel to the optical axis.

Let the focal point be the origin of coordinates, the plane of XY coinciding with the focal plane.

For brevity let the line of intersection of any given plane with the focal plane be called the *trace* of the given plane.

Let the trace (XFX') of the plane containing the optical axis and the direct rays of light parallel to SO be taken as the axis of X, and let the x-coordinate of the image be positive.

Let $PFX = SPF = A$ be the angle which the trace of the plane containing the optical axis and the point O makes with the axis of X: A having any value between 0° and 360°.

Let $SNP = B$ be the angle which the trace of the plane of incidence and reflection makes with the trace PF; then $StX = A + B$. These two planes (traces PF and Sf) intersect in a line NO which is normal to the reflecting surface at the point O, and consequently bisects the angle $POF = v$; and also the angle $SOf = \phi$ included between the direct ray SO and the reflected ray Of.

For a given value of v this normal (NO) makes the fixed angle $ONP = 90° - \frac{1}{2}v$ with the trace PF, and a variable angle $ONS = I$ (which is a function of A, v and θ) with the trace Sf.

It is required to find the rigorous general expression for the focal plane coordinates of the point f, having given A, v, θ and ρ $(= F \sec^2 \frac{1}{2} v)$.

Referring now to Fig. 1, we have

(1) $PO = \rho \cos v$

(2) $SO = \rho \cos v \sec \theta$

(3) $PS = \rho \cos v \tan \theta$

(4) $ON = \rho \cos v \sec \frac{1}{2} v$

(5) $NF = \rho \cos v \, (\tan v - \tan \frac{1}{2} v) = \rho \tan \frac{1}{2} v$

In the plane triangle SPN two sides and the included angle A are given, hence

(6) $SN = \rho \cos v \sqrt{\tan^2 \frac{1}{2} v + \tan^2 \theta - 2 \tan v \tan \theta \cos A}$

We next find the angle B from

(7)
$$\sin B = \frac{PS. \sin A}{NS} = \frac{\tan \theta \sin A}{\sqrt{\tan^2 \frac{1}{2} v + \tan^2 \theta - 2 \tan v \tan \theta \cos A}}$$

As the trace Sf makes the angle B with the trace PF, the expression for the inclination $I = SNO$ of the normal to the trace Sf becomes

(8) $\cos I = \sin \frac{1}{2} v \cos B$

To obtain an expression for the value of the angle $SOF = \phi$ we have given two sides and the included angle I of the triangle NOS, hence

(9)
$$\sin \tfrac{1}{2} \phi = \frac{NS \sin I}{SO}$$
$$= \sin I \cos \theta \sqrt{\tan^2 \tfrac{1}{2} v + \tan^2 \theta - 2 \tan \tfrac{1}{2} v \tan \theta \cos A}$$

In the plane triangle NOf the length of the normal NO and all the angles are known, therefore

(10)
$$NF = \frac{NO \sin \frac{1}{2} \phi}{\sin (I - \frac{1}{2} \phi)} = \rho \frac{\cos v \sec \frac{1}{2} v \sin \frac{1}{2} \phi}{\sin (I - \frac{1}{2} \phi)}$$

By multiplying the second member of eq. (10) by $\cos[180° + (A+B)]$ and $\sin[180° + (A+B)]$, respectively, we obtain the x'' and y'' coordinates of the point f referred to N as an origin.

Let x' and y' denote the coordinates of the point N referred to the point where the optical axis pierces the focal plane, then

(11) $x' = FN \cos A = \rho \tan \frac{1}{2} v \cos A$

(12) $y' = FN \sin A = \rho \tan \frac{1}{2} v \sin A$

Let the focal plane coordinates of the point f referred to the optical axis be x and y, then

(13) $x = x' + x''$

(14) $y = y' + y''$

Substituting, and writing $F \sec^2 \frac{1}{2} v$ in place of ρ, we obtain, finally, the following general and wholly rigorous expressions for the focal plane coordinates of the point f expressed in terms of the principal focal length F.

(15)
$$\frac{x}{F} = \sec^3 \tfrac{1}{2} v \left[\sin \tfrac{1}{2} v \cos A + \frac{\cos v \sin \frac{1}{2} \phi \cos (180° + A + B)}{\sin (I - \frac{1}{2} \phi)} \right]$$

(16)
$$\frac{y}{F} = \sec^3 \tfrac{1}{2} v \left[\sin \tfrac{1}{2} v \sin A + \frac{\cos v \sin \frac{1}{2} \phi \sin (180° + A + B)}{\sin (I - \frac{1}{2} \phi)} \right]$$

I have computed two sets of values of $\frac{x}{F}$ and $\frac{y}{F}$ for every 15° of A from 0° to 360°. The resulting coordinates were plotted and a smooth curve drawn through the several points (see Fig. 2). One set corresponds to the constant values of $v = 3°$ and $\theta = 30'$, and gives the smaller one of the two closed curves. The heavier (larger) curved figure corresponds to the constant values $v = 5°$ and $\theta = 30'$.

Each curve is therefore the locus of the intersections with the focal plane of the rays reflected from the successive points of a narrow annular ring whose angular radius (as seen from the focal point) is v; the direct rays being parallel and inclined 30′ to the optical axis.

A study of the relative size and position of the two curves (with reference to the point where $v = 0°$, $\theta = 30'$) shows in a striking manner the rapid increase in the radial distortion of the image for increasing values of v and the accompanying rapid increase in the transverse distortion when v is large and increasing.

The approximate law of distribution of the intensity of the light in the complete focal plane image can also be deduced from a consideration of the position and relative size of each image (as formed by different concentric rings of the reflecting surface) with reference to the size and position of an assumed unit-area and intensity formed by a small central area of the mirror.

The formulas and figure show that practically all the light coming from a small unit-area having an angular radius of, say, $v = 1°$ (corresponding to a ratio of focal length to aperture of about 28 to 1) can be said to lie within a focal plane area of one square second of arc, which will be taken as the unit-area in the focal plane. Let this particular unit-area have a mean intensity equal to unity.

Now for $v = 3°$ the total increase in the amount of light is approximately $3^2 - 1^2 = 8$; while the total increase in the focal area over which the light is spread (found approximately by adding to the number of squares contained in the smaller closed curve, the number contained between this curve and the two tangent lines drawn from the point for which $v = 0°$) is $9 - 1 = 8$. Therefore, if the light were uniformly distributed, the intensity for $\theta = 30'$ of the focal plane image formed by reflection from the zone of the mirror included between the radii $v = 1°$ and $v = 3°$ would be practically the same as the unit-intensity, although the total amount of light is actually eight times as great. An inspection of the figure, however, plainly shows that for moderate values of v the image formed by an annular ring of sensible width will be brightest on the side towards the optical axis; but as only a portion of the central area is covered by a small but more condensed part of the image formed by reflection from an annular ring-area whose radii are 1° and 3°, only a slight gain in intensity of the central unit-area results from the additional zone.

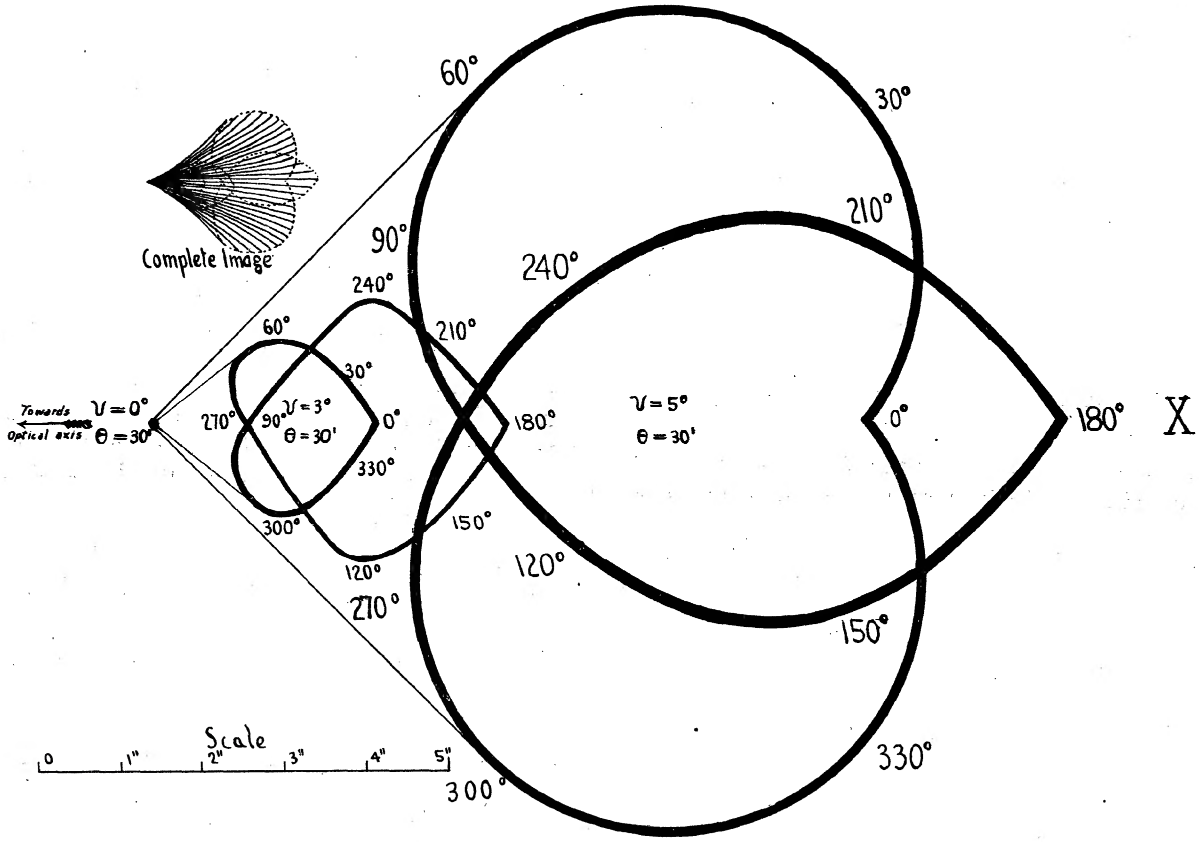

FIGURE 2.

Considering now the image formed by a zone of the mirror's surface included between the radii $v = 3°$ and $v = 5°$, we see at once that no part of the resulting image can overlap the central unit-area when θ is as large as 30'. Consequently no increase in the intensity of the central unit-area can result, however great the value of v may become.

The increase (over the central area) in the total amount of light reflected is $5^2 - 3^2 - 1^2 = 15$, while the increase in the area over which the image is spread is about sixty times the unit-area; giving a mean intensity only one-fourth as great as the unit-intensity, for fifteen times the amount of light.

The typical "complete image" (for $v = 5°$ and $\theta = 30'$) formed by reflection from all the rings of which the mirror's surface may be said to be made up, is approximately of the form and relative intensity shown in the smaller diagram of Fig. 2.

As far as I am aware the coordinates given by equations (15) and (16) differ quite radically from any focal plane coordinates heretofore deduced. It would seem, therefore, that the claim I made, and still maintain, that certain fundamental principles of optics had, up to the present time, been overlooked in the theory of the parabolic reflector, is confirmed by the results of the general theory here developed.* The principal oversight in the past has, according to my view, been the neglect to consider in a consistent manner the relation existing between F and ρ for different values of v.

I shall now discuss, very briefly, equations (15) and (16) for certain special values of v, θ and A.

(*a*) Let $v = 0$. For this condition the points P, N and F must evidently coincide; S must, therefore, lie upon the axis of x, and consequently A must be zero. Equations (7), (8) and (9) also show that $B = A = 0$, $I = 90°$ and $\frac{1}{2}\phi = \theta$, hence (15) and (16) become

$$\frac{x}{F} = \tan\theta \tag{17}$$

$$\frac{y}{F} = 0 \tag{18}$$

The geometrical image in this case is simply the point where a single ray of light pierces the focal plane. It is the only part of the focal plane image of a star which has no aberration, and *it is always nearer to the optical axis*

* This remark is called forth by two papers in *Popular Astronomy* for February, 1898, written by the same author. For my reply to the same see *Popular Astronomy* for March, 1898.

than any other part of the star-image formed by the whole mirror.

(*b*) Let $\theta = 0$. Equations (7), (8) and (9) show that for this case $B = 0$, $I = 90° - \frac{1}{2}v$ and $\frac{1}{2}\phi = \frac{1}{2}v$. These values substituted in the second members of (15) and (16) make the two terms within the brackets respectively equal to each other, hence

$$\frac{x}{F} = 0 \tag{19}$$

$$\frac{y}{F} = 0 \tag{20}$$

All the reflected rays, therefore, pass through the point where the optical axis pierces the focal plane.

(*c*) Let $A = 0$. Equations (7), (8) and (9) now show that for this case $B = 0$, $I = 90° - \frac{1}{2}v$ and $\frac{1}{2}\phi = \frac{1}{2}v - \theta$. Substituting in (15) and (16) we obtain after a few simple transformations,

$$\frac{x}{F} = \sin\theta \sec(v-\theta) \sec^2\tfrac{1}{2}v \tag{20}$$

$$\frac{y}{F} = 0 \tag{21}$$

Referring to Fig. 2 we see that for $v = 3°$ the corresponding value of x is at a secondary maximum; while for $v = 5°$ it corresponds to a secondary minimum, at the apex of a re-entrant curve.

(*d*) Let $A = 180°$. The only difference, in the substitutions, between this case and the previous one is that now $\frac{1}{2}\phi = \frac{1}{2}v + \theta$. The resulting equations are

$$\frac{x}{F} = \sin\theta \sec(v+\theta) \sec^2\tfrac{1}{2}v \tag{22}$$

$$\frac{y}{F} = 0 \tag{23}$$

It is evident that there are other values of x which make $y = 0$ besides those corresponding to $A = 0°$ and $A = 180°$; for as the two terms of equation (16) *always* have opposite signs (except for the special cases $A = 0°$ and $180°$, already considered), and both terms pass through zero, it is evident that there must be at least two additional values of A which make the terms finite and numerically equal to each other. For a particular value $A = A_1$ less than $180°$ fulfilling this condition, the first term will be positive, and the second negative. The substitution of A_1 in (15) will give one value of $x = x_1$ which renders $y = 0$. An identical value $x_2 = x_1$ will result from the substitution of $A_2 = 360° - A_1$ in (15), since this value of A_2 will also make $y = 0$, the first term of (16) now being negative and numerically equal to the positive second term. In general, therefore, the locus of the point f crosses the axis of x at least four times, as y becomes equal to zero for $A = 0°$, $180°$, A_1 and A_2.

Figure 2 shows that both for $v = 3°$ and $v = 5°$ the value of x corresponding to $A = 180°$ is at a maximum. To prove that this condition holds true for all values of v and θ we know first of all that, as the coordinates of every reflected ray from a star exactly on the optical axis are given by equations (19) and (20), it follows that the x-coordinates of every reflected ray from a star at an angular distance θ from the optical axis must be *positive*, as the fixed normal to any point of the reflecting surface must always lie between the direct and reflected ray. Secondly, the two terms of equation (15) always have opposite signs, except when either one is zero, in which case the other is always a positive quantity. Consequently as x cannot be negative we have only to determine that value of A which makes the algebraic sum of the two terms a maximum positive quantity.

A simple inspection of (15) shows the numerator of the second term is a maximum and the denominator a minimum when $A = 180°$. For this value of A the factor $\frac{\cos v \sin\frac{1}{2}\phi}{\sin(I-\frac{1}{2}\phi)}$ is also much greater than $\sin\frac{1}{2}v$. Moreover, as $\cos(180+A+B)$ diminishes, numerically, more rapidly than $\cos A$ for values slightly greater or less than $A = 180°$, it follows that x is at its maximum possible value when $A = 180°$.*

The radial aberration is therefore a maximum in the plane containing the optical axis and the star whose image is under consideration.

The actual amount of the blurring can readily be found by means of the rigorous equation (22). The position of the image formed by the center of the mirror is at once obtained by making $v = 0$. Equation (22) then becomes

$$x_0 = F\tan\theta \tag{23}$$

Subtracting (23) from (22) we obtain the desired expression

$$x - x_0 = F\sin\theta\left[\sec(v+\theta)\sec^2\tfrac{1}{2}v - \sec\theta\right] \tag{24}$$

A more convenient formula is the expression for what I have called the "blurring factor" found by *dividing* (22) by (23).

$$\frac{x}{x_0} = \text{Blurring Factor} = \frac{\cos\theta}{\cos(v+\theta)\cos^2\frac{1}{2}v} \tag{25}$$

Equation (22) can also be written

$$x = \rho\sin\theta\sec(v+\theta) \tag{26}$$

which corresponds with the formula (2) of my paper in *A.J.* 423, except that I there inadvertently wrote $\tan\theta$ instead of $\sin\theta$. With this wholly insensible correction equation (4) of the previously mentioned paper becomes identical with the above rigorous expression (25).

* There are other values of A which give secondary maxima values of x, but these need not be considered in the present paper; an inspection of Fig. 2 will show, in a general way, the location of maximum and minimum points for both x and y.

For telescopes of large angular aperture θ must always be very small if the best possible results are desired, so that for this case equation (25) reduces to the convenient approximate form already given in *A.J.* 413,

$$\text{Blurring Factor} = \frac{1}{\cos v \cos^2 \frac{1}{2} v} \tag{27}$$

The law expressed by equation (24) gives, for any value of v and θ, the maximum distance through which the image, formed by any ring-area of the paraboloid, has shifted with reference to the image formed by the central reflecting area.

When v is large the *rate* of shifting for an infinitesimal value of θ is practically just as great as it is for any value of θ likely to be used in practice. Images of objects having sensible angular magnitudes will therefore necessarily be blurred. The *rate* of blurring is dependent directly upon the value of v, while the *amount* depends upon both v and θ.

Owing to the extremely limited size of the field of view (with the optical axis as a center), which can be used when results of great delicacy are desired, I am inclined to the belief that for mirrors of large angular apertures the Cassegrain form of telescope will eventually supersede the ordinary form for obtaining, on a large scale, photographic representations of celestial areas not much exceeding one or two minutes of arc in diameter. This opinion is based upon actual experimental results obtained at the Lick Observatory with different forms of telescopes.

Lick Observatory, University of California, 1898 *March*.

ON THE STAR-IMAGE FORMED BY A PARABOLIC MIRROR,

By H. C. PLUMMER.

The interesting papers which have been published recently on the subject of the aberration of parabolic mirrors, have only considered the rays which are incident in a plane passing through the axis of the mirror. It is proposed here to find the approximate form of the star-image in the focal plane, and to estimate the distribution of intensity.

For the surface of the mirror we take

$$4fx = y^2 + z^2 = r^2$$

Let the plane ZOX pass through the star, and let θ be its angular distance from the center of the field. Then the ray incident at (xyz) is

$$\frac{X-x}{\cos\theta} = \frac{Y-y}{0} = \frac{Z-z}{\sin\theta}$$

The normal at (xyz) is $\dfrac{X-x}{-2f} = \dfrac{Y-y}{y} = \dfrac{Z-z}{z}$

Let the reflected ray be $\dfrac{X-x}{\lambda} = \dfrac{Y-y}{\mu} = \dfrac{Z-z}{\nu}$

The laws of reflection give $\begin{vmatrix} \lambda & \mu & \nu \\ \cos\theta & 0 & \sin\theta \\ -2f & y & z \end{vmatrix} = 0$

and $\quad -2f\lambda + \mu y + \nu z = -2f\cos\theta + z\sin\theta$

Hence

$$\frac{\lambda - \cos\theta}{2fz\sin\theta + r^2\cos\theta} = \frac{\mu}{2fy\cos\theta - yz\sin\theta}$$
$$= \frac{\nu - \sin\theta}{(y^2+4f^2)\sin\theta + 2fz\cos\theta} = k, \text{ say.}$$

Eliminating λ, μ, ν between these three equations and

$$\lambda^2+\mu^2+\nu^2 = 1 \quad , \quad \text{we get} \quad k = \frac{-2}{4f^2+r^2}$$

Let $(\xi\eta\zeta)$ be the intersection of the reflected ray with the focal plane.

$$\therefore \xi = f \quad ; \quad \eta = y + \frac{\mu}{\lambda}(f-x) \quad ; \quad \zeta = z + \frac{\nu}{\lambda}(f-x)$$

Hence we obtain

$$\eta = \frac{-2yz(f+x)\sin\theta}{(4f^2-r^2)\cos\theta - 4fz\sin\theta}$$
$$\zeta = \frac{-\sin\theta\{4f^3+2fz^2-x(y^2-z^2)\}}{(4f^2-r^2)\cos\theta - 4fz\sin\theta}$$

These expressions are exact, but from this point powers of $\tan\theta$ above the first are neglected. Thus

$$\eta = -\frac{2yz(f+x)}{4f^2-r^2}\tan\theta$$
$$\zeta = -\frac{4f^3+2fz^2-x(y^2-z^2)}{4f^2-r^2}\tan\theta$$

But for the ray reflected at the vertex:

$$\eta_0 = 0 \quad ; \quad \zeta_0 = -f\tan\theta$$

Hence finally, neglecting x, which is small compared with f,

$$\eta - \eta_0 = -\frac{\tan\theta}{2f}yz$$
$$\zeta - \zeta_0 = -\frac{\tan\theta}{4f}(r^2+2z^2)$$

If we eliminate yz from these by means of $y^2+z^2 = r^2$, we obtain

$$(\eta-\eta_0)^2 + \left\{\zeta-\zeta_0+\frac{r^2\tan\theta}{4f}\right\}\left\{\zeta-\zeta_0+\frac{3r^2\tan\theta}{4f}\right\} = 0$$

as the locus of points in which the rays reflected from an infinitely narrow zone of the mirror cut the focal plane. This shows that the locus is a circle whose center is $\left(\eta_0 \cdot \zeta_0 - \dfrac{r^2\tan\theta}{2f}\right)$ and whose radius is $\dfrac{r^2\tan\theta}{4f}$.

Now, if F is the focus of the mirror, and O is the point $(f\eta_0\zeta_0)$, the lines through O which make angles of 30° with FO produced touch this circle, and similarly all the