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Differential Calculus for Engineers

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Preface

This book is based on lectures given by the authors at the Faculty of Geodesy and the Faculty of Civil Engineering from Technical University of Civil Engineering of Bucharest.

The material presented in this book exceeds the content of the spoken lessons and so it is also useful for other engineering specialties and even for students in mathematics.

The book provides the engineering disciplines with necessary information of differential calculus of functions with one and several variables.

The paper includes all the basic notions and results of differential calculus that are taught today in higher technical education, namely: sequences and series of numbers, sequences, and series of functions, power series, elements of topology on n-dimensional space, limits of functions, continuous functions, partial derivatives of functions of several variables, Taylor's formula, extrema of a function of several variables (free or with constrains), change of variables, dependent functions.

The style of the paper is direct and few references are made to previous knowledge. The selection of the material made by the authors, the simplest, often original, versions chosen of the demonstrations presented as well as the numerous examples and exercises completely solved constitute the secret of the success of this work.

We tried to offer the fundamental material concisely and without distracting details. We focused on the presentation of basic ideas in order to make in detailed and a comprehensible as possible. Besides students in technical faculties and those starting a mathematical course, the book may be useful to engineers and scientists who wish to refresh their knowledge about some aspects of differential calculus.

Bucharest, Romania

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1.1 Real Numbers

Further we shall denote by $\mathbb{N}$ the set of **natural** numbers i.e.:

$$\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\} \text{ and by } \mathbb{N}^* = \mathbb{N} \setminus \{0\}.$$

On the set of natural numbers $\mathbb{N}$ are defined two arithmetic operations: the addition noted by "+" and the multiplication noted by "$\cdot$". The other two basic arithmetic operations i.e. subtraction and division are not possible in $\mathbb{N}$.

To make possible the subtraction operation the negative numbers are added to $\mathbb{N}$, thus obtaining the set of **integer**s:

$$\mathbb{Z} = \{\ldots, -n, \ldots, -2, -1, 0, 1, 2, \ldots, n, \ldots\}.$$

The next extension of numbers, which makes possible also the division operation, is the set of **rational** numbers denoted by $\mathbb{Q}$, namely the numbers of the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$, p and q relatively prime (i.e. only integer factor divides both of them is 1). All rational numbers can be expressed as repeating decimal fractions. Also we remark that $(\mathbb{Q}, +, \cdot)$ is an **Abelian (commutative) field**.

From ancient times it has observed that the set of rational numbers is not sufficient rich to allow the express of the measure of any size in nature. For example, the diagonal of a square of side 1 is $\sqrt{2}$, and it is known that $\sqrt{2} \notin \mathbb{Q}$. Thus, it was necessary to extend the set of rational numbers and it was created the set of **real** numbers denoted by $\mathbb{R}$. Real numbers other than rational are called **irrational**. Unlike rational numbers, irrational numbers can be represented by infinite non-repeating decimal fractions.

We do not intend to present here the construction of real numbers. We will just say that we can construct a set $\mathbb{R}$ which contains the rational numbers, on which two operations are definited: the addition noted by "+" and the multiplication noted by "$\cdot$", and an order relation, denoted by "$\leq$", so that the real number system $(\mathbb{R}, +, \cdot, \leq)$

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is a **total ordered commutative field**, which additionally satisfies the following properties:

(A) (**Archimedes' axiom**)
For any $x \in \mathbb{R}$ and any $y \in \mathbb{R}$, $y > 0$, there exists $n \in \mathbb{N}$ such that $ny > x$.

(B) (**Cantor's axiom**)
Every non-empty subset of $\mathbb{R}$ with an upper bound in $\mathbb{R}$, has a least upper bound in $\mathbb{R}$ (see Definition 1.1.1 *below).*

The last property is what differentiates the real numbers from the rational numbers. For example, the set of rational numbers with square less than 2 has a rational upper bound in $\mathbb{Q}$, namely $\frac{3}{2}$, but no rational least upper bound, because this is $\sqrt{2}$, and $\sqrt{2} \notin \mathbb{Q}$.

Therefore from algebraic point of view, $(\mathbb{R}, +)$ is a commutative group with identity element 0, and $(\mathbb{R}\setminus\{0\}, \cdot)$ is a commutative group with identity element 1.In addition we have the distributive law:

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad \forall x, y, z \in \mathbb{R}.$$

The order relation "$\leq$" is total, namely for any $x, y \in \mathbb{R}$ we have either $x \leq y$ or $y \leq x$. Also the order relation is compatible with algebraic srtucture i.e.:

1. if $x' \leq y'$ and $x'' \leq y''$ then $x' + x'' \leq y' + y''$,
2. if $x \leq y$ and $\alpha > 0$ then $\alpha \cdot x \leq \alpha \cdot y$.

From the fact that $(\mathbb{R}, +, \cdot, \leq)$ is a total ordered commutative field results all the known rules of calculation with real numbers.

Remark 1.1.1 It can show that the Archimedes' axiom is equivalent with the following property:

$$\forall x \in \mathbb{R}, \text{ there is } [x] \in \mathbb{Z} \text{ such that } [x] \leq x < [x] + 1.$$

($[x]$ is called the integer part of x).

Proposition 1.1.1 *For any $x, y \in \mathbb{R}$, $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.*

The proof results from Archimedes' axiom.

From Proposition 1.1.1 we deduce that between two real numbers there are an infinitude of ration numbers.

The follwing proposition following from Cantor's axiom.

Proposition 1.1.2 *For any $x, y \in \mathbb{R}$, $x < y$, there exists at last irrational number z such that $x < z < y$.*

From Proposition 1.1.2 it results that between two real numbers there are an infinitude of irrational numbers.

Definition 1.1.1 A set X of real numbers is said to be **bounded above (below)** if there exists a real number b (respectively a) such that each element $x \in X$ satisfies the inequality:

$$x \leq b \, (\text{respectively } a \leq x)$$

In this case, the number b (respectively a) is called the **upper bound (lower bound)**.

Obviously, if b is an upper bound of X, then any number $b' \geq b$ is also an upper bound, therefore every bounded above set has an infinitude of upper bounds. The argument concerning the lower bounds of a set bounded below is similar.

Definition 1.1.2 A set X of real numbers is said to be **bounded** if has both an upper bound and a lower bound, that is, there are two real numbers a and b such that:

$$a \leq x \leq b, \quad \forall x \in X$$

The least of all the upper bounds of a the set X bounded above is called the **supremum** of that set and is denoted by $M = \sup X$.

The greatest of all the lower bounds of a set X bounded below is called the **infimum** of that set and is denoted by $m = \inf X$.

Remark 1.1.2 A real number M is the supremum of the subset X of $\mathbb{R}$ if and only if the following two requirements are satisfied:

1. $x \leq M, \forall x \in X$;
2. $\forall \varepsilon > 0, \exists x_\varepsilon \in X$ such that $M - \varepsilon < x_\varepsilon$.

Proof "$\Rightarrow$" Indeed, if $M = \sup X$, then M is an upper bound of X, whence it results (1). Since M is the least upper bound of X, it follows that for any $\varepsilon > 0$, $M - \varepsilon$ is not an upper bound of X, hence there exists $x_\varepsilon \in X$ such that $M - \varepsilon < x_\varepsilon$.

"$\Leftarrow$" Let $M \in \mathbb{R}$ be with the properties (1) and (2). From (1) it results that M is an upper bound of X. Let $M' < M$ and $\varepsilon = M - M' > 0$. From (2) we deduce that there exists $x_\varepsilon \in X$ such that $x_\varepsilon > M - \varepsilon = M'$. Therefore M' is not an upper bound of X, so $M = \sup X$.

Remark 1.1.3 A real number m is the infimum of the subset X of $\mathbb{R}$, if and only if the following two requirements are satisfied:

1. $m \leq x, \forall x \in X$;
2. $\forall \varepsilon > 0, \exists x_\varepsilon \in X$ such that $x_\varepsilon < m + \varepsilon$.

Remark 1.1.4 It can show that the Cantor's axiom is equivalent with the following property:

If $\{a_n\}$ and $\{b_n\}$ are two rational number sequences with the properties:

1. $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1$,

2. $\lim_{n\to\infty}(b_n - a_n) = 0^{*)}$,

then there is an unique element $c \in \mathbb{R}$ such that $a_n \leq c \leq b_n, \forall n \in \mathbb{N}$.

$^{*)}\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that $b_n - a_n < \varepsilon, \forall n \geq n_\varepsilon$.

1.2 Real Number Sequences

Definition 1.2.1 A real number sequence $\{x_n\}$ is said to be **increasing (decreasing)** if $x_n \leq x_{n+1}, \forall n \geq 1$ (respectively $x_{n+1} \leq x_n, \forall n \geq 1$).

Both increasing and decreasing sequences are referred to as **monotonic (monotone)** sequences.

Definition 1.2.2 A sequence $\{x_n\}$ is said to **bounded above (below)**, if there exists a real numbers M (respectively m) suh that:

$$x_n \leq M(m \leq x_n), \quad \forall n \geq 1.$$

The sequence $\{x_n\}$ is said to be **bounded** if it is bounded both above and below i.e. there exist the numbers m and M such that $m \leq x_n \leq M, \forall n \in \mathbb{N}^*$.

Definition 1.2.3 The sequence $\{x_n\}$ is called **convergent** if there exits a number $l \in \mathbb{R}$ with the property: for any $\varepsilon > 0$, there is a number $n_\varepsilon \in \mathbb{N}$, such that $|x_n - l| < \varepsilon, \forall n \geq n_\varepsilon$.

The number l is called the **limit** of the sequence $\{x_n\}$ and we shall write:

$$\lim_{n\to\infty} x_n = l \text{ or } x_n \xrightarrow[n\to\infty]{\mathbb{R}} l.$$

Remark 1.2.1 A convergent sequence has an unique limit.

Indeed, let l and l' be the limits of the convergence sequence (x_n). If we denote by $y_n = x_n - l$ and by $y'_n = x_n - l'$, then $\lim_{x\to\infty} y_n = \lim_{x\to\infty} y'_n = 0$. On the other hand we have:

$$0 = \lim_{n\to\infty}\left(y_n - y'_n\right) = l' - l,$$

hence $l' = l$.

Definition 1.2.4 Let $\{x_n\}$ be a sequence and let $\{k_n\}$ be an arbitrary increasing sequence of positive integers ($k_1 < k_2 < \ldots < k_n < \ldots$). The sequence $\left\{x_{k_n}\right\}$ will be referred to as a **subsequence of the sequence** $\{x_n\}$. In particular, the sequence $\{x_n\}$ itself can be regarded as a subsequence of the sequence $\{x_n\}$ (for $k_n = n$).

Obviously, if a sequence is convergent and has the limit l, then every subsequence is also convergent and has a same limit l. But if a given sequence $\{x_n\}$ has a convergent subsequence this does not imply that given sequence is itself convergent.

Lemma 1.2.1 (Cesàro). *Every bounded sequence* $\{x_n\}$ *contains a convergent subsequence* $\left\{x_{k_n}\right\}$.

Proof Let $\{x_n\}$ be a bounded real number sequence. Then there exists $a, b \in \mathbb{Q}$ such that $a < x_n < b, \forall n \in \mathbb{N}$. If we denote by c the middle of the interval $[a, b]$, then at least of the intervals $[a, c]$ or $[c, b]$ contains an infinite number of elements x_n. If $[a, c]$ has this property, then we shall denote $a_1 = a$ and $b_1 = c$ and choose $x_{k_1} \in [a_1, b_1]$. If c_1 is the middle of the interval $[a_1, b_1]$ then at least of the intervals $[a_1, c_1]$ respectively $[c_1, b_1]$ contains an infinite terms of the sequence $\{x_n\}$. Let's suppose that $[c_1, b_1]$ has this property. Then we denote by $a_2 = c_1, b_2 = b_1$. Clearly we can choose among the elements of sequence $\{x_n\}$ an element $x_{k_2} \in [a_2, b_2]$, such that $k_2 > k_1$. Continuing in this way are obtained two sequences of rational numbers $\{a_n\}, \{b_n\}$ with the properties:

1. $a_1 \le a_2 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_2 \le b_1$.
2. $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0$.
3. there is a strictly increasing sequence of natural numbers $k_1 < k_2 < \ldots < k_n < \ldots$ such that $x_{k_n} \in [a_n, b_n], \forall n \in \mathbb{N}$.

On the other hand, from Remark 1.1.4 it follows that there exists an unique element $l \in \mathbb{R}$ with the property $a_n \le l \le b_n, \forall n \in \mathbb{N}$. Since

$$\left|x_{k_n} - l\right| \le b_n - a_n = \frac{b-a}{2^n} \text{ and } \lim_{n\to\infty} \frac{b-a}{2^n} = 0,$$

we deduce that the subsequence $\left\{x_{k_n}\right\}$ is convergent and has the limit l.

Definition 1.2.5 A real number sequence $\{x_n\}$ is said to be **fundamental (or Cauchy)**, if for any $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that:

$$|x_m - x_n| < \varepsilon, \quad \forall m \ge n_\varepsilon, \quad \forall n \ge n_\varepsilon$$

If we denote by $p = m - n$ (if $m > n$), respectively $p = n - m$ (if $m < n$) we obtain the following equivalent definition:

The sequence $\{x_n\}$ is fundamental if $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that $\forall n \ge n_\varepsilon$ and $\forall p \in \mathbb{N}^*$ we have $\left|x_{n+p} - x_n\right| < \varepsilon$.

Lemma 1.2.2 *Every fundamental real numbers sequence is bounded.*

Proof Let $\{x_n\}$ be a fundamental real number sequence. For $\varepsilon = 1$, there is $n_1 \in \mathbb{N}^*$ such that $\left|x_{n+p} - x_n\right| < 1, \forall n \ge n_1, \forall p \in \mathbb{N}^*$.

In particular case for $n = n_1$ it results that $x_{n_1} - 1 < x_{n_1+p} < x_{n_1} + 1, \forall p \in \mathbb{N}^*$. If we denote by:

$$a = \min\{x_1, \ldots, x_{n_1-1}, x_{n_1} - 1\} \text{ and by } b = \max\{x_1, \ldots, x_{n_1-1}, x_{n_1} + 1\},$$

then $a \le x_n \le b, \forall n \in \mathbb{N}$, hence $\{x_n\}$ is bounded.

Theorem 1.2.1 (Cauchy's test for convergence of a sequence). *For the real number sequence $\{x_n\}$ to be convergent it is necessary and sufficient that it should be fundamental.*

Proof **Necessity**: Let $\{x_n\}$ be a convergent sequence with the limit l. Then $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that $\forall n \ge n_\varepsilon, |x_n - l| < \frac{\varepsilon}{2}$. Of course, for $m \ge n_\varepsilon$, we have also $|x_m - l| < \frac{\varepsilon}{2}$ and further:

$$|x_m - x_n| = |(x_m - l) + (l - x_n)| \le |x_m - l| + |x_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\forall n, m \ge n_\varepsilon$ we have $|x_m - x_n| < \varepsilon$, hence $\{x_n\}$ is fundamental.

Sufficiency: Let $\{x_n\}$ be a fundamental real number sequences. According to the Definition 1.2.5 for $\forall \varepsilon > 0, \exists n'_\varepsilon \in \mathbb{N}^*$ such that:

$$|x_n - x_m| < \frac{\varepsilon}{2}, \quad \forall n, m \ge n'_\varepsilon.$$

On the other hand, from Lemma 1.2.2 it results that $\{x_n\}$ is bounded, and from Lemma 1.2.1, that it contains a subsequence $\{x_{k_n}\}$ convergent. Let $l = \lim_{n\to\infty} x_{k_n}$ and let $n''_\varepsilon \in \mathbb{N}^*$ such that $|x_{k_n} - l| < \frac{\varepsilon}{2}, \forall n \ge n''_\varepsilon$. If we denote by $n_\varepsilon = \max\{n'_\varepsilon, n''_\varepsilon\}$ then, for $n \ge n_\varepsilon$ we have:

$$|x_n - l| = |x_n - x_{k_n} + x_{k_n} - l| \le |x_n - x_{k_n}| + |x_{k_n} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $|x_n - l| < \varepsilon$, for any $n \ge n_\varepsilon$, hence $\{x_n\}$ is convergent with the limit l.

Cauchy's test for convergence of a sequence allowing the question as to its convergence to be decided only by the basis of the magnitude of the differences between its terms.

Example 1.2.1 Let us use Cauchy's test to establish the convergence of the sequence with the general term:

$$x_n = \frac{\sin x}{5} + \frac{\sin 2x}{5^2} + \cdots + \frac{\sin nx}{5^n}, \text{ for an arbitrary fixed } x \in \mathbb{R}.$$

We shall prove that the sequence $\{x_n\}$ is fundamental. Indeed, we have:

$$\left|x_{n+p} - x_n\right| = \left|\frac{\sin(n+1)x}{5^{n+1}} + \cdots + \frac{\sin(n+p)x}{5^{n+p}}\right| \le \frac{1}{5^{n+1}} + \cdots + \frac{1}{5^{n+p}}$$
$$= \frac{1}{5^{n+1}} \cdot \frac{1 - \frac{1}{5^p}}{1 - \frac{1}{5}} < \frac{1}{4 \cdot 5^n}, \quad p \in \mathbb{N}^*.$$

Since $\lim_{n\to\infty} \frac{1}{4\cdot 5^n} = 0$, it results that $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}^*$ such that $\forall n \ge n_\varepsilon, \forall p \in \mathbb{N}^*$, we have $\left|x_{n+p} - x_n\right| < \frac{1}{4\cdot 5^n} < \varepsilon$.

Therefore $\{x_n\}$ is fundamental, hence $\{x_n\}$ is convergent, according to Theorem 1.2.1.

Theorem 1.2.2 (Weierstrass). *Any increasing (decreasing) real number sequence bounded above (below) is convergent.*

Proof Let us suppose that $\{x_n\}$ is an increasing bounded above sequence. Since the set $A = \{x_n; n \in \mathbb{N}\}$ of terms of the sequnece $\{x_n\}$ is bounded above, from Cantor's axiom it follows that there exists $M = \sup A$. Therefore $x_n \le M$, $\forall n \in \mathbb{N}^*$. According to Remark 1.1.2, for any $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $M - \varepsilon < x_{n_\varepsilon}$. Since the sequence $\{x_n\}$ is increasing we have $x_n \ge x_{n_\varepsilon}$, $\forall n \ge n_\varepsilon$. Consequently:

$$M - \varepsilon < x_n \le M \le M + \varepsilon, \quad \forall n \ge n_\varepsilon,$$

that is:

$$|x_n - M| < \varepsilon, \quad \forall n \ge n_\varepsilon,$$

hence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = M$.

A reasoning for decreasing sequence is quite analogous.

Example 1.2.2 (Number e). The best known application of Theorem 1.2.2 is the one consisting to prove the convergence of the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$, $n \ge 1$.

We shall prove that this sequence is increasing and bounded above. Applying Newton's binomial formula we get:

$$x_n = 1 + n \cdot \frac{1}{n} + \frac{n\cdot(n-1)}{1\cdot 2} \cdot \frac{1}{n^2} + \cdots + \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot[n-(n-1)]}{1\cdot 2\cdot 3\cdot\ldots\cdot n} \cdot \frac{1}{n^n}$$
$$= 2 + \frac{1}{2!}\cdot\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\cdot\left(1 - \frac{1}{n}\right)\cdot\left(1 - \frac{2}{n}\right)\cdot\ldots\cdot\left(1 - \frac{n-1}{n}\right).$$

and, similarly for the term x_{n+1}:

$$x_{n+1} = 2 + \frac{1}{2!}\cdot\left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\cdot\left(1 - \frac{1}{n+1}\right)\cdot\left(1 - \frac{2}{n+1}\right)\cdot\ldots\cdot\left(1 - \frac{n}{n+1}\right)$$

We observe that the terms in x_n are less than the corresponding terms in x_{n+1}, and, beside, x_{n+1} includes an extra positive term (the last one). Therefore:

$$x_n < x_{n+1}, \quad \forall n \in \mathbb{N}^*,$$

hence the sequence (x_n) is increasing.

On the other hand we have:

$$\begin{aligned}
x_n &= 2 + \frac{1}{2!} \cdot \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!} \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{n-1}{n}\right) \\
&< 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\
&= 2 + \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} < 2 + \frac{\frac{1}{2}}{\frac{1}{2}} = 3
\end{aligned}$$

Thus we see that the sequence is bounded above. Consequently the sequence is incresing and bounded above, hence it is convergent according to Theorem 1.2.2.

The limit of this sequence is refered to as the number e. Thus,

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

We shall see later that the number e plays an extremely important role in mathematics.

For the moment we only mention that e is an irrational number, $2 < e < 3$ and $e \approx 2.718281$.

Remark 1.2.2 The statement of Theorem 1.2.2 it is not true for rational number sequences.

For example, the rational numbers sequence $x_n = \left(1 + \frac{1}{n}\right)^n, n \geq 1$ is fundamental, because it is convergent in $\mathbb{R}$, and any convergent sequence is fundamental (as it results from the first part of the proof of the Theorem 1.2.2), but this sequence is not convergent in $\mathbb{Q}$, since according to Remark 1.2.1 its limit is e, and $e \notin \mathbb{Q}$.

Example 1.2.3 Let us use Weierstrass's test (Theorem 1.2.2) to establish the convergence of the sequence with the general term:

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n, \quad n \geq 1.$$

We shall prove that this sequence is decreasing and bounded below. Indeed, from Lagrange's Theorem for function $f(x) = \ln(x)$, $x \in [k, k+1]$, it results that there exists $c_k \in (k, k+1)$ such that $\ln(k+1) - \ln(k) = \frac{1}{c_k}$.

On the other hand, from inequalities:

$$\frac{1}{k+1} < \frac{1}{c_k} < \frac{1}{k}, \quad \forall k \in \mathbb{N}^*$$

we deduce:

$$\frac{1}{k+1} < \ln(k+1) - \ln(k) < \frac{1}{k}, \quad (\forall) k \in \mathbb{N}^*$$

For $k = \overline{1, n}$ we get successively:

$$\begin{array}{l} \frac{1}{2} < \ln(2) - \ln(1) < 1 \\ \frac{1}{3} < \ln(3) - \ln(2) < \frac{1}{2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n} \quad (*) \end{array}$$

Summing the above inequalites it follows:

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Since $\ln(n) < \ln(n+1)$, we deduce that $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n > 0$. On the other hand, taking into account of inequality (*) it results:

$$x_n - x_{n+1} = \ln(n+1) - \ln(n) - \frac{1}{n+1} > 0.$$

Therefore the sequence is decreasing and bounded below, hence it is convergent. The limit will be denoted by C (***Euler's constant***). Thus,

$$C = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$$

On can prove that C is an irrational number and $C \approx 0.57721$.
If we denote by $\varepsilon_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) - C$, then:

$$\varepsilon_n > 0 \text{ and } \lim_{n\to\infty} \varepsilon_n = 0.$$

It results the following interesting identity:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + C + \varepsilon_n, \quad \forall n \geq 1.$$

From this identity we deduce that:

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=\infty.$$

1.3 Extended Real Number Line

The extended real number line is denoted by $\overline{\mathbb{R}}$ and is obtained from the real number line $\mathbb{R}$ by adding two elements $+\infty$ and $-\infty$, i.e. $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$.

The order relation on $\overline{\mathbb{R}}$ is obtained by extension of the order relation from $\mathbb{R}$ so:

$$-\infty<+\infty,\ -\infty<x,\ x<+\infty,\quad \forall x\in\mathbb{R}.$$

In this way, $\left(\overline{\mathbb{R}},\leq\right)$ is a totally ordered set, and every subset of $\overline{\mathbb{R}}$ has a supremum and an infimum. If $A\subset\overline{\mathbb{R}}$ is bounded, then the assertion follows from Cantor's axiom; if $A\subset\overline{\mathbb{R}}$ is unbounded above, then $\sup A=+\infty$ and if $A\subset\overline{\mathbb{R}}$ is unbouded below, then $\inf A=-\infty$.

The algebraic operations from $\mathbb{R}$ extended on $\overline{\mathbb{R}}$ so:

$$x+\infty=\infty+x=\infty,\quad \forall x\in\overline{\mathbb{R}},\ x\neq-\infty$$
$$x+(-\infty)=(-\infty)+x=-\infty,\quad \forall x\in\overline{\mathbb{R}},\ x\neq\infty$$
$$x\cdot\infty=\infty\cdot x=\begin{cases}\infty & \text{dacă } x>0\\ -\infty & \text{dacă } x<0\end{cases},\quad \forall x\in\overline{\mathbb{R}}$$
$$x\cdot(-\infty)=(-\infty)\cdot x=\begin{cases}-\infty & \text{dacă } x>0\\ \infty & \text{dacă } x<0\end{cases},\quad \forall x\in\overline{\mathbb{R}}.$$

The expressions: $\infty-\infty$, $-\infty+\infty$, $0\cdot(\pm\infty)$, $(\pm\infty)\cdot 0$, are usually left undefined.

Definition 1.3.1 A real number sequence $\{x_n\}$ has the limit $+\infty$ ($-\infty$) if:

$$\forall\varepsilon>0,\ \exists n_\varepsilon\in\mathbb{N}^*\text{ such that } x_n>\varepsilon\ (x_n<-\varepsilon),\quad \forall n\geq n_\varepsilon.$$

We shall write: $\lim_{n\to\infty}x_n=+\infty$ (respectively $\lim_{n\to\infty}x_n=-\infty$).

Proposition 1.3.1 *Any monotone real number sequence has a limit in* $\overline{\mathbb{R}}$ *and any real number sequence contains a subsequence which has a limit in* $\overline{\mathbb{R}}$.

Proof Let $\{x_n\}$ be a monotone increasing real number sequence. If $\{x_n\}$ is bounded above then, according to Theorem 1.2.2 it is convergent, hence has a finite limit. If $\{x_n\}$ is unbounded above then $\lim_{n\to\infty}x_n=+\infty$. For decreasing sequence the reasoning is quite analogous.

Let now $\{x_n\}$ be an arbitrary real number sequence. If $\{x_n\}$ is bounded, then from Lemma 1.2.1 it results that it contains a subsequnce $\left\{x_{k_n}\right\}$ convergent. Let us suppose that $\{x_n\}$ is unbounded (for example the sequence is unbounded above). In this case we will prove that there exists a subsequence $\left\{x_{k_n}\right\}$ such that $\lim_{k\to\infty} x_{n_k} = +\infty$. Indeed, since there exist an infinity of terms $x_n > 1$, we can choose $x_{k_1} > 1$. Also, since there exist an infinity of terms $x_n > 2$ we can choose $k_2 > k_1$ with the property $x_{k_2} > 2$, and so on. Thus we can construct a subsequence $\left\{x_{k_n}\right\}$ such that $x_{k_n} > n$, hence $\lim_{n\to\infty} x_{k_n} = +\infty$.

Definition 1.3.2 Let $\{x_n\}$ be a real number sequence. An element $a \in \overline{\mathbb{R}}$ is called the **limit point** of the sequence $\{x_n\}$ if there exists a subsequence $\left\{x_{k_n}\right\}$ such that $a = \lim_{n\to\infty} x_{k_n}$.

Remark 1.3.1 Every sequence which has a limit (finite or infinite), has only a limit point coinciding with the limit of the sequence.

Example 1.3.1

1. The sequence $x_n = (-1)^{n+1}$ has two limit points, namely -1 and 1.
2. The sequence $x_n = n^{(-1)^{n+1}}$ has two limit points, namely 0 and ∞.
3. The sequence $x_n = \sqrt{n}$ has an unique limit point, namely ∞.
4. The sequence $x_n = \frac{(-1)^n}{\sqrt{n}}$ has an unique limit point, namely 0.

Definition 1.3.3 The largest limit point of the sequence $\{x_n\}$ is called **the limit superior** of $\{x_n\}$ and is denoted by $\overline{\lim}_{n\to\infty} x_n$, and the lowest limit point of the sequence $\{x_n\}$ is called **the limit inferior** of $\{x_n\}$ and is denoted by $\underline{\lim}_{n\to\infty} x_n$.

Theorem 1.3.1 *Any sequence of real numbers $\{x_n\}$ has the limit superior (the limit inferior) equal to a finite number or $+\infty$ or to $-\infty$. If the sequence is bounded its limit superior and limit inferior are finite numbers.*

Proof Let us denoted by A the set of all limit points of the sequence $\{x_n\}$. If $\{x_n\}$ is unbounded above then $+\infty \in A$ and $\overline{\lim}_{n\to\infty} x_n = +\infty$. If $\{x_n\}$ is bounded above and $A \cap \mathbb{R} = \phi$, then $A = \{-\infty\}$ and $\overline{\lim}_{n\to\infty} x_n = -\infty$. If the sequence is bounded, then according to Lemma 1.2.1, $A \cap \mathbb{R} \neq \phi$. Since $A \cap \mathbb{R}$ is bounded above it follows from Cantor's axiom that there exists $\overline{a} = \sup A \cap \mathbb{R}$. For any $n \in \mathbb{N}^*$ there is an element $a_n \in A \cap \mathbb{R}$ such that $\overline{a} - \frac{1}{n} < a_n \leq \overline{a}$. Let $\left(x_{k_n}\right)$ be a subsequence of the sequence $\{x_n\}$ such that $x_{k_n} \to a_n$. Obviously, for n sufficient large we can suppose that $\overline{a} - \frac{1}{n} < x_{n_k} < \overline{a} + \frac{1}{n}$, hence $x_{k_n} \to \overline{a}$. Therefore $\overline{a} \in A \cap \mathbb{R}$ and $\overline{\lim}_{n\to\infty} x_n = \overline{a}$. An analogous reasoning is possible for $\underline{\lim}_{n\to\infty} x_n$.

Remark 1.3.2 Let $\{x_n\}$ be a bounded real number sequence, $L = \overline{\lim}_{n\to\infty} x_n$ and $l = \underline{\lim}_{n\to\infty} x_n$. Then:

(a) For any $a < L$, there exist an infinite number of terms $x_n > a$ and for any $b > L$ there are not more than a finite number of terms $x_n > b$.
(b) For any $a < l < b$, then there exist an infinite number of terms $x_n < b$ and not more than a finite number of terms $x_n < a$.

Indeed, we justify the statement for $L = \overline{\lim}_{n\to\infty} x_n$. From (a) it results that for any $\forall n \in \mathbb{N}^*$, there exist an infinite number of terms $x_n \in \left(L - \frac{1}{n}, L + \frac{1}{n}\right)$.

Then, by mathematical induction, we can construct a strictly increasing sequence of natural numbers $\{k_n\}$ such that $x_{k_n} \in \left(L - \frac{1}{n}, L + \frac{1}{n}\right)$, from which we deduce that $\left|x_{k_n} - L\right| < \frac{2}{n}$, and further that $x_{k_n} \to L$. Therefore L is a limit point of $\{x_n\}$. On the other hand, if $L' > L$, then there exists $n \in \mathbb{N}^*$ such that $L' - \frac{1}{n} > L$. Since there are not more than a finite number of terms $x_n > L' - \frac{1}{n}$ it follows that L' can not be a limit point of $\{x_n\}$.

Further, if we denote by $m = \inf\{x_n; n \in \mathbb{N}\}$ and by $M = \sup\{x_n; n \in \mathbb{N}\}$ then we have:

$$-\infty \leq m \leq l \leq L \leq M \leq +\infty$$

Example 1.3.2 Find $m, M,$ and l, L for the sequence with the general term:

$$x_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}.$$

We notice that:

$$x_n = \begin{cases} -\frac{1}{n} & \text{for } n \text{ odd,} \\ \frac{1}{n} + 1 & \text{for } n \text{ even.} \end{cases}$$

Therefore the sequence contains two convergent subsequences i.e.: $-\frac{1}{n} \to 0$ and $\frac{1}{n} + 1 \to 1$, hence $l = 0$ and $L = 1$. The subsequence $\left\{-\frac{1}{n}\right\}$ is increasing and the subsequence $\left\{\frac{1}{n} + 1\right\}$ is decreasing. It follows that $m = -1$ and $M = 2$. Therefore we have: $m = -1 < l = 0 < L = 1 < M = 2$.

Proposition 1.3.2 *The necessary and sufficient condition for the sequence* $\{x_n\}$ *to have a limit (finite or infinite) is that* $L = \overline{\lim}_{n\to\infty} x_n = l = \underline{\lim}_{n\to\infty} x_n$.

Proof **Neccesity**: According to Remark 1.3.1, if there exists $\lim_{n\to\infty} x_n = a$, then the sequence has an unique limit point coinciding with a, hence $L = l = a$.

Sufficiency: First, let us suppose that $L = l = a \in \mathbb{R}$. From Remark 1.3.2 it results that for any $\varepsilon > 0$, the interval $(a - \varepsilon, a + \varepsilon)$ contains an infinite number of terms of the sequence $\{x_n\}$ and outside this interval there is a finite number of terms of the sequence. It results that $a = \lim_{n\to\infty} x_n$. If $L = l = +\infty$, then $\lim_{n\to\infty} x_n = +\infty$, and if $L = l = a = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.

Chapter 2
Real Number Series

2.1 Convergent and Divergent Series

Let us consider a real number sequence $\{u_n\}$. A ***real number series*** is an expression of de form:

$$u_1 + u_2 + \ldots + u_n + \ldots = \sum_{n=1}^{\infty} u_n \tag{2.1}$$

The "infinite sum" (2.1) is purely formal because the addition of an infinite number of summands does not make sense. The real numbers $u_1, u_2, \ldots, u_n, \ldots.$ are called ***the terms*** of the series and the number u_n is called ***the general term*** of the series. The number:

$$S_n = u_1 + u_2 + \ldots + u_n$$

is called the ***nth partial sum*** of the series (1).

Definition 2.1.1 Series (1) is called **convergent** if the sequence $\{S_n\}$ of the partial sums of that series is convergent and the limit S of the sequence $\{S_n\}$ is called the **sum** of the series. In this case we write:

$$S = u_1 + u_2 + \ldots + u_n + \ldots = \sum_{n=1}^{\infty} u_n$$

In the case where $\lim_{n\to\infty} S_n = \pm\infty$ or does not exist, the series is called **divergent.**

Example 2.1.1 Let us decide which is nature (convergent or divergent) of the following series:

G. Paltineanu et al., *Differential Calculus for Engineers*,
https://doi.org/10.1007/978-981-19-2553-5_2

1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$

We notice that the partial sum $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$ can be calculated. Indeed, if in the identity:

$$\frac{1}{k\ (k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

we succesively give to k the values $1, 2, \ldots, n$, we get:

$$\begin{aligned} S_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}. \end{aligned}$$

Since $\lim_{n \to \infty} S_n = 1$, we deduce that the series is convergent and its sum is $S = 1$. Therefore we have $\sum_{n=1}^{\infty} \frac{1}{n\ (n+1)} = 1$.

2. $\sum_{n=1}^{\infty} n = 1 + 2 + \cdots + n + \cdots$

In this case we have $S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Since $\lim_{n \to \infty} S_n = \infty$, the series is divergent.

3. $\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots$

In this case we observe that:

$$S_n = 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$$

Obviously, the limit of the sequence $\{S_n\}$ does not exist, hence the series is divergent.

Example 2.1.2 Geometric series.

The general term of the geometric series is $u_n = a \cdot q^{n-1}$. Therefore the geometric series has the following form:

$$\sum_{n=1}^{\infty} a \cdot\ q^{n-1} = a + a \cdot q + a \cdot q^2 + \cdots + a \cdot q^{n-1} + \cdots$$

The partial sum is $S_n = a + a \cdot q + a \cdot q^2 + \cdots + a \cdot q^{n-1} = a \cdot \frac{1-q^n}{1-q}$, for $q \neq 1$.
If $|q| < 1$, then $\lim_{n \to \infty} q^n = 0$, hence $\lim_{n \to \infty} S_n = \frac{a}{1-q}$.
Therefore, if $|q| < 1$, the geometric series is convergent and has the sum $S = \frac{a}{1-q}$.

If $q = 1$, then $S_n = \underbrace{a + a + a + \cdots + a}_{n \text{ times}} = a \cdot n$ and $\lim\limits_{n \to \infty} S_n = \pm\infty$.

If $q = -1$, then $S_n = a - a + a - \cdots + (-1)^{n-1} \cdot a = \begin{cases} a \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$.

Clearly, in this case the limit of the sequence $\{S_n\}$ does not exist.

If $q > 1$, then $\lim\limits_{n \to \infty} q^n = +\infty$, hence $\lim\limits_{n \to \infty} S_n = \pm\infty$.

Finally, if $q < -1$, then the sequence $\{q^n\}$ does not have the limit, hence neither the sequence $\{S_n\}$ does not have the limit.

In conclusion, the geometric series is convergent if $|q| < 1$ and has the sum $S = \frac{a}{1-q}$ and it is divergent if $|q| \geq 1$.

Further, as an application of the convergent geometric series, we shall present the **conversion of a decimal fraction into a vulgar fraction**.

Case 1. The terminating decimal: $0 \cdot \overline{b_1 b_2 \ldots b_k} = \frac{\overline{b_1 b_2 \ldots b_k}}{10k}$.

For example $0.57\,3 = \frac{573}{1000}$.

Case 2. The pure recurring decimal: $0 \cdot \overline{(b_1 b_2 \ldots b_k)} = \frac{\overline{b_1 b_2 \ldots b_k}}{\underbrace{99\ldots9}_{k \text{ times}}}$.

Indeed,

$$\begin{aligned} 0 \cdot \overline{(b_1 b_2 \ldots b_k)} &= \frac{\overline{b_1 b_2 \ldots b_k}}{10^k} + \frac{\overline{b_1 b_2 \ldots b_k}}{10^{2\ k}} + \cdots \\ &= \frac{\overline{b_1 b_2 \ldots b_k}}{10^k} \left(1 + \frac{1}{10^k} + \frac{1}{10^{2k}} + \cdots\right) \end{aligned}$$

We observe that in parentheses we have a convergent geometric series with the ratio $q = \frac{1}{10^k} \in (-1,\ 1)$. Therefore we have:

$$0.\overline{(b_1\, b_2 \ldots b_k)} = \frac{\overline{b_1\, b_2 \ldots b_k}}{10^k} \cdot \frac{1}{1 - \frac{1}{10^k}} = \frac{\overline{b_1\, b_2 \ldots b_k}}{\underbrace{9\,9 \ldots 9}_{k \text{ times}}}$$

For example $0,\ (573) = \frac{573}{999}$

Case 3. The mixed recurring decimal:

$$0.\overline{b_1\, b_2 \ldots b_k \big(c_1\, c_2 \ldots c_p\big)} = \frac{\overline{b_1\, b_2 \ldots b_k\, c_1\, c_2 \ldots c_p} - \overline{b_1\, b_2 \ldots b_k}}{\underbrace{9\,9 \ldots 9}_{p\ times}\ \underbrace{0\,0 \ldots 0}_{k\ times}}$$

For proof we consider the particular case $k = 1$, $p = 2$:

$$\begin{aligned} 0 \cdot \overline{b_1\,(c_1\, c_2)} &= \frac{b_1}{10} + \frac{\overline{c_1\, c_2}}{10^3} + \frac{\overline{c_1\, c_2}}{10^5} + \cdots \\ &= \frac{b_1}{10} + \frac{\overline{c_1\, c_2}}{10^3} \left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) \end{aligned}$$

$$= \frac{b_1}{10} + \frac{\overline{c_1\,c_2}}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}} = \frac{b_1}{10} + \frac{\overline{c_1\,c_2}}{10} \cdot \frac{1}{99}$$
$$= \frac{99\,b_1 + \overline{c_1\,c_2}}{990} = \frac{\left(10^2 - 1\right) b_1 + 10\,c_1 + c_2}{990}$$
$$= \frac{10^2\,b_1 + 10\,c_1 + c_2 - b_1}{990} = \frac{\overline{b_1\,c_1\,c_2} - b_1}{990}.$$

For example $0,5(7\,3) = \frac{573-5}{990} = \frac{568}{990}$.

Example 2.1.3 Harmonic series.

The harmonic series has the general term $u_n = \frac{1}{n}$, hence the harmonic series is:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

The partial sum is $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln\ n + C - \varepsilon_n$, where $\lim\limits_{n \to \infty} \varepsilon_n = 0$ (Example 1.2.3, Chap. 1). It results that $\lim\limits_{n \to \infty} S_n = +\infty$, hence the harmonic series is divergent.

Proposition 2.1.1 If the series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim\limits_{n \to \infty} u_n = 0$.

Proof Let $S = \lim\limits_{n \to \infty} S_n$. Since $u_n = S_n - S_{n-1}$, it results that:

$\lim\limits_{n \to \infty} u_n = S - S = 0.$

The reverse statement is not true. For example the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent although $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$.

From Proposition 2.1.1 it results the following practical criterion of divergence.

Corollary 2.1.1 *If* $\lim\limits_{n \to \infty} u_n$ *is not equal to zero or does not exist, then the series* $\sum_{n=1}^{\infty} u_n$ *is divergent.*

Thus, when we investigate the convergence of the series $\sum_{n=1}^{\infty} u_n$, you must first check if $\lim\limits_{n \to \infty} u_n = 0$.

Example 2.1.4 The series $\sum_{n=1}^{\infty} \frac{\ln(3 + e^n)}{\ln\left(1 + e^{3\,n}\right)}$ is divergent, because

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\ln e^n\left(1 + 3\ e^{-n}\right)}{\ln e^{3n}\left(1 + e^{-3n}\right)}$$
$$= \lim_{n\to\infty} \frac{n + \ln\left(1 + 3\,e^{-n}\right)}{3n + \ln\left(1 + e^{-3n}\right)} = \frac{1}{3} \neq 0$$

Theorem 2.1.1 (Cauchy criterion) *A necessry and sufficient condition for a number series $\sum_{n=1}^{\infty} u_n$ to be convergent is that for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that for any $n \geq n_\varepsilon$ and any $p \in \mathbb{N}^*$ there holds:*

$$\left|u_{n+1} + u_{n+2} + \ldots + u_{n+p}\right| < \varepsilon$$

Proof The series $\sum_{n=1}^{\infty} u_n$ is convergent if the sequence $\{S_n\}$ of its.

partial sums is convergent. On the other hand, according to Theorem 1.2.1, the sequence $\{S_n\}$ is convergent if it is fundamental i.e.:
$\forall\ \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N}^*$ such that $\forall\ n \geq n_\varepsilon$ and $\forall\ p \in \mathbb{N}^*$ there holds:

$$\left|S_{n+p} - S_n\right| = \left|u_{n+1} + u_{n+2} + \cdots + u_{n+p}\right| < \varepsilon$$

Remark 2.1.1 Deletion of a finite number of terms of a series (or addition of a finite number of terms of a series) does not affect its convergence or divergence.

Indeed, if $\{S_n\}$ is the sequence of partial sums of given series then the sequence of partial sums of modified series is of form $\{S_n \pm c\}$,where c is the sum of terms add or delete.

2.2 Series with Positive Terms

In this section we shall deal with series with positive terms i.e. the series of de form:

$$\sum_{n=1}^{\infty} u_n, u_n > 0 \forall\ n \in \mathbb{N}^*$$

The special place that these series occupy among the numerical series is highlighted by the following theorem.

Theorem 2.2.1 *A necessry and sufficient condition for a series with positive terms to be convergent is that its sequence of partial sums to be bounded above.*

Proof Necessity follows from the fact that if series is convergent then the sequence of partial sums is convergent, hence bounded.

The sufficiency follows from the fact that the sequence of the partial sums of a series with positive terms is increasing and, being also bounded above it is convergent, according to Theorem 1.2.1.

Theorem 2.2.2 (The first comparison test). *Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series with positive terms. Also we suppose that there exists $k \in \mathbb{N}^*$ such that $u_n \leq v_n,\ \ \forall\ n \geq k$. Then:*

(1) *If the series* $\sum_{n=1}^{\infty} v_n$ *is convergent, then the series* $\sum_{n=1}^{\infty} u_n$ *is convergent.*
(2) *If the series* $\sum_{n=1}^{\infty} u_n$ *is divergent, then the series* $\sum_{n=1}^{\infty} v_n$ *is divergent.*

Proof From Remark 2.1.1 it results that deleting possible a finite number of terms of given two series we can suppose that $u_n \leq v_n, \ \forall\, n \in \mathbb{N}^*$.

If we shall denote by $S_n = u_1 + u_2 + \cdots + u_n$ and by $T_n = v_1 + v_2 + \cdots + v_n$, then we deduce that $S_n \leq T_n, \ \forall\, n \in \mathbb{N}^*$. The last inequality implies that the boundedness of the sequence $\{T_n\}$ entails the boundedness of the sequence $\{S_n\}$ and, conversely, the unboundedness of the sequence $\{S_n\}$ entails the unboundedness of the sequence $\{T_n\}$. The Theorem follows now from Theorem 2.2.1.

Example 2.2.1 Find the nature of the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

Since $0 < \ln n < n, \forall n \geq 2$, it results that $\frac{1}{n} < \frac{1}{\ln n}, \ \forall n \geq 2$.

As the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Example 2.1.3), we deduce from Theorem 2.2.2 that the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is also divergent.

Example 2.2.2 **The generalized harmonic series**

The generalized harmonic series is:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = 1 + \frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} + \cdots + \frac{1}{n^{\alpha}} + \cdots, \alpha > 0$$

We shall prove that this series is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

Indeed, if $\alpha \leq 1$ then $n^{\alpha} \leq n$, whence it results $\frac{1}{n} \leq \frac{1}{n^{\alpha}}$.

As the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we deduce, from Theorem 2.2.2, that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is divergent.

If $\alpha > 1$, then there exits $\beta > 0$ such that $\alpha = 1 + \beta$. Further we have:

$$\frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}} < \frac{1}{2^{\alpha}} + \frac{1}{2^{\alpha}} = \frac{2}{2^{\alpha}} = \frac{1}{2^{\beta}}$$
$$\frac{1}{4^{\alpha}} + \frac{1}{5^{\alpha}} + \frac{1}{6^{\alpha}} + \frac{1}{7^{\alpha}} < \frac{1}{4^{\alpha}} + \frac{1}{4^{\alpha}} + \frac{1}{4^{\alpha}} + \frac{1}{4^{\alpha}} = \frac{4}{4^{\alpha}} = \frac{1}{4^{\beta}}$$
$$\ldots$$
$$\frac{1}{\left(2^{k-1}\right)^{\alpha}} + \frac{1}{\left(2^{k-1}+1\right)^{\alpha}} + \cdots + \frac{1}{\left(2^{k}-1\right)^{\alpha}} < \frac{2^{k-1}}{\left(2^{k-1}\right)^{\alpha}} = \frac{1}{\left(2^{k-1}\right)^{\beta}}.$$

Suming the previous inequalities it results:

$$S_{2^k-1} = 1 + \left(\frac{1}{2^{\alpha}} + \frac{1}{3^{\alpha}}\right) + \left(\frac{1}{4^{\alpha}} + \frac{1}{5^{\alpha}} + \frac{1}{6^{\alpha}} + \frac{1}{7^{\alpha}}\right) + \cdots$$

$$+\left(\frac{1}{\left(2^{k}-1\right)^{\alpha}}+\frac{1}{\left(2^{k-1}+1\right)^{\alpha}}+\cdots+\frac{1}{\left(2^{k}-1\right)^{\alpha}}\right)<1$$
$$+\frac{1}{2^{\beta}}+\frac{1}{4^{\beta}}+\cdots+\frac{1}{\left(2^{k-1}\right)^{\beta}}.$$

We observe now that in the right side of the last inequality we have the sum of a geometric progression with the ratio $q = \frac{1}{2^{\beta}}$. Further we have:

$$S_{2^k-1} < \frac{1-\left(\frac{1}{2^{\beta}}\right)^k}{1-\frac{1}{2^{\beta}}} < \frac{1}{1-\frac{1}{2^{\beta}}} = M$$

Therefore the subsequence of the partial sums $\left\{S_{2^k-1}\right\}$ is bounded above.

On the other hand, it is clear that for any $n \in \mathbb{N}^*$, there exits $k \in \mathbb{N}^*$ such that $n < 2^k - 1$, whence it results that $S_n < S_{2^k-1} < M$, $\forall\ n \in \mathbb{N}^*$.

In conclusion, the sequence of partial sums $\{S_n\}$ is bounded above, hence, according to Theorem 2.2.1, the general harmonic series is convergent.

Theorem 2.2.3 (The second comparison test). Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series with positive terms. Also we suppose that there exists $k \in \mathbb{N}^*$ such that:

$$\frac{u_{n+1}}{u_n} \le \frac{v_{n+1}}{v_n}, \quad \forall\ n \ge k$$

Then:

(1) *If the series $\sum_{n=1}^{\infty} v_n$ is convergent, then the series $\sum_{n=1}^{\infty} u_n$ is convergent.*
(2) *If the series $\sum_{n=1}^{\infty} u_n$ is divergent, then the series $\sum_{n=1}^{\infty} v_n$ is divergent.*

Proof From Remark 2.1.1 it results that we can suppose that $\frac{u_{n+1}}{u_n} \le \frac{v_{n+1}}{v_n}$, $\forall n \ge 1$. Therefore we have $\frac{u_{n+1}}{v_{n+1}} \le \frac{u_n}{v_n}$, $\forall\ n \in \mathbb{N}^*$, and further: $\frac{u_n}{v_n} \le \frac{u_{n-1}}{v_{n-1}} \le \cdots \le \frac{u_2}{v_2} \le \frac{u_1}{v_1}$,

whence it results $u_n \le \frac{u_1}{v_1} \cdot v_n$, $\forall n \in \mathbb{N}^*$.

The assertion follows now from Theorem 2.2.2.

Theorem 2.2.4 (The third comparison test). *Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series with positive terms, and we suppose that there exists:*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l \in [0,\ \infty]$$

Then:

(1) *If $l \in (0,\ \infty)$ then those two series have the same nature.*
(2) *If $l = 0$ and the series $\sum_{n=1}^{\infty} v_n$ is convergent then the series $\sum_{n=1}^{\infty} u_n$ is convergent.*

(3) *If $l = +\infty$ and the series $\sum_{n=1}^{\infty} v_n$ is divergent, then the series $\sum_{n=1}^{\infty} u_n$ is divergent.*

Proof

(1) Let us suppose $l \in (0, \infty)$. Since $l > 0$ there exists $\varepsilon > 0$ such that $l - \varepsilon > 0$ and let $n_\varepsilon \in \mathbb{N}^*$ be with the property $l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon, \ \forall\, n \geq n_\varepsilon$.

As $v_n > 0$, $n \geq 1$, it results $(l - \varepsilon) \cdot v_n < u_n < (l + \varepsilon) \cdot v_n, \ \forall\, n \geq n_\varepsilon$. The assertion follows now from Theorem 2.2.2.

(2) $l = 0$, then, $\forall\, \varepsilon > 0, \ \exists\, n_\varepsilon \in \mathbb{N}^*$ such that:

$$-\varepsilon \cdot v_n < u_n < \varepsilon \cdot v_n, \ \forall\, n \geq n_\varepsilon$$

The assertion follows now from Theorem 2.2.2.

(3) If $l = +\infty$, then, $\forall\, \varepsilon > 0, \ \exists\, n_\varepsilon \in \mathbb{N}^*$ such that $u_n > \varepsilon\, v_n, \forall\, n \geq n_\varepsilon$.

The assertion follows now from Theorem 2.2.2.

Example 2.2.3 Find the nature of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot \sqrt[n]{n}}$.

Let us denote by $u_n = \frac{1}{n^2 \cdot \sqrt[n]{n}}$ and by $v_n = \frac{1}{n^2}$. Since $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$ it results $\lim\limits_{n \to \infty} \frac{u_n}{v_n} = 1 \in (0, \infty)$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Example 2.2.2, $\alpha = 2 > 1$) it results, from Theorem 2.2.4, that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot \sqrt[n]{n}}$ is also convergent.

Theorem 2.2.5 (Cauchy's root test). *Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let.*

$L = \overline{\lim\limits_{n \to \infty}} \sqrt[n]{u_n}$*.Then:*

(1) If $L < 1$, then the series is convergent.
(2) If $L > 1$, then the series is divergent.

Proof

(1) Let $q \in \mathbb{R}$ such that $L < q < 1$. From Remark 1.3.2 it results that there exists only a finite number of terms $\sqrt[n]{u_n} > q$. In other words, there exists $k \in \mathbb{N}^*$ such that $\sqrt[n]{u_n} < q$, $\forall\, n \geq k$. Therefore we have $u_n < q^n$, $\forall\, n \geq k$.

As the series $\sum_{n=1}^{\infty} q^n$ is convergent, being a geometric series with the ratio $0 < q < 1$, from Theorem 2.2.2 it follows that the series $\sum_{n=1}^{\infty} u_n$ is convergent.

(2) If $L > 1$, then there exists an infinite term $\sqrt[n]{u_n} > 1$. Thus $u_n > 1$ for an infinity of indices, hence the series $\sum_{n=1}^{\infty} u_n$ is divergent according to Proposition 2.1.1.

Corollary 2.2.1 *Let* $\sum_{n=1}^{\infty} u_n$ *be a series with positive terms, and we suppose that there exists* $l = \lim_{n \to \infty} \sqrt[n]{u_n}$. *Then*:

(1) If $l < 1$ then the series is convergent.
(2) If $l > 1$ then the series is divergent.

Proof The assertion follows from Theorem 2.2.5.

Example 2.2.4 Find the nature of the series $\sum_{n=1}^{\infty} \left[3 + (-1)^n\right]^n \cdot \frac{1}{a^n}$, for $a > 0$.

If we denote by $u_n = \left[3 + (-1)^n\right]^n \cdot \frac{1}{a^n} > 0, \; n \geq 1$, then

$$L = \overline{\lim_{n \to \infty}} \sqrt[n]{u_n} = \overline{\lim_{n \to \infty}} \frac{1}{a} \cdot \left[3 + (-1)^n\right] = \frac{4}{a}$$

From Theorem 2.2.5 it follows that the series is convergent if $a > 4$ and divergent if $a < 4$.

If $a = 4$, then $u_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$.

Since $\lim_{n \to \infty} u_n \neq 0$, according to Proposition 2.1.1, it results that the series is divergent.

Example 2.2.5 Find the nature of the series $\sum_{n=1}^{\infty} \frac{n^3}{\left(e + \frac{1}{n}\right)^n}$.

Since $\lim_{n \to \infty} \sqrt[n]{n^3} = 1$, it results that $\lim_{n \to \infty} \sqrt[n]{u_n} = \lim_{n \to \infty} \frac{\sqrt[n]{n^3}}{e + \frac{1}{n}} = \frac{1}{e} < 1$.
From Corollary 2.2.1 we deduce that the series is convergent.

Theorem 2.2.6 (D'Alembert's test)**.** *Let* $\sum_{n=1}^{\infty} u_n$ *be a series with positive terms, and we suppose that there exists* $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$. *Then*:

(1) *If* $l < 1$ *the series is convergent.*
(2) *If* $l > 1$ *the series is divergent.*

Proof Since $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$, for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that:

$$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon \tag{2.2}$$

(1) As $l < 1$, we can choose $\varepsilon > 0$ with the property $q = l + \varepsilon < 1$. From (2.2) it results that:

$$u_{n+1} < q \cdot u_n, \ \forall n \geq n_\varepsilon \tag{2.3}$$

According to the Remark 2.1.2 we can suppose that $n_\varepsilon = 1$. If in the inequality (2.3) we successively give to n the values $1, 2, \ldots, n-1, \ldots$ we get:

$$\begin{aligned}
&u_2 \leq q \cdot u_1 \\
&u_3 \leq q \cdot u_2 \leq q^2 \cdot u_1 \\
&\ldots \\
&u_n \leq q \cdot u_{n-1} \leq q^{n-1} \cdot u_1 \\
&\ldots
\end{aligned}$$

Therefore we have $u_n \leq u_1 \cdot q^{n-1}$, $\forall n \in \mathbb{N}^*$. The assertion follows now from Theorem 2.2.2, since the series $\sum_{n=1}^{\infty} u_1 \cdot q^{n-1}$ is convergent, being a geometric series with the ratio $q \in (0, 1)$.

(2) As $l > 1$, we can choose $\varepsilon > 0$ such that $l - \varepsilon > 1$. From (2.2) it follows that:

$$\frac{u_{n+1}}{u_n} > l - \varepsilon > 1, \forall n \geq n_\varepsilon.$$

From Remark 2.1.1 we can suppose that $n_\varepsilon = 1$. Therefore we have:

$$u_n < u_{n+1}, \ \forall n \geq 1$$

Thus, the sequence $\{u_n\}$ is strictly increasing and so does not have the limit 0. The assertion follows now from Corollary 2.1.1.

Example 2.2.6 Find the nature of the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Since $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{(n+1)\, n^n}{(n+1)^{n+1}} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$, it results, according to previous theorem, that the series is convergent.

Theorem 2.2.7 (Raabe's test). *Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms, and we suppose that there exists $\lim\limits_{n\to\infty} n \cdot \left(\frac{u_n}{u_{n+1}} - 1\right) = l$. Then*:

(1) *If $l > 1$ the series is convergent.*
(2) *If $l < 1$ the series is divergent.*

Proof Since $\lim\limits_{n\to\infty} n \cdot \left(\frac{u_n}{u_{n+1}} - 1\right) = l$ it results that $\forall \varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}^*$ such that:

$$l - \varepsilon < n \cdot \left(\frac{u_n}{u_{n+1}} - 1\right) < l + \varepsilon, \forall n \geq n_\varepsilon \tag{2.4}$$

(1) As $l > 1$ we can choose $\varepsilon > 0$ such that $\alpha = l - \varepsilon > 1$. Deleting (if necessary) a finite number of terms of the series we can suppose, according to Remark 2.1.2, that $n_\varepsilon = 1$. Further, from (2.4) we deduce:

$n \cdot u_n - n \cdot u_{n+1} \geq \alpha \cdot u_{n+1}, \forall n \geq 1.$
If we successively give to n the values $1, 2, \ldots, n$ we get:

$$\begin{aligned}
&u_1 - u_2 \geq \alpha \cdot u_1\\
&2 \cdot u_2 - 2 \cdot u_3 \geq \alpha \cdot u_3\\
&3 \cdot u_3 - 3 \cdot u_4 \geq \alpha \cdot u_4\\
&\ldots\\
&n \cdot u_n - n \cdot u_{n+1} \geq \alpha \cdot u_{n+1}
\end{aligned}$$

By suming the previous inequalities it results:

$$S_n = u_1 + u_2 + \cdots + u_n \geq \alpha \cdot (S_n - u_1 + u_{n+1}) > \alpha \cdot (S_n - u_1)$$

and further:

$$S_n \leq \frac{\alpha \cdot u_1}{\alpha - 1}, \forall n \in \mathbb{N}^*$$

Therefore the sequence of the partial sums of the series is bounded above, hence the series is convergent.

(2) Since $l < 1$, we can choose $\varepsilon > 0$ such that $l + \varepsilon < 1$. From (2.4) it follows that:

$n \cdot \left(\frac{u_n}{u_{n+1}} - 1\right) < 1$, hence $n \cdot u_n \leq (n + 1) \cdot u_{n+1}$, $\forall n \geq 1$.
Further we have:

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} \leq \frac{u_{n+1}}{u_n}, \quad \forall\, n \geq 1$$

The assertion follows now from Theorem 2.2.3, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Example 2.2.7 Find the nature of the series:

$$\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \cdot \ldots \cdot (3n)}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)} \cdot \frac{1}{n+2}$$

If we denote the general term of the series by:

$$u_n = \frac{3 \cdot 6 \cdot 9 \cdot \ldots \cdot (3n)}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)} \cdot \frac{1}{n+2}$$

then we have:

$$\lim_{n\to\infty} n \cdot \left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left[\frac{(3n+2)\cdot(n+3)}{(3n+3)\cdot(n+2)} - 1\right]$$
$$= \lim_{n\to\infty} \frac{2n^2}{3n^2+9n+6} = \frac{2}{3} < 1.$$

From the previous theorem it resuts that the series is divergent.

2.3 Series with Arbitrary Terms

In this section we investigate series whose terms are real numbers of any sign. The series that have only a finite number of positive (negative) terms can be assimilated with a series with positive terms.

For the series with arbitrary terms we already have a convergence test, namely Cauchy's test (Theorem 2.1.1). Further we present another important test.

Theorem 2.3.1 (Dirichlet-Abel's test). *Let $\{a_n\}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$, and let $\{v_n\}$ be a sequence such that there exists $M > 0$ with the property:*

$$|v_1 + v_2 + \ldots + v_n| \le M,\ \forall n \in \mathbb{N}^*.$$

Then the series $\sum_{n=1}^{\infty} a_n \cdot v_n$ is convergent.

Proof If we denote by $S_n = v_1 + v_2 + \ldots + v_n$, $\forall n \in \mathbb{N}^*$, then by hypothesis there exists $M > 0$ such that $|S_n| \le M$, $\forall n \in \mathbb{N}^*$. Also, we shall denote by $\{T_n\}$ the sequence of the partial sums of the series $\sum_{n=1}^{\infty} a_n \cdot v_n$, i.e.:

$$T_n = a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n, \quad \forall \in \mathbb{N}^*$$

First we remark that, since the sequence $\{a_n\}$ is decreasing, it follows:

$$|a_k - a_{k+1}| = a_k - a_{k+1}, \text{ for any } k \in \mathbb{N}^*.$$

Futher we have:

$$\left|T_{n+p} - T_n\right|$$

$$
\begin{aligned}
&= \left|a_{n+1} \cdot v_{n+1} + a_{n+2} \cdot v_{n+2} + \cdots + a_{n+p} \cdot v_{n+p}\right| \\
&= \left|a_{n+1} \cdot (S_{n+1} - S_n) + a_{n+2} \cdot (S_{n+2} - S_{n+1}) + \cdots + a_{n+p} \cdot \left(S_{n+p} - S_{n+p-1}\right)\right| \\
&= |-a_{n+1} \cdot S_n + (a_{n+1} - a_{n+2}) \cdot S_{n+1} + \cdots \\
&\quad + \left(a_{n+p-1} - a_{n+p}\right) \cdot S_{n+p-1} + a_{n+p} \cdot S_{n+p}| \\
&\le a_{n+1} \cdot |S_n| + (a_{n+1} - a_{n+2}) \cdot |S_{n+1}| + \cdots \\
&\quad + \left(a_{n+p-1} - a_{n+p}\right) \cdot \left|S_{n+p-1}\right| + a_{n+p} \cdot \left|S_{n+p}\right| \\
&\le M \cdot \left(a_{n+1} + a_{n+1} - a_{n+2} + \cdots + a_{n+p-1} - a_{n+p} + a_{n+p}\right) = 2 \cdot M \cdot a_{n+1}.
\end{aligned}
$$

Therefore for any n and $p \in \mathbb{N}^*$ we have $\left|T_{n+p} - T_n\right| \le 2 \cdot M \cdot a_{n+1}$. As $\lim\limits_{n \to \infty} a_n = 0$, it follows that for any $\forall\ \varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N}^*$ such that:

$$a_n < \frac{\varepsilon}{2 \cdot M},\ \forall\ n \ge n_\varepsilon$$

In conclusion, for any $n \ge n_\varepsilon$ and any $p \in \mathbb{N}^*$ we have

$$\left|T_{n+p} - T_n\right| \le 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$$

From Theorem 2.1.1 we deduce now that the series $\sum_{n=1}^{\infty} a_n \cdot v_n$ is convergent.

Example 2.3.1 Find the nature of the series $\sum_{n=1}^{\infty} \frac{\sin n^2 \cdot \sin n}{\sqrt{n+1}}$.

Taking into account that $\sin n^2 \cdot \sin n = \frac{\cos n(n-1) - \cos n(n+1)}{2}$, the given series can be written also in the form:

$$\sum_{n=1}^{\infty} \frac{\cos n(n-1) - \cos n(n+1)}{2\sqrt{n+1}}$$

Let us choose $a_n = \frac{1}{2\sqrt{n+1}} > 0$ and $v_n = \cos n(n-1) - \cos n(n+1)$. Clearly, the sequence $\{a_n\}$ is decreasing and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2\sqrt{n+1}} = 0$. On the other hand the sequence $\{S_n\}$ of the partial sums of the series $\sum_{n=1}^{\infty} v_n$ has the property:
$|S_n| = \left|\sum_{k=1}^{n} v_k\right| = |1 - \cos n(n+1)| \le 2, \quad \forall\ n \in \mathbb{N}^*$, hence it is bounded.
From Theorem 2.3.1 it follows now that the series is convergent.

Definition 2.3.1 A series is called **alternating** if its terms are alternately positive and negative.

Therefore an alternating series has the form:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot u_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n-1} u_n + \cdots,\ u_n > 0, n \in \mathbb{N}^*$$

Theorem 2.3.2 (Leibniz's test). *Any alternatimg series* $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot u_n$ *such that the sequence* $\{u_n\}$ *is decreasing and convergent to* 0*, is convergent.*

Proof The assertion follows immediate from Theorem 2.3.1 for:

$a_n = u_n$ and $v_n = (-1)^{n-1}$.

Indeed, by hypothesis, the sequence $\{a_n\}$ is decreasing and $\lim\limits_{n\to\infty} a_n = 0$. On the other hand.

$$S_n = \sum_{k=1}^{n} v_k = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$$

hence $|S_n| \leq 1, \ \forall n \in \mathbb{N}^*$.

A series satisfying the conditions of Theorem 2.3.2 is often called ***Leibniz's series***.

Example 2.3.2 Find the nature of the series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\cdot\frac{1}{n} + \cdots$$

The series is convergent, since the sequence $\left\{\frac{1}{n}\right\}$ is decreasing, and $\lim\limits_{n\to\infty}\frac{1}{n} = 0$.

For calculate the sum of this series we establish in advance the following equality:

$$\begin{aligned} S_{2\cdot n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - 2\cdot\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\ &= \ln(2\cdot n) + C + \varepsilon_{2\cdot n} - (\ln n + C + \varepsilon_n) = \ln 2 + \varepsilon_{2\cdot n} - \varepsilon_n. \end{aligned}$$

We mention that C is Euler's constant and $\lim\limits_{n\to\infty}\varepsilon_{2n} = \lim\limits_{n\to\infty}\varepsilon_n = 0$ (Example 1.2.3).

Therefore $\lim\limits_{n\to\infty} S_{2n} = \ln 2$. As the sequence $\{S_n\}$ is convergent it follows $\lim\limits_{n\to\infty} S_n = \ln 2$.

In conclusion we have:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\cdot\frac{1}{n} + \cdots$$

2.4 Approximating the Sum of a Leibniz's Series

Since the exact compute of the sum of a convergent series is generally not possible, it is customary to approximate the sum S by a partial sum S_n. The absolute error is:

$$|R_n| = |S - S_n|$$

In the particular case of a Leibniz's series we have $|R_n| = |S - S_n| < u_{n+1}$.

Indeed, let $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot u_n$ be a Leibniz's series. Since the sequence $\{u_n\}$ is decreasing it follows:

$$S_{2n} = S_{2n-2} + (u_{2n-1} - u_{2n}) \geq S_{2n-2}$$
$$S_{2n+1} = S_{2n-1} - (u_{2n} - u_{2n+1}) \leq S_{2n-1}.$$

If we denote by S the sum of the series then $S_{2n} \nearrow S$ and $S_{2n+1} \searrow S$. Thus, we have:

$$S_2 < S_4 < \cdots < S_{2n} < \cdots < S < \cdots < S_{2n+1} < \cdots < S_3 < S_1$$

whence it results:

$$0 < S_{2n+1} - S < S_{2n+1} - S_{2n+2} = u_{2n+2}$$

$$0 < S - S_{2n} < S_{2n+1} - S_{2n} = u_{2n+1}$$

Therefore we have:

$$|R_n| = |S - S_n| < u_{n+1}.$$

The absolute error due to replacement of the sum of a Leibniz's series by its n-th partial sum is not larger, in absolute value, than the first discarded terms.

Example 2.4.1 Compute to exact four decimal the sum of the series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^n} = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

According to the above must be that $u_{n+1} = \frac{1}{(n+1)^{n+1}} < 10^{-}$, whence it results $n \geq 5$. Therefore we put approximately:

$$S \simeq S_5 = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} = 0.78345$$

2.5 Absolutely and Conditionally Convergent Series

Definition 2.5.1 A series with arbitrary terms $\sum_{n=1}^{\infty} u_n$ is said to be **absolutely convergent** if the series $\sum_{n=1}^{\infty} |u_n| = |u_1| + |u_2| + \cdots + |u_n| + \cdots$ is convergent.

Theorem 2.5.1 *Any absolutely convergent series is convergent.*

Proof Indeed, if we suppose that the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent; then the series $\sum_{n=1}^{\infty} |u_n|$ is convergent and, by Cauchy's test (Theorem 2.1.1) it results that $\forall \varepsilon > 0, \exists\ n_\varepsilon \in \mathbb{N}^*$ such that:

$|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \varepsilon, \quad \forall\ n \geq n_\varepsilon, \quad \forall\ p \in \mathbb{N}^*$
On the other hand we have:

$$\left|u_{n+1} + u_{n+2} + \cdots + u_{n+p}\right| \leq |u_{n+1}| + |u_{n+2}| + \cdots + \left|u_{n+p}\right| < \varepsilon, \quad \forall n \geq n_\varepsilon, \forall\ p \in \mathbb{N}^*.$$

From Cauchy's criterion it follows now that the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Remark 2.5.1 The reverse statement is generally not true. There are series (with arbitrary terms) which are convergent but not absolutely convergent. This can be demonstrated by the example of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ which is convergent (Example 2.3.2) but the series $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Example 2.1.3).

Definition 2.5.2 The series $\sum_{n=1}^{\infty} u_n$ is called **conditionally convergent** if it is convergent and the series $\sum_{n=1}^{\infty} |u_n|$ is divergent.

Remark 2.5.2 The series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + (-1)^n \cdot \frac{1}{n} + \cdots$$

is conditionally convergent.

One of the most important properties of the sum of a finite number of real summands is the ***commutativity property*** which asserts that a rearrangement of the summands does not affect the sum. A question naturally arises whether this property remains valid for the sum of a convergent series. The answer generally is negative.

Example 2.5.2 Let us consider the following Leibniz's series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n+1} - \frac{1}{2n} + \cdots$$

This series is convergent and has the sum $S = \ln 2$(Example 2.3.2). If $\{S_n\}$ denoted the sequence of the partial sums of this series then $\lim\limits_{n\to\infty} S_n = \ln 2$.

Let us now rearrange the terms of series so:

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots$$
$$+ \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots$$

If we denote by $\{S'_n\}$ the sequence of the partial sums of the modified series we have:

$$S'_{3\cdot n} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}\right) = \sum_{k=1}^{n} \left(\frac{1}{4k-2} - \frac{1}{4k}\right)$$
$$= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k}\right) = \frac{1}{2} \cdot S_{2n}.$$

Obviously we have also the relations:

$$S'_{3n-1} = \frac{1}{2} S_{2n} + \frac{1}{4n}$$
$$S'_{3n-2} = S'_{3n-1} + \frac{1}{4n-2}.$$

Therefore there exists $\lim\limits_{n\to\infty} S'_n = \frac{1}{2} \ln 2 \neq S = \ln 2$. Thus we proved that, by rearrangement of the terms of initial series the sum is modified. The concrete example we have considered shows that a conditionally convergent series does not posse the commutativity property.

Further we present, without proof, the following theorem:

Theorem 2.5.2 (Riemann's theorem). *If a series is conditionally convergent, then, for any specified number A, the terms of the series can be rearranged so that the resultant series will have the sum A.*

Moreover, the terms of a conditionally convergent series can be rearranged so that the resultant series will be divergent.

From Riemann's theorem we deduce that a conditionally convergent series does not posse the commutativity property.

Further we shall prove that the commutativity property is valid for every absolutely convergent series.

Theorem 2.5.2 *Any series with positive terms convergent has the commutativity property.*

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms convergent, and let S be its sum. Let $\sum_{n=1}^{\infty} v_n$ be the series obtaining from the series $\sum_{n=1}^{\infty} u_n$ by rearranging the terms. Obviously for any $n \in \mathbb{N}^*$ there exists $k_n \in \mathbb{N}^*$ such that $v_n = u_{k_n}$.

Since $T_n = v_1 + v_2 + \cdots + v_n \leq S$ it follows that the series $\sum_{n=1}^{\infty} v_n$ is convergent and that its sum $T \leq S$. On the other hand, the series $\sum_{n=1}^{\infty} u_n$ can be seen as a series obtaining from the series $\sum_{n=1}^{\infty} v_n$ by rearranging the terms, where from it follows that $S \leq T$, hence $S = T$.

Theorem 2.5.4 (Cauchy's theorem). *If the terms of an absolutely convergent series are arbitrarily rearranged, then the series ramain absolutely convergent, and its sum remains the same.*

Proof Let us consider the absolutely convergent series $\sum_{n=1}^{\infty} u_n$ and let S be its sum.

Let us put:

$$u_n^+ = \begin{cases} u_n \text{ if } u_n > 0 \\ 0 \text{ if } u_n < 0 \end{cases} \text{ and } u_n^- = \begin{cases} -u_n \text{ if } u_n < 0 \\ 0 \text{ if } u_n > 0 \end{cases}$$

We observe that we have $u_n = u_n^+ - u_n^-$ and $|u_n| = u_n^+ + u_n^-$.

Let us consider the series with the positive terms $\sum_{n=1}^{\infty} u_n^+$and $\sum_{n=1}^{\infty} u_n^-$. We remark that this two series are convergent because the series $\sum_{n=1}^{\infty} |u_n|$ is convergent and $u_n^+ \leq |u_n|$, $u_n^- \leq |u_n|$ (See Theorem 2.2.2).

If we denote by $\{S_n\}$ the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n$ then we have:

$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} \left(u_k^+ - u_k^-\right) = \sum_{k=1}^{n} u_k^+ - \sum_{k=1}^{n} u_k^-$$

Taking into account that the series $\sum_{n=1}^{\infty} u_n^+$and $\sum_{n=1}^{\infty} u_n^-$ are convergent it rsults:

$$\sum_{n=1}^{\infty} u_n = S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} u_k^+ - \lim_{n\to\infty} \sum_{k=1}^{n} u_k^- = \sum_{n=1}^{\infty} u_n^+ - \sum_{n=1}^{\infty} u_n^-$$

Therefore we have:

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} u_n^+ - \sum_{n=1}^{\infty} u_n^- \tag{2.5}$$

Let $\sum_{n=1}^{\infty} v_n$ be the series obtained by rearranging the terms of the initial series $\sum_{n=1}^{\infty} u_n$. Obviously we can construct, as above, the corresponding numbers v_n^+ and v_n^-. We remark that the series $\sum_{n=1}^{\infty} v_n$ is also absolutely convergent because:

$$|v_1| + |v_2| + \cdots + |v_n| = |u_{k_1}| + |u_{k_2}| + \cdots + |u_{k_n}| \le \overline{S} = \sum_{n=1}^{\infty} |u_n|$$

Then we have a similar formula with (1):

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} v_n^+ - \sum_{n=1}^{\infty} v_n^- \tag{2.6}$$

Of course, the series $\sum_{n=1}^{\infty} v_n^+$ and $\sum_{n=1}^{\infty} v_n^-$ are series obtained from the series $\sum_{n=1}^{\infty} u_n^+$and $\sum_{n=1}^{\infty} u_n^-$ by rearranging the terms. According to Theorem 2.5.3 it results that $\sum_{n=1}^{\infty} v_n^+ = \sum_{n=1}^{\infty} u_n^+$ and $\sum_{n=1}^{\infty} v_n^- = \sum_{n=1}^{\infty} u_n^-$. Then from (1) and (2) we deduce that $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} u_n$ and so the proof is finished.

2.6 Operations on Convergent Series

We shall discuss here the problem of addition and multiplication of convergent series.

Theorem 2.6.1 *If the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent, having the sums U respectively V, then the series $\sum_{n=1}^{\infty} (\alpha \cdot u_n + \beta \cdot \mathrm{v}_n)$ is convergent for any $\alpha, \beta \in \mathbb{R}$, and has the sum $\alpha \cdot U + \beta \cdot V$.*

Proof The assertion follows immediate from the equality:

$$\sum_{k=1}^{n} (\alpha \cdot\ u_k + \beta \cdot v_k) = \alpha \cdot \sum_{k=1}^{n} u_k + \beta \cdot \sum_{k=1}^{n} v_k$$

By the product of the series $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ is meant any series of the form $\sum_{k,l} u_k \cdot v_l$. Therefore there are an infinity ways to multiply two series. Of these, the best known are the following two:

$$u_1 \cdot v_1 + (u_1 \cdot v_2 + \ u_2 \cdot v_1) + \cdots + (u_1 \cdot v_n + u_2 \cdot v_{n-1} + \cdots + u_n \cdot v_1) + \cdots \tag{2.7}$$

$$\begin{aligned} &u_1 \cdot v_1 + (u_1 \cdot v_2 + u_2 \cdot v_2 + u_2 \cdot v_1) + \cdots \\ &\quad + (u_1 \cdot v_n + u_2 \cdot v_n + \cdots + u_n \cdot v_n \ + \cdots + u_n \cdot v_1) + \cdots . \end{aligned} \tag{2.8}$$

We mention that the product of two convergent series, generally, is not convergent.

Theorem 2.6.2 *If the series $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} u_n$ are absolutely convergent, having the sums U respectively V, then any series product of these series is also absolutely convergent and has the sum $U \cdot V$.*

Proof Let $\sum_{k=1}^{\infty} u_{i_k} \cdot v_{j_k}$ be an arbitrary series product of these given series. Then we have:

$$\left|u_{i_1} \cdot v_{j_1}\right| + \left|u_{i_2} \cdot v_{j_2}\right| + \cdots + \left|u_{i_n} \cdot v_{j_n}\right| \\ \leq (|u_1| + \cdots + |u_m|) \cdot (|v_1| + \cdots + |v_m|)$$

where $m = \max\{i_1, \ldots, i_n; j_1, \ldots, j_n\}$. Since the series $\sum_{n=1}^{\infty} |u_n|$, $\sum_{n=1}^{\infty} |v_n|$ are convergent it rsults that the series $\sum_{k=1}^{\infty} u_{i_k} v_{j_k}$ is absolutely convergent, hence convergent.

On the other hand, according to Theorem 2.5.4, the series absolutely convergent are commutative, thus the sum of the series $\sum_{k=1}^{\infty} u_{i_k} \cdot v_{j_k}$ is equal to the sum of the series product (4). But we observe immediate that the sequence $\{P_n\}$ of partial sum of the series (4) is:

$$P_n = (u_1 + u_2 + \cdots + u_n) \cdot (v_1 + v_2 + \cdots + v_n)$$

Therefore the sum of any series product of the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ is equal to $\lim_{n \to \infty} P_n = U \cdot V$.

2.7 Sequences and Series of Complex Numbers

Further we shall denote by $\mathbb{C}$ the set of complex number i.e.:

$\mathbb{C} = \left\{z = a + i \cdot b \mid a, b \in \mathbb{R}, i = \sqrt{-1}\right\}$.

The modulus of the complex number $z = a + i \cdot b$ is denoted by $|z|$ and is definite so:

$|z| = \sqrt{a^2 + b^2}$.

Definition 2.7.1 A sequence of complex numbers $\{z_n\}$ is **convergent in** $\mathbb{C}$ and has the limit $z \in \mathbb{C}$, if $\lim_{n \to \infty} |z_n - z| = 0$. We shall denote this by $z_n \xrightarrow{\mathbb{C}} z$.

Therefore $z_n \xrightarrow{\mathbb{C}} z$ if $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}^*$ such that $|z_n - z| < \varepsilon, \forall n \geq n_\varepsilon$.

Theorem 2.7.1 *A sequence of complex numbers* $\{z_n\}$,

$z_n = a_n + i \cdot b_n$, $\forall n \in \mathbb{N}^*$, *is convergent in* $\mathbb{C}$, *and has the limit* $z = a + i\, b$, *if and only if,* $a_n \xrightarrow{\mathbb{R}} a$ and $b_n \xrightarrow{\mathbb{R}} b$.

Proof The assertion follows immediate from the inequalities:

$$|a_n - a| \leq |z_n - z| = \sqrt{(a_n - a)^2 + (b_n - b)^2} \leq |a_n - a| + |b_n - b|$$

$$|b_n - b| \le |z_n - z| = \sqrt{(a_n - a)^2 + (b_n - b)^2} \le |a_n - a| + |b_n - b|$$

Example 2.7.1 The sequence:

$$z_n = \frac{4 \cdot n + 5}{2 \cdot n - 3} + i \cdot \frac{n^3 + 3 \cdot n - 7}{7 \cdot n^3 + 1} \xrightarrow{\mathbb{C}} z = \frac{1}{2} + i \cdot \frac{1}{7}$$

Definition 2.7.2 A sequence of complex numbers $\{z_n\}$, is **fundamental in** $\mathbb{C}$ if:

$\forall\, \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N}^*$ such that $\left|z_{n+p} - z_n\right| < \varepsilon,\ \forall n \ge n_\varepsilon,\ \forall\, p \in \mathbb{N}^*$.

Theorem 2.7.2 *A sequence of complex numbers* $\{z_n\}$.

$z_n = a_n + i\, b_n,\ \forall\, n \in \mathbb{N}^*$ *is fundamental in* $\mathbb{C}$ *if the sequences of real numbers* $\{a_n\}, \{b_n\}$ *are fundamental in* $\mathbb{R}$.

Proof The assertion follows immediate from the inequalities:

$$\left|a_{n+p} - a_n\right| \le \left|z_{n+p} - z_n\right| = \sqrt{\left(a_{n+p} - a_n\right)^2 + \left(b_{n+p} - b_n\right)^2} \le \left|a_{n+p} - a_n\right| + \left|b_{n+p} - b_n\right|.$$

$$\left|b_{n+p} - b_n\right| \le \left|z_{n+p} - z_n\right| = \sqrt{\left(a_{n+p} - a_n\right)^2 + \left(b_{n+p} - b_n\right)^2} \le \left|a_{n+p} - a_n\right| + \left|b_{n+p} - b_n\right|.$$

Theorem 2.7.3 *A sequence of complex numbers* $\{z_n\}$ *is convergent in* $\mathbb{C}$ *if it is fundamental in* $\mathbb{C}$.

The assertion follows from Theorems 2.7.1 and 2.7.2 and the general Cauchy's test for convergence of real numbers sequences (Theorem 1.2.1).

Definition 2.7.3 A series of complex numbers $\sum_{n=1}^{\infty} z_n$ is called **convergent in** $\mathbb{C}$ if the sequence of its partial sums $\{S_n\}$, $S_n = z_1 + z_2 + \ldots + z_n$, $\forall n \in \mathbb{N}^*$ is convergent in $\mathbb{C}$. If $S = \lim\limits_{n \to \infty} S_n$, then S is the sum of the series and we shall denote this by $S = \sum_{n=1}^{\infty} z_n$.

Theorem 2.7.4 *For the series of complex numbers* $\sum_{n=1}^{\infty} z_n$ *to be convergent it is necessary and sufficient that* $\forall\, \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ *such that* $\forall n \ge n_\varepsilon$ *and* $\forall p \in \mathbb{N}^*$ *we have*:

$$\left|z_{n+1} + z_{n+2} + \cdots + z_{n+p}\right| < \varepsilon$$

The proof is similar to the proof of Theorem 2.1.1.

Definition 2.7.4 A series of complex numbers $\sum_{n=1}^{\infty} z_n$ is said to be **absolutely convergent** if the series of positive numbers $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Theorem 2.7.5 *Any series of complex numbers absolutely convergent is convergent.*

The proof follows from Theorem 2.7.4 and the inequality:

$$\left|z_{n+1} + z_{n+2} + \cdots + z_{n+p}\right| \leq |z_{n+1}| + |z_{n+2}| + \cdots + \left|z_{n+p}\right|$$

As an application of the last theorem we present the exponential function $e^z e^z$.

Let us consider the series $\sum_{n=0}^{\infty} \frac{z^n}{n\ !} = 1 + \frac{z}{1\ !} + \frac{z^2}{2\ !} + \cdots + \frac{z^n}{n!} + \cdots, z \in \mathbb{C}$.

From Theorem 2.2.6 it results that this series is absolutely convergent. Indeed, we have:

$$\lim_{n\to\infty} \frac{\frac{|z|^{n+1}}{(n+1)!}}{\frac{|z|^n}{(n)!}} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0 < 1$$

By Theorem 2.7.5 we deduce that the series is convergent. We will denote by e^z its sum.

Therefore we have:

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n\ !} = 1 + \frac{z}{1!} + \frac{z^2}{2\ !} + \cdots + \frac{z^n}{n\ !} + \cdots, z \in \mathbb{C}$.

Remark 2.7.1 The statement and the proof of Theorem 2.6.2 are true also for the series of complex numbers.:

Theorem 2.7.6 $e^z \cdot e^u = e^{z+u}, \ \forall z, u \in \mathbb{C}$.

Proof According to the definition of the **exponential function** we have:

$$e^z = 1 + \frac{z}{1\ !} + \frac{z^2}{2\ !} + \cdots + \frac{z^n}{n!} + \cdots$$

and

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots + \frac{u^n}{n\ !} + \cdots$$

From Remark 2.7.1 it follows that every series product of these series is absolutely convergent and has the sum $e^z \cdot e^u$. Using the product of the series (3)we have:

$$e^z \cdot e^u = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{z^{n-k} \cdot u^k}{(n-k)! \cdot k\ !} \right) = \sum_{n=0}^{\infty} \left(\frac{1}{n\ !} \cdot \sum_{k=0}^{n} \frac{n\ !}{(n-k)! \cdot k\ !} \cdot z^{n-k} \cdot u^k \right)$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \cdot \sum_{k=0}^{n} C_n^k \cdot z^{n-k} \cdot u^k \right) = \sum_{n=0}^{\infty} \frac{(z+u)^n}{n!} = e^{z+u}.$$

We mention also the following immediate properties of the exponential function:

(1) $e^0 = 1$.
(2) $e^z \cdot e^{-z} = 1, \quad \forall z \in \mathbb{C}$.
(3) $e^{-z} = \frac{1}{e^z}, \forall \; z \in \mathbb{C}$ if $e^z \neq 0$.

Chapter 3
Sequences of Functions (Functional Sequences)

3.1 Simple and Uniformly Convergence

Let us consider a sequence of real valued functions $\{f_n\}$ defined on the set $D \subset \mathbb{R}$ and let also be a function $f : D \to \mathbb{R}$.

Definition 3.1.1 We say that the sequence of functions $\{f_n\}$ is **simple (pointwise) convergent to the function** f **on the set** D, if for any $x \in D$, the real number sequence $\{f_n(x)\}$ converge to $f(x)$, i.e. for any $x \in D$ there exists $\lim_{n\to\infty} f_n(x) = f(x)$. In this case we denote:

$$f_n \xrightarrow[{[D]}]{s} f$$

Remark 3.1.1 Obviously, if the point $x \in D$ changes, then also the number sequence $\{f_n(x)\}$ changes, hence $f_n \xrightarrow[D]{s} f$ if $\forall x \in D$ and $\forall \varepsilon > 0$, $\exists n_{x,\varepsilon} \in \mathbb{N}^*$ such that:

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq n_{x,\varepsilon}.$$

Example 3.1.1 Let us consider the following sequence of functions:

$$f_n(x) = x^n, \quad x \in [0, 1], \quad n \in \mathbb{N}^*.$$

We observe that for any $x \in [0, 1)$, $\lim_{n\to\infty} x^n = 0$ and $f_n(1) = 1$, $\forall n \in \mathbb{N}^*$. Therefore $f_n \xrightarrow[{[0,1]}]{s} f$, where $f(x) = \begin{cases} 0 \text{ for } x \in [0, 1) \\ 1 \text{ for } x = 1 \end{cases}$.

Definition 3.1.2 We say that the sequence of functions $\{f_n\}$ is **uniformly convergent to the function** f **on the set** D, if $\forall \varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that:

$$|f_n(x) - (x)| < \varepsilon, \quad \forall n \geq n_\varepsilon \text{ and } \forall x \in D.$$

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In this case we denote: $f_n \xrightarrow[D]{u} f$.

Remark 3.1.3 What is very essential to Definition 3.1.2, in contrast to Definition 3.1.1 is that n_ε depend only on ε and is independent of $x \in D$.

Example 3.1.2 Study the pointwise convergence and uniformly convergence of the following sequence of functions:

$$f_n : [-\pi, \pi] \to \mathbb{R},\ f_n(x) = \frac{\cos n \ \cdot x}{2 \cdot n^2}, \quad n \in \mathbb{N}^*, \quad x \in [-\pi, \pi].$$

For any $x \in [-\pi, \pi]$, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cos n \cdot x}{2 \cdot n^2} = 0$.
If we denote by f the function $f(x) = 0$, $\forall x \in [-\pi, \pi]$, then $f_n \xrightarrow[[-\pi,\pi]]{s} f$.
We shall prove that $f_n \xrightarrow[[-\pi,\pi]]{u} f$.
Indeed, for any $x \in [-\pi, \pi]$ we have:

$$|f_n(x) - f(x)| = \left|\frac{\cos n\, x}{2 \cdot n^2}\right| = \frac{|\cos n\, x|}{2 \cdot n^2} \le \frac{1}{2 \cdot n^2}.$$

Since $\lim_{n\to\infty} \frac{1}{2 \cdot n^2} = 0$, it results that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ (we can choose for example $n_\varepsilon = \left[\sqrt{\frac{1}{2\varepsilon}}\right] + 1$), such that $\frac{1}{2 \cdot n^2} < \varepsilon$, $\forall n \ge n_\varepsilon$.
Therefore, for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that:

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \ge n_\varepsilon, \quad \forall x \in [-\pi, \pi],$$

hence $f_n \xrightarrow[[-\pi,\pi]]{u} f \equiv 0$.

Remark 3.1.4 The geometrical interpretation of the uniform convergence is the following: for $n \ge n_\varepsilon$, the graph of the function f_n is between the graphs of the functions $f - \varepsilon$ and $f + \varepsilon$ (Fig. 3.1).

Further we shall denote by $\mathscr{B}(D)$ the vector space of all bounded real valued functions defined on D and for any $f \in \mathscr{B}(D)$ we note with:

$$\|f\| = \sup\{|f(x)|; \forall x \in D\}.$$

Remark 3.1.5 We observe that the uniform convergence of the sequence $\{f_n\}$ on D to f is equivalent to the convergence of the number sequence $\{\|f_n - f\|\}$ to zero. That is:

$$f_n \xrightarrow[D]{u} f \text{ if } \lim_{n\to\infty} \|f_n - f\| = 0.$$

Indeed, the assertion follows immediate from the equivalence:

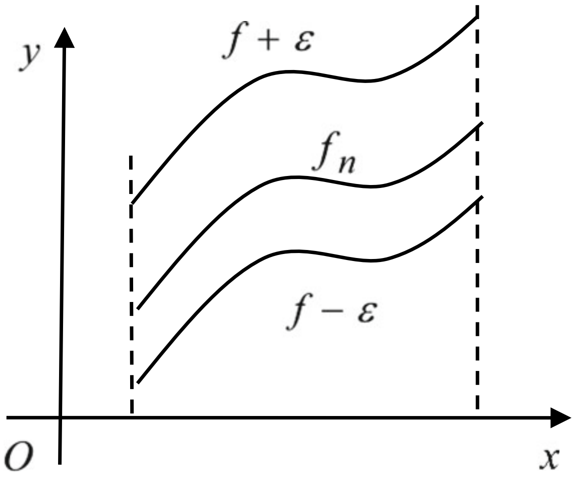

Fig. 3.1 Uniform convergence of a sequence of functions

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in D \Leftrightarrow \|f_n - f\| < \varepsilon$$

Remark 3.1.6 Obviously the uniform convergence involves the simple (pointwise) convergence. The reverse assertion, generally, is not true.

Example 3.1.3 Let us consider the sequence of functions $\{f_n\}$, $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n, n \in \mathbb{N}^*$ and let $f(x) = \begin{cases} 0 \text{ dac } \breve{a} \ x \in [0, 1] \\ 1 \text{ dac } \breve{a} \ x = 1 \end{cases}$.

In Example 3.1.1 we proved that $f_n \xrightarrow[{[0,1]}]{S} f$. On the other hand, we have:

$$\begin{aligned}\|f_n - f\| &= \sup\{|f_n(x) - f(x)|; \forall x \in [0, 1]\} \\ &= \sup\big\{\{x^n; \forall x \in [0, 1)\}; 0\big\} = 1, \quad \forall n \in \mathbb{N}^*.\end{aligned}$$

It results that $\lim_{n\to\infty} \|f_n - f\| = 1 \neq 0$, hence the sequence $\{f_n\}$ is not uniformly convergent to f on $[0, 1]$.

Example 3.1.4 Study the uniformly convergence of the sequence of functions $\{f_n\}$, $f_n(x) = \frac{x}{x+n}$, $n \in \mathbb{N}^*$, on the following sets:

(a) the interval $[a, b]$, $0 < a < b$;
(b) the interval $[0, \infty)$.

(a) Since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0, \forall x \in \mathbb{R}$, it results that

$$f_n \xrightarrow[\mathbb{R}]{s} f \equiv 0,$$

For any $0 < a \le x \le b$, and any $n \in \mathbb{N}^*$, we have:

$$\frac{a}{b+n} \leq \frac{x}{b+n} \leq \frac{x}{x+n} \leq \frac{b}{x+n} \leq \frac{b}{a+n},$$

from which we deduce that:

$$\begin{aligned}\|f_n - f\| &= \sup\{|f_n(x) - f(x)|; x \in [a, b]\} \\ &= \sup\left\{\frac{x}{x+n}; x \in [a, b]\right\} \leq \frac{b}{a+n}.\end{aligned}$$

As $\lim_{n\to\infty} \frac{b}{a+n} = 0$, it follows that $\lim_{n\to\infty}\|f_n - f\| = 0$, hence $f_n \xrightarrow[{[a,b]}]{u} 0$.

(b) For any $x \in [0, \infty)$ and any $n \in \mathbb{N}^*$, we have:

$$\|f_n - f\| = \sup\left\{\frac{x}{x+n}; \forall x \in [0, \infty)\right\} \geq \frac{n}{n+n} = \frac{1}{2},$$

from which results that $\lim_{n\to\infty}\|f_n - f\| \neq 0$, hence the sequence $\{f_n\}$ is not uniformly convergent to f on $[0, \infty)$.

Theorem 3.1.1 (Cauchy's uniformly convergence criterion). *The necessary and sufficient condition for the sequence of functions* $\{f_n\}$ *to be uniformly convergent on the set D to the function f is that for any* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}^*$ *such that:*

$$\left|f_{n+p}(x) - f_n(x)\right| < \varepsilon, \quad \forall x \in D, \quad \forall n \geq n_\varepsilon, \quad \forall p \in \mathbb{N}^*.$$

Proof **Necessity**. If $f_n \xrightarrow[D]{u} f$, then $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \forall n \geq n_\varepsilon, \quad \forall x \in D.$$

If $p \in \mathbb{N}^*$, then all the more so:

$$\left|f_{n+p}(x) - f(x)\right| < \frac{\varepsilon}{2}, \quad \forall n \geq n_\varepsilon, \quad \forall x \in D.$$

Further, for any $n \geq n_\varepsilon$, $p \in \mathbb{N}^*$ and $x \in D$, we have:

$$\left|f_{n+p}(x) - f_n(x)\right| \leq \left|f_{n+p}(x) - f(x)\right| + |f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Sufficiency. By hypothesis it results that $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that:

$$\left|f_{n+p}(x) - f_n(x)\right| < \varepsilon, \quad \forall n \geq n_\varepsilon, \quad \forall p \in \mathbb{N}^*, \quad \forall x \in D. \tag{3.1}$$

On the other hand, it is obvious that for any fixed point $x \in D$, the real number sequence $\{f_n(x)\}$ is fundamental, hence convergent, according to Cauchy's test for convergence of real number sequences (Theorem 1.2.1).

Let us denote by $f(x) = \lim_{n\to\infty} f_n(x)$. If in the inequality (3.1) we pass to the limit on p (i.e., $p \to \infty$), it results:

$$|f(x) - f_n(x)| < \varepsilon, \quad \forall n \geq n_\varepsilon, \quad \forall x \in D,$$

hence $f_n \xrightarrow[D]{u} f$.

The following result establishes a sufficient condition for the uniformly convergence of a functional sequence.

Proposition 3.1.1 *If there exists a sequence of positive numbers* $\{a_n\}$ *with the property* $\lim_{n\to\infty} a_n = 0$, *there exists also* $n_0 \in \mathbb{N}^*$ *such that:*

$$|f_n(x) - f(x)| \leq a_n, \quad \forall n \geq n_0, \quad \forall x \in D,$$

then $f_n \xrightarrow[D]{u} f$.

Proof Obviously we have:

$$\|f_n - f\| = \sup\{|f_n(x) - f(x)|; x \in D\} \leq a_n, \quad \forall n \geq n_0.$$

As $\lim_{n\to\infty} a_n = 0$ it follows that $\lim_{n\to\infty} \|f_n - f\| = 0$, hence $f_n \xrightarrow[D]{u} f$.

Example 3.1.5 Study the pointwise convergence and uniformly convergence of the functional sequence $\{f_n\}$, where:

$$f_n(x) = \frac{n^2 + \cos n \cdot x}{3 \cdot n^2}, \quad n \in \mathbb{N}^*, \quad x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2+\cos n\cdot x}{3\cdot n^2} = \frac{1}{3}$, so $f_n \xrightarrow[\mathbb{R}]{s} \frac{1}{3}$.

Also for any $x \in \mathbb{R}$, we have:

$$\left| f_n(x) - \frac{1}{3} \right| = \left| \frac{n^2 + \cos n \cdot x}{3 \cdot n^2} - \frac{1}{3} \right| = \left| \frac{\cos n \cdot x}{3 \cdot n^2} \right|$$
$$= \frac{|\cos n \cdot x|}{3 \cdot n^2} \leq \frac{1}{3 \cdot n^2} = a_n, \quad \forall n \geq 1$$

Since $a_n = \frac{1}{3\cdot n^2} > 0, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3\cdot n^2} = 0$, from Proposition 3.1.1 it results that $f_n \xrightarrow[\mathbb{R}]{u} \frac{1}{3}$.

3.2 The Properties of the Uniformly Convergent Functional Sequences

Further, we will examine the conditions under which a certain common property (continuity, derivability, integrability) of the terms of a sequence of functions is also transmitted to the limit of this sequence. We note that, as a rule, simple convergence is insufficient to achieve this transfer.

Indeed, resuming Example 3.1.1, we find that although the functions f_n are continuous on the interval [0, 1], its limit f is not a continuous function on [0, 1].

Theorem 3.2.1 *If a functional sequence $\{f_n\}$ converges uniformly on the set D to the function f and if each term f_n of the sequence is continuous on D, then f also is continuous on D.*

Proof Let $x_0 \in D$ be an arbitrary fixed point. For any $x \in D$ and $n \in \mathbb{N}^*$, we have:

$$\begin{aligned} |f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| \\ + |f_n(x_0) - f(x_0)| \end{aligned} \tag{3.2}$$

Since $f_n \xrightarrow[D]{u} f$, then $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*$ such that:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \quad \forall n \geq n_\varepsilon, \quad \forall x \in D.$$

On the other hand, taking into account that f_n is continuous at point x_0 it results that $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$, such that, for any $x \in D$ with the property $|x - x_0| < \delta_\varepsilon$ we have: $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$.

If in the inequality (3.2) we suppose that $n \geq n_\varepsilon$ and $|x - x_0| < \delta_\varepsilon$, then it results:

$$|f(x) - f(x_0)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

hence f is continuous at the point $x = x_0$.

Remark 3.2.1 If we suppose that x_0 is an accumulation (limit) point of the set D (i.e. for any neighborhood V of x_0 there exists $x \in V \cap D, x \neq x_0$), then from Theorem 3.2.1 it follows:

$$\lim_{x \to x_0} \left[\lim_{n \to \infty} f_n(x) \right] = \lim_{n \to \infty} \left[\lim_{x \to x_0} f_n(x) \right].$$

Indeed, taking into account of the continuity of f (respectively f_n) at the point x_0, it results: $\lim_{x \to x_0} f(x) = f(x_0)$, respectively, $\lim_{x \to x_0} f_n(x) = f_n(x_0)$.

Therefore we have:

$$\lim_{x\to x_0}\left[\lim_{n\to\infty} f_n(x)\right] = \lim_{x\to x_0} f(x) = f(x_0) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty}\left[\lim_{x\to x_0} f_n(x)\right].$$

Remark 3.2.2 Theorem 3.2.1 can be used in applications to show that a sequence of functions does is not uniformly convergent.

Example 3.2.1 Show that the sequence:

$$f_n(x) = \frac{1}{1+n^2\cdot x^2}, \quad x\in\mathbb{R}, \quad \forall n\in\mathbb{N}^*,$$

is not uniformly convergent on $\mathbb{R}$.

If we denote by $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} 0 \text{ if } x\neq 0\\ 1 \text{ if } x=0\end{cases}$, then we observe that $f_n \xrightarrow[\mathbb{R}]{s} f$.

Therefore the functions, $f_n, n\in\mathbb{N}^*$, are continuous on $\mathbb{R}$, and the limit function f is not continuous on $\mathbb{R}$. From Theorem 3.2.1 it results that $\{f_n\}$ is not uniformly convergent on $\mathbb{R}$.

Theorem 3.2.2 *Let $\{f_n\}$ be a sequence of differentiable (derivable) functions on the interval $I\subset\mathbb{R}$ with the properties that $f_n \xrightarrow[I]{u} f$, and the sequence of derivatives $\{f_n'\}$ is uniformly convergent on I to the function g. Then function f is differentiable on I and $f'(x)=g(x), \forall x\in I$.*

Proof For any $h\in\mathbb{R}$ such that $x+h\in I$, we have:

$$\begin{aligned}\frac{f_{n+p}(x+h)-f_{n+p}(x)}{h} - g(x) &= \frac{\left(f_{n+p}-f_n\right)(x+h)-\left(f_{n+p}-f_n\right)(x)}{h}\\ &\quad + \left[\frac{f_n(x+h)-f_n(x)}{h} - f_n'(x)\right]\\ &\quad + \left[f_n'(x)-g(x)\right], \quad (\forall) n, p\in\mathbb{N}^*.\end{aligned}$$

From Lagrange's theorem it results that there exists a point c between x and $x+h$ such that:

$$\left(f_{n+p}-f_n\right)(x+h)-\left(f_{n+p}-f_n\right)(x) = \left(f_{n+p}'-f_n'\right)(c)\cdot h.$$

Therefore $\forall x\in I$ and $\forall n, p\in\mathbb{N}^*$ we have:

$$\begin{aligned}\left|\frac{f_{n+p}(x+h)-f_{n+p}(x)}{h} - g(x)\right| &\le \left|f_{n+p}'(c)-f_n'(c)\right|\\ &\quad + \left|\frac{f_n(x+h)-f_n(x)}{h} - f_n'(x)\right|\\ &\quad + \left|f_n'(x)-g(x)\right|. \end{aligned} \tag{3.3}$$

Since $f_n' \xrightarrow[I]{u} g$, from Theorem 3.1.1, it results that $\forall \varepsilon > 0$, $\exists n_\varepsilon' \in \mathbb{N}^*$ such that:

$$\left|f_{n+p}'(c) - f_n'(c)\right| + \left|f_n'(x) - g(x)\right| < \frac{\varepsilon}{2},$$

$\forall n \geq n_\varepsilon'$, $\forall x \in I$ and $\forall p \in \mathbb{N}^*$.

On the other hand, since $\lim_{h\to 0} \frac{f_n(x+h)-f_n(x)}{h} = f_n'(x)$, it follows that there exists $\delta_\varepsilon > 0$ such that:

$$\left|\frac{f_n(x+h) - f_n(x)}{h} - f_n'(x)\right| < \frac{\varepsilon}{2},$$

$\forall x \in I$, $\forall h \in \mathbb{R}$, with the property $x + h \in I$ and $|h| < \delta_\varepsilon$.

If in the inequality (3.3) we suppose $n \geq n_\varepsilon'$, $p \in \mathbb{N}^*$ and $|h| < \delta_\varepsilon$, it results:

$$\left|\frac{f_{n+p}(x+h) - f_{n+p}(x)}{h} - g(x)\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.4}$$

If in the inequality (3.4) we pass to the limit on p (i.e. $p \to \infty$), we obtain:

$$\left|\frac{f(x+h) - f(x)}{h} - g(x)\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\forall x \in I$, $\forall h \in \mathbb{R}$, with $x + h \in I$ and $|h| < \delta_\varepsilon$, where we deduce that there exists

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = g(x), \quad \forall x \in I$$

and thus, the proof is finished.

Example 3.2.2 Establish the nature of the convergence of the sequences $\{f_n\}$ and $\{f_n'\}$, where $f_n(x) = \frac{\arctan n\cdot x}{n}$, $n \in \mathbb{N}^*$, $x \in \mathbb{R}$.

Since $\lim_{n\to\infty} \frac{\arctan n\cdot x}{n} = 0$, $\forall x \in \mathbb{R}$, it results that $f_n \xrightarrow[\mathbb{R}]{s} f \equiv 0$.

We observe also that:

$$|f_n(x) - f(x)| = \left|\frac{\arctan n \cdot x}{n} - 0\right| = \frac{|\arctan n \cdot x|}{n} \leq \frac{\pi}{2 \cdot n}, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}^*$$

As $\lim_{n\to\infty} \frac{\pi}{2\cdot n} = 0$, from Proposition 3.1.1 we deduce that $f_n \xrightarrow[\mathbb{R}]{u} 0$.

On the other hand we have $f_n'(x) = \frac{1}{1+n^2\cdot x^2}$, $n \in \mathbb{N}^*$, $x \in \mathbb{R}$, and $f_n' \xrightarrow[\mathbb{R}]{s} g$, where:

$$g(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}, \quad x \in \mathbb{R}.$$

Since the functions f_n' are continuous on $\mathbb{R}$ and the limit function g is not continuous on $\mathbb{R}$, we deduce from Theorem 3.2.2 that the sequence $\{f_n'\}$ is not uniformly convergent on $\mathbb{R}$.

Moreover, we have:

$$(\lim_{n\to\infty} f_n(x))' = f'(x) = 0,$$

hence $(\lim_{n\to\infty} f_n(x))' \neq \lim\limits_{n\to\infty} f_n'(x)$, since $f'(0) = 0$ and $g(0) = 1$.

Theorem 3.2.3 *Let $\{f_n\}$ be a sequence of continuous functions on the interval $[a, b]$ such that $f_n \xrightarrow[{[a,b]}]{u} f$. Then, there exist:*

$$\lim_{n\to\infty}\left[\int_a^b f_n(x)\mathrm{d}x\right] = \int_a^b f(x)\mathrm{d}x = \int_a^b \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x.$$

Proof From Theorem 3.2.1 it follows that f is a continuous function on $[a, b]$, hence f is an integrable function on $[a, b]$. Further we have:

$$\begin{aligned}\left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)\mathrm{d}x\right| &= \left|\int_a^b [f_n(x) - f(x)]\mathrm{d}x\right| \\ &\leq \int_a^b |f_n(x) - f(x)|\mathrm{d}x \\ &\leq \|f_n - f\| \cdot \int_a^b \mathrm{d}x = (b-a)\cdot\|f_n - f\|.\end{aligned}$$

As $\lim_{n\to\infty}\|f_n - f\| = 0$, it results that $\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$.

Remark 3.2.3 The assertion of Theorem 3.2.3 remains true if it is replaced the continuity by the integrability of functions f_n, $n \in \mathbb{N}^*$.

Example 3.2.3 Establish the nature of the convergence of the sequences $\{f_n\}$, where:

$$f_n(x) = \frac{n\cdot x}{\mathrm{e}^{n\cdot x^2}}, \quad n \in \mathbb{N}^*, \quad x \in [0, 1]$$

Since for any $x \in [0, 1]$, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{n\cdot x}{\mathrm{e}^{n\cdot x^2}} = 0$, we deduce that:

$$f_n \xrightarrow[{[0,1]}]{s} f \equiv 0.$$

Obviously, the functions, f and $f_n, n \in \mathbb{N}^*$, are continuous functions on the interval [0, 1].

On the other hand we have:

$$\int_0^1 f_n(x)\mathrm{d}x = \int_0^1 \frac{n \cdot x}{\mathrm{e}^{n \cdot x^2}}\mathrm{d}x = -\frac{1}{2 \cdot \mathrm{e}^{n \cdot x^2}}\bigg|_0^1 = \frac{1}{2} - \frac{1}{2 \cdot \mathrm{e}^n} \text{ and}$$

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0.$$

Therefore:

$$\lim_{n\to\infty}\int_0^1 f_n(x)\mathrm{d}x = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2 \cdot \mathrm{e}^n}\right) = \frac{1}{2} \neq 0$$

$$= \int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x.$$

From Theorem 3.2.3 it follows that the sequence $\{f_n\}$ is not uniformly convergent to 0 on the interval [0, 1].

Chapter 4
Series of Functions (Functional Series)

4.1 Simple and Uniform Convergence

Definition 4.1.1 Let $\{u_n\}$ be a sequence of real valued functions defined on the set $D \subset \mathbb{R}$. The series of functions $\sum_{n=1}^{\infty} u_n$ is called **simply (uniformly) convergent** on the set D if the functional sequence of partial sums $\{S_n\}$, defined by $S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n$; $\forall n \in \mathbb{N}^*$, is simply (uniformly) convergent on D. If $S(x) = \lim_{n\to\infty} S_n(x)$, $x \in D$, then the function $S : D \to \mathbb{R}$ is called the sum of the series and we will use notation:

$$S = \sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

The series of functions $\sum_{n=1}^{\infty} u_n$ is called **absolutely simply (uniformly) convergent** on D if the series of functions $\sum_{n=1}^{\infty} |u_n|$ is simply (uniformly) convergent on the set D.

Remark 4.1.1 Any series of functions, uniformly convergent on the set D is simply convergent on D. The reverse assertion, generally, is not true.

Theorem 4.1.1 (Cauchy's uniform convergence criterion). *A necessary and sufficient condition for the series of functions $\sum_{n=1}^{\infty} u_n$ to be uniformly convergent on the set D, is that for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that:*

$$\left|u_{n+1}(x) + \cdots + u_{n+p}(x)\right| < \varepsilon, \quad \forall n \geq n_\varepsilon, \quad \forall p \in \mathbb{N}^* \text{ and } \forall x \in D.$$

Proof The series of functions $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on D, if the sequence of partial sequence $\{S_n\}$ is uniformly convergent on D. According to Cauchy's uniform convergence criterion for sequences of functions (Theorem 3.1.1), the sequence $\{S_n\}$ is uniformly convergent on D, if for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that:

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$$\begin{aligned}&\left|S_{n+p}(x)-S_n(x)\right|=\left|u_{n+1}(x)+\cdots+u_{n+p}(x)\right|<\varepsilon,\\&\quad\forall n\ge n_\varepsilon,\forall p\in\mathbb{N}^*,\forall x\in D.\end{aligned}$$

Further we present a practical criterion of uniform convergence for series of functions.

Theorem 4.1.2 (Weierstrass' uniform convergence criterion). *Let $\sum_{n=1}^{\infty}u_n$ be a series of functions defined on the set D. If there exists $n_0\in\mathbb{N}^*$, and a convergent number series $\sum_{n=1}^{\infty}a_n$ such that:*

$$|u_n(x)|\le a_n,\quad\forall n\ge n_0,\quad\forall x\in D,$$

then the series of functions $\sum_{n=1}^{\infty}u_n$ is uniformly and absolutely convergent on D.

Proof Since the number series $\sum_{n=1}^{\infty}a_n$ is convergent, according to Theorem 2.1.1, it results that for any $\varepsilon>0$, there exists $n'_\varepsilon\in\mathbb{N}^*$ such that:

$$a_{n+1}+\cdots+a_{n+p}<\varepsilon,\quad\forall n\ge n'_\varepsilon\text{ and }\forall p\in\mathbb{N}^*.$$

Let $n_\varepsilon=\max\left\{n'_\varepsilon,n_0\right\}$, $n\ge n_\varepsilon$ and $p\in\mathbb{N}^*$. Then we have:

$$\begin{aligned}&\left|u_{n+1}(x)+\cdots+u_{n+p}(x)\right|\le|u_{n+1}(x)|+\cdots+\left|u_{n+p}(x)\right|\\&\quad\le a_{n+1}+\cdots+a_{n+p}<\varepsilon,\quad\forall x\in D.\end{aligned}$$

The assertion results now from Theorem 4.1.1.

Example 4.1.1 Study the uniform convergence of the series:

$$\sum_{n=1}^{\infty}\frac{\sin(n+1)\cdot x}{x^4+n\cdot\sqrt{n}},\ \text{ on }\mathbb{R}.$$

Since we have:

$$\left|\frac{\sin(n+1)\cdot x}{x^4+n\cdot\sqrt{n}}\right|=\frac{|\sin(n+1)\cdot x|}{x^4+n\cdot\sqrt{n}}\le\frac{1}{x^4+n\cdot\sqrt{n}}\le\frac{1}{n\cdot\sqrt{n}},\quad\forall x\in\mathbb{R},\forall\,n\in\mathbb{N}^*$$

and the series with positive terms $\sum_{n=1}^{\infty}\frac{1}{n\cdot\sqrt{n}}$ is convergent (Example 2.2.2; $\alpha=\frac{3}{2}>1$), according to Theorem 4.1.2, the given series is uniformly and absolutely convergent on $\mathbb{R}$.

4.2 Properties of the Uniformly Convergent Series of Functions

For the sum of a finite number of functions, the following properties are known:

1. The sum of a finite number of continuous functions is also a continuous function.
2. The sum of a finite number of differentiable functions is also a differentiable function and the derivative of the sum is the sum of the derivatives.
3. The sum of a finite number of integrable functions is also an integrable function and the integral of the sum is the sum of the integrals.

Further we will show under what conditions these properties are kept for the sum of a series of functions (the "sum" of an infinity number of functions).

Theorem 4.2.1 *If the series of functions $\sum_{n=1}^{\infty} u_n$ converges uniformly on the set $D \subset \mathbb{R}$ and all its terms are continuous functions on D, then its sum S is also a continuous function on D.*

Proof Since any sum of a finite number of continuous functions is also a continuous function, it results that the sequence $\{S_n\}$ of partial sums of the given series is a sequence of continuous functions on D. As $S_n \xrightarrow[D]{u} S$, from Theorem 3.2.1 we deduce that S is a continuous function on D.

Remark 4.2.1 If the series $\sum_{n=1}^{\infty} u_n$, whose terms are continuous on D, but which do not uniformly converge on D, may have as its sum a discontinuous function.

Example 4.2.1 Consider the series $\sum_{n=1}^{\infty} (1-x) \cdot x^{n-1}$, $x \in [0, 1]$.

We have:

$$S_n(x) = \sum_{k=1}^{n} (1-x) \cdot x^{k-1} = (1-x) \cdot (1 + x + x^2 + \cdots + x^{n-1})$$
$$= (1-x) \cdot \frac{1-x^n}{(1-x)} = 1 - x^n, \quad \forall x \in [0, 1)$$

and $S_n(1) = 0$. Obviously, $S_n \xrightarrow[{[0,1]}]{s} S$, where $S(x) = \begin{cases} 1 \text{ if } x \in [0, 1) \\ 0 \text{ if } x = 1 \end{cases}$.

Therefore the sum S is a discontinuous function on $[0, 1]$.

Theorem 4.2.2 (Term-by-term differentiation theorem). *Let $\sum_{n=1}^{\infty} u_n$ be a series of functions uniformly convergent on the interval $I \subset \mathbb{R}$, and all its terms are differentiable on I. Suppose that the series $\sum_{n=1}^{\infty} u_n'$ of these derivatives is also uniformly convergent on I. If S is the sum of the series $\sum_{n=1}^{\infty} u_n$, and T is the sum of the series $\sum_{n=1}^{\infty} u_n'$, then S is differentiable on I and:*

$$S'(x) = \left(\sum_{n=1}^{\infty} u_n(x) \right)' = \sum_{n=1}^{\infty} u_n'(x) = T(x), \quad \forall\, x \in I.$$

Proof If $S_n(x) = \sum_{k=1}^{n} u_k(x), \forall x \in I$, then:

$$S_n'(x) = \sum_{k=1}^{n} u_k'(x) = T_n(x), \quad \forall x \in I$$

On the other hand, by hypothesis $S_n \xrightarrow[I]{u} S$ and $S_n' = T_n \xrightarrow[I]{u} T$. From Theorem 3.2.2 we deduce that S is differentiable on I and $S' = T$ on I.

Example 4.2.2 Study the differentiability of the function:

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos n \cdot x}{n^2 \cdot (n+1)}, \quad x \in \mathbb{R}.$$

According to Theorem 4.1.2, the functional series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{\cos n \cdot x}{n^2 \cdot (n+1)}$ converges uniformly on $\mathbb{R}$ because $\left|\frac{\cos n \cdot x}{n^2 \cdot (n+1)}\right| \le \frac{1}{n^2 \cdot (n+1)} \le \frac{1}{n^3}, \forall x \in \mathbb{R}, \forall n \ge 1$ and the number series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent (Example 2.2.2; $\alpha = 3 > 1$).

The series of derivatives $\sum_{n=1}^{\infty} u_n'(x) = \sum_{n=1}^{\infty} \frac{-\sin n \cdot x}{n \cdot (n+1)}$ is also uniformly convergent on $\mathbb{R}$, because:

$$\left|\frac{-\sin n \cdot x}{n \cdot (n+1)}\right| \le \frac{1}{n \cdot (n+1)} \le \frac{1}{n^2}, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}^*$$

and the series with positive terms $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Example 2.2.2; $\alpha = 2 > 1$).

From Theorem 4.2.2 it results that f is differentiable on $\mathbb{R}$ and:

$$f'(x) = \sum_{n=1}^{\infty} \frac{-\sin n \cdot x}{n \cdot (n+1)}, \quad x \in \mathbb{R}.$$

Theorem 4.2.3 (Term-by-term integration theorem). *Let* $\sum_{n=1}^{\infty} u_n$ *be a series of continuous functions, uniformly convergent on the interval* $[a, b]$*. Then its sum* S *is integrable on* $[a, b]$ *and we have:*

$$\int_a^b S(x)\mathrm{d}x = \int_a^b \left(\sum_{n=1}^{\infty} u_n(x)\right)\mathrm{d}x = \sum_{n=1}^{\infty} \left(\int_a^b u_n(x)\mathrm{d}x\right) \tag{4.1}$$

Proof From Theorem 4.2.1, it results that S is continuous on $[a, b]$, hence it is integrable on $[a, b]$.

If we denote by $\{S_n\}$ the sequence of the partial sums of the series $\sum_{n=1}^{\infty} u_n$, then $S_n \xrightarrow[{[a,b]}]{u} S$, and according to the Theorem 3.2.3 we deduce that:

$$\int_a^b S(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b S_n(x)\mathrm{d}x.$$

On the other hand, we have:

$$\int_a^b S_n(x)\mathrm{d}x = \int_a^b \left(\sum_{k=1}^{n} u_k(x)\right)\mathrm{d}x = \sum_{k=1}^{n} \int_a^b u_k(x)\mathrm{d}x.$$

Therefore:

$$\int_a^b \left(\sum_{n=1}^{\infty} u_n(x)\right)\mathrm{d}x = \int_a^b S(x)\mathrm{d}x$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b u_k(x)\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b u_n(x)\mathrm{d}x.$$

Remark 4.2.2 The assertion of Theorem 4.2.3 rests true if the continuity of the functions u_n, $n \in \mathbb{N}^*$, is replaced with the integrability of this functions.

Example 4.2.3 Show that the function $f(x) = \sum_{n=1}^{\infty} \frac{\sin n\cdot x}{n\cdot(n+1)}$ is continuous on $\mathbb{R}$ and compute $\int_0^{\frac{\pi}{2}} f(x)\mathrm{d}x$, using Euler's result $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

According to the Example 4.2.1, the series of continuous functions $\sum_{n=1}^{\infty} \frac{\sin n\cdot x}{n\cdot(n+1)}$ is uniformly convergent on $\mathbb{R}$ and its sum f is continuous on $\mathbb{R}$ (Theorem 4.2.1). Further, taking into account from the Euler's result and from the Example 2.1.1 we obtain:

$$\int_0^{\frac{\pi}{2}} f(x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} \frac{\sin n\cdot x}{n\cdot(n+1)}\right)\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{\sin n\cdot x}{n\cdot(n+1)}\mathrm{d}x = \sum_{n=1}^{\frac{\pi}{2}} -\frac{\cos n\cdot x}{n^2\cdot(n+1)}\Bigg|_0^{\frac{\pi}{2}}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2\cdot(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n\cdot(n+1)} = \frac{\pi^2}{6} - 1.$$

4.3 Power Series

A power series is a particular case of a functional series, so all results obtained for functional series rest valid also for power series.

Definition 4.3.1 A **power series** is a series of functions of the type:

$$\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n + \cdots, \quad x \in \mathbb{R} \tag{4.2}$$

where $\{a_n\}$ is a sequence of real numbers.

The set M_c of all points $x_0 \in \mathbb{R}$ such that the number series $\sum_{n=0}^{\infty} a_n \cdot x_0^n$ is convergent is called ***the set of convergence of power series*** (4.2); Therefore:

$$M_c = \left\{ x_0 \in \mathbb{R}; \sum_{n=0}^{\infty} a_n \cdot x_0^n \text{ is convergent} \right\}.$$

We observe that $0 \in M_c$, hence $M_c \neq \varnothing$.

Lemma 4.3.1 *If the series* (4.2) *is convergent at the point* $x_0 \in \mathbb{R}$, *then this series is absolutely convergent at every point* $x \in \mathbb{R}$ *with the property* $|x| < |x_0|$.

Proof Since the number series $\sum_{n=0}^{\infty} a_n \cdot x_0^n$ is convergent, from Proposition 2.1.1 it follows that the sequence $\left\{a_n \cdot x_0^n\right\}$ is convergent and has the limit 0. Because any convergent sequence is bounded, it results that there exists $M > 0$ such that $\left|a_n \cdot x_0^n\right| \leq M, \forall n \in \mathbb{N}$.

On the other hand we have:

$$\left|a_n \cdot x^n\right| = \left|a_n \cdot x_0^n\right| \cdot \left|\frac{x}{x_0}\right|^n \leq M \cdot \left|\frac{x}{x_0}\right|^n = v_n, \quad \forall n \in \mathbb{N}.$$

If $|x| < |x_0|$, then the series $\sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} M \cdot \left|\frac{x}{x_0}\right|^n$ is convergent because is a geometric series with the ratio $q = \left|\frac{x}{x_0}\right| < 1$ (Example 2.1.2). According to Theorem 2.2.2 it results that the series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is absolutely convergent.

Theorem 4.3.1 (Abel's first theorem). *For any power series* $\sum_{n=0}^{\infty} a_n \cdot x^n$ *there exists* $0 \leq R \leq \infty$ *with the properties:*

1. *The series* $\sum_{n=0}^{\infty} a_n \cdot x^n$ *is absolutely convergent for any* $x \in \mathbb{R}$, $|x| < R$.
2. *The series* $\sum_{n=0}^{\infty} a_n \cdot x^n$ *is divergent if* $x \in \mathbb{R}$, $|x| > R$.
3. *The series* $\sum_{n=0}^{\infty} a_n \cdot x^n$ *is uniformly and absolutely convergent on the interval* $[-r, r]$, $\forall 0 < r < R$.

R is called ***the radius of convergence*** *of the power series and the interval* $(-R, R)$ *is called* ***the convergence interval*** *of the series.*

Proof If the convergence set $M_c = \{0\}$, then $R = 0$ and the proof is finished.

Suppose that $M_c \neq \{0\}$.

Case 1. If M_c is unbounded, then $M_c = \mathbb{R}$ and $R = +\infty$.

Indeed, let $x_1 \in \mathbb{R}$, be an arbitrary point. Since the set M_c is unbounded, there exists $x_0 \in M_c$ such that $|x_1| < x_0 = |x_0|$, hence the series $\sum_{n=0}^{\infty} a_n \cdot x_1^n$ is absolutely convergent according to Lemma 4.3.1.

Case 2. If the set M_c is bounded above, then $R = \sup M_c > 0$.

Indeed, if $|x| < R$ then there exists $x_0 \in M_c$ such that $|x| < x_0 = |x_0| < R$. According to Lemma 4.3.1, the power series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is convergent. Let $x \in \mathbb{R}$ such that $|x| > R$. Obviously, there exists $y > 0$ such that $|x| > y > R$. If we suppose that the series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is convergent, then from Lemma 4.3.1 it results that $y \in M_c$ and this contradicts the definition of $R = \sup M_c$. Therefore the series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is divergent for $|x| > R$.

Finally, the fact that the series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is uniformly and absolutely convergent on every interval $[-r, r], 0 < r < R$, follows from Weierstrass's test (Theorem 4.1.2), because $|a_n \cdot x^n| < |a_n \cdot r^n|, \forall x \in [-r, r]$, and the number series $\sum_{n=0}^{\infty} |a_n \cdot r^n|$ is convergent.

The next theorem gives a practical computational formula for the convergence radius.

Theorem 4.3.2 (Cauchy–Hadamard). *Let* $\sum_{n=0}^{\infty} a_n \cdot x^n$ *be a power series and let* $\omega = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$. *Then:*

1. $R = \frac{1}{\omega}$, if $0 < \omega < \infty$,
2. $R = 0$, if $\omega = \infty$,
3. $R = \infty$, if $\omega = 0$.

Proof Applying Cauchy's test (Theorem 2.2.5) to the number series with positive terms $\sum_{n=0}^{\infty} |a_n \cdot x^n|$ we get:

$$L = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n \cdot x^n|} = \omega \cdot |x|.$$

1. If $0 < \omega < \infty$ and $|x| < \frac{1}{\omega}$, then $L < 1$, and according to Cauchy's test the series $\sum_{n=0}^{\infty} |a_n \cdot x^n|$ is convergent, hence the series $\sum_{n=0}^{\infty} a_n \cdot x^n$ is absolutely convergent.

   If $|x| > \frac{1}{\omega}$ then there exists $|x| > y > \frac{1}{\omega}$. If we suppose that $\sum_{n=0}^{\infty} a_n \cdot x^n$ is convergent, then from Lemma 4.3.1 it results that the series $\sum_{n=0}^{\infty} |a_n \cdot y^n|$ is convergent. On the other hand, we have $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n| \cdot y^n} = \omega \cdot y > 1$, where results, according to Cauchy's test, that the series $\sum_{n=0}^{\infty} |a_n| \cdot y^n$ is divergent. We come thus to a contradiction.
2. Let $\omega = \infty$ and let $x_0 \neq 0$ be arbitrary. We will show that the series $\sum_{n=0}^{\infty} a_n \cdot x_0^n$ is divergent. Indeed, let $0 < y < |x_0|$. If we suppose that the series $\sum_{n=0}^{\infty} a_n \cdot x_0^n$ is convergent, then from Lemma 4.3.1 it results that the series $\sum_{n=0}^{\infty} |a_n| \cdot y^n$ is convergent. On the other hand we have $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n| \cdot y^n} = \omega \cdot y > 1$, whence it results that the series $\sum_{n=0}^{\infty} |a_n| \cdot y^n$ is divergent. Thus we come to a contradiction. Therefore, $M_c = \{0\}$, hence $R = 0$.
3. If $\omega = 0$, then $L = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n \cdot x^n|} = \omega \cdot |x| = 0 < 1, \forall x \in \mathbb{R}$, and according to Cauchy's test, the series $\sum_{n=0}^{\infty} |a_n \cdot x^n|$ is convergent. Therefore $M_c = \mathbb{R}$, and $R = \infty$.

Remark 4.3.1 The convergence radius R of the power series $\sum_{n=0}^{\infty} a_n \cdot x^n$ can be compute by the formula:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

if this limit exists.

Proof Indeed, if there exists $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$, then there exists $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ (with the conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$). But it is known that if there exists $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$, then there exists $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l$.

Example 4.3.1 Find the radius of convergence R and the set of convergence M_c for the following power series:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}}{n} \cdot x^n + \cdots$.

   Since $a_n = \frac{(-1)^{n-1}}{n}$, $n \geq 1$, by Remark 4.3.1 we deduce:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

   From Abel's theorem (Theorem 4.3.1), it results that the series is absolutely convergent on the interval $(-1, 1)$ and divergent on the set $(-\infty, -1) \cup (1, \infty)$.

   For $x = R = 1$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which is convergent according to Leibniz's test for alternating series (Theorem 2.3.2).
   For $x = -R = -1$, we get the series $\sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.

   Therefore the set of convergence of the series is $M_c = (-1, 1]$.

2. $\sum_{n=0}^{\infty} n! \cdot x^n = 1 + 1! \cdot x + 2! \cdot x^2 + \cdots + n! \cdot x^n + \cdots$.

   Since $a_n = n!$, $n \geq 0$ we have:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

   Therefore $M_c = \{0\}$.

3. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$.

   Since $a_n = \frac{1}{n!}$, $n \geq 0$, we have:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = +\infty.$$

   Therefore $M_c = \mathbb{R}$.

4. Series $\sum_{n=0}^{\infty} \frac{x^{3n}}{2^n} = 1 + \frac{x^3}{2} + \frac{x^6}{2^2} + \cdots + \frac{x^{3n}}{2^n} + \cdots$.

We notice that in this case $a_n = \begin{cases} 0 & \text{dacă } n \neq 3k \\ 2^{-n} & \text{dacă } n = 3k \end{cases}, n \geq 0.$

The sequence $\left\{\sqrt[n]{a_n}\right\}$ is composed from the subsequences $\{0\}$ and $\left\{\sqrt[3n]{2^{-n}}\right\}$ hence: $\omega = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = \max\left\{0, \frac{1}{\sqrt[3]{2}}\right\} = \frac{1}{\sqrt[3]{2}}$, whence it follows that $R = \frac{1}{\omega} = \sqrt[3]{2}$.

From Abel's theorem (Theorem 4.3.1), it results that the series is absolutely convergent on the interval $\left(-\sqrt[3]{2}, \sqrt[3]{2}\right)$, and divergent on the set $(-\infty, -\sqrt[3]{2}) \cup (\sqrt[3]{2}, \infty)$.

For $x = R = \sqrt[3]{2}$, we obtain the series $\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$, which is divergent.

For $x = -R = -\sqrt[3]{2}$, we get the divergent series $\sum_{n=0}^{\infty} (-1)^n$, hence $M_c = \left(-\sqrt[3]{2}, \sqrt[3]{2}\right)$.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (x-3)^n = (x-3) - \frac{1}{2} \cdot (x-3)^2 + \cdots + \frac{(-1)^{n-1}}{n} \cdot (x-3)^n + \cdots$

If we use the notation $x - 3 = y$, then we obtain the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot y^n$.

According to Example 4.3.1 (1), the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot y^n$ is convergent on the interval $(-1, 1]$.

Since $y \in (-1, 1] \Leftrightarrow x \in (2, 4]$, we deduce that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot (x-3)^n$ is convergent for $x \in (2, 4]$, hence $M_c = (2, 4]$.

Theorem 4.3.3 *The sum of a power series is a continuous function on the convergence interval of the series.*

Proof Let $x \in (-R, R)$ be an arbitrary fixed point. Obviously, there exists $r \in \mathbb{R}$, such that $|x| < r < R$. According to Theorem 4.3.1, the power series converges uniformly on the interval $[-r, r]$. Taking into account to Theorem 4.2.1, we deduce that the sum S of the series is a continuous function on the interval $[-r, r]$. Particularly, the sum S is continuous in the point x.

Theorem 4.3.4 *Any power series can be differentiated term-by-term at any point x of its convergence interval $(-R, R)$ and the series of derivatives has the same radius of convergence as the initial series.*

Proof The series of derivatives is:

$$\begin{aligned} &a_1 + 2 \cdot a_2 \cdot x + 3 \cdot a_3 \cdot x^2 + \cdots + n \cdot a_n \cdot x^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot x^n \end{aligned}$$

If we denote by R' the radius of convergence of the series of derivatives, then we have:

$$\begin{aligned}R' &= \overline{\lim_{n\to\infty}} \sqrt[n]{(n+1)\cdot|a_{n+1}|}\\ &= \lim_{n\to\infty} \sqrt[n]{(n+1)} \cdot \limsup_{n\to\infty} \sqrt[n]{|a_{n+1}|} = 1\cdot R = R.\end{aligned}$$

Therefore, from Theorem 4.2.2 it results that:

$$\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right)' = \sum_{n=1}^{\infty} n\cdot a_n \cdot x^{n-1}, \quad \forall x \in (-R, R).$$

Theorem 4.3.5 *Any power series can be integrated term-by-term in its convergence interval* $(-R, R)$ *and the radius of convergence of the series obtained by term wise integration is also R. Specifically, there holds:*

$$\int_0^x \left(\sum_{n=1}^{\infty} a_n \cdot x^n\right) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot x^{n+1}, \quad x \in (-R, R)$$

Proof Let $S(x) = \sum_{n=0}^{\infty} a_n \cdot x^n, \forall x \in (-R, R)$. Since the series $\sum_{n=0}^{\infty} a_n \cdot x^n$ converges uniformly on the interval $[-r, r]$, for any $0 < r < R$, it results, from Theorem 4.2.3, that:

$$\int_0^x S(t)\mathrm{d}t = \sum_{n=0}^{\infty} a_n \cdot \int_0^x t^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot x^{n+1}, \quad \forall x \in (-R, R).$$

On the other hand we have:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{a_n}{n+1}\right|} = \frac{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}{\lim\limits_{n\to\infty} \sqrt[n]{n+1}} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = R,$$

whence it results that the radius of convergence of the series obtained by integration is also *R*.

Remark 4.3.2 A power series can be differentiated and integrated term-by-term on its convergence interval, whenever we want, and the radius of convergence of the new obtained series is equal with the radius of convergence of the initial series.

Can prove the following theorem ([10], Theorem 2.4.6):

Theorem 4.3.6 (Abel's second theorem). *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with the radius of convergence* $R < \infty$ *and with the sum S. If this power series converges at the point* $x = R$ *(respectively* $x = -R$*), then* $S(R) = \lim_{x\to R_{x<R}} S(x)$ *(respectively* $S(-R) = \lim_{x\to -R_{x>-R}} S(x)$*).*

Example 4.3.2 (*Logarithmic series*). Find the set of convergence and the sum of the logarithmic series i.e.:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}}{n} \cdot x^n + \cdots .$$

According to Example 4.3.1 (1), the set of convergence is $M_c = (-1, 1]$.
Let

$$\begin{aligned} S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\ &+ \frac{(-1)^{n-1}}{n} \cdot x^n + \cdots , \quad x \in (-1, 1] \end{aligned}.$$

From Theorem 4.3.4 it results:

$$S'(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}, \quad \forall x \in (-1, 1)$$

and further:

$$S(x) = \int \frac{1}{1+x} \mathrm{d}x = \ln(1+x) + C, \quad \forall x \in (-1, 1).$$

Since $S(0) = 0$, it follows that $C = 0$, hence $S(x) = \ln(1+x)$, $\forall x \in (-1, 1)$.
Because $x = 1 \in M_c$, from Theorem 4.3.6, we deduce that:

$$S(1) = \lim_{\substack{x \to 1 \\ x < 1}} \ln(x+1) = \ln 2.$$

Therefore we have:

$$\begin{aligned} \ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}}{n} \cdot x^n \\ &+ \cdots , \quad \forall x \in (-1, 1]. \end{aligned}$$

On the other hand, for $x = 1$ we get:

$$\ln 2 = S(1) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} + \cdots .$$

Example 4.3.3 Find the set of convergence and the sum of the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n}{2n+1} \cdot x^{2n+1} + \cdots .$$

We have $a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{(-1)^n}{2n+1} & \text{if } n \text{ is odd} \end{cases}, n \geq 0$, whence it results: $\omega = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = \overline{\lim}_{n\to\infty} \left(\sqrt[2n+1]{\frac{1}{2n+1}} \right) = 1$, hence $R = 1$.

For $x = R = 1$, we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which is convergent, according to Leibniz's test for alternating series.
For $x = -R = -1$ the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$, which is also convergent.

Therefore the set of convergence is $M_c = [-1, 1]$.
Let $S(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n}{2n+1} \cdot x^{2n+1} + \cdots, \forall x \in [-1, 1]$.
From Theorem 4.3.4, we deduce:

$$S'(x) = 1 - x^2 + x^4 - \cdots + (-1)^n \cdot x^{2n} + \cdots, \quad \forall x \in (-1, 1).$$

Since the series of the right part is a geometric series with the ratio $q = -x^2$ we get:

$$S'(x) = \frac{1}{1+x^2}, \quad \forall x \in (-1, 1).$$

Integrating the last relation it results:

$$S(x) = \tan^{-1} x + C, \quad \forall x \in (-1, 1)$$

Taking into account that $S(0) = 0$ it follows that $C = 0$, hence:

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \quad \forall x \in (-1, 1).$$

On the other hand, since $x = \pm 1 \in M_c$, from Theorem 4.3.6 we get:

$$S(1) = \lim_{\substack{x\to 1 \\ x<1}} S(x) = \lim_{\substack{x\to 1 \\ x<1}} \tan^{-1} x = \tan^{-1} 1 = \frac{\pi}{4}$$

and:

$$S(-1) = \lim_{\substack{x\to -1 \\ x>-1}} S(x) = \lim_{\substack{x\to -1 \\ x>-1}} \tan^{-1} x = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

Therefore $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^{2n+1}, \forall x \in [-1, 1]$.

As $S(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, we deduce that:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

4.4 Taylor's Formula

Taylor's formula is one of the basic formulas of mathematical analysis which has numerous applications in the analysis itself and in the adjacent fields of mathematics concerning especially to the approximation of functions by polynomials.

Definition 4.4.1 Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a function which is n-times differentiable at the point $a \in I$. **The Taylor polynomial of order n** is denoted by T_n and is defined as:

$$\begin{aligned} T_n(x) = f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 \\ + \cdots + \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n, \quad x \in I. \end{aligned}$$

The *n*th remainder of Taylor's formula is denoted by R_n and is defined as:

$$R_n(x) = f(x) - T_n(x), \quad \forall x \in I.$$

Therefore $f(x) = T_n(x) + R_n(x)$ and so, the Taylor's formula is:

$$\begin{aligned} f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x). \end{aligned} \tag{4.3}$$

Definition 4.4.2 The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be of class C^p on the interval I if it is p-times differentiable on the interval I and its derivatives are continuous on I. We will use the notation $f \in C^p(I)$, $p \in \mathbb{N}^*$.

Theorem 4.4.1 (Taylor's formula with Lagrange's remainder). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function of class C^{n+1} on the open interval I, and let $a \in I$ be an arbitrary fixed point. Then, for any $x \in I$, there exists ξ between a and x such that:*

$$\begin{aligned} f(x) = f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-a)^{n+1}. \end{aligned} \tag{4.4}$$

Proof We will search the nth remainder of Taylor's formula of the form:

$$R_n(x) = (x-a)^{n+1} \cdot K(x). \tag{4.5}$$

With this choise, the formula (4.3) becomes:

$$\begin{aligned} f(x) = f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n + (x-a)^{n+1} \cdot K(x). \end{aligned} \tag{4.6}$$

For determine the function K we consider the following auxiliary function:

$$\begin{aligned} g(t) = f(t) + \frac{f'(t)}{1!} \cdot (x-t) + \frac{f''(t)}{2!} \cdot (x-t)^2 + \cdots \\ + \frac{f^{(n)}(t)}{n!} \cdot (x-t)^n + (x-t)^{n+1} \cdot K(x). \end{aligned}$$

Let $x \in I, x > a$ be an arbitrary fixed point..

We remark that g is continuous on the interval $[a, x]$, differentiable on the open interval (a, x) and $g(a) = g(x) = f(x)$. From Rolle's Theorem it results that there exists $\xi \in (a, x)$ such that:

$$g'(\xi) = 0. \tag{4.7}$$

Further we have:

$$\begin{aligned} g'(t) &= f'(t) + \frac{f''(t)}{1!} \cdot (x-t) - \frac{f'(t)}{1!} + \cdots + \frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n \\ &- \frac{f^{(n)}(t)}{n!} \cdot n \cdot (x-t)^{n-1} - (n+1) \cdot (x-t)^n \cdot K(x) \\ &= \frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n - (n+1) \cdot (x-)^n \cdot K(x). \end{aligned}$$

According to (4.7) it results:

$$K(x) = \frac{f^{(n+1)}(\xi)}{(n+1) \cdot n!} = \frac{f^{(n+1)}(\xi)}{(n+1)!}. \tag{4.8}$$

From (4.5) and (4.8) we deduce **the Lagrange's remainder of Taylor's formula**:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \tag{4.9}$$

and so the proof of the theorem is finished.

Remark 4.4.1 If f is a polynomial function of order n, then the nth remainder $R_n(x) = 0$, $\forall x \in I$ and Taylor's formula becomes:

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\quad + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n. \end{aligned}$$

Thus, in the case of a polynomial function, the Taylor's formula returns to its representation as a polynomial in its powers $(x-a)$.

Theorem 4.4.2 (Taylor's formula with Cauchy's remainder). *Let $f \in C^{n+1}(I)$ and $a \in I$. Then, for any $x \in I$, there exists ξ between a and x such that:*

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n + \frac{f^{(n+1)}(\xi)}{n!}\cdot(x-a)\cdot(x-\xi)^n. \end{aligned} \tag{4.10}$$

Proof The proof is similar to the proof of Theorem 4.4.1 except that we will search the remainder of Taylor's formula (4.3) of the form:

$$R_n(x) = (x-a)\cdot K(x).$$

Proceeding as in the proof of Theorem 4.4.1 we obtain the following expresion for **Cauchy's remainder**:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}\cdot(x-a)\cdot(x-\xi)^n. \tag{4.11}$$

Remark 4.4.2 Since ξ is between a and x, there exists $0 < \theta < 1$ such that $\xi - a = \theta\cdot(x-a)$. If we denote by $h = x - a$, it results that $x = h + a$, $\xi = a + \theta\cdot h$ and $x - \xi = (1-\theta)\cdot h$. Thus the Taylor's formula (4.3) is written:

$$\begin{aligned} f(a+h) &= f(a) + \frac{f'(a)}{1!}\cdot h + \frac{f''(a)}{2!}\cdot h^2 \\ &\quad + \cdots + \frac{f^{(n)}(a)}{n!}\cdot h^n + R_n(x) \end{aligned}$$

where the remainder R_n has one of the forms:

$R_n(x) = \frac{f^{(n+1)}(a+\theta\cdot h)}{(n+1)!}\cdot h^{n+1}$ (**Lagrange's form**)
$R_n(x) = \frac{f^{(n+1)}(a+\theta\cdot h)}{n!}\cdot h^{n+1}\cdot(1-\theta)^n$ (**Cauchy's form**).

If $a = 0 \in I$, then the Taylor's formula is called ***Maclaurin's formula***.

Therefore, the Maclaurin's formula with the Lagrange's remainder is:

$$f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \cdots + \frac{f^{(n)}(0)}{n} \cdot x^n \\ + \frac{f^{(n+1)}(\theta \cdot x)}{(n+1)!} \cdot x^{n+1}.$$

and the Maclaurin's formula with the Cauchy's remainder is:

$$f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \cdots + \frac{f^{(n)}(0)}{n!} \cdot x^n \\ + \frac{f^{(n+1)}(\theta \cdot x)}{n!} \cdot x^{n+1} \cdot (1-\theta)^n.$$

Further we will write the Maclaurin's formula, with the Lagrange's remainder, for some basic functions:

1. $f(x) = \mathrm{e}^x, x \in \mathbb{R}$.

   Since $f^{(n)}(x) = \mathrm{e}^x$ and $f^{(n)}(0) = 1, \forall n \in \mathbb{N}$, we obtain:

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \cdot \mathrm{e}^{\theta \cdot x}, \quad 0 < \theta < 1.$$

2. $f(x) = \sin x, x \in \mathbb{R}$.

   We have:

$$f^{(n)}(x) = \sin\left(x + \frac{n \cdot \pi}{2}\right), \quad \forall n \in \mathbb{N},$$

and

$$f^{(n)}(0) = \sin \frac{n \cdot \pi}{2} = \begin{cases} 0 & \text{if}\, n = 2 \cdot k \\ (-1)^{\frac{n-1}{2}} & \text{if}\, n = 4 \cdot k + 1 (\text{or } 4 \cdot k + 3) \end{cases}.$$

Therefore we obtain:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \\ + \frac{x^{2n+2}}{(2n+2)!} \cdot \sin(\theta \cdot x + (n+1) \cdot \pi)$$

3. $f(x) = \cos x, x \in \mathbb{R}$.

We have:

$$f^{(n)}(x) = \cos\left(x + \frac{n \cdot \pi}{2}\right) \text{ and}$$

$$f^{(n)}(0) = \cos\frac{n \cdot \pi}{2} = \begin{cases} 0 & \text{if } n = 2 \cdot k + 1 \\ (-1)^k & \text{if } n = 4 \cdot k \end{cases}.$$

Therefore we obtain:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \cdot \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} \cdot \sin\left(\theta \cdot x + (2n+1) \cdot \frac{\pi}{2}\right).$$

4. $f(x) = \ln(1 + x), x \in (-1, \infty)$

Since $f^{(n)}(x) = (-1)^{n-1} \cdot \frac{(n-1)!}{(1+x)^n}$, $f(0) = 0$, and $f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$, we obtain:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \cdot \frac{x^n}{n} + (-1)^n \cdot \frac{x^{n+1}}{n+1} \cdot \frac{1}{(1 + \theta \cdot x)^n}.$$

Further we will present two important applications of Taylor's formula for study of real functions.

Let $I \subset \mathbb{R}$ be an open interval, $a \in I$ and let $f : I \to \mathbb{R}$, be a function differentiable at $x = a$. It is well known that a necessary condition for the point $x = a$ to be a local extremum point for f is that $f'(a) = 0$.

The next theorem states a sufficient condition for the existence of the local extremum points of a function.

Theorem 4.4.3 *Let $I \subset \mathbb{R}$ be an open interval, $a \in I$ and $f \in C^n(I)$, such that:*

$$f'(a) = f''(a) = f'''(a) = \cdots = f^{(n-1)}(a) = 0 \text{ and } f^{(n)}(a) \neq 0.$$

If n is an even number, then $x = a$ is a point of local extremum point for f, namely, $x = a$ is a point of local maximum if $f^{(n)}(a) < 0$ and a point of local minimum if $f^{(n)}(a) > 0$.

If n is an odd number, then $x = a$ isn't a point of local extremum for f.

Proof From Taylor's formula with Lagrange's remainder it results:

$$f(x) = f(a) + \frac{f^{(n)}(\xi)}{n!}(x - a)^n, \quad (\forall)x \in I$$

where ξ is between a and x.

Suppose that n is an even number and $f^{(n)}(a) < 0$. Since $f^{(n)}$ is a continuous function at the point in $x = a$, it results that there exists an open interval J such that $a \in J \subset I$ and $f^{(n)}(t) < 0, \forall t \in J$. Therefore, if $x \in J$ we have:

$$f(x) - f(a) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n \leq 0,$$

whence it results that $f(x) - f(a) \leq 0, \forall x \in J$, hence $x = a$ is a point of local maximum for f. Similarly, we prove that if $f^{(n)}(a) > 0$, then $x = a$ is a point of local minimum for f.

If n is an odd number, then the expression $f(x) - f(a)$ has no constant sign on any neighborhood of the point $x = a$, hence $x = a$ does not be a point of local extremum for f.

Corollary 4.4.1 *If $f'(a) = 0$ and $f''(a) \neq 0$, then $x = a$ is a point of local extremum for f, namely, a point of local minimum if $f''(a) > 0$, respectively a point of local maximum if $f''(a) < 0$.*

If $f'(a) = f''(a) = 0$ and $f'''(a) \neq 0$, then $x = a$ isn't point of local extremum for f (it is a point of inflexion for f).

Definition 4.4.3 A function $f \in C^2(I)$ is said to be **convex (concave) on the interval** I if for any $a, x \in I$ we have:

$$f(x) \geq f(a) + f'(a)(x-a),$$

respectively:

$$f(x) \leq f(a) + f'(a)(x-a).$$

From a geometric point of view, the function is convex (respectively, concave) if its graph is located above (respectively, below) to the tangent of the graph at any its point.

Proposition 4.4.1 *If $f \in C^2(I)$ and $f''(x) \geq 0$ $\big(f''(x) \leq 0\big)$ for any $x \in I$, then f is convex (concave) on the interval I.*

Proof Let $a, x \in I$. From Taylor's formula with Lagrange's remainder and $n = 1$, it results:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(\xi)}{2!}(x-a)^2,$$

with ξ between a and x.

If $f'' \geq 0$ on I, it results:

$$f(x) - \big(f(a) + f'(a)(x-a)\big) = \frac{f''(\xi)}{2!}(x-a)^2 \geq 0, \quad \forall x \in I,$$

therefore f is convex function on the interval I.

Similarly, if $f'' \le 0$ on I it results:

$$f(x) \le f(a) + f'(a)(x-a), \quad \forall x \in I,$$

hence, f is concave function on the interval I.

4.5 Taylor's and Maclaurin's Series

Further we will denote by $C^\infty(I)$—the class of all real valued functions indefinitely differentiable on the open interval I (i.e. f has derivatives of all orders on I).

Definition 4.5.1 Let I be an open interval, $a \in I$ and $f \in C^\infty(I)$. The power series:

$$\begin{aligned} f(a) &+ \frac{f'(a)}{1!}\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \cdots \\ &+ \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n + \cdots \end{aligned} \tag{4.12}$$

is called the **Taylor's series attached to the function** f **at the point** $x = a$.

In the particular case $a = 0 \in I$, the series (4.12) becomes:

$$f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\cdot x^n + \cdots \tag{4.13}$$

and is called the **Maclaurin's series attached to the function** f.

If we denote by M_c the set of convergence of series (4.12), then we remark that $a \in M_c$, hence $M_c \cap I \neq \phi$ (because $a \in M_c \cap I$).

Definition 4.5.2 A function $f \in C^\infty(I)$ is said to be **expanded into the Taylor's series around point** $x = a$ **on the set** $A \subset M_c \cap I$, if $\forall x \in A$ we have:

$$f(x) = f(a) + \frac{f'(a)}{1!}\cdot(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n + \cdots \tag{4.14}$$

Remark 4.5.1 Let $f \in C^\infty(I)$ and $a \in I$. Generally is not true that the function f can be expanded into the Taylor's series around point $x = a$ on a set $A \subset M_c \cap I$, as it results from the following Cauchy's example.

Example 4.5.1 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

We will show that f is indefinitely differentiable on $\mathbb{R}$ and that for any $n \in \mathbb{N}^*$ we have:

$$f^{(n)}(0) = 0.$$

Indeed, if $x \neq 0$, then $f'(x) = \frac{2}{x^3} \cdot \mathrm{e}^{-\frac{1}{x^2}}$. Since f is continuous at the point $x = 0$, we have:

$$f'(0) = \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{\frac{2}{x^3}}{\mathrm{e}^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{-\frac{6}{x^4}}{-\frac{2}{x^3} \cdot \mathrm{e}^{\frac{1}{x^2}}} = 3 \cdot \lim_{x\to 0} \frac{\frac{1}{x}}{\mathrm{e}^{\frac{1}{x^2}}}$$
$$= 3 \cdot \lim_{x\to 0} \frac{-\frac{1}{x^2}}{-\frac{2}{x^3} \cdot \mathrm{e}^{\frac{1}{x^2}}} = \frac{3}{2} \cdot \lim_{x\to 0} \frac{x}{\mathrm{e}^{\frac{1}{x^2}}} = 0.$$

Generally we have, $f^{(n)}(x) = \frac{P(x)}{x^m}\mathrm{e}^{-\frac{1}{x^2}}$, $x \neq 0$, where P is a polynomial function. Applying L'Hospital rule a sufficient number of times, it results:

$$f^{(n)}(0) = \lim_{x\to 0} f^{(n)}(x) = P(0) \cdot \lim_{x\to 0} \frac{\mathrm{e}^{-\frac{1}{x^2}}}{x^m} = P(0) \cdot \lim_{x\to 0} \frac{\frac{1}{x^m}}{\mathrm{e}^{\frac{1}{x^2}}} = 0, \quad n \geq 1.$$

The Maclaurin's series (4.13) attached to f, is convergent on $\mathbb{R}$, and has the sum $S(x) = f(0) = 0$, $\forall x \in \mathbb{R}$.

It results that $f(x) \neq S(x)$, $\forall x \in \mathbb{R}\setminus\{0\}$, hence, f is not expanded into Maclaurin's series on any set $A \subset \mathbb{R}$, $A \neq \{0\}$.

Theorem 4.5.1 *The necessary and sufficient condition for a function $f \in C^\infty(I)$ to be expanded into the Taylor's series around the point $x = a \in I$ on the set $A \in M_c \cap I$ is $\lim_{n\to\infty} R_n(x) = 0, \forall x \in A$.*

Proof We remark that the Taylor's polynomial T_n, of order n, attached to function f at the point $x = a$, coincides with the partial sum S_n of the Taylor's series (4.12). If we denote by S the sum of the series (4.12), then:

$$S(x) = \lim_{n\to\infty} T_n(x), \quad \forall x \in M_c.$$

On the other hand, from Taylor's formula (4.3) we have:

$$f(x) = T_n(x) + R_n(x), \quad \forall x \in I.$$

The fact that f can be expanded into Taylor's series on the set $A \subset M_c \cap I$, arround the point $x = a$, means that $f(x) = S(x)$, $\forall x \in A$. Obviously this happens if and only if $\lim_{n\to\infty} R_n(x) = 0$, $\forall \in A$.

Corollary 4.5.1 *A sufficient condition for a function $f \in C^\infty(I)$ to be expanded into the Taylor's series around the point $x = a \in I$, on the set $A \subset M_c \cap I$ is there exists $M > 0$ such that:*

$$\left|f^{(n)}(x)\right| \leq M, \quad \forall n \in \mathbb{N}^*, \quad \forall x \in A \subset M_c \cap I.$$

Proof For any fixed point $x \in A$, from the formula (4.9) we have:

$$|R_n(x)| = \frac{|x-a|^{n+1}}{(n+1)!} \cdot \left| f^{(n+1)}(\xi) \right| \le M \cdot \frac{|x-a|^{n+1}}{(n+1)!} = u_n,$$

where ξ is between a and x.

We observe that $\lim_{n\to\infty} u_n = 0$, since the number series with positive terms $\sum_{n=1}^{\infty} M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$ is convergent, as it results immediate from Theorem 2.2.6 because:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{|x-a|}{n+2} = 0 < 1.$$

Therefore, $\lim_{n\to\infty} R_n(x) = 0, \forall x \in A$ and the assertion follows now from Theorem 4.5.1.

Example 4.5.2 Maclaurin's expansion for some elementary functions.

1. $f(x) = \mathrm{e}^x, x \in \mathbb{R}$.

For any $r > 0$, we have:

$$\left| f^{(n)}(x) \right| = \mathrm{e}^x \le \mathrm{e}^r = M, \quad \forall x \in [-r, r], \quad \forall n \in \mathbb{N}.$$

From Corollary 4.5.1 it results that the function $f(x) = \mathrm{e}^x$ can be expanded into Taylor's series on the interval $[-r, r]$ around any point $a \in [-r, r]$. As $r > 0$ was arbitrary, we deduce that the expansion is valid on $\mathbb{R}$. Therefore we have:

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}.$$

2. $f(x) = \sin x, x \in \mathbb{R}$.

We have:

$$\left| f^{(n)}(x) \right| = \left| \sin\left(x + \frac{n \cdot \pi}{2} \right) \right| \le 1, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}^*.$$

From Corollary 4.5.1 it results that the function $f(x) = \sin x$ can be expanded into Taylor's series on $\mathbb{R}$. The Maclaurin's series for the function $f(x) = \sin x, x \in \mathbb{R}$ is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \quad \forall x \in \mathbb{R}.$$

3. $f(x) = \cos x, x \in \mathbb{R}$, can be expanded into Taylor's series on $\mathbb{R}$ around any point $a \in \mathbb{R}$, because:

$$\left|f^{(n)}(x)\right| = \left|\cos\left(x + \frac{n\pi}{2}\right)\right| \leq 1, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}^*.$$

The Maclaurin's series for the function $f(x) = \cos x, x \in \mathbb{R}$ is:

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \cdot \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}, \quad \forall x \in \mathbb{R}.\end{aligned}$$

4. The function $f(x) = (1+x)^{\alpha}, \alpha \in \mathbb{R}\backslash\mathbb{N}, x \in (-1, \infty)$, can be expanded into Maclaurin's series on the interval $(-1, 1)$.

According to Theorem 4.5.1, it is sufficient to prove that:

$$\lim_{n\to\infty} R_n(x) = 0, \quad \forall x \in (-1, 1).$$

Since $f^{(n)}(x) = \alpha \cdot (\alpha - 1) \cdot \ldots \cdot (\alpha - n + 1) \cdot (1+x)^{\alpha - n}, n \in \mathbb{N}^*$, it results from (4.11) that the Cauchy's remainder is:

$$R_n(x) = \frac{\alpha \cdot (\alpha - 1) \cdot \ldots \cdot (\alpha - n) \cdot x^{n+1}}{n!} \cdot \left(\frac{1-\theta}{1+\theta x}\right)^n \cdot (1+\theta x)^{\alpha - 1},$$

where $\theta \in (0, 1)$ and, generally, depends of n.

If we denote by $u_n = \left|\frac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{n!} \cdot x^{n+1}\right|$ and suppose that $|x| < 1$, then the series $\sum_{n=1}^{\infty} u_n$ is convergent as it results immediate from Theorem 2.2.6. Indeed we have:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{|\alpha - n - 1|}{n+1}|x| = |x| < 1.$$

From Proposition 2.1.1, it results that $\lim_{n\to\infty} u_n = 0$. On the other hand, if $|x| < 1$ we have $0 < 1 - \theta < 1 + \theta \cdot x$, whence we deduce that:

$$0 < \frac{1-\theta}{1+\theta \cdot x} < 1, \text{ hence } \lim_{n\to\infty} \left(\frac{1-\theta}{1+\theta\cdot x}\right)^n = 0.$$

Finally, we remark that if $-1 < x < 1$ then:

$$1 - |x| \leq 1 - \theta \cdot |x| \leq |1 + \theta \cdot x| \leq 1 + \theta \cdot |x| \leq 1 + |x|,$$

whence it results:

$$\begin{cases} |1+\theta x|^{\alpha-1} \le (1+|x|)^{\alpha-1} \text{ if } \alpha > 1 \\ |1+\theta x|^{\alpha-1} \le (1-|x|)^{\alpha-1} \text{ if } \alpha < 1 \end{cases}. \tag{4.15}$$

From (4.15) we obtain:

$$|R_n(x)| = u_n \cdot \left(\frac{1-\theta}{1+\theta x}\right)^n \cdot (1+\theta x)^{\alpha-1} < u_n \cdot (1+\theta x)^{\alpha-1} \tag{4.16}$$

Taking into account to (4.16) and to the fact that $\lim_{n\to\infty} u_n = 0$, we deduce now that $\lim_{n\to\infty} R_n(x) = 0$, $\forall x \in (-1, 1)$, and so the proof is finished.

Therefore we have the expansion:

$$\begin{aligned}(1+x)^\alpha &= 1 + \frac{\alpha}{1!} \cdot x + \frac{\alpha \cdot (\alpha-1)}{2!} \cdot x^2 + \cdots + \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-n+1)}{n!} \cdot x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-n+1)}{n!} \cdot x^n, \quad x \in (-1, 1)\end{aligned}$$

also known as ***the binomial series***.

In the particular case $\alpha = -\frac{1}{2}$ we have:

$$\begin{aligned}\frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2} \cdot x + \frac{1 \cdot 3}{2 \cdot 4} \cdot x^2 + \cdots + (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \cdot x^n + \ldots, \\ &\forall x \in (-1, 1).\end{aligned}$$

At the end of this section we recapitulate the Maclaurin's expansions of the basic functions:

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}.$
2. $$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}.\end{aligned}$$
3. $$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \cdot \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}.\end{aligned}$$
4. $$\begin{aligned}(1+x)^\alpha &= 1 + \frac{\alpha}{1!} \cdot x + \frac{\alpha \cdot (\alpha-1)}{2!} \cdot x^2 \\ &\quad + \cdots + \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-n+1)}{n!} \cdot x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-n+1)}{n!} \cdot x^n, \quad x \in (-1, 1).\end{aligned}$$

(***binomial series***)

5. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \cdot \frac{x^n}{n} + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n}, \quad x \in (-1, 1].$$
(***logarithmic series***) (i.e. Example 4.3.2).

6. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1).$
(***geometric series***)

4.6 Elementary Functions. Euler's Formulas. Hyperbolic Trigonometric Functions

Euler's formulas play an important role in mathematics, especially in the theory of complex functions and the differential equations. We recall (see Chap. 2, Sect. 2.7) that the exponential function $z \to e^z : \mathbb{C} \to \mathbb{C}$ is definite by:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots, \quad z \in \mathbb{C}.$$

For $z = i \cdot x$, $x \in \mathbb{R}$, $i = \sqrt{-1}$ we get:

$$e^{i \cdot x} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Similarly, for $z = -i \cdot x$, $x \in \mathbb{R}$ we get:

$$e^{-i \cdot x} = 1 - i\frac{x}{1!} - \frac{x^2}{2!} + i\frac{x^3}{3!} + \frac{x^4}{4!} - \cdots.$$

Summing and subtracting this above relations it results:

$$e^{i \cdot x} + e^{-i \cdot x} = 2 \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2 \cdot \cos x$$
$$e^{i \cdot x} - e^{-i \cdot x} = 2 \cdot i \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 2 \cdot i \cdot \sin x.$$

From the last relations we deduce the **Euler's formulas**:

$$\cos x = \frac{e^{i \cdot x} + e^{-i \cdot x}}{2}$$
$$\sin x = \frac{e^{i \cdot x} - e^{-i \cdot x}}{2 \cdot i}. \tag{4.17}$$

An equivalent form of formulas (4.17), are the following formulas:

$$\begin{aligned} e^{i\cdot x} &= \cos x + i\cdot \sin x \\ e^{-i\cdot x} &= \cos x - i\cdot \sin x \end{aligned}$$

In particular, from Euler's formulas follow this two somewhat surprising results:

$$e^{i\cdot k\cdot\pi} = e^{-i\cdot k\cdot\pi} = -1 \text{ and } e^{i\cdot 2\cdot k\cdot\pi} = e^{-i\cdot 2\cdot k\cdot\pi} = 1, \quad \forall k\in\mathbb{Z}$$

In the following we will define two other important elementary functions, namely:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2\cdot n}}{(2\cdot n)!} + \cdots, \quad z\in\mathbb{C}, \tag{4.18}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^{n-1} \frac{z^{2\cdot n-1}}{(2\cdot n-1)!} + \cdots, \quad z\in\mathbb{C}. \tag{4.19}$$

The definitions make sense because the series on the right side is absolutely convergent as it immediately follows from Cauchy's root test.

Taking into account the Maclaurin's expansions of the basic real functions presented in the previous paragraph we notice that the complex variable functions (4.18), (4.19) are generalizations of the real variable functions:

$$x \to \cos x : \mathbb{R}\to\mathbb{R}, \; x \to \sin x : \mathbb{R}\to\mathbb{R}.$$

From Euler's formulas and the property of the exponential function $e^{z+u} = e^z \cdot e^u$ it results:

$$\begin{aligned} \sin(z\pm u) &= \sin z\cdot\cos u \pm \sin u\cdot\cos z. \\ \cos(z\pm u) &= \cos z\cdot\cos u \mp \sin z\cdot\sin u. \end{aligned}$$

Indeed, for example:

$$\begin{aligned} \sin z\cdot\cos u + \sin u\cdot\cos z &= \frac{e^{i\cdot z} - e^{-i\cdot z}}{2\cdot i}\cdot\frac{e^{i\cdot u} + e^{-i\cdot u}}{2} + \frac{e^{i\cdot u} - e^{-i\cdot u}}{2\cdot i}\cdot\frac{e^{i\cdot z} + e^{-i\cdot z}}{2} \\ &= \frac{2\cdot\left(e^{i\cdot(z+u)} - e^{-i(z+u)}\right)}{4\cdot i} = \sin(z+u). \end{aligned}$$

By analogy with formulas (4.17), are definite the **hyperbolic trigonometric functions**:

$$\cosh : \mathbb{R}\to(0,\infty), \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh : \mathbb{R} \to \mathbb{R}, \quad \sinh x = \frac{e^x - e^{-x}}{2}. \tag{4.20}$$

The main property of this functions is:

$$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1, \quad \forall x \in \mathbb{R} \tag{4.21}$$

The hyperbolic trigonometric functions are used for parametric equations of the hyperbola. Indeed, the canonical equation of the hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0.$$

We observe that the parametric equations of the branch of the hyperbola situated into the first quadrant are:

$$\begin{cases} x = a \cdot \mathrm{ch} t \\ y = b \cdot \mathrm{sh} t \end{cases}, \quad t \geq 0,$$

because:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \mathrm{ch}^2 t - \mathrm{sh}^2 t = 1.$$

The following properties of the hyperbolic trigonometric functions are easily verified:

$$\cosh(0) = 1, \cosh(-x) = \cosh(x), (\cosh x)' = \sinh x, \quad \forall x \in \mathbb{R}$$
$$\sinh(0) = 0, \sin\mathrm{h}(-x) = -\sin\mathrm{h}(x), (\sinh x)' = \cosh x, \quad \forall x \in \mathbb{R}.$$

The Maclaurin's expansions of this functions are:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2\cdot n}}{(2\cdot n)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2\cdot n}}{(2\cdot n)!}, \quad x \in \mathbb{R}$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2\cdot n+1}}{(2\cdot n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2\cdot n+1}}{(2\cdot n+1)!}, \quad x \in \mathbb{R}.$$

Chapter 5
Functions of Several Variables

5.1 Vector Space $\mathbb{R}^n$. Basic Notions and Notations

The n***-dimensional space*** $\mathbb{R}^n$ is$n-\mathbb{R}^n$ defined as follows:

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{x = (x_1, \ldots, x_n); x_1, \ldots, x_n \in \mathbb{R}\}$$

.

It is easy to verify that $\mathbb{R}^n$ is a real vector space with respect to the operations:

$$x + y = (x_1 + y_1, \ldots, x_n + y_n), \quad \forall\ x = (x_1, \ldots, x_n), \forall y = (y_1, \ldots, y_n) \in \mathbb{R}^n$$
$$\lambda \cdot\ x = (\lambda \cdot x_1, \ldots, \lambda \cdot x_n), \forall \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}.$$

Any ordered system $x = (x_1, x_2, \ldots, x_n)$ is called ***vector (point)*** of $\mathbb{R}^n$, and $x_1, \ldots, x_n$ are called the coordinates (components) of the vector x.

The vector (point) $0_{\mathbb{R}^n} = (0, 0, \ldots, 0)$ is called ***the zero vector of*** $\mathbb{R}^n$ and

$$x + x + 0_{\mathbb{R}^n} = x, \quad \forall x \in \mathbb{R}^n$$

.

Definition 5.1.1 The scalar (inner) product of two vectors $x, y \in \mathbb{R}^n$ is defined as:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i = x_1 \cdot y_1 + \cdots + x_n \cdot y_n,$$
$$\forall \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \forall y = (y_1, ..., y_n) \in \mathbb{R}^n.$$

The scalar product obviously possesses the following properties:

(i) $\langle x,\ y\rangle = \langle y,\ x\rangle, \forall\, x\ ,\ y\ \in \mathbb{R}^n;$

G. Paltineanu et al., *Differential Calculus for Engineers*,
https://doi.org/10.1007/978-981-19-2553-5_5

(ii) $\langle \lambda \cdot x + \mu \cdot y, z\rangle = \lambda \cdot \langle x, z\rangle + \mu \cdot \langle y, z\rangle, \forall\ x, y, z \in \mathbb{R}^n, \forall \lambda, \mu \in \mathbb{R}$;
(iii) $\langle x, x\rangle \geq 0, \forall x \in \mathbb{R}^n$ *and* $\langle x, x\rangle = 0$ iff $x = 0_{\mathbb{R}^n}$.

Theorem 5.1.1 (Inequality Cauchy–Buniakovski–Schwarz)**.** *For any $x, y \in \mathbb{R}^n$ we have:*

$$\begin{aligned}\langle x, y\rangle^2 &= \left(\sum_{i=1}^{n} x_i \cdot y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right). \\ &= \langle x, x\rangle \cdot \langle y, y\rangle\end{aligned}$$

Proof From the property (iii) of the scalar product it results:

$$\langle \lambda \cdot x + y, \lambda \cdot x + y\rangle \geq 0, \forall x, y \in \mathbb{R}^n \text{ and } \forall \lambda \in \mathbb{R}.$$

Using also the other properties of the scalar product we obtain:

$$\lambda^2 \cdot \langle x, x\rangle + 2 \cdot \lambda \cdot \langle x, y\rangle + \langle y, y\rangle \geq 0, \forall \lambda \in \mathbb{R}.$$

For this inequality to be true for any $\lambda \in \mathbb{R}$, it must that:
$\langle x, y\rangle^2 - \langle x, x\rangle \cdot \langle y, y\rangle \leq 0$, hence $\langle x, y\rangle^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$ i.e.:

$$\left(\sum_{i=1}^{n} x_i \cdot y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right).$$

Definition 5.1.2 Any map $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, with the properties:

(i) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|, \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$;
(ii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in \mathbb{R}^n$;
(iii) $\|x\| \geq 0, \forall x \in \mathbb{R}^n$ and $\|x\| = 0$ if $x = 0_{\mathbb{R}^n}$,

is called a **norm** on $\mathbb{R}^n$.

Several norms can be introduced on the space $\mathbb{R}^n$. Two of these are most commonly used, namely

$$\|x\|_\infty = \max\{|x_i|;\ 1 \leq i \leq n\}, \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n \tag{5.1}$$

and

$$\begin{aligned}&\|x\|_2 = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + \ldots + x_n^2}, \\ &\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n\end{aligned} \tag{5.2}$$

(The norm defined by formula (5.2) is also called the Euclidean norm).

The fact that formula (5.1) defines a norm results immediately from the properties of the module. Further we will prove that formula (5.2) defines a norm on $\mathbb{R}^n$. Since the properties (i) and (ii) are obvious, it remains to be proved the property (iii). For this we use the Cauchy–Buniakovski–Schwarz's inequality:

$$\begin{aligned}\|x+y\|_2^2 &= \langle x, x\rangle + 2\cdot\langle x, y\rangle + \langle y, y\rangle \le \langle x, x\rangle + 2\cdot|\langle x, y\rangle| + \langle y, y\rangle \\ &\le \|x\|_2^2 + 2\cdot\|x\|_2\cdot\|y\|_2 + \|y\|_2^2 = \left(\|x\|_2 + \|y\|_2\right)^2.\end{aligned}$$

Therefore:

$$\|x + y\|_2 \le \|x\|_2 + \|y\|_2,\ \forall\, x,\ y \in \mathbb{R}^n.$$

Proposition 5.1.1 *The two norms are equivalent because*:

$$\|\, x\,\|_\infty \le \|\, x\,\|_2 \le \sqrt{n}\cdot \|\, x\,\|_\infty,\ \forall\, x \in \mathbb{R}^n.$$

Proof Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be an arbitrary fixed vector. Obviously there is $i_0 \in \{1, 2, \ldots, n\}$ such that $\|\, x\,\|_\infty = \max\{|x_i|;\ 1 \le i \le n\} = \left|x_{i_0}\right|$.

Further we have:

$$\|x\|_\infty = \left|x_{i_0}\right| \le \|x\|_2 = \sqrt{x_1^2 + \cdots + x_{i_0}^2 + \cdots + x_n^2} \le \sqrt{n\cdot\left|x_{i_0}\right|^2} = \sqrt{n}\cdot\|x\|_\infty$$

5.2 Convergent Sequences of Vectors in $\mathbb{R}^n$

Definition 5.2.1 The $\mathbb{R}^n$ sequence of vectors $\{x_k\}$ from $\mathbb{R}^n$ is said to be **convergent in** $\mathbb{R}^n$ if there exists a vector $x \in \mathbb{R}^n$ such that $\lim\limits_{k\to\infty}\|x_k - x\|_\infty = 0$. We will denote this by:

$$x_k \xrightarrow[k\to\infty]{\mathbb{R}^n} x.$$

Therefore $x_k \xrightarrow[k\to\infty]{\mathbb{R}^n} x$, if $\forall\,\varepsilon > 0,\ \exists\, k_\varepsilon \in \mathbb{N}^*$ such that:

$$\|x_k - x\|_\infty < \varepsilon,\ \forall k \ge k_\varepsilon.$$

From Proposition 5.1.1 it results that:

Remark 5.2.1 The sequence of vectors $\{x_k\}$ converges to the vector x in $\mathbb{R}^n$ if and only if $\lim\limits_{k\to\infty}\|x_k - x\|_2 = 0$.

Theorem 5.2.1 *The sequence of vectors* $\{x_k\}$ *from* $\mathbb{R}^n$*,where for any* $k \in \mathbb{N}^*$, $x_k = (x_{k\ 1}, x_{k\ 2}, \ldots, x_{k\ n})$, $x_{k\ i} \in \mathbb{R}$, $\forall\, i = \overline{1, n}$, *converges in* $\mathbb{R}^n$ *to the vector*

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, *if and only if,* $\forall\, i = \overline{1\,,\,n}$ *the sequence of real numbers* $\{x_{k\quad i}\}$ *converges in* $\mathbb{R}$ *to* x_i.

Proof The assertion follows immediately from (5.1) and Definition 5.2.1.

From Theorem 5.2.1 we deduce that the convergence of a sequence of vectors from $\mathbb{R}^n$ is reduced to the study of the convergence of the real number sequences formed by their components.

Example 5.2.1

(1) The sequence of vectors $\left\{\left(\frac{n^2+1}{n^2}, \frac{1}{\sqrt{n}}\right)\right\} \xrightarrow{\mathbb{R}^2} (1,\ 0)$.

(2) The sequence $\left\{\left(\frac{n}{3n-1}, \frac{n^2}{5n^2-4n+1}, \left(1+\frac{1}{n}\right)^n\right)\right\} \xrightarrow{\mathbb{R}^3} \left(\frac{1}{3}, \frac{1}{5}, e\right)$

Remark 5.2.2 From Theorem 5.2.1 it results also:

(1) Any convergent sequence of vectors from $\mathbb{R}^n$ has a unique limit.
(2) Any subsequence of a convergent sequence of vectors from $\mathbb{R}^n$ is also convergent and has the same limit with the initial sequence.

5.3 Topology Elements on $\mathbb{R}^n$

Definition 5.3.1 Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be o fixed point and $r > 0$.The set of all $x \in \mathbb{R}^n$ such that $\|x - a\| < r$ (respectively$\|x - a\| \le r$) is calledar **the n-dimensional open (respectively, closed) ball with center at a and radius r.**

Further we will denote the n-dimensional open ball by:

$$B(a, r) = \left\{x \in \mathbb{R}^n;\ \|x - a\| < r\right\},$$

respectively, the n-dimensional closed ball by:

$$\widehat{B}(a, r) = \left\{x \in \mathbb{R}^n;\ \|x - a\| \le r\right\}$$

As we introduced on the space $\mathbb{R}^n$ two norms ($\|\cdot\|_2$ and $\|\cdot\|_\infty$), it results that we have two types of open (respectively, closed) balls, namely

$$\begin{aligned} B_2(a, r) &= \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ \|x - a\|_2 < r\right\} \\ &= \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < r^2\right\} \end{aligned}$$

respectively:

$$\begin{aligned} B_\infty(a, r) &= \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ \|x - a\|_\infty < r\right\} \\ &= \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ |x_1 - a_1| < r, \ldots, |x_n - a_n| < r\right\}. \end{aligned}$$

In a similar way we have:

$$\hat{B}_2[a, r] = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 \le r^2\right\}$$
$$\hat{B}_\infty[a, r] = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ |x_1 - a_1| \le r, \ldots, |x_n - a_n| \le r\right\}.$$

Example 5.3.1

(1) Since on $\mathbb{R}$ we have $\|x\|_2 = \|x\|_\infty = |x|$, $\forall x \in \mathbb{R}$, it results that for any $a \in \mathbb{R}$ and $r > 0$, we have:

$$B_2(a, r) = B_\infty(a, r) = \{x \in \mathbb{R};\ |x - a| < r\} = (a - r, a + r)$$

$$\widehat{B}_2(a, r) = \widehat{B}_\infty(a, r) = \{x \in \mathbb{R};\ |x - a| \le r\} = [a - r, a + r]$$

From a geometric point of view, the open (respectively, closed) ball with center at a and radius r represents the symmetric open (closed) interval $(a - r, a + r)$. (respectively, $[a - r, a + r]$).

(2) Let $a = (a_1, a_2) \in \mathbb{R}^2$ be a fixed point and $r > 0$. Then we have:

$$B_2(a, r) = \left\{x = (x_1, x_2) \in \mathbb{R}^2;\ (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\right\}$$

and

$$B_\infty(a, r) = \left\{x = (x_1, x_2) \in \mathbb{R}^2;\ |x_1 - a_1| < r,\ |x_2 - a_2| < r\right\}.$$

From geometric point of view, $B_2(a, r)$ represents the interior of the circle with center at a and radius r (without circumference) (Fig. 5.1), and $B_\infty(a, r)$ is the interior of the square with center at a and side $2 \cdot r$ (without sides) (Fig. 5.2).

(3) On the space $\mathbb{R}^3$ let $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $r > 0$.Then we have:

$$B_2(a, r) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 < r^2\}$$

and

$$B_\infty(a, r) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; |x_1 - a_1| < r, |x_2 - a_2| < r, |x_3 - a_3| < r\}.$$

Geometrically speaking, $B_2(a, r)$ represents the interior (inside) of the sphere with the center at $a = (a_1, a_2, a_3)$ and the radius r and $B_\infty(a, r)$ represents the interior (inside) of the cube with the center at $a = (a_1, a_2, a_3)$ and the length of the edge $2 \cdot r$.

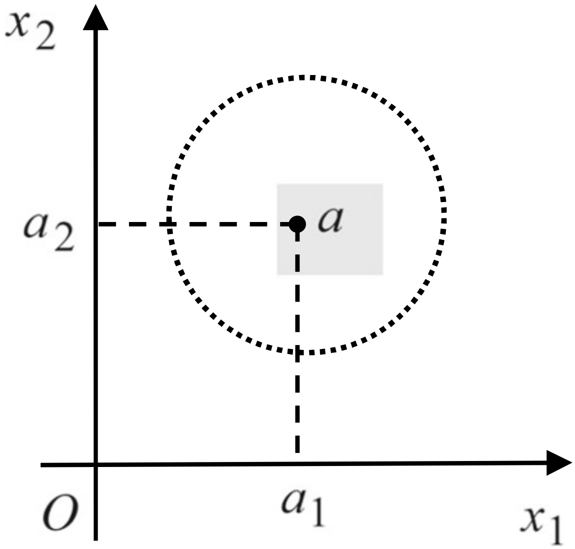

Fig. 5.1 $B_2(a,\ r)$ origin and radious $R = B_2(a, r)$

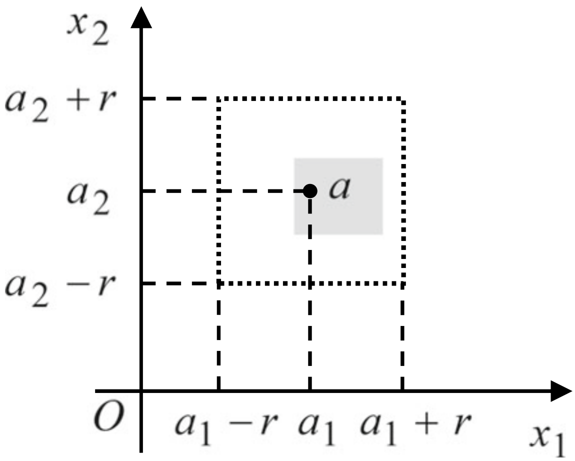

Fig. 5.2 $B_\infty(a,\ r)$

Generally, on $\mathbb{R}^n$ we will call $B_2(a,\ r)$ the n-***dimensional open sphere*** and $B_\infty(a,\ r)$ the n-***dimensional open cube***.

Remark 5.3.1 The two types of balls of $\mathbb{R}^n$ satisfy the following inclusions:

$$B_\infty\left(a,\ \frac{r}{\sqrt{n}}\right) \subset B_2(a,\ r) \subset B_\infty(a,\ r)$$

Indeed, if $x = (x_1, \ldots, x_n) \in B_\infty\left(a, \frac{r}{\sqrt{n}}\right)$, then $|x_i - a_i| < \frac{r}{\sqrt{n}}, \forall i$, whence it results $(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2 < n \cdot \frac{r^2}{n} = r^2$, hence $x \in B_2(a, r)$.

On the other hand, if $x = (x_1, \ldots, x_n) \in B_2(a, r)$, then:

$$(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2 < r^2.$$

Taking into account that $|x_i - a_i| \le \sqrt{(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2} < r, \forall i = \overline{1, n}$, we deduce that $x \in B_\infty(a, r)$.

In the particular case $n = 2$ we obtain the inclusions:

$$B_\infty\left(a, \frac{r}{\sqrt{2}}\right) \subset B_2(a, r) \subset B_\infty(a, r),$$

which are represented geometrically in Fig. 5.3.

Definition 5.3.2 Any subset $V \subset \mathbb{R}^n$ with the property that there exists $r > 0$ such that $V \supset B(a, r)$ is called a **neighborhood of the point** $a \in \mathbb{R}^n$.

The family of all neighborhoods of the point $a \in \mathbb{R}^n$ will be denoted by $\mathcal{V}$(a).

According to Definition 5.3.2, on $\mathbb{R}$, any subset $V \subset \mathbb{R}$ which contains an open interval $(a - r, a + r)$, $r > 0$, is a neighborhood of the point $a \in \mathbb{R}$. In particular the interval $(a - r, a + r)$ itself is a neighborhood of the point $a \in \mathbb{R}$.

It would seem that on $\mathbb{R}^n$ $(n \ge 2)$ there are two types of neighborhoods for a point $a \in \mathbb{R}^n$, namely neighborhoods containing open balls of the type $B_2(a, r)$, respectively, neighborhoods containing open balls of type $B_\infty(a, r)$. From Remark 5.3.1, it results that the two types of neighborhoods coincide.

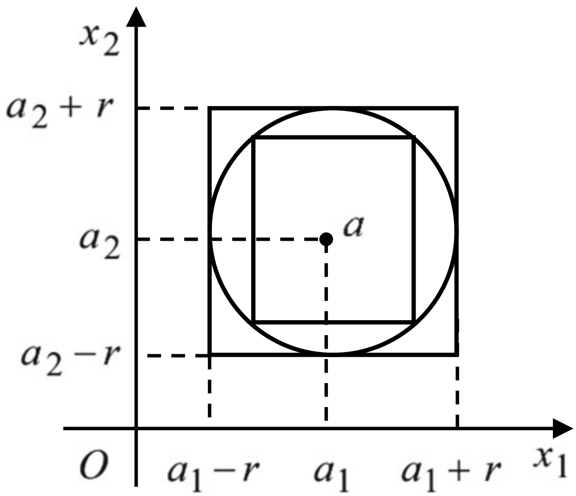

Fig. 5.3 $B_\infty(a, \frac{r}{\sqrt{2}}) \subset B_2(a, r) \subset B_\infty(a, r)$

Therefore, further, by the neighborhood of the point $a \in \mathbb{R}^n$ we mean any subset of $\mathbb{R}^n$ which contains either an open n-dimensional sphere or an open n-dimensional cube.

Proposition 5.3.1 *The family* $\mathcal{V}(\mathrm{a})$ *of all neighborhoods of the point* $a \in \mathbb{R}^n$ *has the following properties:*

(1) $a \in V$ for any $V \in \mathcal{V}(\mathrm{a})$.
(2) If $V \in \mathcal{V}(\mathrm{a})$ and $U \supset V$, then $U \in \mathcal{V}(\mathrm{a})$.
(3) If $a \neq b \in \mathbb{R}^n$, then there are $V \in \mathcal{V}(\mathrm{a})$ and $W \in \mathcal{V}(\mathrm{b})$ such that $V \cap W = \emptyset$.
(4) If $V_i \in \mathcal{V}(\mathrm{a})$, $\mathrm{i} \in \{1, 2, ..., \mathrm{m}\}$, then $\bigcap_{i=1}^{m} V_i \in \mathcal{V}(\mathrm{a})$.
(5) For any $V \in \mathcal{V}(\mathrm{a})$, there exists $W \in \mathcal{V}(\mathrm{a})$, such that $V \in \mathcal{V}(\mathrm{b})$, for any $b \in W$.

Proof The properties (1) and (2) are obvious. If $a \neq b$, then $\|a - b\| = r > 0$. It is immediate that $B\left(a, \frac{r}{3}\right) \cap B\left(b, \frac{r}{2}\right) = \emptyset$, hence (3) is true. Let $V_i \in \mathcal{V}(\mathrm{a})$ and $r_i > 0$ such that $V_i \supset B(a, r_i)$, $i = \overline{1, m}$. If we denote by: $r = \min\{r_i : 1 \leq i \leq m\}$, then $\bigcap_{i=1}^{m} V_i \supset B(a, r)$, whence it results that $\bigcap_{i=1}^{m} V_i \in \mathcal{V}(\mathrm{a})$.

Finally, let $V \in \mathcal{V}(\mathrm{a})$ and $r > 0$ such that $V \supset B(a, r)$. If we denote by $W = B\left(a, \frac{r}{2}\right)$, then for any $b \in W$ and any $x \in B\left(b, \frac{r}{2}\right)$ we have:

$$\|x - a\| \leq \|x - b\| + \|b - a\| < \frac{r}{2} + \frac{r}{2} = r,$$

whence it results that $B\left(b, \frac{r}{2}\right) \subset B(a, r) \subset V$, hence $V \in \mathcal{V}(\mathrm{b})$.

Definition 5.3.3 A point $a \in A \subset \mathbb{R}^n$ is called **an interior point of the set** A if there exists $V \in \mathcal{V}(\mathrm{a})$ such that $V \subset A$. The set of all interior points of the set A is called **the interior of set** A, and is denoted by $\overset{\circ}{A}$.Obviously, $\overset{\circ}{A} \subset A$. The reverse inclusion, generally, is not true. The set $A \subset \mathbb{R}^n$ is called **open set** if $A \subset \overset{\circ}{A}$, i.e. $A = \overset{\circ}{A}$.

Proposition 5.3.2 *The open ball* $B(a, r)$ *is an open set.*

Proof Let $b \in B(a, r)$ and let $0 < \varepsilon < r - \|b - a\|$. If $x \in B(b, \varepsilon)$, then:

$$\| x - a\| \leq \| x - b\| + \| b - a\| < \varepsilon + \| b - a\| < r,$$

whence it results that $x \in B(a, r)$, hence that $B(b, \varepsilon) \subset B(a, r)$.

Therefore any point $b \in B(a, r)$ is an interior point of the set $B(a, r)$, therefore $B(a, r)$ is an open set.

Example 5.3.2

(1) On $\mathbb{R}$, any open symmetric interval $(a - r, a + r)$, $a \in \mathbb{R}$, $r > 0$ is an open set.

Let $(\alpha\,,\,\beta) \subset \mathbb{R}$ be an arbitrary open interval. If we denote by $a = \frac{\alpha+\beta}{2}$ and by $r = \frac{\beta-\alpha}{2}$, then $(\alpha\,,\,\beta) = (a - r\,,\, a + r)$. It results that any open interval from $\mathbb{R}$ is an open set.

(2) On $\mathbb{R}^2$, the interior of any circle (square) is an open set.
(3) On $\mathbb{R}^3$, the interior of any sphere (cube) is an open set.

Proposition 5.3.3 *The open sets have the following properties:*

(1) *Any union of open sets is also an open set.*
(2) *Any finite intersection of open sets is also an open set.*
(3) *The sets $\mathbb{R}^n$ and ϕ(empty set) are open sets.*

Proof

(1) Let $\{D_i\}_{i\,\in I}$ be an arbitrary family of open sets and let $D = \bigcup_{i\,\in I} D_i$

If $a \in D$, then there exists $i_0 \in I$ such that $a \in D_{i_0}$. As $D_{i\,0}$ is an open set, it results that there exists a neighborhood $V \in \mathcal{V}(\mathrm{a})$ such that $V \subset D_{i_0}$. Obviously $V \subset D$, whence it results that a is an interior point of D, hence D is an open set.

(2) Let $D_1,\, \ldots,\, D_m$ be a family of open sets and let $D = \bigcap_{i=1}^m D_i$. If $a \in D$, then $a \in D_i, \forall\, i \in \{1,\, 2,\, \ldots,\, m\}$. Since D_i is an open set, it follows that there exists a neighborhood $V_i \in \mathcal{V}(\mathrm{a})$ such that $V_i \subset D_i$. If we denote by $V = \bigcap_{i=1}^m V_i$, then $V \in \mathcal{V}(\mathrm{a})$ and $V \subset D$. It results that a is an interior point of D, hence D is an open set.

The property (3) is obvious.

Proposition 5.3.2 allows us to give more varied examples of open sets. For example, on $\mathbb{R}$, any union and any finite intersection of open intervals there is an open set. On $\mathbb{R}^2$, various unions and finite intersections of the interiors of circles or squares are examples of open sets and so on.

Definition 5.3.4 A point $a \in \mathbb{R}^n$ is called an **adherent point of the set** $A \subset \mathbb{R}^n$ if for any neighborhood $V \in \mathcal{V}(\mathrm{a})$ we have $V \cap A \neq \emptyset$.

The set of all adherent points of the set A is denoted by $\overline{A}$ and is called the ***closure of the set** A.* Obviously $A \subset \overline{A}$. The reverse inclusion, generally, is not true. The set $A \subset \mathbb{R}^n$ is called ***closed*** if $\overline{A} \subset A$, i.e. $A = \overline{A}$.

Remark 5.3.2 Any point $a \in A$ is an adherent point of the set A; there may be adherent points of A which belong not to A.

Theorem 5.3.1 *The necessary and sufficient condition for a set $A \subset \mathbb{R}^n$ to be a closed set is that its complement set $\complement A = \mathbb{R}^n \setminus A$ to be an open set.*

Proof. **Necessity**:

Suppose that $A \subset \mathbb{R}^n$ is a closed set, i.e. $A = \overline{A}$. $V \subset \complement A$ then $b \notin A = \overline{A}$, thus b is not an adherent point of A. It results that there exists a neighborhood $V \in \mathcal{V}(\mathrm{b})$ such that $V \cap A = \emptyset$, hence $V \subset \complement A$. Therefore b is an interior point of $\complement A$ so $\complement A$ is an open set.

Sufficiency: Suppose that $\complement A$ is an open set. $If b \in \complement A$, , then there exists a neighborhood $V \in \mathcal{V}$(b) such that $V \subset \complement A$. Then we have $V \cap A = \emptyset$, whence it results that b is not an adherent point of A, hence $b \in \complement \overline{A}$.

Therefore we proved that $\complement A \subset \complement \overline{A}$ whence we deduce $\overline{A} \subset A$, hence $A = \overline{A}$.

Proposition 5.3.4 *The closed ball* $\hat{B}[a, r]$ *is a closed set.*

Proof From Theorem 5.3.1 it follows that it is sufficient to show that the set $\complement \hat{B}[a, r] = \{x \in \mathbb{R}^n; \|x - a\| > r\}$ is an open set.

Let $b \in \complement \hat{B}[a, r]$ and let $0 < \varepsilon < \|b - a\| - r$. If $x \in B(b, \varepsilon)$, then $\|x - b\| < \varepsilon$.

Further we have $\|b - a\| \le \|b - x\| + \|x - a\| < \|b - a\| - r + \|x - a\|$, whence it results that $\|x - a\| > r$, hence $x \in \complement \hat{B}[a, r]$.Therefore, $B(b, \varepsilon) \subset n\complement \hat{B}[a, r]$, so b is an interior point of $\complement \hat{B}[a, r]$. As b was arbitrary, it results that $\complement \hat{B}[a, r]$. is an open set.

Example 5.3.3

(1) On $\mathbb{R}$, for any $a \in \mathbb{R}$ and $r > 0$, the closed ball.

$$\hat{B}[a, r] = [a - r , a + r].$$

From Proposition 5.3.4 we deduce that any symmetric closed interval is a closed set. Since any closed interval $[\alpha , \beta] \subset \mathbb{R}$ can be represented as a symmetric closed interval, it follows that any closed interval of $\mathbb{R}$ is a closed set.

(2) On $\mathbb{R}^2$, for any $a = (a_1, a_2) \in \mathbb{R}^2$ and $r > 0$ we have:

$$\hat{B}_2[a, r] = \left\{x = (x_1, x_2) \in \mathbb{R}^2;\ (x_1 - a_1)^2 + (x_2 - a_2)^2 \le r^2\right\},$$

which geometrically represents the closed disk with center at a and radius r(consisting of the interior of the circle and its circumference, Fig. 5.4).and

$$\hat{B}_\infty[a, r] = \left\{x = (x_1, x_2) \in \mathbb{R}^2;\ |x_1 - a_1| \le r , |x_2 - a_2| \le r\right\},$$

which geometrically represents the closed square with center at a and side $2 \cdot \ r$.

(containing besides the interior of the square and its sides, Fig. 5.5).

Both the sets $\widehat{B}_2(a, r)$ and $\widehat{B}_\infty(a, r)$ are closed sets.

(3) On $\mathbb{R}^3$, examples of closed sets are $\hat{B}_2[a, r]$ which represents the solid sphere with center at a and radius r (including the all interior points of the sphere and all points that constitute the sphere) and $\hat{B}_\infty[a, r]$ represents the solid cube with center at a and side $2 \cdot r$(including the interior of the cube and its faces).

Proposition 5.3.3 *The family of closed sets has the following properties:*

(1) *Any finite union of closed sets is also a closed set.*

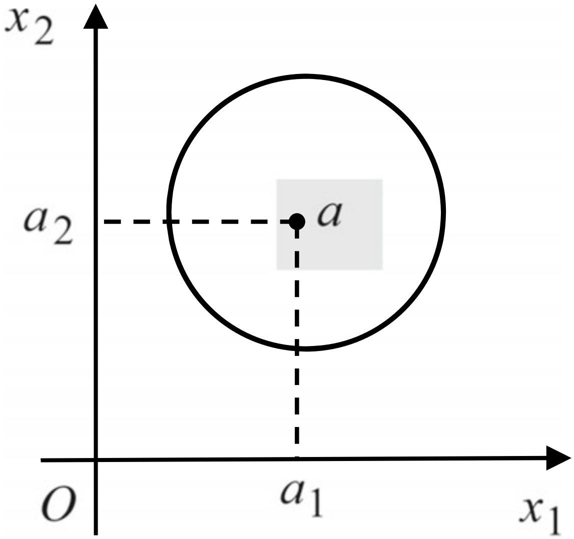

Fig. 5.4 $\hat{B}_2(a, r)$—the closed ball centered in origin and radius R

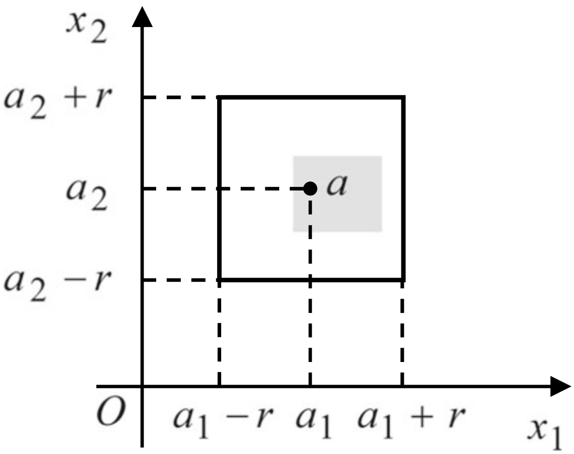

Fig. 5.5 $\hat{B}_\infty(a, r)$—the closed square with center at a and sides $2r.2r = \hat{B}_\infty(a, r)$

(2) *Any intersection of closed sets is also a closed set.*
(3) *The set $\mathbb{R}^n$ and the empty set $\emptyset$ are closed sets.*

Proof The assertions follow from Theorem 5.3.1, Proposition 5.3.2, and De Morgan relations. For example:

(1) Let $F_1, \ldots, F_m$ be a finite number of closed sets and let $F = \bigcup_{i=1}^m F_i$. From Theorem 5.3.1, it results that it is sufficient to prove that the complement set $\complement F$ is an open set. On the other hand, from De Morgan relation we have $\complement F = \bigcap_{i=1}^m \complement F_i$. As $\complement F_i$ is an open set for any $i \in \{1, 2, \ldots, m\}$, from Proposition 5.3.2 it results that $\complement F = \bigcap_{i=1}^m \complement F_i$ is an open set.

Definition 5.3.5 is called **the boundary of the set** $A \subset \mathbb{R}^n$ and is denoted by Fr A the following set:

$$\operatorname{Fr} A = \overline{A} \cap \overline{\complement A}$$

Example 5.3.4 (1) On $\mathbb{R}$, for any closed interval $[\alpha\,,\,\beta]$, we have $\operatorname{Fr}([\alpha, \beta]) = \{\alpha; \beta\}$.

(2) On $\mathbb{R}^2$, for any closed ball $\hat{B}_2[a,\, r]$ we have:

$$\operatorname{Fr}\Big(\hat{B}_2[a,\, r]\Big) = \big\{(x_1, x_2) \in \mathbb{R}^2;\; x_1^2 + x_2^2 = r^2\big\},$$

which represents the circumference of the circle with center at a and radius r, and

$$\operatorname{Fr}\Big(\hat{B}_\infty[a, r]\Big) = \left\{\begin{array}{l}(x_1, a_2 - r) \cup (x_1, a_2 + r) \cup (a_1 - r, x_2) \cup (a_1 + r, x_2); \\ |x_1 - a_1| \le r, |x_2 - a_2| \le r\end{array}\right\}$$

which represents the union of the sides of the square:

$$\big\{x = (x_1,\, x_2) \in \mathbb{R}^2;\; |x_1 - a_1| \le r\,,\, |x_2 - a_2| \le r\big\}.$$

(3) On $\mathbb{R}^3$, the boundary of the solid sphere $\hat{B}_2[a,\, r]$ is the sphere $\big\{(x_1, x_2, x_3) \in \mathbb{R}^3;\; x_1^2 + x_2^2 + x_3^2 = r^2\big\}$ and the boundary of the solid cube $\hat{B}_2[a,\, r]$ consists in the union of its faces.

Lemma 5.3.1 *For any subset* $A \subset \mathbb{R}^n$ *we have*:

$$\complement \overset{\circ}{A} = \overline{\complement A} \text{ and } \complement \overline{A} = \overset{\circ}{\widehat{\complement A}}.$$

Proof If $b \in \complement \overset{\circ}{A}$, then $b \notin \overset{\circ}{A}$ and so for any neighborhood $V \in \mathcal{V}(\mathrm{b})$ we have $V \cap \complement A \neq \emptyset$, whence it results that $b \in \overline{\complement A}$. Reciprocal, if $b \in \overline{\complement A}$, then for any neighborhood $V \in \mathcal{V}(\mathrm{b})$ we have avem $V \cap \complement A \neq \emptyset$, whence it results that $b \notin \overset{\circ}{A}$, hence $\overline{\complement A} \subset \complement \overset{\circ}{A}$. The other equality is similarly proved.

Proposition 5.3.4 *Let* A *be an arbitrary subset of* $\mathbb{R}^n$. *Then:*

(i) $\overset{\circ}{A}$, *the interior of* A, *is an open set.*
(ii) $\overline{A}$, *the closure of* A, *is a closed set.*
(iii) Fr A, *the boundary of* A, *is a closed set and* $\operatorname{Fr} A = \overline{A} \setminus \overset{\circ}{A}$.

Proof

(i) If $a \in \overset{\circ}{A}$, then there exists $r > 0$ such that $V = B(a,\, r) \subset A$. According to Remark 5.3.2, V is an open set. Therefore we have $V = \overset{\circ}{V} \subset \overset{\circ}{A}$, whence it results that a is an interior point of $\overset{\circ}{A}$, hence $\overset{\circ}{A}$ is an open set.

(ii) From Lemma 5.3.1 it results that $\complement\overline{A} = \overset{\circ}{\widehat{\complement A}}$. On the other hand, according to (i) $\overset{\circ}{\widehat{\complement A}}$ is an open set, hence $\complement\,\overline{A}$ is an open set. From Theorem 5.3.1, it follows that $\overline{A}$ is a closed set.

(iii) From Lemma 5.3.1 it results that:

$$\operatorname{Fr} A = \overline{A} \cap \overline{\complement A} = \overline{A} \cap \complement \overset{\circ}{A} = \overline{A} \setminus \overset{\circ}{A}$$

As, according to (ii), $\overline{A}$ and $\overline{\complement A}$ are closed sets, we deduce from Proposition 5.3.3 that Fr A is a closed set.

Theorem 5.3.2 *Let $A \subset \mathbb{R}^n$ be an arbitrary subset. Then:*

(i) A point $b \in \mathbb{R}^n$ is an adherent point for A iff there exists a convergent sequence $\{a_k\}$, $a_k \in A$, $\forall k \in \mathbb{N}^*$, such that $a_k \xrightarrow{\mathbb{R}^n} b$.

(ii) The subset A is closed iff the limit of any convergent sequence of vectors of A belongs to A.

Proof

(i) If $b \in \overline{A}$, then $A \cap \ B(b, r) \neq \emptyset, \forall r > 0$. In particular, for any $k \in \mathbb{N}^*$ there exists $a_k \in A \cap B\left(a, \frac{1}{k}\right)$. Therefore $a_k \in A$ and $\|a_k - a\| < \frac{1}{k}, \forall k \in \mathbb{N}^*$, whence it results that $a_k \xrightarrow{\mathbb{R}^n} b$.

Reciprocal, if $a_k \in A, \forall k \in \mathbb{N}^*$ and şi $a_k \xrightarrow{\mathbb{R}^n} b$, then $\forall r > 0, \exists k_r \in \mathbb{N}^*$ such that $\|a_k - b\| < r$, $\forall k \geq k_r$, whence it results that $a_k \in A \ \cap B(b, r)$, $\forall k \geq k_r$, hence $b \in \overline{A}$.

(ii) Let A be a closed set and let $\{a_k\}$ be a convergent sequence of vectors of A, such that $a_k \xrightarrow{\mathbb{R}^n} b$. From (i) it results that $b \in \overline{A} = A$.

Reciprocal, if $b \in \overline{A}$, then from (i) it results that there exists a convergent sequence of vectors $\{a_k\}$, $a_k \in A$, $\forall k \in \mathbb{N}^*$, such that cu $a_k \xrightarrow{\mathbb{R}^n} b$. By hypothesis it follows that $b \in A$, whence we deduce that A is closed.

Remark 5.3.3 From Theorem 5.3.2, (ii), it results that a set is closed if it contains all of its limit points, in other words if it is "**closed**" with respect to the limit crossing operation.

Definition 5.3.6 A point $b \in \mathbb{R}^n$ is called an **accumulation point** of the set $A \subset \mathbb{R}^n$ if for any neighborhood $V \in \mathcal{V}$(b) there exists $a \in V \ \cap A$ such that $b \neq a$.

The set of all accumulations points of A is called the **derived set** of A and is denoted by A'. Obviously $A' \subset \overline{A}$.

Remark 5.3.4 An accumulation point of A may or may not belong to A.

Theorem 5.3.3 *Let $A \subset \mathbb{R}^n$ be an arbitrary subset. Then:*

(i) *A point $b \in \mathbb{R}^n$ is an accumulation point of A iff there exists a convergent sequence of vectors $\{a_k\}$, $a_k \in A$, $\forall k \in \mathbb{N}^*$,$a_k \neq a_m$ if $k \neq m$, such that $a_k \xrightarrow{\mathbb{R}^n} b$.*

(ii) *A is closed iff $A' \subset A$.*

Proof

(i) If $b \in A'$, then there exists $a_1 \in A \cap B(b, 1)$, $a_1 \neq b$.

Let $r_1 == \|a_1 - b\| < 1$ and $a_2 \in A \cap B\left(b, \frac{r_1}{2}\right)$. Obviously $a_2 \neq a_1$ and $\|a_2 - b\| < \frac{1}{2}$. Let $r_2 == \|a_2 - b\|$ and $a_3 \in A \cap B\left(b, \frac{r_2}{2}\right)$, $a_3 \neq b$. Obviously $a_3 \neq a_2$, $a_3 \neq a_1$ şi $\|a_3 - b\| < \frac{r_2}{2} < \frac{1}{2^2}$. By complete mathematical induction it can be shown that there exists a sequence $\{a_k\}$, $a_k \in A$, $\forall k \in \mathbb{N}^*$,$a_k \neq a_m$ if $k \neq m$, such that $\|a_k - b\| < \frac{1}{2^{k-1}}$, whence it results that $a_k \xrightarrow{\mathbb{R}^n} b$.

The reverse assertion is obvious.

(ii) If A is closed, then $\overline{A} \subset A$. As $A' \subset \overline{A}$ it follows that $A' \subset A$.

Reciprocal, suppose that $A' \subset A$ and let $b \in \overline{A}$. From Theorem 5.3.2, it results that there exists a sequence $\{a_k\}$, $a_k \in A$, $\forall k \in \mathbb{N}^*$ such that $a_k \xrightarrow{\mathbb{R}^n} b$. If the sequence $\{a_k\}$ has an infinity number of terms different two by two, it results according to (i) that $b \in A'$. Since $A' \subset A$, it follows that $b \in A$. If $\{a_k\}$ does not have an infinity number of terms different two by two, then $a_k = b$ for an infinity of indices k, hence $b \in A$. Therefore $\overline{A} \subset A$, and so A is closed.

Remark 5.3.5 From Theorem 5.3.3 it follows that, if b is an accumulation point of A, then every neighborhood $V \in \mathcal{V}$(b) contains an infinity number of different elements of A. It results that the finite sets do not have accumulation points, so, according to Theorem 5.3.3, any finite set is a closed set. The sets which do not have accumulation points are called ***discrete sets.*** There are infinite discrete sets. For example, the set $\mathbb{Z}$ of all integers is a discrete subset of $\mathbb{R}$.

Definition 5.3.7 A subset $A \subset \mathbb{R}^n$ is called **bounded** if there exists $M > 0$ such that $\|x\| \leq M, \forall x \in A$.

Lemma 5.3.2 (Cesàro). *Any bounded sequence of vectors from $\mathbb{R}^n$ contains a convergent subsequence.*

Proof We present the proof for the particular case $n = 2$.

Let $\{z_k\} = \{(x_k\,,\ y_k)\}$, $k \in \mathbb{N}^*$, be a bounded sequence of vectors$\mathbb{R}^2$ of $\mathbb{R}^2$. Then there exists $M > 0$ such that $\|z_k\|_\infty \leq M$, $\forall\, k \in \mathbb{N}^*$. Since $|x_k|, |y_k| \leq \|z_k\|_\infty \leq M, \forall k \in \mathbb{N}^*$, it results that the real number sequences $\{x_k\}$ and $\{y_k\}$ is bounded. From Cesàro's lemma for real number sequences (Lemma 1.2.1), it follows that there exists a convergent subsequence $\left\{x_{k_p}\right\}$. Let $x = \lim\limits_{p\to\infty} x_{k_p}$. Applying again Lemma 1.2.1 for sequence $\left\{y_{k_p}\right\}$ it results that there exists a subsequence $\left\{y_{k_{p_l}}\right\}$ which converges in $\mathbb{R}$

to y. From Theorem 5.2.1 we deduce that the sum sequence $\{z_{k_{p_l}}\} = \{(x_{k_{p_l}}, y_{k_{p_l}})\}$ is convergent in $\mathbb{R}^2$ and has the limit $z = (x, y)$.

For the theory of limits of functions of several variables, it is important to know when a subset $A \subset \mathbb{R}^n$ has accumulation points. The following theorem gives us a sufficient condition for a subset of $\mathbb{R}^n$ to have accumulation points.

Theorem 5.3.4 (Weierstrass–Bolzano). *Every bounded and infinite set of $\mathbb{R}^n$ has at least one accumulation point.*

Proof Let $A \subset \mathbb{R}^n$ be a bounded and infinite subset. Since A is infinite, it results that there exists a sequence $\{x_k\}$,$x_k \notin A$, $x_k \neq x_l$ if $k \neq l$. Taking into account that A is bounded, it results that the sequence $\{x_k\}$ is bounded. From Lemma 5.3.2, it follows that there exists a convergent subsequence $\{x_{k_p}\}$. If we denote by $b = \lim\limits_{p \to \infty} x_{k_p}$, then, according to Theorem 5.3.2, b is an accumulation point of A.

Definition 5.3.8 A subset $K \subset \mathbb{R}^n$ is called a **compact set**$\mathbb{R}^n$ if it is bounded and closed.

From Proposition 5.3.3, it follows that a finite union of compact sets is also a compact set and any intersection of compact set is also a compact set. It is also obvious that any finite set is a compact set (Remark 5.3.7).

5.4 Limits of Functions of Several Variables

Definition 5.4.1 By **vector function** we mean any function F defined on a set $A \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$.

So, for any $x = (x_1, \ldots, x_n) \in A \subset \mathbb{R}^n$, the image $y = F(x) \in \mathbb{R}^m$ it is a vector of form $F(x) = (y_1, \ldots, y_m) \in \mathbb{R}^m$,

If we denote by $f_i(x) = y_i$, $\forall x = (x_1, \ldots, x_n) \in A$, $\forall i = \overline{1, m}$, then we obtain m scalar functions $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$, $i = \overline{1, m}$, which we call **the scalar components** of the vector function F. Therefore, we have:

$F = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $F(x) = (f_1(x), \ldots, f_m(x))$, $\forall x \in A$.

Example 5.4.1

(1) The function $r : [0, 2\pi] \subset \mathbb{R} \to \mathbb{R}^2$ defined by:

$r(t) = (a \cdot \cos t, a \cdot \sin t)$, $t \in [0, 2\pi]$, $a > 0$,

is a vector function. The image of this function is the circle with center at origin and radius a.

(2) The function $r : [0, 2\pi] \times [0, \pi] \subset \mathbb{R}^2 \to \mathbb{R}^3$,

$$r(u, v) = (a \cdot \sin u \cdot \cos v, a \cdot \sin u \cdot \sin v, a \cdot \cos u),$$
$$u \in [0, \pi], v \in [0, 2\pi] a > 0$$

is a vector function, whose image represents the sphere with center at origin and radius a.

Definition 5.4.2 Let $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector function, $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ an accumulation point of A and $L \in \mathbb{R}^m$. It is said that L is **the limit of function** F **at** the **point** a, and we will note this by $\lim_{x \to a} F(x) = L$, if for any neighborhood $U \subset \mathbb{R}^m$ of L, there exists a neighborhood $V \subset \mathbb{R}^n$ of a, such that $F(x) \in U, \forall x \in V \cap A$, $x \neq a$.

Theorem 5.4.1 *Let* F, a *and* L *be as in* Definition 5.4.2. *Then the following statements are equivalent:*

(1) $\lim_{x \to a} F(x) = L$;

(2) *For any* $\varepsilon > 0$, *there is* $\delta_\varepsilon > 0$ *such that for any* $x \in A$, $x \neq a$ *with the property* $\|x - a\| < \delta_\varepsilon$, *it results that* $\|F(x) - L\| < \varepsilon$ (*here the norm* $\|\cdot\|$ *is any of the norms* $\|\cdot\|_\infty$ *sau* $\|\cdot\|_2$).

(3) *For any sequence of vectors* $\{x_k\}$, $x_k \in A$ $x_k \neq a$, $\forall k \in \mathbb{N}^*$, $x_k \xrightarrow{\mathbb{R}^n} a$, *it follows that* $F(x_k) \xrightarrow{\mathbb{R}^m} L$.

Proof

(1)⇒(2). If $\varepsilon > 0$ and $U = B(L, \varepsilon)$, then obviously $U \in \mathcal{V}$(L) and according to (1), there exists a neighborhood $V \in \mathcal{V}$(a)(which generally depends on ε) such that $F(x) \in U$, $\forall x \in V \cap A, x \neq a$.

Since $V \in \mathcal{V}$(a), there exists $\delta_\varepsilon > 0$ with the property $V \supset B(a, \delta_\varepsilon)$. If $x \in A$, $x \neq a$, and $\|x - a\| < \delta_\varepsilon$, it results that $x \in B(a, \delta_\varepsilon)$, hence $x \in V \cap A$, and $x \neq a$. According to (1) we have $F(x) \in U$, so $\|F(x) - L\| < \varepsilon$

(2)⇒(3) Let $\varepsilon > 0$ and $\delta_\varepsilon > 0$ with the properties from (2). If $\{x_k\}$ is a convergent sequence of vectors from A, $x_k \neq a$, $\forall k \in \mathbb{N}^*$ and $x_k \xrightarrow{\mathbb{R}^n} a$, then there exists $k_\varepsilon \in \mathbb{N}^*$ such that $\|x_k - a\| < \delta_\varepsilon$, for any $k \geq k_\varepsilon$. From (2) we deduce that $\|F(x_k) - L\| < \varepsilon, \forall k \geq k_\varepsilon$, hence $F(x_k) \xrightarrow{\mathbb{R}^m} L$.

(3)⇒(1) If we suppose that (1) is not true, then there is a neighborhood $U_0 \in \mathcal{V}$(L) such that for any neighborhood $V \in \mathcal{V}$(a), there exists.

$x\_V \in V \cap A, x_V \neq a$, with the property $F(x_V) \notin U_0$.

In particular, for $V_k = B\left(a, \frac{1}{k}\right), \forall k \in \mathbb{N}^*$, it results that there exists $x_k \in A \cap B\left(a, \frac{1}{k}\right)$, $x_k \neq a$, such that $F(x_k) \notin U_0$, $\forall k \in \mathbb{N}^*$.

As $x_k \in B\left(a, \frac{1}{k}\right)$, $\forall k \in \mathbb{N}^*$, it follows that $\|x_k - a\| < \frac{1}{k}$, hence that $x_k \xrightarrow{\mathbb{R}^n} a$. From (3) it results that $F(x_k) \xrightarrow{\mathbb{R}^m} L$ and this contradicts the fact that $F(x_k) \notin U_0$, $\forall k \in \mathbb{N}^*$.

Remark 5.4.1

(1) $\lim_{x \to a} F(x) = L$ can also be written in the form:

$$\lim_{\substack{x_1 \to a_1 \\ \cdots \\ x_n \to a_n}} F(x_1, \ldots, x_n) = L.$$

In the second notation it is highlighted the fact that the components x_i of the vector $x = (x_1, x_2, \ldots, x_n)$ converge simultaneously and independently to the components a_i of the vector $a = (a_1, \ldots, a_n)$.

(2) For to prove that the function $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ does not have limit at the point $a \in A'$, it is sufficient to find two sequences $\{x_k\}$ and $\{y_k\}$ of vectors from A, such that $x_k \xrightarrow{\mathbb{R}^n} a$, $y_k \xrightarrow{\mathbb{R}^n} a$, and the sequences $\{F(x_k)\}$ and $\{F(y_k)\}$ have different limits.

Remark 5.4.2 If $f : A \subset \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ is an accumulation point of A and $l \in \mathbb{R}$, then, from Theorem 5.4.1, it results that $\lim\limits_{x \to a} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for any $x \in A$, $x \neq a$, with the property $|x - a| < \delta_\varepsilon$ it follows that $|f(x) - l| < \varepsilon$ (here we use the fact that on $\mathbb{R}$, $\|x\|_2 = \|x\|_\infty = |x|$).

We thus found the definition of the limit of a function of one variable.

Theorem 5.4.2 *Let $F = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector function, $a \in \mathbb{R}^n$ be an accumulation point of A and $L = (l_1, \ldots, l_m) \in \mathbb{R}^m$. Then, the necessary and sufficient condition for $\lim\limits_{x \to a} F(x) = L$ is that $\lim\limits_{x \to a} f_i(x) = l_i$, for any $i = \overline{1, m}$.*

Proof Let $\{x_k\}$ be a sequence of vectors from A, $x_k \neq a$, $\forall k \in \mathbb{N}^*$, $x_k \xrightarrow{\mathbb{R}^n} a$. From Theorem 5.4.1, it follows that $\lim\limits_{x \to a} F(x) = L$ if and only if:

$$F(x_k) = (f_1(x_k), \ldots, f_m(x_k)) \xrightarrow{\mathbb{R}^m} L = (l_1, \ldots, l_m)$$

On the other hand, according to Theorem 5.2.1, this is equivalent with the fact that $\forall i = \overline{1, m}$ $f_i(x_k) \xrightarrow{\mathbb{R}} l_i$.

Applying again Theorem 5.4.1, it results that $\lim\limits_{x \to a} f_i(x) = l_i$.

Remark 5.4.3 From Theorem 5.4.2 it follows that the study of the limit of a vector function returns to the study of the limits of its scalar components. For this reason, it is sufficient to further study only limits of scalar functions.

Remark 5.4.4 Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar function, $a = (a_1, \ldots, a_n)$ an accumulation point of A and $l \in \mathbb{R}$. If we use the norm $\|\cdot\|_\infty$, then from Theorem 5.4.1, it results that the following statements are equivalent with $\lim\limits_{x \to a} f(x) = l$:

(1) For any neighborhood $U \in \mathcal{V}(\mathrm{l})$, $U \subset \mathbb{R}$, there is a neighborhood $V \in \mathcal{V}(\mathrm{a})$, $V \subset \mathbb{R}^n$ such that $f(x) \in U$, $\forall x \in V \cap A$, $x \neq a$;

(2) $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\forall x = (x_1, \ldots, x_n) \in A$, $x \neq a$, with the property $|x_1 - a_1| < \delta_\varepsilon$, ..., $|x_n - a_n| < \delta_\varepsilon$, it results $|f(x_1, \ldots, x_n) - l| < \varepsilon$;

(3) For any sequence $\{x_k\}$ of vectors from A, $x_k \neq a$, $\forall k \in \mathbb{N}^*$, $x_k \xrightarrow{\mathbb{R}^n} a$, we have $f(x_k) \xrightarrow{\mathbb{R}} l$.

Let us now consider the case even simpler when $n = 2$.

Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a scalar function of two variables and $l \in \mathbb{R}$. We will use the notations (x, y) instead (x_1, x_2) and (a, b) instead (a_1, a_2), with $(a, b) \in A'$. It follows from the above that the following statements are equivalent:

(2) $\to$ (1) $\lim\limits_{(x,y)\to(a,b)} f(x, y) = \lim\limits_{\substack{x\to a\\ y\to b}} f(x, y) = l$;

(3) $\to$ (2) $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for any $(x, y) \in A$ with the properties $|x - a| < \delta_\varepsilon$, $|y - b| < \delta_\varepsilon$, it follows $|f(x, y) - l| < \varepsilon$.

(4) For any sequence $\{(x_n, y_n)\}$ of elements of A,

$(x_n, y_n) \neq (a, b)\, \forall n \in \mathbb{N}^*$, $(x_n, y_n) \xrightarrow{\mathbb{R}^2} (a, b)$, it results that: $f(x_n, y_n) \xrightarrow{\mathbb{R}} l$.

Example 5.4.2 Compute $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$, where:

$$f(x, y) = \frac{\sin\left(x^3 + y^3\right)}{x^2 + y^2}, \ (x, y) \in \mathbb{R}^2\backslash\{(0, 0)\}$$

Since $|\sin x| \leq |x|$, $\forall x \in \mathbb{R}$, it results:

$$\begin{aligned}|f(x, y)| &= \left|\frac{\sin\left(x^3 + y^3\right)}{x^2 + y^2}\right| \leq \frac{\left|x^3 + y^3\right|}{x^2 + y^2} \leq \frac{\left|x^3\right| + \left|y^3\right|}{x^2 + y^2} \\ &= \frac{x^2|x| + y^2|y|}{x^2 + y^2} \\ &= |x|\frac{x^2}{x^2 + y^2} + |y|\frac{y^2}{x^2 + y^2} \leq |x| + |y|.\end{aligned}$$

For any sequence $\{(x_n, y_n)\}$, $x_n \neq 0$, $y_n \neq 0$, $(x_n, y_n) \xrightarrow{\mathbb{R}^2} (0, 0)$, we have $|x_n| + |y_n| \xrightarrow{\mathbb{R}} 0$ and so $f(x_n, y_n) \xrightarrow{\mathbb{R}} 0$.

Therefore $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y) = 0$.

Example 5.4.3 Let the scalar function:

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \ (x, y) \in \mathbb{R}^2\backslash\{(0, 0)\}$$

To show that does not exist $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x, y)$

Consider the sequences $\left\{\left(\frac{1}{n},\,-\frac{1}{n}\right)\right\}$, respectively, $\left\{\left(\frac{2}{n},\,-\frac{1}{n}\right)\right\}$ from $\mathbb{R}^2$.

Obviously $\left(\frac{1}{n},\,-\frac{1}{n}\right)\xrightarrow{\mathbb{R}^2}(0,\,0)$ and $\left(\frac{2}{n},\,-\frac{1}{n}\right)\xrightarrow{\mathbb{R}^2}(0,\,0)$.On the other hand we have $f\left(\frac{1}{n},\,-\frac{1}{n}\right)\xrightarrow{\mathbb{R}} 0$ and $f\left(\frac{2}{n},\,-\frac{1}{n}\right)\xrightarrow{\mathbb{R}} \frac{3}{5}$, whence it results that does not exists $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2}$.

Theorem 5.4.3 (Cauchy–Bolzano). *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar function and let $a = (a_1,\,\ldots,\,a_n) \in \mathbb{R}^n$ be an accumulation point of A.*

The necessary and sufficient condition to exist $\lim\limits_{x\to a} f(x) = l \in \mathbb{R}$ is:

for any $\varepsilon > 0$ there exists a neighborhood $V_\varepsilon \in \mathcal{V}(\mathrm{a})$ such that $\forall x',\, x'' \in A \cap \mathcal{V}_\varepsilon$, $x' \neq a$, $x'' \neq a$ we have $\left|f\left(x'\right) - f\left(x''\right)\right| < \varepsilon$.

Proof. Necessity.

If $l = \lim\limits_{x\to a} f(x) \in \mathbb{R}$, then $\forall\,\varepsilon > 0$ there exists a neighborhood $V_\varepsilon \in \mathcal{V}(\mathrm{a})$ such that $|f(x) - l| < \frac{\varepsilon}{2}$ for any $x \in V_\varepsilon \cap A$, $x \neq a$.

For any $x',\, x'' \in V_\varepsilon \cap A$, $x' \neq a$, $x'' \neq a$, we have:

$$\left|f\left(x'\right) - f\left(x''\right)\right| \le \left|f\left(x'\right) - l\right| + \left|l - f\left(x''\right)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Sufficiency: Let $\varepsilon > 0$ and $V_\varepsilon \in \mathcal{V}(\mathrm{a})$ be as in the statement of the Theorem, and let $\{x_k\}$ be a sequence of vectors from A, $x_k \neq a$, $\forall k \in \mathbb{N}^*$,$x_k \xrightarrow{\mathbb{R}^n} a$. Then there exists a rank $k_\varepsilon \in \mathbb{N}^*$ (this rank depends on V_ε, which in turn depends on ε) such that $x_k \in V_\varepsilon$, $\forall\,k \ge k_\varepsilon$. It follows that $\left|f(x_k) - f\left(x_p\right)\right| < \varepsilon$, for any $k,\; p \ge k_\varepsilon$, whence we deduce that the sequence $\{f(x_k)\}$ is a fundamental sequence of real numbers. From Cauchy's criterion for sequences of real numbers (Theorem 1.2.1), it results that the sequence $\{f(x_k)\}$ converges in $\mathbb{R}$ to a number $l \in \mathbb{R}$. If $\left\{x_k'\right\}$ is an another sequence of vectors from A, $x_k' \neq a$, $\forall k \in \mathbb{N}^*$, $x_k' \xrightarrow{\mathbb{R}^n} a$, then the sequence $\left\{f(x_k')\right\}$ is also convergent. Let us note with l' its limit. On the other hand it is obvious that the sequence:

$$\left\{x_k,\, x_k'\right\}_k = \left\{x_1,\, x_1',\, \ldots,\, x_k,\, x_k',\, \ldots\right\} \xrightarrow{\mathbb{R}^n} a \text{ and } x_k \neq a,\; x_k' \neq a.$$

From the above we deduce that the sequence $\left\{f(x_1),\, f(x_1'),\, \ldots,\, f(x_k),\, f(x_k'),\, \ldots\right\}$ is convergent. As the sequences $\{f(x_k)\}$ and $\left\{f\left(x_k'\right)\right\}$ are subsequences of this convergent sequence, it follows that they have the same limit so that $l = l'$.

According to Theorem 5.4.1, (3), it follows that there exists $\lim\limits_{x\to a} f(x) = l$.

For a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ we can consider, in addition to the previously defined limit, in which the variables $x_1,\, x_2\,,\,\ldots\,,\, x_n$ tend independently, but simultaneously to their limiting values $a_1,\, a_2,\,\ldots\,,\, a_n$, and ***iterated limits***, in which the variables $x_1,\, x_2\,,\,\ldots\,,\, x_n$ tend to their limiting values $a_1,\, a_2,\,\ldots\,,\, a_n$ in succession.

By way of illustration we consider a scalar function of two variables.

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, where $D = \left\{(x,\ y) \in \mathbb{R}^2;\ |x - a| < h,\ |y - b| < k\right\}$ is a rectangle.

Suppose that for any $x \in (a - h,\ a + h)$ there exists $\lim\limits_{y \to b} f(x,\ y)$. Obviously this limit depends on x. We will denote by $\varphi(x) = \lim\limits_{y \to b} f(x,\ y)$, $\forall x \in (a - h,\ a + h)$.

If we assume in addition that there exists $\lim\limits_{x \to a} \varphi(x)$, then this limit is denoted by $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x,\ y)$ and is called the $(x, y)-$ ***iterated (repeated) limit of the function*** f ***at point*** $(a,\ b)$. Similarly is defined $(y, x)-$ ***iterated (repeated) limit of the function*** f ***at point*** $(a,\ b)$ which is denoted by $\lim\limits_{y \to b} \lim\limits_{x \to a} f(x,\ y)$.

The iterated limits are generally not equal, according to the following examples:

Example 5.4.4 For the function:

$$f(x,\ y) = \frac{x^2 - y^2}{x^2 + y^2}\ ,\ (x,\ y) \in \mathbb{R}^2 \backslash \{(0,\ 0)\}$$

we have $\lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x,\ y) = 1 \neq \lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x,\ y) = -1$

We remark that in this case the $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exists (see Example 5.4.3).

Example 5.4.5 Consider the function:

$$f(x,\ y) = x\ \sin\frac{1}{y}\ ,\ (x,\ y) \in \mathbb{R} \times \mathbb{R}^*$$

Since $\left|x\ \sin\frac{1}{y}\right| \leq |x|$, it follows that $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y) = 0$.

We notice that $\lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x,\ y) = 0$, while the limit

$$\lim_{x \to 0} \lim_{y \to 0} f(x,\ y)$$

does not exist.

The connection between the iterated limits and the limit with respect to both variables, i.e. $\lim\limits_{\substack{x \to \infty \\ y \to b}} f(x, y)$ is highlighted by the following proposition.

Proposition 5.4.1 *Let* $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ *be a scalar function and let* $(a\ ,\ b)$ *be an accumulation point of* A*. If we suppose that there exists* $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = l$ *and for any* $x \in (a - h,\ a + h)$ *there exists also* $\varphi(x) = \lim\limits_{y \to b} f(x,\ y)$*, then there exists the* $(x, y)-$ *iterated limit* $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x,\ y)$ *and* $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x,\ y) = \lim\limits_{x \to a} \varphi(x) = l$.

Proof Since $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = l$, it results that for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that $\forall\, (x\,,\; y) \in A$, with $|x - a| < \delta_\varepsilon$, $|y - b| < \delta_\varepsilon$ we have:

$$|f(x\,,\; y) - l\,| < \varepsilon.$$

When $y \to b$ we obtain $|\varphi(x) - l\,| \leq \varepsilon$, for any $x \in (a - h, a + h)$, with the property $|x - a| < \delta_\varepsilon$. Therefore there exists $\lim\limits_{x \to a} \varphi(x) = l$, so there exists $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x, y) = l$.

Corollary 5.4.1 *If the iterated limits exist and are different then* $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y)$ *does not exists.*

Corollary 5.4.2 *If there exists only one of the three limits* $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y)$, $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x, y)$ or $\lim\limits_{y \to b} \lim\limits_{x \to a} f(x, y)$, *then it does not results that there are the other two.*

5.5 Continuous Functions of Several Variables

For any vector function $F : \mathbb{R}^n \to \mathbb{R}^m$ and any subset $A \subset \mathbb{R}^n$, we will denote by $F(A) = \{F(x)\,;\; x \in A\}$. Obviously $F(A) \subset \mathbb{R}^m$ and is called ***the direct image of*** A ***under*** F.

For any subset $B \subset \mathbb{R}^m$ we denote by $F^{-1}(B) = \{x \in \mathbb{R}^n\,;\; F(x) \in B\}$. The subset $F^{-1}(B)$ is called ***the inverse image of*** B ***under*** F.

Definition 5.5.1 Let $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector function and $a \in A$. On said that the function F is **continuous at the point** a if for any neighborhood U of $F(a)$, there exists a neighborhood V of a, such that $F(V \cap A) \subset U$.

The function F is **continuous on** A if it is continuous at every point of A.

Remark 5.5.1 If $a \in A$ is an accumulation point of A, then F is continuous at the point a iff there exists $\lim\limits_{x \to a} F(x)$ and $\lim\limits_{x \to a} F(x) = F(a)$. If $a \in A$ does not an accumulation point of A (i.e.*a is an* ***isolated point***), then there is a neighborhood $V \in \mathcal{V}(\mathrm{a})$ such that $V \cap A = \{a\}$ and is obvious that $F(V \cap A) \subset U, \forall\, U \in \mathcal{V}(\mathrm{F(a)})$, whence it results that F is continuous at a.

Therefore, any function is continuous at every isolated point of its domain.

Theorem 5.5.1 *Let* $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ *be a vector function and* $a \in A$.*The following statements are equivalent*:

(1) *F is continuous at a;*

(2) *For any* $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ *such that* $\forall x \in A$ *with* $\|x - a\| < \delta_\varepsilon$,*it results* $\|F(x) - F(a)\| < \varepsilon$*;*

(3) *For any convergent sequence* $\{x_k\}$ *of vectors from* A, $x_k \xrightarrow{\mathbb{R}^n} a$*, it results* $F(x_k) \xrightarrow{\mathbb{R}^m} F(a)$.

Proof The proof results from Theorem 5.4.1 and Observation 5.5.1, nothing that if a is an isolated point, then any of the statements (1)–(3) is obvious.

Theorem 5.5.2 *The vector function* $F = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ *is continuous at the point* $a \in A$ *iff each of its scalar component* $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$ *is continuous at* a*, for any* $i \in \{1, 2, \ldots, m\}$.

Proof The proof follows from Theorem 5.4.2 and Remark 5.5.1.

From above theorem we deduce that it is sufficient to study the continuity of scalar functions.

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar function and $a = (a_1, \ldots, a_n) \in A$. If we use the norm $\|\cdot\|_\infty$, then the function f is continuous at the point a if $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for any $x = (x_1, \ldots, x_n) \in A$ with property

$$|x_1 - a_1| < \delta_\varepsilon, \ldots, |x_n - a_n| < \delta_\varepsilon$$

it results:

$$|f(x_1, \ldots, x_n) - f(a_1, \ldots, a_n)| < \varepsilon$$

Further we consider the case of functions of two variables and we use the notations (x, y) instead of (x_1, x_2) and (a, b) instead of (a_1, a_2).

Remark 5.5.2 If $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in A$, then the following statements are equivalent:

(1) f is continuous at (a, b);

(2) $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for any $(x, y) \in A$ with the property $|x - a| < \delta_\varepsilon$, $|y - b| < \delta_\varepsilon$, it results $|f(x, y) - f(a, b)| < \varepsilon$;

(3) For any sequence $\{(x_n, y_n)\}$, $(x_n, y_n) \in A$, $\forall n \in \mathbb{N}^*$,

$$(x_n, y_n) \xrightarrow{\mathbb{R}^2} (a, b)$$

it results:

$$f(x_n, y_n) \xrightarrow{\mathbb{R}} f(a, b)$$

Example 5.5.1

(1) The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - 3\,x \cdot y + y + 1$ is continuous on $\mathbb{R}^2$, because $\forall (a, b) \in \mathbb{R}^2$ and $\forall (x_n, y_n) \xrightarrow{\mathbb{R}^2} (a, b)$, we have:

$$f(x_n, y_n) \xrightarrow{\mathbb{R}} f(a, b)$$

(2) The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{\sin(x^3+y^3)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is continuous on $\mathbb{R}^2$.

Indeed, if $(a, b) \neq (0, 0)$, then the statement follows from Remark 5.5.2, (3). The continuity at the origin follows since, as we proved in Example 5.4.2,

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(x^3 + y^3)}{x^2 + y^2} = 0 = f(0, 0).$$

(3) The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} \text{ dac } \breve{\text{a}} \ (x, y) \neq (0, 0) \\ 0 \quad \text{dac } \breve{\text{a}} \ (x, y) = (0, 0) \end{cases}$

is not continuous at the origin, since, as we proved in Example 5.4.3, this function does not have limit at origin.

Definition 5.5.2 Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar function,$a \in A$ where $a = (a_1, \ldots, a_n)$ and $A_i = \{t \in \mathbb{R};\ (a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n) \in A\}$, $i = \overline{1, n}$.

The function f is said to be **continuous at the point** $x = a$ **with respect to the variable** x_i**,** if the function of one variable $\varphi_i : A_i \to \mathbb{R}$ defined by:

$$\varphi_i(t) = f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$$

is continuous at the point $t = a_i$.

Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, be a function of two variables and $(a, b) \in A$.

Then f is continuous at the point (a, b) with respect to the variable x (respectively, y) if the function of one variable $\varphi(x) = f(x, b)$, $\forall x \in \mathbb{R}$, is continuous at the point $x = a$ (respectively, if the function $\psi(y) = f(a, y)$, $\forall y \in \mathbb{R}$, is continuous at the point $y = b$).

Remark 5.5.3 If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous at the point $a \in A$ with respect to all variables (Definitia 5.5.1), then f is continuous at the point $x = a$ with respect to each variable x_i, $i = \overline{1, n}$. The reverse statement, generally, is not true.

Example 5.5.2 Let:

$$f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & if \ (x, y) \neq (0, 0) \\ 0 & if \ (x, y) = (0, 0) \end{cases}.$$

We remark that this function is not continuous at the origin $(0, 0)$(with respect to the both variables), because $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist.

Indeed, the following sequences $\left\{\left(\frac{1}{n}, \frac{1}{n^2}\right)\right\}$ and $\left\{\left(\frac{2}{n}, \frac{1}{n^2}\right)\right\}$ converge in $\mathbb{R}^2$ to the origin $(0, 0)$, but $\lim\limits_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \frac{1}{2} \neq \lim\limits_{n \to \infty} f\left(\frac{2}{n}, \frac{1}{n^2}\right) = \frac{4}{17}$.

On the other hand, we have $\varphi(x) = f(x, 0) = 0$, $\forall x \in \mathbb{R}$, whence it results that the function $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$, hence f is continuous at the point $(0, 0)$ with respect to the variable x.

Similarly it turns out that f is continuous at the point $(0, 0)$ with respect to the variable y.

Theorem 5.5.3 *Assume that the function $F : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$ is continuous at the point $a \in A$ and the function $G : B \subset \mathbb{R}^m \to \mathbb{R}^p$ is continuous at the point $b = F(a) \in B$. Then the composite function $H = G \circ F : A \subset \mathbb{R}^n \to \mathbb{R}^p$ is continuous at the point a.*

Proof The proof results from Theorem 5.5.1, (3). Indeed, let $\{x_k\}$ be a sequence of elements of A, with $x_k \xrightarrow{\mathbb{R}^n} a$.Then, taking into account of the continuity of the functions F and G, it results that $F(x_k) \xrightarrow{\mathbb{R}^m} F(a) = b$ and $G(F(x_k)) \xrightarrow{\mathbb{R}^p} G(F(a))$. Therefore $H(x_k) \xrightarrow{\mathbb{R}^p} H(a)$, hence H is continuous at the a.

Theorem 5.5.4 *If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous at the point $a \in A$ and $f(a) > 0$ (respectively, $f(a) < 0$), then there exists a neighborhood $V \in \mathcal{V}(\mathrm{a})$ such that $f(x) > 0$ (respectively, $f(x) < 0$), for any $x \in V \cap A$.*

Proof Suppose $f(a) > 0$ and denote by $\varepsilon = \frac{1}{2} \cdot f(a) > 0$. Since f is continuous at a, it results that there is $\delta_\varepsilon > 0$ such that for any $x \in A$, with $\|x - a\| < \delta_\varepsilon$ we have $|f(x) - f(a)| < \varepsilon$. If we will denote by $V = B(a, \delta_\varepsilon)$, then $V \in \mathcal{V}(\mathrm{a})$ and for any $x \in V \cap A$ we have:

$\frac{1}{2} \cdot f(a) = f(a) - \varepsilon < f(x) < f(a) + \varepsilon = \frac{3}{2} \cdot f(a)$,

hence $f(x) > 0$.

Theorem 5.5.5 *Assume that the functions, $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$, are continuous at the point $a \in A$.Then, for any α, $\beta \in \mathbb{R}$ the functions $\alpha \cdot f + \beta \cdot g$ and $f \cdot g$ are continuous at a. If $g(a) \neq 0$, then the function $\frac{f}{g}$ is also continuous at a.*

Proof Let $\{x_k\}$ be a sequence of elements of A, $x_k \xrightarrow{\mathbb{R}^n} a$. By hypothesis it results that $f(x_k) \xrightarrow{\mathbb{R}} f(a)$ and $g(x_k) \xrightarrow{\mathbb{R}} g(a)$. Taking into account by the properties of the real number sequences, it results:

$$\begin{aligned}(\alpha \cdot f + \beta \cdot g)(x_k) &= \alpha \cdot f(x_k) + \beta \cdot g(x_k)\\ &\xrightarrow{\mathbb{R}} \alpha \cdot f(a) + \beta \cdot g(a)\\ &= (\alpha \cdot f + \beta \cdot g)(a).\end{aligned}$$

$$(f \cdot g)(x_k) = f(x_k) \cdot g(x_k) \xrightarrow{\mathbb{R}} f(a) \cdot g(a) = (f \cdot g)(a)$$

whence it follows that că $\alpha \cdot f + \beta \cdot g$ and $f \cdot g$ are continuous at a.

If $g(a) \neq 0$, then, according to Theorem 5.5.4, it results that there exists a neighborhood $V \in \mathcal{V}(a)$ such that $g(x) \neq 0, \forall x \in V \cap A$. Possibly removing a finite number of terms of the sequence $\{x_k\}$, and denoting its terms, we can assume that $x_k \in V \cap A, \forall k \in \mathbb{N}^*$, hence that $g(x_k) \neq 0, \forall k \in \mathbb{N}^*$. Then we have:

$\left(\frac{f}{g}\right)(x_k) = \frac{f(x_k)}{g(x_k)} \xrightarrow{\mathbb{R}} \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$,

whence it results that the function $\frac{f}{g}$ is continuous at the point a.

Theorem 5.5.6 *The following statements are equivalent:*

(i) $F : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous* on $\mathbb{R}^n$;
(ii) $F^{-1}(D)$ *is an open set for any open set* $D \subset \mathbb{R}^m$;
(iii) $F^{-1}(B)$ *is a closed set for any closed set* $B \subset \mathbb{R}^m$.

Proof (i)⇒(ii) Let $D \subset \mathbb{R}^m$ be an open set and let $a \in F^{-1}(D)$ be an arbitrary point. Then $b = F(a) \in D$ and because D is open there exists a neighborhood $U \in \mathcal{V}(b)$ such that $U \subset D$. As F is continuous at a, it results that there exists a neighborhood $V \in \mathcal{V}(a)$ with the property $F(V) \subset U \subset D$, so $V \subset F^{-1}(U) \subset F^{-1}(D)$. Therefore a is an interior point of $F^{-1}(D)$, hence $F^{-1}(D)$ is an open set.

(ii)⇒(i) Let $a \in \mathbb{R}^n$ be arbitrary and let $b = F(a)$. If $U \in \mathcal{V}(b)$, then there exists $r > 0$ such that $U \supset B(b, r)$. As $B(b, r)$ is open (Proposition 5.3.2), from (ii) it results that $V = F^{-1}(B(b, r))$ is also open. Obviously $V \in \mathcal{V}(a)$ and $F(V) \subset U$, hence F is continuous at the point a.

(ii)⇔(iii) This equivalence follows from Theorem 5.3.1 and from the immediate remark $\complement F^{-1}(B) = F^{-1}\big(\complement B\big)$, for any $B \subset \mathbb{R}^m$.

5.6 Properties of Continuous Functions Defined on Compact or Connected Sets

We recall that a subset of $\mathbb{R}^n$ is compact if it is closed and bounded.

Theorem 5.6.1 *Let* $F : \mathbb{R}^n \to \mathbb{R}^m$ *be a continuous vector function. If* $K \subset \mathbb{R}^n$ *is a compact set, then its direct image* $F(K)$ *is also a compact set.*

Proof We will prove that the set $F(K) = \{F(x); x \in K\}$ is closed and bounded. First we shall prove that the set $F(K)$ is bounded.

If us suppose that $F(K)$ is unbounded, then for any $M > 0$ there exists $x_M \in K$ such that $\|F(x_M)\| > M$. In particular, for $M = p$, $p \in \mathbb{N}^*$ there is $x_p \in K$ such that:

$$\left\|F\left(x_p\right)\right\| > p \tag{5.1}$$

Since the subset K is bounded, it follows that the sequence $\{x_p\}$ is bounded; hence it contains a convergent subsequence $\{x_{p_l}\}$ (Lemma 5.3.2). Let $a = \lim_{l\to\infty} x_{p_l}$. As K is closed, it results that $a \in K$, according to Theorem 5.3.2.

Finally, taking into account from the continuity of F we deduce, from Theorem 5.5.1, that $\lim_{l\to\infty} F(x_{p_l}) = F(a)$.On the other hand, from (5.1) we have $\|F(x_{p_l})\| > p_l$, whence it results that $\lim_{l\to\infty} \|F(x_{p_l})\| = \infty$. We have thus come to a contradiction, because $\|F(a)\|$ it is a finite number.

We will prove now that $F(K)$ is a closed set. Let $b \in \overline{F(K)}$ be an arbitrary point. From Theorem 5.3.2, it results that there exists a sequence $\{y_p\}$ of elements of $F(K)$ such that $y_p \xrightarrow{\mathbb{R}^m} b$.

Let $x_p \in K$ be with the property that $y_p = F(x_p)$. As we proved in the first part of the proof, there exists a convergent subsequence (x_{p_l}), $x_{p_l} \xrightarrow{\mathbb{R}^n} a$ and $a \in K$.

As F is a continuous function, it follows that $y_{p_l} = F(x_{p_l}) \xrightarrow{\mathbb{R}^m} F(a)$ On the other hand, $y_{p_l} \xrightarrow{\mathbb{R}^m} b$, whence we deduce that $b = F(a) \in F(K)$.

Therefore $\overline{F(K)} \subset F(K)$, hence $F(K)$ is a closed set.

Definition 5.6.1 Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded scalar function. We will say that f **attains its supremum and its infimum on** A, if there are $x_M \in A$ and $x_m \in A$ such that:

$f(x_M) = M = \sup f(A)$ *and* $f(x_m) = m = \inf f(A)$.

Theorem 5.6.2 *Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If K is a compact subset then f is bounded on K and attains its supremum and infimum on K.*

Proof The fact that f is bounded on K follows from Theorem 5.6.1.

If we denote by $M = \sup f(K)$ and by $m = \inf f(K)$, then $M, m \in \overline{f(K)}$. Indeed, if V is a neighborhood of M, then there exists $\varepsilon > 0$ such that:

$$V \supset (M - \varepsilon, M + \varepsilon).$$

From the definition of the supremum it results that there exists $x_\varepsilon \in K$ such that $M - \varepsilon < f(x_\varepsilon)$. (Remark 1.1.2). Therefore, $V \cap f(K) \neq \emptyset$. As V was an arbitrary neighborhood of M, it results that M is an adherent point for $f(K)$, i.e.$M \in \overline{f(K)}$. Analogously it is shown that $m \in \overline{f(K)}$. On the other hand, according to Theorem 5.6.1, $f(K)$ is closed, hence $M, m \in f(K)$. Therefore there are $x_M \in K$ and $x_m \in K$ such that $M = f(x_M)$ and $m = f(x_m)$.

Definition 5.6.2 A function $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called **uniformly continuous** on the set A if for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that $\forall x', x'' \in A$ with the property $\|x' - x''\| < \delta_\varepsilon$ it results that $\|F(x') - F(x'')\| < \varepsilon$.

We recall that the function F is continuous on the set A if it is continuous at each point x of A, hence for any $x \in A$ and any $\varepsilon > 0$ there exists $\delta_{x,\varepsilon} > 0$ such that for any $x' \in A$ with the property $\|x' - x\| < \delta_{x,\varepsilon}$ it results that:

$\|F(x') - F(x)\| < \varepsilon$.

Therefore, the difference between uniformly continuity and continuity of the function F on the set A is that in the case of uniform continuity, for any $\varepsilon > 0$, there is a number $\delta_\varepsilon > 0$, ***the same for all points*** $x \in A$, such that $\|F(x') - F(x'')\| < \varepsilon$ for any $x', x'' \in A$ satisfies the condition $\|x' - x''\| < \delta_\varepsilon$.

Obviously any uniformly continuous function on the set A is a continuous function on A. The reverse statement, generally, is not true.

Example 5.6.1 The function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$ is continuous $\mathbb{R}$, but is not uniformly continuous on $\mathbb{R}$.

For to prove that f is not uniformly continuous on $\mathbb{R}$, we will have to show that there exists $\varepsilon_0 > 0$ such that $\forall\, \delta > 0$, $\exists x'_\delta, x''_\delta \in \mathbb{R}$ with the properties:

$$\left|x'_\delta - x''_\delta\right| < \delta \text{ and } \left|f(x'_\delta) - f(x''_\delta)\right| \geq \varepsilon_0.$$

Let $\varepsilon_0 = \frac{1}{2}$ and $\delta > 0$ arbitrary. We observe that if $x' = \sqrt{n+1}$.
and $x'' = \sqrt{n}$, then $\left|f(x') - f(x'')\right| = 1, \quad \forall\, n \in \mathbb{N}^*$.
On the other hand, we have:

$$x' - x'' = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

whence it follows that there exists $n_\delta \in \mathbb{N}^*$ such that:

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \delta, \ \forall n \geq n_\delta$$

Now if we choose $x'_\delta = \sqrt{n_\delta + 1}$ and $x''_\delta = \sqrt{n_\delta}$, then we observe that:

$$\left|x'_\delta - x''_\delta\right| < \delta \text{ and } \left|f(x'_\delta) - f(x''_\delta)\right| = 1 > \varepsilon_0 = \frac{1}{2}.$$

Therefore, the function f is not uniformly continuous on $\mathbb{R}$.

Proposition 5.6.1 *If $f : I \subset \mathbb{R} \to \mathbb{R}$ is differentiable on the interval I, and has bounded derivative on this interval I, then f is uniformly continuous on I.*

Proof Let $M > 0$ be such that $\left|f'(x)\right| \leq M$, $\forall\, x \in I$. From Lagrange's theorem it results that for any two points $x, y \in I$, there exists a point ξ between x and y such that:

$$f(x) - f(y) = f'(\xi)(x - y).$$

Further we have:

$$|f(x) - f(y)| = \left|f'(\xi)\right||x - y| \leq M|x - y|, \quad \forall x, y \in I$$

Let us denote by $\delta_\varepsilon = \frac{\varepsilon}{M} > 0$ for any $\varepsilon > 0$.We observe now that if $|x - y| < \delta_\varepsilon$ then:

$$|f(x) - f(y)| \le M \cdot \frac{\varepsilon}{M} = \varepsilon, \forall \quad x, y \in I$$

so f is uniformly continuous on I.

Example 5.6.2 Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 5 \cdot x - \cos^2 x$ is uniformly continuous on $\mathbb{R}$.

Since $\left|f'(x)\right| = |5 + \sin 2x| \le 6$, $\forall x \in \mathbb{R}$, from Proposition 5.6.1, we deduce that f is uniformly continuous on $\mathbb{R}$.

The following theorem shows us, that on compact sets, the notions of continuity and uniformly continuity are equivalent.

Theorem 5.6.3 (Cantor's theorem). *Let $F : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function on K. If $K \subset \mathbb{R}^n$ is a compact set, then F is uniformly continuous on K.*

Proof If us suppose that F does not be uniformly continuous on K, then there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, $\exists\, x'_\delta$, $x''_\delta \in K$ with the properties:

$$\left\|x'_\delta - x''_\delta\right\| < \delta \text{ and } \left\|F\left(x'_\delta\right) - F\left(x''_\delta\right)\right\| \ge \varepsilon_0.$$

In particular, for $\delta = \frac{1}{p}$, $p \in \mathbb{N}^*$, it results that there are two points x'_p, $x''_p \in K$ with the properties:

$$\left\|x'_p - x''_p\right\| < \frac{1}{p} \tag{5.2}$$

and

$$\left\|F\left(x'_p\right) - F\left(x''_p\right)\right\| \ge \varepsilon_0 \tag{5.3}$$

Since K is bounded, it results that the sequence $\left\{x'_p\right\}$ of elements of K is also bounded so, according to Lemma 5.3.2, there exists a convergent subsequence $x'_{p_l} \xrightarrow{\mathbb{R}^n} a$. Taking into account that K is closed, it follows that $a \in K$. (Theorem 5.3.2). On the other hand, from (5.2) we deduce that $x''_{p_l} \xrightarrow{\mathbb{R}^n} a$. As F is continuous, from Theorem 5.5.1 we have:

$$F\left(x'_{p_l}\right) \xrightarrow{\mathbb{R}^m} F(a) \text{ and } F\left(x''_{p_l}\right) \xrightarrow{\mathbb{R}^m} F(a).$$

Therefore:

$$F\left(x'_{p_l}\right) - F\left(x''_{p_l}\right) \xrightarrow{\mathbb{R}^m} F(a) - F(a) = 0.$$

We have thus come to a contradiction, because according to (5.3), $\left\| F\left(x'_{p_l}\right) - F\left(x''_{p_l}\right) \right\| \geq \varepsilon_0\,,\ \forall\, l \in \mathbb{N}^*$.

Definition 5.6.3 A subset $A \subset \mathbb{R}^n$ is called **connected** if does not exist two non-empty open sets D_1 and D_2 with properties:

$$A \subset D_1 \cup D_2\,,\ D_1 \cap D_2 \cap A = \emptyset\,,\ D_1 \cap A \neq \emptyset\,,\ D_2 \cap A \neq \emptyset$$

Example 5.6.3 The set $A = (0, 1) \cup (2, 3)$ is disconnected, while the set $A = (-1, 1)$ is connected.

The following characterization theorem for connected subsets of $\mathbb{R}$ can be proved.

Theorem 5.6.4 *A subset non-void $A \subset \mathbb{R}$ is connected iff it is an interval, namely if it has the property: $\forall\, x\,,\ y \in A$ and any $x < z < y$, it results that $z \in A$.*

Theorem 5.6.5 *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. If the subset $A \subset \mathbb{R}^n$ is connected, then $F(A) \subset \mathbb{R}^m$ is connected.*

Proof If we suppose that $F(A) \subset \mathbb{R}^m$ is disconnected, then there exist two non-void open sets $D_i \subset \mathbb{R}^m$, $i = 1, 2$ with properties:

$$F(A) \subset D_1 \cup D_2,\ D_1 \cap D_2 \cap F(A) = \emptyset,$$
$$D_1 \cap F(A) \neq \emptyset,\ D_2 \cap F(A) \neq \emptyset$$

From Theorem 5.5.6 it results that the sets $F^{-1}(D_i)$, $i = 1, 2$ are open sets. It easy to see that $F^{-1}(D_i) \neq \phi$, $i = 1, 2$ and

$$A \subset F^{-1}(D_1) \cup F^{-1}(D_2),\ F^{-1}(D_1) \cap F^{-1}(D_2) \cap A = \emptyset,$$
$$F^{-1}(D_i) \cap A \neq \emptyset, \quad i = 1, 2$$

As $F^{-1}(D_i)$, $i = 1, 2$ are open sets, it results that A is disconnected which is a contradiction.

Definition 5.6.4 Let $I \subset \mathbb{R}$ be an interval. We say that the function $f : I \to \mathbb{R}$ **has the Darboux's property on** I if for any interval $J \subset I$ its image $f(J)$ is also an interval.

Corollary 5.6.1 *If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a continuous function on the interval I, then f has the Darboux's property on I.*

Proof If $J \subset I$ is an interval, then from Theorem 5.6.4 it follows that J is a connected subset of $\mathbb{R}$. Taking into account to Theorem 5.6.5, it results that $f(J)$ is also a connected subsets of $\mathbb{R}$. Applying again Theorem 5.6.4, we deduce that $f(J)$ is an interval, so f has Darboux's property on I.

Corollary 5.6.2 (Bolzano's theorem). *If $f : [a\,,\ b] \to \mathbb{R}$ is continuous and $f(a) \cdot f(b) < 0$, then there exists a point $a < c < b$ such that $f(c) = 0$.*

Proof From Corollary 5.6.1 it results that the set $J = f([a\ ,\ b]) \subset \mathbb{R}$ is an interval. Since $f(a) \cdot f(b) < 0$ we can suppose, for example, that $f(a) < 0 < f(b)$. As J is an interval and $0 \in J$, it follows that there exists $c \in (a\ ,\ b)$ that that $f(c) = 0$.

We remark that Corollary 5.6.2 is useful in solving equations. Suppose that the equation $f(x) = 0$ has an unique root α in the interval $(a\ ,\ b)$. For to make a choice we shall assume that $f(a) > 0$ and $f(b) < 0$ (Fig. 5.1). The simplest method of approximating this root is the **bisection method** which consists of the following: We divide the segment $[a\ ,\ b]$ into two equal parts by the point $c = \frac{a+b}{2}$. If $f(c) = 0$, then $\alpha = c$ is the sought-after root.

If $f(c) < 0$ than the root $\alpha \in [a, c]$ and we shall denote by $a_1 = a$ and $b_1 = c$. Let $c_1 = \frac{a_1+b_1}{2}$ be the middle point of the interval $[a_1, b_1]$. If $f(c_1) = 0$, then $\alpha = c_1$ and the algorithm terminates. Let us suppose that $f(c_1) < 0$. Then the required root $\alpha \in [a_1, c_1]$ and we shall denote by $a_2 = a_1$ and $b_2 = c_1$.

Now we shall deal with the interval $[a_2, b_2]$ in the same way as with the interval $[a_1, b_1]$. Let c_2 be the middle point of the interval $[a_2, b_2]$. If $f(c_2) = 0$, then $\alpha = c_2$ and the algorithm terminates.

If $f(c_2) > 0$, then the required root $\alpha \in [c_2, b_2]$, and we shall denote by $a_3 = c_2$ and $b_3 = b_2$.Proceeding in the same way, we shall have two possibilities:

1 Either the algorithm described above will terminate because in the middle point c_n of the interval $[a_n, b_n]$ we have $f(c_n) = 0$ (in this case $\alpha = c_n$).
2 Or the process described can be continued indefinitely and we shall receive a nested system of closed intervals $[a_1, b_1] \supset [a_2, b_2] \supset .... \supset [a_n, b_n] \supset ....$ with $f(a_n) \cdot f(b_n) < 0$.This nested system has a common point α to which each of the sequences $\{a_n\}$ and $\{b_n\}$ converges.

Since f is continuous, it follows that $f(a_n) \to f(\alpha)$ and $f(b_n) \to f(\alpha)$. The length of the interval $[a_n, b_n]$ being equal to $\frac{b-a}{2^n}$, the number $c_n = \frac{a_n+b_n}{2}$ differs from the root α by not more than $\frac{b-a}{2^{n+1}}$. Therefore it possible to compute the root α with any pre-assigned accuracy.

Example 5.6.4 Find the root of the equation $x^3 - x - 2 = 0$, lies in the interval $[1, 2]$.

We have $f(x) = x^3 - x - 2$, $f(1) = -2$, $f(2) = 4$, so the equation $f(x) = 0$ has a root α in the closed interval closed $[1, 2]$. We observe that α is the unique root of the equation in the interval $[1, 2]$, because the function f is strictly increasing on this interval. Indeed, $f'(x) = 3x^2 - 1 > 0, \forall x \in [1, 2]$.

According to the algorithm of bisection method we have:

$$a = 1 \quad b = 2 \quad c = 1.5 \quad f(c) = -0.125$$

$$a_1 = 1.5 \quad b_1 = 2 \quad c_1 = 1.75 \quad f(c_1) = 1.6093750$$

$$a_2 = 1.5 \quad b_2 = 1.75 \quad c_2 = 1.625 \quad f(c_2) = 0.6660156$$

$$a_3 = 1.5 \quad b_3 = 1.625 \quad c_3 = 1.5625 \quad f(c_3) = 0.2521973$$

$$a_4 = 1.5 \quad b_4 = 1.5625 \quad c_4 = 1.53125 \quad f(c_4) = 0.0591125$$

$$a_5 = 1.5 \quad b_5 = 1.53125 \quad c_5 = 1.515625 \quad f(c_5) = -0.0340538$$

$$a_6 = 1.515625 \quad b_6 = 1.53125 \quad c_6 = 1.5234375 \quad f(c_6) = 0.0122504$$

$$\ldots$$

$$a_{15} = 1.5213623 \quad b_{15} = 1.5214233 \quad c_{15} = 1.5213928 \quad f(c_{15}) = 0.0000780$$

5.7 Linear Continuous Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition 5.7.1 A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear map (transformation)** if it satisfies the following condition:

$$T(\lambda \cdot x + \mu \cdot y) = \lambda \cdot T(x) + \mu \cdot T(y), \; \forall x, y \in \mathbb{R}^n, \; \forall \lambda, \mu \in \mathbb{R}.$$

Further we will denote by $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proposition 5.7.1 *The set $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ has the following properties:*

(i) $T(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}, \; \forall T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.
(ii) *If $T, U \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then:*

$$\alpha \cdot T + \beta \cdot U \in \mathcal{L}\big(\mathbb{R}^n, \mathbb{R}^m\big), \; \forall \alpha, \beta \in \mathbb{R}$$

(iii) *If* $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *and* $U \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$*, then:*

$$U \circ T \in \mathcal{L}\big(\mathbb{R}^n, \mathbb{R}^p\big)$$

The verification is immediate.

The following theorem for characterizing linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is also well known.

Theorem 5.7.1 *The map* $T : \mathbb{R}^n \to \mathbb{R}^m$ *is linear iff there exists a matrix* A *with* m *rows and* n *columns* $\big(A \in \mathcal{M}_{m,n}(\mathbb{R})\big)$ *such that:*

$$T(x) = x \quad A^t, \forall x \in \mathbb{R}^n$$

where A^t *denoted the transpose of the matrix* A.

(A *is called* ***the matrix associated with the linear map*** T***, with respect to the canonical bases from*** $\mathbb{R}^n$ *and* $\mathbb{R}^m$).

If:

$$A = \big(\lambda_{ij}\big)_{\substack{1 \le i \le m \\ 1 \le j \le n}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} \dots & \lambda_{2n} \\ \dots & \dots\dots & \dots \\ \lambda_{m1} & \lambda_{m2} \dots & \lambda_{mn} \end{pmatrix} \in \mathcal{M}_{m,n}(\mathbb{R})$$

and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then:

$$T(x) = (\lambda_{11}x_1 + \lambda_{12}x_2 + \dots + \lambda_{1n}x_n, \dots, \lambda_{m1}x_1 + \lambda_{m2}x_2 + \dots + \lambda_{mn}x_n).$$

Corollary 5.7.1 *A map* $T : \mathbb{R} \to \mathbb{R}$ *is linear iff there exists* $\lambda \in \mathbb{R}$ *such that:*
$T(x) = \lambda \cdot x, \forall x \in \mathbb{R}$.

Corollary 5.7.2 *A map* $T : \mathbb{R}^n \to \mathbb{R}$ *is linear iff there are* $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ such that:

$$T(x) = \lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \dots + \lambda_n \cdot x_n, \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let A be its matrix with respect to the canonical bases from $\mathbb{R}^n$ and $\mathbb{R}^m$.

According to (5.4), for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we have:

$$T(x) = (\lambda_{11} \cdot x_1 + \dots + \lambda_{1n} \cdot x_n, \dots, \lambda_{m1} \cdot x_1 + \dots + \lambda_{mn} \cdot x_n)$$

whence it results that:

$$\|T(x)\|_2 = \sqrt{(\lambda_{11} \cdot x_1 + \dots + \lambda_{1n} \cdot x_n)^2 + \dots + (\lambda_{m1} \cdot x_1 + \dots + \lambda_{mn} \cdot x_n)^2} \tag{5.4}$$

Further, taking into account to inequality Cauchy–Buniakovski, we have:

$$\|T(x)\|_2 \le \sqrt{\sum_{j=1}^{n} \lambda_{1j}^2 \cdot \sum_{j=1}^{n} x_j^2 + \cdots + \sum_{j=1}^{n} \lambda_{m\ j}^2 \cdot \sum_{j=1}^{n} x_j^2}$$
$$= \|x\|_2 \cdot \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{i\ j}^2}. \tag{5.5}$$

Also we have:

$$\|T(x)\|_\infty = \max(|\lambda_{11} \cdot x_1 + \cdots + \lambda_{1\ n} \cdot x_n|, \cdots, |\lambda_{m\ 1} \cdot x_1 + \cdots + \lambda_{m\ n} \cdot x_n|)$$
$$\le \max(|\lambda_{11}| \cdot |x_1| + \cdots + |\lambda_{1\ n}| \cdot |x_n|, \ldots, |\lambda_{m\ 1}| \cdot |x_1| + \cdots + |\lambda_{m\ n}| \cdot |x_n|)$$
$$\le \|x\|_\infty \cdot \max(|\lambda_{11}| + \cdots + |\lambda_{1n}|, \ldots, |\lambda_{m1}| + \cdots + |\lambda_{m\ n}|). \tag{5.6}$$

Several norms can be entered on the set $\mathcal{M}_{m\,,\,n}(\mathbb{R})$ of all matrices with m rows and n columns. Further, we consider the following two norms for any

$$A = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1\ n} \\ \cdots & \cdots & \cdots \\ \lambda_{m\ 1} & \cdots & \lambda_{m\ n} \end{pmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}) :$$

$$\|A\|_2 = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{i\ j}^2} \tag{5.7}$$

$$\|A\|_\infty = \max(|\lambda_{11}| + \cdots + |\lambda_{1\ n}|, \ldots, |\lambda_{m\ 1}| + \cdots + |\lambda_{mn}|). \tag{5.8}$$

From the relations (5.5)–(5.8), we deduce the following result:

Theorem 5.7.2 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let A be its matrix with respect to the canonical bases from $\mathbb{R}^n$ and $\mathbb{R}^m$. Then we have:*

$\|T(x)\|_2 \le \|A\|_2 \cdot \|x\|_2$ *and* $\|T(x)\|_\infty \le \|A\|_\infty \cdot \|x\|_\infty$, $\forall\, x \in \mathbb{R}^n$.

(Therefore, $\|T(x)\| \le \|A\| \cdot \|x\|$, $\forall\, x \in \mathbb{R}^n$, for any of the norms $\|\cdot\|_2$ or $\|\cdot\|_\infty$).

Corollary 5.7.3 *Any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$.*

Proof Let $a \in \mathbb{R}^n$ be arbitrary and let $\{x_k\}$ be a sequence of vectors of $\mathbb{R}^n$ such that $x_k \xrightarrow{\mathbb{R}^n} a$.

From Theorem 5.7.2 and from the linearity of T it results:

$$\|T(x_k) - T(a)\| = \|T(x_k - a)\| \le \|A\| \cdot \|x_k - a\|.$$

As $\lim\limits_{k\to\infty} \|x_k - a\| = 0$, it follows that $T(x_k) \xrightarrow{\mathbb{R}^m} T(a)$, hence T is continuous at the point $x = a$.

Chapter 6
Differential Calculus of Functions of Several Variables

6.1 Partial Derivatives. Differentiability of a Function of Several Variables

Definition 6.1.1 Let$A \subset \mathbb{R}^2$ be an open subset and let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. **The partial derivative of f, with respect to x, at the point** $(a, b) \in A$ is defined as the limit:

$$\lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},$$

provided that it exists and is finite.

We shall denote the partial derivative of the function f at the point (a, b) by $\frac{\partial f}{\partial x}(a, b)$ or by $f'_x(a, b)$.

If there exists $\frac{\partial f}{\partial x}(a, b)$ at any point $(a, b) \in A$, then the function of two variables $(a, b) \to \frac{\partial f}{\partial x}(a, b) : A \to \mathbb{R}$ is denoted by $\frac{\partial f}{\partial x}$ and named **the partial derivative of the function f with respect to x on A.**

Similarly defined **the partial derivative of the function f with respect to y at the point** (a, b), namely:

$$\frac{\partial f}{\partial y}(a, b) = f'_y(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b}$$
$$= \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}.$$

Remark 6.1.1 The partial derivative $\frac{\partial f}{\partial x}(a, b)$ is nothing but the ordinary derivative of function of on variable $t \to f(t, b)$ at the point $t = a$.Therefore the partial derivative $\frac{\partial f}{\partial x}$ is an ordinary derivative with respect to x by regarding y as a constant and similarly, the partial derivative $\frac{\partial f}{\partial y}$ is an ordinary derivative with respect to y computed by regarding x as a constant.Therefore, partial derivatives can be

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compute in concordance with the rules governing the camputation of the derivatives of functions of one variable.

Example 6.1.1 Let $A = \left\{(x\,,\ y) \in \mathbb{R}^2\,;\ x > 0\,,\ y > 0\right\}$ and $f : A \to \mathbb{R}$, defined by:

$$f(x,\ y) = x^2 \cdot y^3 \ - \ 2 \cdot y \cdot \ \ln x \ + \ y^x.$$

Compute the partial derivatives $\frac{\partial\ f}{\partial\ x}$ and $\frac{\partial\ f}{\partial\ y}$ on A.
Taking into account of Remark 6.1.1 we have:

$$\frac{\partial\ f}{\partial\ x}(x,\ y) = 2 \cdot\ x\ \cdot y^3\ -\ 2\ \cdot\ \frac{y}{x}\ +\ y^x \ln y.$$

And

$$\frac{\partial\ f}{\partial\ y}(x,\ y)\ =\ 3\ \cdot\ x^2 \cdot y^2\ -\ 2 \cdot\ \ln x\ +\ x\ \cdot\ y^{x\,-\,1}.$$

For an arbitrary function of several variable we have the following definition:

Definition 6.1.2 Let $A \subset \mathbb{R}^n$ be an open subset, $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a real function of n variables and $a = (a_1,\ \ldots\ ,\ a_n) \in A$.

The partial derivative of function f with respect to the variable x_i at the point a is defined by:

$$\begin{aligned}
&\frac{\partial f}{\partial x_i}(a) = \frac{\partial f}{\partial x_i}(a_1, \ldots, a_i, \ldots, a_n)\\
&= \lim_{x_i \to a_i} \frac{f(a_1,\ \ldots,\ a_{i\,-1},\ x_i\ ,\ a_{i\,+1},\ \ldots,\ a_n)\ -\ f(a_1,\ \ldots, a_i, \ldots\ ,\ a_n)}{x_i\ -\ a_i}\\
&= \lim_{h \to 0} \frac{f(a_1,\ \ldots,\ a_{i\,-1},\ a_i +\ h,\ a_{i\,+1},\ \ldots,\ a_n) - f(a_1,\ \ldots\ , a_i, \ldots,\ a_n)}{h}\\
&= f'_{x_i}(a).
\end{aligned}$$

provided this limit exists and is finite.

Remark 6.1.2 The partial derivative $\frac{\partial\ f}{\partial\ x_i}$ is the ordinary derivative of function f with respect to x_i computed by regarding the other variables $x_j,\ \overline{j = 1, n},\ j \neq i$ as constants.

Proposition 6.1.1 *Let $A \subset \mathbb{R}^n$ be an open subset, $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a real function of n variables and $a = (a_1,\ \ldots\ ,\ a_n) \in A$. If $\frac{\partial f}{\partial x}(a)$ exists, then f is continuous with respect to x_i at the point a.*

Proof Since the function of one variable:

$$x_i\ \to f(a_1,\ \ldots,\ a_{i\,-1},\ x_i\ ,\ a_{i\,+1},\ \ldots,\ a_n)$$

is derivable at the point $a_i \in \mathbb{R}$, it results that this function is continuous at the point a_i, hence the function of n variables f is continuous with respect to x_i at the point a.

Remark 6.1.3 The existence of all partial derivatives of a function at a given point does not imply, in general, the continuity of the function at that point. Indeed, we proved that the function:

$$f(x, y) = \begin{cases} \frac{x^2 \cdot y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

does not continuous at the origin $(0, 0)$,because $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exists(Example 5.5.2). On the other hand it is clearly that:

$$\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0.$$

Further, we will present the notion of derivative of a function of one variable in an equivalent form, that will allow the generalization of this notion for vector functions.

Let $I \subset \mathbb{R}$ be on open iterval, $a \in I$ and $f : I \to \mathbb{R}$.We recall that the function f is derivable at the point a if the following limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a),$$

exists and is finite.

Proposition 6.1.2 *Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $a \in I$ an interior point. The following statements are equivalent:*

(i) *f is derivable (differentiable) at the point a.*
(ii) *There exists a linear map $T = T_a : \mathbb{R} \to \mathbb{R}$ such that:*

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - T(h)}{h} = 0.$$

Proof (i) $\Rightarrow$(ii) Since f is derivable at the point a, it results that the limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

exists and it is finite.

If we denote by $T(h) = f'(a) \cdot h$, for any $h \in I$, then $T : \mathbb{R} \to \mathbb{R}$ is a linear map (obviously continuous) on $\mathbb{R}$ and we have:

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - T(h)}{h} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} - f'(a) = 0.$$

(ii) $\Rightarrow$(i) According to the hypothesis there exists a linear map $T : \mathbb{R} \to \mathbb{R}$ such that:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0 \tag{6.1}$$

On the other hand, from Corollary 5.7.1 it results that there exists $\lambda \in \mathbb{R}$ such that $T(h) = \lambda \cdot h, \ \forall h \in \mathbb{R}$.

Taking into account of (6.1) we deduce that:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lambda \in \mathbb{R},$$

hence f is derivable at a and $f'(a) = \lambda$.

Remark 6.1.4 From the proof of Proposition 6.1.2, it follows that if $f : I \to \mathbb{R}$ is derivable at the point a, then there exists an unique linear map $T : \mathbb{R} \to \mathbb{R}$ for which it takes place (6.1), namely $T(h) = f'(a) \cdot h, \ h \in \mathbb{R}$.

This linear map is called the differential of f at the point a and is denoted by $df(a)$.

Definition 6.1.3 Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $a \in I$ an interior point. It's called the **differential of f at the point** a, the following linear map on $\mathbb{R}$:

$$df(a) : \mathbb{R} \to \mathbb{R}, \ df(a)(h) = f'(a) \cdot h, \ \forall h \in \mathbb{R}.$$

Further we present the geometric interpretation of the differential. Let (C) be the graph of the function f and $M_0(a, f(a)) \in C$(Fig. 6.1). It is known that $f'(a) = \operatorname{tg}\alpha$(the slope of the tangent line to the graph at the point M_0). From the triangle M_0PT we have:

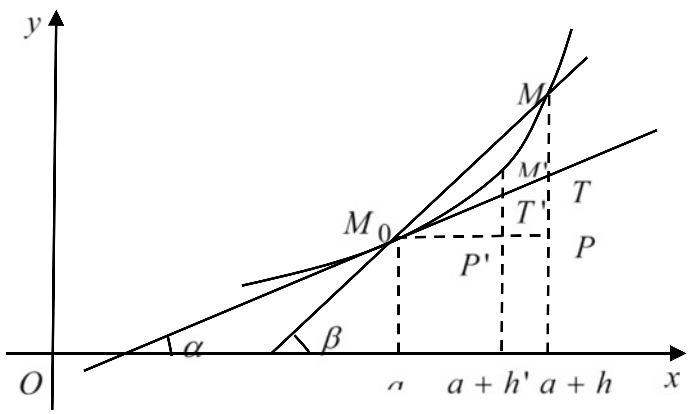

Fig. 6.1 The geometrical interpretation of differential

$$f'(a) = \operatorname{tg}\alpha = \frac{PT}{M_0P} = \frac{PT}{h},$$

whence it results that $PT = f'(a) \cdot h = df(a)(h)$.

Therefore, $df(a)(h)$ is equal with the segment length PT, where T is the point belongs to the the tangent line to the graph at the point M_0, corresponding to the abscissa $a + h$.

On the other hand, the increment of the function f i.e.,$f(a + h) - f(a)$ is equal to the the segment lenght PM, where $M(a+h, f(a+h))) \in C$.

Clearly, $PM = f(a+h) - f(a) \neq PT = df(a)(h)$, but we observe that for a small increment h' of the argument x, the increment:

$$f\left(a + h'\right) - f(a) = P'M' \approx P'T' = df(a)(h').$$

Therefore, for small values ($|h| << 1$) we have:

$$f(a + h) - f(a) \approx df(a)(h) = f'(a) \cdot h.$$

Exemple 6.1.2 Compute $df(2)(0.001)$ for the function:

$$f(x) = x^2, \forall x \in \mathbb{R}.$$

We have:

$$df(2)(h) = f'(2) \cdot h = 4 \cdot h, \ \forall\, h \in \mathbb{R},$$

hence:

$$df(2)(0.001) = 0.004.$$

On the other hand, the increment of f is:

$$f(2.001) - f(2) = 0.004001,$$

so:

$$f(2.001) - f(2) \cong df(2)(0.001).$$

Further,for any function $f : I \to \mathbb{R}$, derivable at the interior point $a \in I$,we shall denote by:

$$\omega(h) = \omega_a(h) = \begin{cases} \dfrac{f(a+h) - f(a)}{h} - f'(a), \text{ if } h \neq 0 \text{ and } a+h \in I, \\ 0\,, \text{ if } h = 0. \end{cases}$$

Obvious, $\lim_{h \to 0} \omega(h) = \omega(0) = 0$, whence it results that ω is continuous at 0.
On the other hand, for any $h \in \mathbb{R}$ such that $a + h \in I$, we have:

$$f(a + h) - f(a) = f'(a) \cdot h + \omega(h) \cdot h,$$

or,

$$f(a + h) - f(a) = d f(a)(h) + \varphi(h)$$

where, we denoted by $\varphi(h) = \omega(h) \cdot h$, $\forall h \in \mathbb{R}$, with $a + h \in I$. We observe that:

$$\lim_{h \to 0} \varphi(h) = \lim_{h \to 0} \omega(h) \cdot h = 0 \text{ and } \lim_{h \to 0} \frac{\varphi(h)}{h} = \lim_{h \to 0} \omega(h) = 0,$$

so, the function φ is "two times small" at 0. Such a function is also called ***infinitely small function*** and is denoted by $o(h)$. Geometrically, $\varphi(h)$ is equal with segment length TM. Now we better understand why for small values of the increment h of x we can approximate the increment $f(a + h) - f(a)$ with $d f(a)(h)$, because for such values, the term $\varphi(h)$ is very small in relation to the term $d f(a)(h)$.

Then the following statements are equivalent:

Proposition 6.1.3 *Let $f : I \to \mathbb{R}$ be a real function, $a \in I$ an interior point and $J = \{h \in \mathbb{R};\ a + h \in I\}$.*

(1) *f is derivable (differentiable) at the point a.*
(2) *There exist a linear map $T = T_a : \mathbb{R} \to \mathbb{R}$ and a function*

$\varphi = \varphi_a : J \to \mathbb{R}$ continuous at 0, with the properties $\lim_{h \to 0} \varphi(h) = 0$ and $\lim_{h \to 0} \frac{\varphi(h)}{h} = 0$ such that:

$$f(a + h) - f(a) = T(h) + \varphi(h),\ \forall h \in J \tag{6.2}$$

We mention that:
$T(h) = d f(a)(h) = f'(a) \cdot h$ *and* $\varphi(h) = \omega(h) \cdot h$, i.e.,$\varphi = o(h)$.

Definition 6.1.4 Let $A \subset \mathbb{R}^n$ an open subset and $f : A \subset \mathbb{R}^n \to \mathbb{R}$.The function f is said to be differentiable at the point $a \in A$ if there exists a linear map $T = T_a : \mathbb{R}^n \to \mathbb{R}$, such that:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0.$$

If f is differentiable at any point $a \in A$, then we say that f is differentiable on A.

Theorem 6.1.1 *If the function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at the interior point $a \in A$, then there exist the partial derivatives $\frac{\partial f}{\partial x_i}(a)$, $\forall\ i = \overline{1,\ n}$. Moreover, the linear map $T : \mathbb{R}^n \to \mathbb{R}$ is unique and is defined as follows:*

$$T(h) = \frac{\partial f}{\partial x_1}(a) \cdot h_1 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot h_n,\ \forall\ h = (h_1, \ldots, h_n) \in \mathbb{R}^n.$$

Further we will call this map ***the differential of first order of function*** f ***at the point*** a and denoted it with $d f(a)$.

Therefore $d f(a) : \mathbb{R}^n \to \mathbb{R}$ is the following linear map:

$$df(a)(h) = \frac{\partial f}{\partial x_1}(a) \cdot h_1 + \frac{\partial f}{\partial x_2}(a) \cdot h_2 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot h_n, \forall h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$$

Proof By hypothesis, there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\lim_{h \to 0} \frac{|f(a + h) - f(a) - T(h)|}{\| h \|} = 0.$$

On the other hand, according to Corollary 5.7.2, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that:

$$T(h) = \lambda_1 \cdot h_1 + \cdots + \lambda_n \cdot h_n,\ \forall\ h = (h_1, \ldots, h_n) \in \mathbb{R}^n.$$

For $h = (0, \ldots, h_i, \ldots, 0) \in \mathbb{R}^n$, it results that:

$$\lim_{h_i \to 0} \frac{|f(a_1, \ldots, a_i + h_i, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n) - \lambda_i h_i|}{|h_i|} = 0$$

and further that:

$$\lim_{h_i \to 0} \left| \frac{f(a_1, \ldots, a_i + h_i, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h_i} - \lambda_i \right| = 0,$$

which is equivalent with the fact that there exists $\frac{\partial f}{\partial x_i}(a) = \lambda_i$.

Therefore, the linear map T is uniquely determined, namely:

$$T(h) = T(h_1, \ldots, h_n) = \frac{\partial f}{\partial x_1}(a) \cdot h_1 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot h_n, .$$
$$\forall h = (h_1, \ldots, h_n) \in \mathbb{R}^n$$

Theorem 6.1.2 *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and let $a \in A$ be an interior point.The following properties are equivalent:*

(i) *f is differentiabe at the point a.*
(ii) *There exist a linear map $T : \mathbb{R}^n \to \mathbb{R}$, a neighborhood V of the*

origine and an infinitely small function $\varphi = \varphi_a : V \to \mathbb{R}$ (i.e.,$\varphi = o(h)$) *such that:*

$$f(a + h) - f(a) = T(h) + \varphi(h), \ \forall\, h \in V. \tag{6.3}$$

(*we recall that* $\varphi = o(h)$ *if* $\lim\limits_{h \to 0} \varphi(h) = \varphi(0) = 0$ *and* $\lim\limits_{h \to 0} \frac{|\varphi(h)|}{\| h \|} = 0$).

Proof (i) $\Rightarrow$ (ii) We denote by $V = \{h \in \mathbb{R}^n ;\ a + h \in A\}$ and we consider the function $\omega : V \to \mathbb{R}$ given by:

$$\omega(h) = \begin{cases} \dfrac{f(a+h) - f(a) - T(h)}{\|h\|}, & \text{if } h \neq 0,\ h \in V \\ 0\,, \text{if } h = 0 & \end{cases}.$$

Obvious we have:

$$f(a + h) - f(a) = T(h) + \omega(h) \cdot \| h \|, \ \forall\, h \in V.$$

If we denote by $\varphi(h) = \omega(h) \cdot \| h \|,\ h \in V$, then:

$$\lim_{h \to 0} \varphi(h) = \varphi(0) = 0, \ \lim_{h \to 0} \frac{|\varphi(h)|}{\| h \|} = 0$$

and:

$$f(a + h) - f(a) = T(h) + \varphi(h).$$

The implication (ii)$\Rightarrow$(i) is obvious.
From Theorems 6.1.1 and 6.1.2 it results:

Remark 6.1.5 The function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at the interior point $a \in A$ if there are a neighborhood $V \in \mathcal{V}(0)$ and a function $\varphi = 0(h),\varphi : V \to \mathbb{R}$. such that:

$$f(a + h) - f(a) = d f(a)(h) + \varphi(h), \ \forall\, h \in V.$$

Remark 6.1.6 Let $A \subset \mathbb{R}^2$ be an open subset and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$.The function f is differentiable at the point $(a, b) \in A$, if there exist two real numbers $\lambda, \mu \in \mathbb{R}$ and a function $\omega : A \to \mathbb{R}$, continuous at the point (a, b), with $\omega(a, b) = 0$, such that:

$$\begin{aligned} f(x, y) - f(a, b) &= \lambda \cdot (x - a) + \mu \cdot (y - b) \\ &+ \omega(x, y) \cdot \sqrt{(x - a)^2 + (y - b)^2}, \\ &\forall (x, y) \in A \end{aligned}$$

Proposition 6.1.4 *If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at the interior point $a \in A$, then f is continuous at a.*

Proof If f is differentiable at the point $a \in A$, then according to Theorem 6.1.2, the properties from (ii) are satisfied.

If we cross to the limit in (6.3) we obtain:

$$\lim_{h \to 0} f(a + h) - f(a) = \lim_{h \to 0} T(h) + \lim_{h \to 0} \varphi(h)$$
$$= T(0_{\mathbb{R}^n}) + 0 = 0 + 0 = 0$$

Therefore there exists $\lim_{h \to 0} f(a + h) = f(a)$, hence f is continuous at the point a.

We are now able to show that the reciprocal statement of Theorem 6.1.1 is not generally true. The existence of the partial derivatives and the continuity of a function at a point is necessary but not sufficient condition for the differentiability of the function at that point, as will be shown in the following example:

Exemple 6.1.3 Let $f(x, y) = \begin{cases} \frac{x^2 \cdot y}{x^4 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$.

Since $f(x, 0) = f(0, y) = 0$, it results that $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$.

On the other hand, f does not differentiable at the origin $(0, 0)$, because f does not continuous at this point (Exemple 5.5.2 and Proposition 6.1.4).

Further we presents sufficient conditions that a function to be differentiable. First we give the following definition:

Definition 6.1.5 Let $A \subset \mathbb{R}^n$ be an open subset.

A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be of C^1 class on A, if there exist the continuous partial derivatives $\frac{\partial f}{\partial x_i}$ on A, $i = \overline{1, n}$.We will use the notation $f \in C^1(A)$.

Theorem 6.1.3 *Let $A \subset \mathbb{R}^n$ be an open subset. If $f \in C^1(A)$, then f is differentiable on A.*

Proof To simplify the writing, we present the proof in the particular case $n = 2$. Let $a = (a_1, a_2) \in A$ and $r > 0$ such that $B(a, r) \subset A$.

For any $h = (h_1, h_2) \in \mathbb{R}^2$ such that $a + h \in A$ we have:

$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2)$$
$$= f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2 + h_2)$$
$$+ f(a_1, a_2 + h_2) - f(a_1, a_2).$$

From Lagrange's Theorem it results that there are $0 < \theta_i < 1$, $i = \overline{1, 2}$, such that:

$$\begin{aligned} & f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) \\ & = \frac{\partial f}{\partial x}(a_1 + \theta_1 \cdot h_1, a_2 + h_2) \cdot h_1 + \frac{\partial f}{\partial y}(a_1, a_2 + \theta_2 \cdot h_2) \cdot h_2 . \end{aligned} \tag{6.4}$$

If we denote by:

$$\omega_1(h_1, h_2) = \frac{\partial f}{\partial x}(a_1 + \theta_1 \cdot h_1, a_2 + h_2) - \frac{\partial f}{\partial x}(a_1, a_2),$$

and

$$\omega_2(h_1, h_2) = \frac{\partial f}{\partial y}(a_1, a_2 + \theta_2 \cdot h_2) - \frac{\partial f}{\partial y}(a_1, a_2),$$

then, from (6.4) it follows:

$$\begin{aligned} & f(a + h) - f(a) \\ & = \frac{\partial f}{\partial x}(a) \cdot h_1 + \frac{\partial f}{\partial y}(a) \cdot h_2 + \omega_1(h) \cdot h_1 + \omega_2(h) \cdot h_2 \end{aligned}$$

Finally, if we denote by $\varphi(h) = \omega_1(h) \cdot h_1 + \omega_2(h) \cdot h_2$, then:

$$f(a + h) - f(a) = d f(a)(h) + \varphi(h). \tag{6.5}$$

If we will prove that φ is an infinitely small ($\varphi = o(h)$) then will results that f is differentiable at the point a.

Since $\frac{\partial f}{\partial x_i}$ are continuous we deduce that $\lim\limits_{h \to 0} \omega_i(h) = 0, i = \overline{1, 2}$.

On the other hand, we have:

$$\frac{|\varphi(h)|}{\| h \|} \leq |\omega_1(h)| \cdot \frac{|h_1|}{\| h \|} + |\omega_2(h)| \cdot \frac{|h_2|}{\| h \|} \leq |\omega_1(h)| + |\omega_2(h)|,$$

whence it results that $\lim\limits_{h \to 0} \frac{|\varphi(h)|}{\| h \|} = 0$, and so the proof is finished.

Remark 6.1.7 From Theorem 6.1.3 it results that the elementary functions of several variables are differentiable on any open subset in their domain of definition.

6.2 Differentiability of Vector Functions. Jacobian Matrix

Definition 6.2.1 Let $F = (f_1, \dots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector function of n variables and let $a \in A$ be an interior point. We say that F is **differentiable at the point** a if there exists a linear map $T = T_a : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{h \to 0} \frac{\|F(a + h) - F(a) - T(h)\|}{\| h \|} = 0.$$

where $\| \cdot \|$ is any of the norms $\| \cdot \|_\infty$ or $\| \cdot \|_2$.

Similar to the statement in the Theorem 6.1.2. We have:

Remark 6.2.1 A vector function $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at the interior point $a \in A$ iff there exist a linear map $T = T_a : \mathbb{R}^n \to \mathbb{R}^m$, a neighborhood V of the origine and an infinitely small function $\Phi : V \to \mathbb{R}^m$(i.e.$\Phi = 0(h)$) such that:

$$F(a + h) - F(a) = T(h) + \Phi(h), \ \forall\, h \in V.$$

We specify that $\Phi = 0(h)$ means:

$$\lim_{h \to 0_{\mathbb{R}^n}} \Phi(h) = \Phi(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m} \text{ and } \lim_{h \to 0_{\mathbb{R}^n}} \frac{\Phi(h)}{\|h\|} = 0_{\mathbb{R}^m}.$$

Theorem 6.2.1 *The vector function $F = (f_1, \dots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at the interior point $a \in A$, if and only if, any its scalar component $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$, $i = \overline{1, m}$, is differentiable at the point a.*

Proof Since F is differentiable at the point a, then there exists a linear map $T = T_a : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{h \to 0} \frac{\|F(a + h) - F(a) - T(h)\|_\infty}{\| h \|_\infty} = 0. \tag{6.6}$$

On the other hand, from Theorem 5.7.1, it results that there exists a matrix $A = \left(\lambda_{i\,j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in \mathcal{M}_{m\,,\,n}(\mathbb{R})$ such that:

$$T(h) = h \cdot A^t = (\lambda_{11} \cdot h_1 + \cdots + \lambda_{1\,n} \cdot h_n\,, \dots, \lambda_{m\,1} \cdot h_1 + \cdots + \lambda_{m\,n} \cdot h_n)$$

for any $h = (h_1\,, \dots, h_n) \in \mathbb{R}^n$.

If we denote by $t_i(h) = \lambda_{i\,1} \cdot h_1 + \cdots + \lambda_{i\,n} \cdot h_n\,,\ \forall\, h = (h_1\,, \dots, h_n) \in \mathbb{R}^n$, then for any $i = \overline{1, m}$, $t_i : \mathbb{R}^n \to \mathbb{R}$ is a linear map and $T = (t_1, \dots, t_m)$. Because we have:

$$|f_i(a + h) - f_i(a) - t_i(h)| \le \|F(a + h) - F(a) - T(h)\|_\infty,$$

it result from (6.6) that:

$$\lim_{h \to 0} \frac{|f_i(a + h) - f_i(a) - t_i(h)|}{\| h \|_\infty} = 0, \ (\forall)\ i = \overline{1,\ m} \tag{6.7}$$

Therefore, the scalar component f_i is differentiable at the point a, for any $i = \overline{1,\ m}$.

Reciprocal, if any scalar component f_i of the vector function F is differentiable at the point a, then the relation (6.7) is true for any $i = \overline{1,\ m}$.

As:

$$\|F(a + h) - F(a) - T(h)\|_\infty = \max_{1 \le i \le m} |f_i(a + h) - f_i(a) - t_i(h)|$$

it follows that:

$$\lim_{h \to 0} \frac{\|F(a + h) - F(a) - T(h)\|_\infty}{\| h \|_\infty} = 0,$$

hence the vector function F is differentiable at the point a.

Remark 6.2.2 From Theorem 6.1.1, we deduce that t_i is the differential of first order of the scalar function f_i at the point a, so for any $i = \overline{1,\ m}$ we have:

$$t_i(h) = d\, f_i(a)(h) = \frac{\partial f_i}{\partial x_1}(a) \cdot h_1 + \cdots + \frac{\partial f_i}{\partial x_n}(a) \cdot h_n\,,$$
$$\forall\ h = (h_1, \ldots, h_n) \in \mathbb{R}^n.$$

Therefore, if the vector function $F = (f_1, \ldots, f_m)$ is differentiable at the point a, then linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined, namely:

$$T(h) = (d\, f_1(a)(h), \ldots, d\, f_m(a)(h)),\ (\forall)\ h \in \mathbb{R}^n.$$

We will call this linear map ***the differential of first order of the vector function*** F ***at the point*** a and we will denote them by $d\,F(a)$.

Therefore $d\,F(a) : \mathbb{R}^n \to \mathbb{R}^m$,

and:

$$d\,F(a)(h) = (d\, f_1(a)(h), \ldots, d\, f_m(a)(h))$$
$$= \left(\frac{\partial f_1}{\partial x_1}(a) \cdot h_1 + \cdots + \frac{\partial f_1}{\partial x_n}(a) \cdot h_n, \ldots, \frac{\partial f_m}{\partial x_1}(a) \cdot h_1 + \ldots + \frac{\partial f_m}{\partial x_n}(a) \cdot h_n\right),$$
$$\forall\, h = (h_1, \ldots, h_n) \in \mathbb{R}^n.$$

Definition 6.2.2 The matrix associated with the linear application $d\,F(a) : \mathbb{R}^n \to \mathbb{R}^m$, with respect to the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^m$ is denoted by:

$$J_F(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_m}{\partial x_1}(a) & \ldots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

and is called the jacobian matrix associated with the vector function F at the point a.

We remark that we have:

$$dF(a)(h) = h \cdot (J_F(a))^t$$

where, for any B we denoted by B^t the transpose of the matrix B.

Remark 6.2.3. From Theorem 6.2.1 it results that the study of the differentiability of a vector function returns to the study of the differentiability of its scalar components. Therefore it is sufficient to consider functions of the form $f : A \subset \mathbb{R}^n \to \mathbb{R}$.

If $m = n$, then the jacobian matrix $J_F(a)$ is a square matrix and its determinant is called ***jacobian determinant*** *(or simply* ***jacobian****) and is denoted by:*

$$\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(a) = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}(a).$$

Therefore we have:

$$\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(a) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1}(a) & \ldots & \frac{\partial f_n}{\partial x_n}(a) \end{vmatrix}.$$

6.3 Differentiability of Composite Functions

Theorem 6.3.1 *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two open subsets. If $F : A \to B$ is differentiable at the point $a \in A$, and $G : B \to \mathbb{R}^p$ is differentiable at the point $b = F(a) \in B$, then the composite function $H = G \circ F : A \subset \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at the point $a \in A$ and:*

$$dH(a) = dG(b) \circ dF(a).$$

$$J_H(a) = J_G(b) \cdot J_F(a).$$

Proof By Remark 6.2.1 we have:

$$F(a + h) - F(a) = dF(a)(h) + \Phi(h), . \tag{6.8}$$

where $\lim\limits_{h \to 0} \Phi(h) = \Phi(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}, \lim\limits_{h \to 0} \frac{\Phi(h)}{\|h\|} = 0_{\mathbb{R}^m}$.

and:

$$G(b + k) - G(b) = dG(b)(k) + \Psi(k) \tag{6.9}$$

where $\lim\limits_{k\to 0} \Psi(k) = \Psi(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}, \lim\limits_{k\to 0} \frac{\Psi(k)}{\|k\|} = 0_{\mathbb{R}^m}$.

Since $H = G \circ F$, further it results:

$$H(a + h) - H(a) = G(F(a + h)) - G(F(a)) =$$
$$= G(F(a) + dF(a)(h) + \Phi(h)) - G(b).$$

If we denote by $k(h) = dF(a)(h) + \Phi(h)$ then, from (6.9) we deduce that:

$$H(a + h) - H(a) = G(b + k(h)) - G(b) = dG(b)(k(h)) + \Psi(k(h)).$$

Taking into account that $dG(b)$ is a linear map it results:

$$H(a + h) - H(a) = dG(b)(dF(a)(h)) + dG(b)(\Phi(h)) + \Psi(k(h))$$
$$= (dG(b) \circ dF(a))(h) + \chi(h).$$

where we denoted by:

$$\chi(h) = dG(b)(\Phi(h)) + \Psi(k(h)).$$

As $dG(b) \circ dF(a) : \mathbb{R}^n \to \mathbb{R}^p$ is also a linear map it results that it is sufficient to prove that $\chi = 0(h)$, i.e.χ is an infinitely small function.

Obviously $\lim\limits_{h\to 0} \chi(h) = \chi(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$. It remains to show that $\lim\limits_{h\to 0} \frac{\chi(h)}{\|h\|} = 0_{\mathbb{R}^m}$.

Using Theorem 5.7.2, and taking into account that $J_G(b)$ is the matrix associated with the linear map $dG(b) : \mathbb{R}^m \to \mathbb{R}^p$, and $J_F(a)$ is the matrix associated with the linear map $dF(a) : \mathbb{R}^n \to \mathbb{R}^m$, it results:

$$\frac{\|\chi(h)\|}{\|h\|} \le \|J_G(b)\| \cdot \frac{\|\Phi(h)\|}{\|h\|} + \frac{\|\Psi(k(h))\|}{\|k(h)\|} \cdot \frac{\|k(h)\|}{\|h\|}$$
$$\le \|J_G(b)\| \cdot \frac{\|\Phi(h)\|}{\|h\|} + \frac{\|\Psi(k(h))\|}{\|k(h)\|} \cdot \left(\|J_F(a)\| + \frac{\|\Phi(h)\|}{\|h\|}\right).$$

Since $\lim\limits_{h\to 0} \frac{\|\Phi(h)\|}{\|h\|} = 0$, $\lim\limits_{h\to 0} k(h) = 0$ and $\lim\limits_{k\to 0} \frac{\|\Psi(k)\|}{\|k\|} = 0$, we deduce that: $\lim\limits_{h\to 0} \frac{\|\chi(h)\|}{\|h\|} = 0$, hence $\lim\limits_{h\to 0} \frac{\chi(h)}{\|h\|} = 0_{\mathbb{R}^m}$.

Therefore we proved that:

$$H(a + h) - H(a) = (dG(b) \circ dF(a))(h) + \chi(h),$$

where χ is an infinitely small, whence it results that H is differentiable at the point a and:

$$d\,H(a) \;=\; d\,G(b)\;\circ\; d\,F(a).$$

As to the operation of composing linear maps corresponds the operation of multiplying their associated matrices, it follows that:

$$J_H(a) \;=\; J_G(b)\;\cdot\; J_F(a).$$

Exemple 6.3.1 Let's consider the subset:

$$A = \left\{(x\,,\; y) \in \mathbb{R}^2\,;\; x > 0\,,\; y > -2\right\} \subset \mathbb{R}^2$$

and the following functions:

$$F : A \subset \mathbb{R}^2 \to \mathbb{R}^3,\; F(x,\; y) = \left(\sqrt{x}\,,\; \sqrt{x^2 +\; 3\cdot y^2}\,,\; \sqrt{y\; +\; 2}\right),$$

$$G : \mathbb{R}^3 \to \mathbb{R}^2,\; G(u,\; v\,,\; w) \;=\; \left(u^2 +\; v^2 +\; 2\cdot\; w^2\,,\; u^2 -\; v^2\right)$$

$$H \;=\; G\;\circ\; F : A \subset \mathbb{R}^2 \to \mathbb{R}^2.$$

Let also be the points: $a \;=\; (1\,,\; -1) \in A$ and $b \;=\; F(a) \;=\; (1\,,\; 2,\; 1) \in \mathbb{R}^3$. Prove that:

$$J_H(a) \;=\; J_G(b)\;\cdot\; J_F(a).$$

We have successive:

$$J_F(x\,,\; y) \;=\; \begin{pmatrix} \frac{1}{2\cdot\,\sqrt{x}} & 0 \\ \frac{x}{\sqrt{x^2+\,3\cdot y^2}} & \frac{3\cdot y}{\sqrt{x^2+\,3\cdot\,y^2}} \\ 0 & \frac{1}{2\cdot\,\sqrt{y\,+\,2}} \end{pmatrix},$$

$$J_F(1\,,\; -1) \;=\; \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -\,\frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix},$$

$$J_G(u,\; v\,,\; w) = \begin{pmatrix} 2\,u & 2\,v & 4w \\ 2\,u & -2\,v & 0 \end{pmatrix};\; J_G(1\,,\; 2,\; 1) = \begin{pmatrix} 2 & 4 & 4 \\ 2 & -4 & 0 \end{pmatrix}$$

and:

$$J_G(1,\; 2\,,\; 1)\;\cdot\; J_F(1\,,\; -1) \;=\; \begin{pmatrix} 3 & -4 \\ -1 & 6 \end{pmatrix}.$$

On the other hand, for any point $(x,\ y) \in A$, we have:

$$H(x,\ y) = G(F(x,\ y)) = G\left(\sqrt{x},\ \sqrt{x^2 + 3 \cdot y^2},\ \sqrt{y+2}\right)$$
$$= \left(x^2 + 3 \cdot y^2 + x + 2 \cdot y + 4,\ -x^2 - 3y^2 + x\right).$$

The jacobian matrix associated with the function $H = G \circ F$ at the point $(x,\ y) \in A$ is:

$$J_H(x,\ y) = \begin{pmatrix} 2 \cdot x + 1 & 6 \cdot y + 2 \\ -2 \cdot x + 1 & -6 \cdot y \end{pmatrix} \text{ and so } J_H(1,\ -1) = \begin{pmatrix} 3 & -4 \\ -1 & 6 \end{pmatrix}$$

Theorem 6.3.2 *Let $A,\ B \subset \mathbb{R}^2$ be two open subsets,*
$F = (u,\ v) : A \subset \mathbb{R}^2 \to B \subset \mathbb{R}^2$ be a vector function of C^1 class on A and let $f : B \subset \mathbb{R}^2 \to \mathbb{R}$ be a scalar function of C^1 class on B.Then the composite function $h = f \circ F : A \subset \mathbb{R}^2 \to \mathbb{R}$, given by:

$$h(x,\ y) = (f \circ F)(x,\ y) = f(F(x,\ y)) = f(u(x,\ y),\ v(x,\ y)),$$
$$\forall (x,\ y) \in A.$$

is of C^1 class on A and we have the following formulas:

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial h}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}. \end{aligned} \tag{6.10}$$

Proof The scalar functions $u,\ v$ and f are differentiable, because, by hypothesis they are of C^1 class (Theorem 6.1.3). From Theorem 6.2.1, it results that the vector function F is differentiable on A, and from Theorem 6.3.1 we deduce that the composite function $h = f \circ F$ is differentiable on A.

Let $(a,\ b) \in A$, be an arbitrary point. If we denote by $c = u(a,\ b)$ and by $d = v(a,\ b)$, then $(c,\ d) \in B$. From Theorem 6.3.1, it results:

$$J_h(a,\ b) = J_f(c,\ d) \cdot J_F(a,\ b),$$

that is:

$$\begin{pmatrix} \frac{\partial h}{\partial x}(a,\ b) & \frac{\partial h}{\partial y}(a,\ b) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u}(c,\ d) & \frac{\partial f}{\partial v}(c,\ d) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x}(a,\ b) & \frac{\partial u}{\partial y}(a,\ b) \\ \frac{\partial v}{\partial x}(a,\ b) & \frac{\partial v}{\partial y}(a,\ b) \end{pmatrix}.$$

Following the computes we obtain:

$$\frac{\partial h}{\partial x}(a,\ b) = \frac{\partial f}{\partial u}(c,\ d)\cdot\frac{\partial u}{\partial x}(a,\ b) + \frac{\partial f}{\partial v}(c,\ d)\cdot\frac{\partial v}{\partial x}(a,\ b)$$
$$\frac{\partial h}{\partial y}(a,\ b) = \frac{\partial f}{\partial u}(c,\ d)\cdot\frac{\partial u}{\partial y}(a,\ b) + \frac{\partial f}{\partial v}(c,\ d)\cdot\frac{\partial v}{\partial y}(a,\ b)\ .$$

Example 6.3.2 Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function of C^1 class on $\mathbb{R}^2$ and

$$h(x,\ y) = f\big(2\cdot x + y^2,\ 1 - y\cdot \cos x\big),\ (x,\ y)\in\mathbb{R}^2.$$

Compute $\frac{\partial h}{\partial x}(x,\ y)$ and $\frac{\partial h}{\partial y}(x,\ y)$.

If we denote by $u(x,\ y) = 2\cdot x + y^2$ and by $v(x,\ y) = 1 - y\cdot\cos x$, then $h(x,\ y) = f(u(x,y),\ v(x,y))$ and:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\cdot 2 + \frac{\partial f}{\partial v}\cdot y\cdot \sin x$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\cdot 2\cdot y + \frac{\partial f}{\partial v}\cdot(-\cos x).$$

Remark 6.3.1 The formulas (6.10) generalizes the known derivation formula of composite functions of ane variable, i.e. if $h(x) = f(u(x))$ then:

$$h'(x) = f'(u(x))\cdot u'(x).$$

Remark 6.3.2 The formulas (6.10) admit the following generalization:

If $h(x_1,\ \ldots,\ x_n) = f(u_1(x_1,\ \ldots,\ x_n),\ \ldots,\ u_m(x_1,\ \ldots,\ x_n))$,then:

$$\begin{aligned} \frac{\partial h}{\partial x_1} &= \frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial f}{\partial u_m}\cdot\frac{\partial u_m}{\partial x_1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial h}{\partial x_n} &= \frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_n} + \ldots + \frac{\partial f}{\partial u_m}\cdot\frac{\partial u_m}{\partial x_n}\ . \end{aligned} \tag{6.11}$$

Definition 6.3.1 *Let $K\subset\mathbb{R}^n$ be an* ***open cone*** *(i.e. a subset with the property that for any $x\in K$ and any $t\in\mathbb{R},\ t\neq 0$, it results that $t\cdot x\in K$). A function $f:K\to\mathbb{R}$ is called homogeneousp of order p,where $p\in\mathbb{R}$, if:*

$$f(t\cdot x_1,\ \ldots,\ t\cdot x_n) = t^p\cdot f(x_1,\ \ldots,\ x_n),\ \forall\, x = (x_1,\ \ldots,\ x_n)\in K.$$

Exemple 6.3.3

1. The function $f:\mathbb{R}^2\to\mathbb{R}$, given by:

$$f(x,\ y) = x^2 + 4\cdot y^2,\ \forall(x,y)\in\mathbb{R}^2$$

is homogeneous of order 2 este $\mathbb{R}^2$, because:

$$f(t\cdot x,\ t\cdot y) = (t\cdot x)^2 + 4\cdot(t\cdot y)^2 = t^2\cdot\left(x^2 + 4\cdot y^2\right)$$
$$= t^2\cdot f(x,\ y),\ \forall\ (x,\ y)\in\mathbb{R}^2$$

2. The function $f : \left\{(x,\ y)\in\mathbb{R}^2;\ x\neq 0\right\}\to\mathbb{R}$, given by:

$$f(x,\ y) = (x+y)\cdot\ \operatorname{arctg}\frac{y}{x},$$

is homogeneous of order 1, because: $f(t\cdot x,\ t\cdot y) = (t\cdot x + t\cdot y)\cdot \operatorname{arctg}\left(\frac{t\cdot y}{t\cdot x}\right) = t\cdot (x+y)\cdot \operatorname{arctg}\frac{y}{x} = t\cdot f(x,\ y)$.

3. The function $f : \mathbb{R}^2\to\mathbb{R}$, given by: $f(x,\ y) = \sqrt[3]{x}+\sqrt[3]{y}$
is omogeneous of order $\frac{1}{3}$, because:

$$f(t\cdot x,\ t\cdot y) = \sqrt[3]{t\cdot x} + \sqrt[3]{t\cdot y} = \sqrt[3]{t}\cdot\left(\sqrt[3]{x} + \sqrt[3]{y}\right) = t^{\frac{1}{3}}\cdot f(x,\ y).$$

Theorem 6.3.3 (Euler) *Let $K\subset\mathbb{R}^n$ be an open cone and $f : K\to\mathbb{R}$ be a function differentiable and homogeneuos of order p.Then we have:*

$$x_1\cdot\frac{\partial f}{\partial x_1}(x) + \cdots + x_n\cdot\frac{\partial f}{\partial x_n}(x) = p\cdot f(x),\ \forall\ x = (x_1,\ \ldots,\ x_n)\in K.$$

Proof By the hypothesis we have:

$$f(t\cdot x_1,\ \ldots,\ t\cdot x_n) = t^p\cdot f(x_1,\ \ldots,\ x_n),$$
$$\forall\ x = (x_1,\ \ldots,\ x_n)\in K,\ \ t\in\mathbb{R}^*. \tag{6.12}$$

We notice that the function from left member can be seen as a composed function, if we denote by:
$u_1(t) = t\cdot x_1,\ \ldots,\ u_n(t) = t\cdot x_n$ and $h(t) = f(u_1(t),\ \ldots,\ u_n(t))$.
Taking into account the rules of derivation (6.11) it results:

$$h'(t) = \frac{\partial f}{\partial u_1}(t\cdot x)\cdot u_1'(t) + \cdots + \frac{\partial f}{\partial u_n}(t\cdot x)\cdot u_n'(t) = p\cdot t^{p-1} f(x)$$

and further:

$$x_1\cdot\frac{\partial f}{\partial u_1}(t\cdot x) + \cdots + x_n\cdot\frac{\partial f}{\partial u_n}(t\cdot x) = p\cdot t^{p-1}\cdot f(x).$$

In particular case $t = 1$ we obtain:

$$x_1\cdot\frac{\partial f}{\partial x_1}(x) + \cdots + x_n\cdot\frac{\partial f}{\partial x_n}(x) = p\cdot f(x),\ \forall\ x = (x_1,\ \ldots,\ x_n)\in K.$$

Example 6.3.4 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by:

$$f(x,\ y) = \sqrt[3]{x} + \sqrt[3]{y}.$$

From Example 6.3.3, we know that the function is f homogeneous of order $\frac{1}{3}$ on $\mathbb{R}^2$, hance satisfaces the following relation:

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = \frac{1}{3} \cdot f.$$

This also results from a direct computation:

$$\frac{\partial f}{\partial x} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}};\ \frac{\partial f}{\partial y} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{y^2}}$$

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = x \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}} + y \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{y^2}}$$

$$= \frac{1}{3} \cdot (\sqrt[3]{x} + \sqrt[3]{x}) = \frac{1}{3} \cdot f(x, y).$$

6.4 The First Order Differenential and Its Invariance Form

Definition 6.4.1 Let $A \subset \mathbb{R}^n$ be an open subset and let $f : A \to \mathbb{R}$ be a differentiable function on A. The first order differential of the function f at the point $a \in A$ is the linear map $d f(a) : \mathbb{R}^n \to \mathbb{R}$, given by:

$$d f(a)(h) = \frac{\partial f}{\partial x_1}(a) \cdot h_1 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot h_n,\ \forall\ h = (h_1,\ \ldots,\ h_n) \in \mathbb{R}^n.$$

We consider now the projection functions $p_i : \mathbb{R}^n \to \mathbb{R}$, $i = \overline{1,\ n}$, given by:

$$p_i(x) = p_i(x_1,\ \ldots,\ x_i,\ \ldots,\ x_n) = x_i,\ \forall\ x = (x_1,\ \ldots,\ x_n) \in \mathbb{R}^n.$$

Since $\frac{\partial p_i}{\partial x_j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$, it results that $d p_i(a)(h) = h_i,\ \forall\ i = \overline{1,\ n}$.
Therefore, the first order differentials of the functions p_i do not depend on the point a. Hence we have:

$$d p_i(h) = d x_i(h) = h_i,\ \forall i = \overline{1,\ n}.$$

With this remark, the first-order differential of the function f at the point a is written:

$$df(a)(h) = \frac{\partial f}{\partial x_1}(a) \cdot dx_1(h) + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot dx_n(h),\ \forall h \in \mathbb{R}^n. \tag{6.13}$$

If we write the previous relation as an equality of functions we have:

$$d\,f(a) = \frac{\partial f}{\partial x_1}(a) \cdot d\,x_1 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot d\,x_n$$

where $d\,x_i$ is the first order differential of the project function p_i, $i = \overline{1,\ n}$.

Remark 6.4.1 Let us consider the particular case $n = 2$.

Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a function differentiable at the interior point $(a,\ b) \in A$.

The first ordin differential of function f at the point $(a,\ b) \in A$ is the linear map $d\,f(a,\ b)\ : \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$\begin{aligned} d\,f(a,\ b)(h,\ k) &= \frac{\partial f}{\partial x}(a,\ b) \cdot h + \frac{\partial f}{\partial y}(a,\ b) \cdot k \\ &= \frac{\partial f}{\partial x}(a,\ b) \cdot d\,x(h,\ k) + \frac{\partial f}{\partial y}(a,\ b) \cdot d\,y(h,\ k), \\ &\forall\ (h,\ k) \in \mathbb{R}^2 \end{aligned}$$

Therefore:

$$d\,f(a,\ b) = \frac{\partial f}{\partial x}(a,\ b) \cdot d\,x + \frac{\partial f}{\partial y}(a,\ b) \cdot d\,y\,.$$

Example 6.4.1 Let $f : \left\{(x,\ y) \in \mathbb{R}^2;\ x > 0,\ y > 0\right\} \subset \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,\ y) = x^2 \cdot y^3 - 2 \cdot y \cdot \ln x + y^x.$$

Compute $d\,f(1,\ 1)$ and $d\,f(1,\ 1)(2,\ -1)$.

Taking into account to Example 6.1.1, we have:

$$\frac{\partial f}{\partial x}(x,\ y) = 2 \cdot x \cdot y^3 - 2 \cdot \frac{y}{x} + y^x \cdot \ln y$$

and

$$\frac{\partial f}{\partial y}(x,\ y) = 3 \cdot x^2 \cdot y^2 - 2 \cdot \ln x + x \cdot y^{x-1},$$

so;

$$\frac{\partial f}{\partial x}(1,\ 1) = 0;\ \frac{\partial f}{\partial y}(1,\ 1) = 4.$$

Then $d f(1, 1) : \mathbb{R}^2 \to \mathbb{R}$ is:

$$d f(1, 1) = \frac{\partial f}{\partial x}(1, 1) \cdot d x + \frac{\partial f}{\partial y}(1, 1) \cdot d y = 4 \cdot d y$$

and

$$d f(1, 1)(2, -1) = 0 \cdot 2 + 4 \cdot (-1) = -4.$$

Let us consider two open subsets $A, B \subset \mathbb{R}^n$, a vector function $F = (u_1, \ldots, u_n) : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^n$ differentiable on A, a scalar function $f : B \subset \mathbb{R}^2 \to \mathbb{R}$ differentiable on B and let $h = f \circ F : A \subset \mathbb{R}^n \to \mathbb{R}$ be the composite function defined by:

$$h(x) = h(x_1, \ldots, x_n) = f(u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)),$$
$$\forall x = (x_1, \ldots, x_n) \in A$$

As $f = f(u_1, \ldots, u_n), (u_1, \ldots, u_n) \in B$, it results:

$$df = \frac{\partial f}{\partial u_1} \cdot du_1 + \cdots + \frac{\partial f}{\partial u_n} \cdot du_n. \tag{6.14}$$

On the other hand we have:

$$dh = \frac{\partial h}{\partial x_1} \cdot d x_1 + \cdots + \frac{\partial h}{\partial x_n} \cdot d x_n.$$

Taking into account that:

$$h(x) = h(x_1, \ldots, x_n) = f(u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)),$$
$$\forall x = (x_1, \ldots, x_n) \in A$$

from the derivation formulas (6.11), for composite function, it results:

$$dh = \left(\frac{\partial f}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial f}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_1}\right)dx_1 + \cdots +$$
$$+\left(\frac{\partial f}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_n} + \cdots + \frac{\partial f}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_n}\right)dx_n$$
$$=\frac{\partial f}{\partial u_1} \cdot \left(\frac{\partial u_1}{\partial x_1} \cdot dx_1 + \cdots + \frac{\partial u_1}{\partial x_n} \cdot dx_n\right) + \cdots + \frac{\partial f}{\partial u_n} \cdot$$ Therefore, we have:
$$\left(\frac{\partial u_n}{\partial x_1} \cdot dx_1 + \cdots + \frac{\partial u_n}{\partial x_n} \cdot dx_n\right)$$
$$=\frac{\partial f}{\partial u_1} \cdot du_1 + \cdots + \frac{\partial f}{\partial u_n} \cdot du_n.$$

$$dh = \frac{\partial f}{\partial u_1} \cdot du_1 + \cdots + \frac{\partial f}{\partial u_n} \cdot du_n \tag{6.15}$$

The formula (6.14) represents the expression of the first order differential of the function f seen as a function that depends on the variables $u_1, \ldots, u_n$. The formula (6.15) represents the expression of the first order differential of the function h regarded as the function that depends on the dependent variables:

$$u_i = u_i(x_1, \ldots, x_n), \ i = \overline{1, n}.$$

This formal equality $dh = df$ is called ***the invariance property of the first-order differential at a change of independent variables.***

Example 6.4.2 Let $f \in C^1(\mathbb{R}^2)$ be an arbitrary function and.

$$h(x, y) = f\left(2 \cdot x + y^2, 1 - y \cdot \cos x\right)$$

Compute $d h(x, y)$ and verify the invariance property of the first-order differential.

If we denote by $u(x, y) = 2 \cdot x + y^2$ and by $v(x, y) = 1 - y \cdot \cos x$, then $h(x, y) = f(u, v)$ and according to Example 6.3.2 we have:

$$\begin{aligned}
dh(x, y) &= \frac{\partial h}{\partial x} \cdot dx + \frac{\partial h}{\partial y} \cdot dy \\
&= \left(\frac{\partial f}{\partial u} \cdot 2 + \frac{\partial f}{\partial v} \cdot y \cdot \sin x\right) dx \\
&+ \left(\frac{\partial f}{\partial u} \cdot 2 \cdot y + \frac{\partial f}{\partial v} \cdot (-\cos x)\right) dy \\
&= \frac{\partial f}{\partial u} \cdot \underbrace{(2 \cdot dx + 2y \cdot dy)}_{du} \\
&+ \frac{\partial f}{\partial v} \cdot \underbrace{(y \cdot \sin x dx - \cos x dy)}_{dv} \\
&= \frac{\partial f}{\partial u} \cdot du + \frac{\partial f}{\partial v} \cdot dv = df(u, v).
\end{aligned}$$

6.5 The Directional Derivative. The Differential Operators: Gradient, Divergence, Curl and Laplacian

Let $A \subset \mathbb{R}^3$ be an open subset, let $a = (a_1, a_2, a_3) \in A$ be a fixed point and let $\overrightarrow{l} = \cos\alpha \cdot \overrightarrow{i} + \cos\beta \cdot \overrightarrow{j} + \cos\gamma \cdot \overrightarrow{k}$ be an unit vector i.e.:

$$\left\| \vec{l} \right\|^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Since $a \in A$ is an interior point, it results that there exists $r > 0$ such that $B(a, r) \subset A$. For any $t \in (-r, r)$, the point $x = a + t \cdot l \in B(a, r) \subset A$, because:

$$\|x - a\| = \|a + t \cdot l - a\| = |t| \cdot \|l\| = |t| < r.$$

(here $l = (\cos\alpha, \cos\beta, \cos\gamma)$).

Definition 6.5.1 We say that the function $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ **is derivable at the point** $a \in A$ **along the vector** $\vec{l}$ if the following limit exists and is finite:

$$\lim_{\substack{x \to a \\ x = a + t \cdot l}} \frac{f(x) - f(a)}{\|x - a\|} = \lim_{t \to 0} \frac{f(a + t \cdot l) - f(a)}{|t|}.$$

This limit is denoted by $\frac{\partial f}{\partial \vec{l}}(a)$ and is called ***the directional derivative of function f along the vector $\vec{l}$ at the point*** a.

From a geometric point of view, the set of points $x = a + t \cdot l$, $t \in \mathbb{R}$ represents the line that passes through a and has the director vector $\vec{l}$ (Fig. 6.2).

The positive half-axis coresponds to the values of the parameter $t > 0$, and the negative half-axis coresponds to values $t < 0$. Further we will note with:

$$\frac{\partial f}{\partial \vec{l}_+}(a) = \lim_{t \to 0^+} \frac{f(a + t \cdot l) - f(a)}{|t|} = \lim_{t \to 0} \frac{f(a + t \cdot l) - f(a)}{t}$$

and with:

$$\frac{\partial f}{\partial \vec{l}_-}(a) = \lim_{t \to 0^-} \frac{f(a + t \cdot l) - f(a)}{|t|} = \lim_{t \to 0} \frac{f(a + t \cdot l) - f(a)}{-t}$$
$$= -\frac{\partial f}{\partial \vec{l}_+}(a).$$

Teorema 6.5.1 If $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable at the interior point $a \in A$, then f is derivable at the point $a \in A$ along the vector $\vec{l} = \cos\alpha \cdot \vec{i} + \cos\beta \cdot \vec{j} + \cos\gamma \cdot \vec{k}$ and:

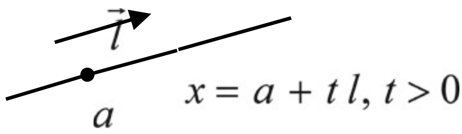

Fig. 6.2 The line passing through a and has the director vector l

$$\frac{\partial f}{\partial \overrightarrow{l}_+}(a) = \frac{\partial f}{\partial x}(a) \cdot \cos\alpha + \frac{\partial f}{\partial y}(a) \cdot \cos\beta + \frac{\partial f}{\partial z}(a) \cdot \cos\gamma.$$

Proof Let $r > 0$ be such that $B(a, r) \subset A$. If $t \in (0, r)$, then $a + t \cdot l \in A$, and taking into account that f is differentiable at the point a it results:

$$f(a + t \cdot l) - f(a) = df(a)(t \cdot l) + \varphi(t \cdot l)$$

where $\varphi = o(h)$(i.e., $\lim\limits_{h\to 0} \varphi(h) = \varphi(0) = 0$ and $\lim\limits_{h\to 0} \frac{|\varphi(h)|}{\|h\|} = 0$).

Since $\|t \cdot l\| = t$ and $df(a)(t \cdot l) = t \cdot df(a)(l)$, further we have:

$$\frac{f(a + t \cdot l) - f(a)}{t} = df(a)(l) + \frac{\varphi(t \cdot l)}{\|t \cdot l\|}.$$

As $\lim\limits_{t\to 0} \frac{\varphi(t \cdot l)}{\|t \cdot l\|} = 0$,it results that there exists:

$$\lim_{t\to 0} \frac{f(a + t \cdot l) - f(a)}{t} = df(a)(l)$$
$$= \frac{\partial f}{\partial x}(a) \cdot \cos\alpha + \frac{\partial f}{\partial y}(a) \cdot \cos\beta + \frac{\partial f}{\partial z}(a) \cdot \cos\gamma.$$

hence:

$$\frac{\partial f}{\partial \overrightarrow{l}_+}(a) = \frac{\partial f}{\partial x}(a) \cdot \cos\alpha + \frac{\partial f}{\partial y}(a) \cdot \cos\beta + \frac{\partial f}{\partial z}(a) \cdot \cos\gamma.$$

Example 6.5.1 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function given by $f(x, y, z) = x \cdot y \cdot z$, the point $a = (1, -1, 1) \in \mathbb{R}^3$ and the unit vector $\overrightarrow{l} = \frac{1}{\sqrt{14}}\overrightarrow{i} - \frac{3}{\sqrt{14}}\overrightarrow{j} - \frac{2}{\sqrt{14}}\overrightarrow{k}$. Compute $\frac{\partial f}{\partial \overrightarrow{l}}(1, -1, 1)$.

We have: $\frac{\partial f}{\partial x} = y \cdot z$, $\frac{\partial f}{\partial y} = x \cdot z$, $\frac{\partial f}{\partial z} = x \cdot y$,
$\frac{\partial f}{\partial x}(1, -1, 1) = -1$, $\frac{\partial f}{\partial y}(1, -1, 1) = 1$, $\frac{\partial f}{\partial z}(1, -1, 1) = -1$ and so:

$$\frac{\partial f}{\partial \overrightarrow{l}}(1, -1, 1) = -\frac{1}{\sqrt{14}} - \frac{3}{\sqrt{14}} + \frac{2}{\sqrt{14}} = -\frac{2}{\sqrt{14}}.$$

Remark 6.5.1 Dfinition 6.5.1 generalizes the definition of the partial derivative. Indeed, if $\overrightarrow{l} = \overrightarrow{i}$, namely $l = (1, 0, 0)$, then:

$$\frac{\partial f}{\partial \overrightarrow{i}_+}(a) = \frac{\partial f}{\partial x}(a) \cdot 1 + \frac{\partial f}{\partial y}(a) \cdot 0 + \frac{\partial f}{\partial z}(a) \cdot 0,$$

hence

$$\frac{\partial f}{\partial \overrightarrow{i}_+} = \frac{\partial f}{\partial x}.$$

Similarly we have:

$$\frac{\partial f}{\partial \overrightarrow{j}_+} = \frac{\partial f}{\partial y}; \frac{\partial f}{\partial \overrightarrow{k}_+} = \frac{\partial f}{\partial z},$$

where $j = (0, 1, 0)$ and $k = (0, 0, 1)$.

Definition 6.5.2 Let $D \subset \mathbb{R}^3$ be an open subset. **The scalar field on** D is a fancy name for any scalar function $u : D \subset \mathbb{R}^3 \to \mathbb{R}$. If in addition $u \in C^k(D)$, then we say that u **is a scalar field of class** C^k **on** D**.** By **vector field on** D is meant any vectorial function $\overrightarrow{V} = (P, Q, R) : D \subset \mathbb{R}^3 \to \mathbb{R}^3$. If $P, Q, R \in C^k(D)$,we say that $\overrightarrow{V}$ is a **vector field of class** C^k **on** D.

Examples of scalar fields are the temperature field, the pressure field, the density field and so on. A typical example of a vector field is the particle velocity field of a moving fluid.

Further we consider a fix point $O \in \mathbb{R}^3$ and the standard basis $\left\{\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\right\}$ consisting of the unit vectors of the coordinate axes. If we identify any point $M(x, y, z) \in \mathbb{R}^3$ with its position vector $\overrightarrow{OM}$, then the vector field $\overrightarrow{V} = (P, Q, R) : D \to \mathbb{R}^3$ can be written in the form:

$$\overrightarrow{V}(x, y, z) = P(x, y, z) \cdot \overrightarrow{i} + Q(x, y, z) \cdot \overrightarrow{j} \\ + R(x, y, z) \cdot \overrightarrow{k}, \ \forall (x, y, z) \in D$$

Definition 6.5.3 If $u : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a scalar field, $u \in C^1(D)$, then the vector field on D, defined by $\frac{\partial u}{\partial x} \cdot \vec{i} + \frac{\partial u}{\partial y} \cdot \vec{j} + \frac{\partial u}{\partial x} \cdot \vec{k}$ is called **the gradient of** u**.**Thus we have:

$$\text{grad } u = \frac{\partial u}{\partial x} \cdot \vec{i} + \frac{\partial u}{\partial y} \cdot \vec{j} + \frac{\partial u}{\partial z} \cdot \vec{k}$$

Definition 6.5.4 A vector field $\vec{V} = P \overrightarrow{i} + Q \overrightarrow{j} + R \overrightarrow{k}$ is said to be **a potential field** if there exists a scalar field $u : D \subset \mathbb{R}^3 \to \mathbb{R}$, $u \in C^1(D)$, such that $\vec{V} = \text{grad } u$. This means that:

$$P = \frac{\partial u}{\partial x}, \ Q = \frac{\partial u}{\partial y}, \ R = \frac{\partial u}{\partial z}.$$

Example 6.5.2 The vector field:

$$\vec{V} = x \cdot z^2 \cdot \vec{i} + y \cdot z^2 \cdot \vec{j} + z \cdot \left(x^2 + y^2\right) \cdot \vec{k}, \ \forall (x, y, z) \in \mathbb{R}^3$$

is a potential vector field, since $\overrightarrow{V} = \text{grad } u$, where:

$$u(x, y, z) = \frac{1}{2} \cdot (x^2 + y^2) \cdot z^2, (x, y, z) \in \mathbb{R}^3.$$

Definition 6.5.5 Let $\overrightarrow{V} = (P, Q, R) : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of C^1-class on D. Is called **the divergence of the vector field** $\vec{V}$ the following scalar field:

$$\text{div } \overrightarrow{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

A vector field $\vec{V}$ whose divergence is identically equal to zero is said to be **solenoidal**. Therefore $\vec{V}$ is solenoidal if div $\vec{V} = 0$ on D.

Example 6.5.3 The vector field:

$$\vec{V} : \mathbb{R}^3 \to \mathbb{R}^3, \ \vec{V} = x^2 \cdot y \cdot z \cdot \vec{i} + x \cdot y^2 \cdot z \cdot \vec{j} - 2 \cdot x \cdot y \cdot z^2 \cdot \vec{k},$$

is solenoidal on $\mathbb{R}^3$, because div $\vec{V} = 2 \cdot x \cdot y \cdot z + 2 \cdot x \cdot y \cdot z - 4 \cdot x \cdot y \cdot z = 0.$

Definition 6.5.6 Let $\vec{V} = (P, Q, R) : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of C^1-class on D.

Is called **the curl of the vector field** $\overrightarrow{V}$ the following vector fied:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \cdot \overrightarrow{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \cdot \overrightarrow{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot \overrightarrow{k}.$$

The curl of the vector field $\overrightarrow{V}$ is denoted by **curl** $\overrightarrow{V}$.Therefore we have:

$$\overline{V} \text{ curl} \overrightarrow{V} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \cdot \overrightarrow{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \cdot \overrightarrow{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot \overrightarrow{k}.$$

The expression of the curl of a vector field $\overrightarrow{V} = (P, Q, R)$ can be conveniently written as the symbolic determinant:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix},$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors in the directions of the coordinate axes and the symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are understood in the sense that the multiplication of such a symbol by a function means the derivation with respect to the corresponding variable, i.e., $\frac{\partial}{\partial x} \cdot Q = \frac{\partial Q}{\partial x}$.

Indeed, expanding (formally) the previous determinant in minors of the first row we obtain:

$$\begin{vmatrix} \vec{i}\,\vec{j}\,\vec{k} \\ \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z} \\ P\,Q\,R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \cdot \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \cdot \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot \vec{k}.$$

Example 6.5.4 If $\vec{V} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{V} = x \cdot z^2 \cdot \vec{i} + y \cdot z^2 \cdot \vec{j} + z \cdot \left(x^2 + y^2\right) \vec{k}$, then:

$$\text{curl } \vec{V} = (2 \cdot y \cdot z - 2 \cdot y \cdot z) \cdot \vec{i} + (2 \cdot x \cdot z - 2 \cdot xz) \cdot \vec{j} + 0 \cdot \vec{k} = \vec{0}.$$

Definition 6.5.7 Is called the **Hamiltonian (nabla) operator** the following symbolic operator:

$$\nabla = \frac{\partial}{\partial x} \cdot \vec{i} + \frac{\partial}{\partial y} \cdot \vec{j} + \frac{\partial}{\partial z} \cdot \vec{k}.$$

Remark 6.5.2 Let $u : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a scalar field of C^1- class on D and let $\vec{V} = (P, Q, R) : D \to \mathbb{R}^3$ be a vector field also of C^1- class on D.The application of the nabla operator ∇ to the scalar function u can be performed as formal multiplication of the "vector" ∇ by the scalar u:

$$\nabla u = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}\right) u = \frac{\partial u}{\partial x} \cdot \vec{i} + \frac{\partial u}{\partial y} \cdot \vec{j} + \frac{\partial u}{\partial z} \cdot \vec{k} = \text{grad } u,$$

and the divergence of a vector field $\vec{V}$ is the formal scalar product of the symbolic vector ∇ by the vector $\vec{V}$.

$$\text{div } \vec{V} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}\right) \cdot \left(P \vec{i} + Q \vec{j} + R \vec{k}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{V}$$

Similarly we have:

$$\text{curl } \vec{F} = \nabla \times \vec{V} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}\right) \times \left(P \vec{i} + Q \vec{j} + R \vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

i.e.,the curl of the vector field $\overrightarrow{V}$ is the formal vector product of the symbolic vector ∇ by the vector $\overrightarrow{V}$.

Definition 6.5.8 Is called the **Laplacian operator** and denoted by Δ the following differential operator.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad {}^{*)}$$

Let $D \subset \mathbb{R}^3$ be a connected open subset and let $u : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a scalar field of C^2 class.Then:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

($^{*)}$ for the partial derivatives of the second order see Definition 6.6.2 below).

The name "laplacian" comes from the French mathematician Pierre Laplace (1749–1827).

The connections between the differential operators Δ, grad, div and curl are highlighted by the following remark:

Remark 6.5.3 Let $u : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a scalar field of C^2- class on the connected, open subset D, and let $\overrightarrow{V} = (P\,,\ Q\,,\ R) : D \to \mathbb{R}^3$ be a vector field also of C^2- class on D.Then:

1. $\operatorname{div}(\operatorname{grad} u) = \Delta\, u$
2. $\operatorname{curl}(\operatorname{grad} u) = \overrightarrow{0}\,;$
3. $\operatorname{div}\left(\operatorname{curl} \overrightarrow{V}\right) = 0.$

6.6 Partial Derivatives and Differentials of Higher Orders

Definition 6.6.1 Let $A \subset \mathbb{R}^2$ be an open subset and let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a real valued function of two variables. We assume that there exist the first partial derivatives $\frac{\partial f}{\partial x} : A \to \mathbb{R}$ and $\frac{\partial f}{\partial y} : A \to \mathbb{R}$. Obviously, these derivatives are in turn real functions of two variables. If there exist the partial derivatives of the functions $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, then they are called **the second-order partial derivatives of the function** f and are denoted as follows:

$$f''_{x^2} = \left(f'_x\right)'_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2};$$

$$f''_{yx} = \left(f'_x\right)'_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$f''_{xy} = \left(f'_y\right)'_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \, \partial y};$$

$$f''_{y^2} = \left(f'_y\right)'_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example 6.6.1 Let $f(x, \ y) = x^2 \cdot y^3 - 2 \cdot y \cdot \ln x + y^x$, $\forall x > 0$, $\forall y > 0$. Compute.

$$\frac{\partial^2 f}{\partial x^2}; \quad \frac{\partial^2 f}{\partial y \, \partial x}; \quad \frac{\partial^2 f}{\partial x \, \partial y}; \quad \frac{\partial^2 f}{\partial y^2}.$$

We have successive:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2 \cdot x \cdot y^3 - 2 \cdot \frac{y}{x} + y^x \cdot \ln y, \ \frac{\partial f}{\partial y} = 3 \cdot x^2 \cdot y^2 \\
&- 2 \cdot \ln x + x \cdot y^{x-1}. \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(2 \cdot x \cdot y^3 - 2 \cdot \frac{y}{x} + y^x \cdot \ln y\right) \\
&= 2 \cdot y^3 + 2 \cdot \frac{y}{x^2} + y^x \cdot \ln^2 y. \\
\frac{\partial^2 f}{\partial y \, \partial x} &= \frac{\partial}{\partial y}\left(2x \cdot y^3 - 2 \cdot \frac{y}{x} + y^x \cdot \ln y\right) \\
&= 6 \cdot x \cdot y^2 - 2 \cdot \frac{1}{x} + x \cdot y^{x-1} \cdot \ln y + y^{x-1} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(3 \cdot x^2 y^2 - 2 \cdot \ln x + x \cdot y^{x-1}\right) \\
&= 6 \cdot x^2 \cdot y + x \cdot (x - 1) \cdot y^{x-2} \\
\frac{\partial^2 f}{\partial x \, \partial y} &= \frac{\partial}{\partial x}\left(3 \cdot x^2 \cdot y^2 - 2 \cdot \ln x + x \cdot y^{x-1}\right) \\
&= 6 \cdot x \cdot y^2 - 2 \cdot \frac{1}{x} + y^{x-1} + x \cdot y^{x-1} \cdot \ln y.
\end{aligned}$$

Remark 6.6.1 A real function of two variables has four second-order partial derivatives i.e.: $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y \, \partial x}$; $\frac{\partial^2 f}{\partial x \, \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$. Two of these, namely $\frac{\partial^2 f}{\partial x \, \partial y}$, $\frac{\partial^2 f}{\partial y \, \partial x}$ are called ***th mixed partial derivative of the second order*** and are generally not equal as shown in the following example:

Example 6.6.2 Consider the function:

$$f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = \begin{cases} x \cdot y \, \frac{x^2 - y^2}{x^2 + y^2} & \text{, if } (x, \ y) \neq (0, \ 0) \\ 0 & \text{, if } (x, \ y) = (0, \ 0) \end{cases}.$$

After easy-to-follow computations we obtain:

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} y \cdot \left(\frac{x^2 - y^2}{x^2 + y^2} + \frac{4 \cdot x^2 \cdot y^2}{(x^2 + y^2)^2} \right) & , \text{if } (x, y) \neq (0, 0) \\ 0 & , \text{if } (x, y) = (0, 0) \end{cases}.$$

and further:

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} x \cdot \left(\frac{x^2 - y^2}{x^2 + y^2} - \frac{4 \cdot x^2 \cdot y^2}{(x^2 + y^2)^2} \right) & , \text{if } (x, y) \neq (0, 0) \\ 0 & , \text{if } (x, y) = (0, 0) \end{cases}.$$

Then we have:

$$\frac{\partial f}{\partial x}(0, y) = -y; \quad \frac{\partial^2 f}{\partial y \, \partial x}(0, y) = -1; \quad \frac{\partial^2 f}{\partial y \, \partial x}(0, 0) = -1$$

$$\frac{\partial f}{\partial y}(x, 0) = x; \quad \frac{\partial^2 f}{\partial x \, \partial y}(x, 0) = 1; \quad \frac{\partial^2 f}{\partial x \, \partial y}(0, 0) = 1.$$

Therefore, $\frac{\partial^2 f}{\partial y \, \partial x}(0, 0) \neq \frac{\partial^2 f}{\partial x \, \partial y}(0, 0)$.

Definition 6.6.2 Let $A \subset \mathbb{R}^n$ be an open subset and let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a real valued function of n variables. We assume that there are $\frac{\partial f}{\partial x_i} : A \to \mathbb{R}$, $\forall\, i = \overline{1, n}$. Obvious, $\frac{\partial f}{\partial x_i}$ is his turn a real valued function of n variables for any $i = \overline{1, n}$. If there exists $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ then, this is called the partial derivative of second-order of the function f and is denoted by $\frac{\partial^2 f}{\partial x_j \, \partial x_i}$ or $f''_{x_j x_i}$. If $i = j$ we will note $\frac{\partial^2 f}{\partial x_i^2}$ or $f''_{x_i^2}$ instead of $\frac{\partial^2 f}{\partial x_i \, \partial x_i}$, and if $i \neq j$, the partial second-order derivatives are called mixed.

Remark 6.6.2 A real valued function of n variables has n^2 partial second-order derivatives.

A sufficient condition for mixed second-order derivatives to be equal is given by the following theorem:

Teorema 6.6.1 (Schwarz). *Let $A \subset \mathbb{R}^2$ be an open subset, $(a, b) \in A$ and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$. If there exist the mixed second-order derivatives $\frac{\partial^2 f}{\partial x \, \partial y}$ and $\frac{\partial^2 f}{\partial y \, \partial x}$ on an open neighborhood V of the point (a, b), $V \subset A$, and they are continuous at the point (a, b), then:*

$$\frac{\partial^2 f}{\partial x \, \partial y}(a, b) = \frac{\partial^2 f}{\partial y \, \partial x}(a, b).$$

Proof Since A is an open subset, there exists a $r > 0$ such that :

$$V = \left\{(x,\ y) \in \mathbb{R}^2;\ (x - a)^2 + (y - b)^2 < r^2\right\} \subset A.$$

Let $R(x,\ y) = f(x,\ y) - f(x,\ b) - f(a,\ y) + f(a,\ b),\ \forall\ (x,\ y) \in V$. For any fixed point $(x,\ y) \in V$ we denote by I (respectively J) the closed interval of ends a and x (respectively b and y). If we denote by:

$$g(t) = f(t,\ y) - f(t,\ b),\forall\ t \in I,$$

then $R(x,\ y) = g(x) - g(a)$. From Lagrange's theorem it results that there exists a point ξ between a and x such that:

$$R(x,\ y) = g'(\xi)(x - a) = \left(f'_x(\xi,\ y) - f'_x(\xi,\ b)\right)(x - a).$$

Applying again Lagrange's theorem to the function f'_x on J, it results that there exists a point η betweeen b and y such that:

$$R(x,\ y) = f'_{yx}(\xi,\ \eta)(x - a)(y - b). \tag{6.16}$$

Similarly, if we denote by $h(t) = f(x,\ t) - f(a,\ t),\ t \in J$, then:

$$R(x,\ y) = h(y) - h(b).$$

Applying Lagrange's theorem twice, it results that there exists η' between b and y and ξ' between a and x such that:

$$R(x,\ y) = f''_{xy}\left(\xi',\ \eta'\right)(x - a)(y - b). \tag{6.17}$$

Therefore, from (6.16) and (6.17) we have:

$$f''_{yx}(\xi,\ \eta) = f''_{xy}\left(\xi',\ \eta'\right). \tag{6.18}$$

Taking into account that f''_{yx} and f''_{xy} are continuous at the point $(a\ ,\ b)$ it follows:

$$f''_{yx}(a,b) = \lim_{\substack{x\to a\\ y\to b}} f''_{yx}(\xi,\eta) = \lim_{\substack{x\to a\\ y\to b}} f''_{xy}\left(\xi',\eta'\right) = f''_{xy}(a,b).$$

Corollary 6.6.1 *Let $A \subset \mathbb{R}^2$ be an open subset and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$. If $f''_{x\,y}$* and *$f''_{y\,x}$ are continuous on A, then $f''_{x\,y}(a) = f''_{y\,x}(a)$, $\forall a \in A$.*

Remark 6.6.3 Theorem 6.6.1 can be generalized to real functions of n variables as follows: if $f : A \subset \mathbb{R}^n \to \mathbb{R}$, where $A \subset \mathbb{R}^n$ is an open subset and the mixed second-order derivatives $\frac{\partial^2 f}{\partial\ x_i\ \partial\ x_j}$ and $\frac{\partial^2 f}{\partial\ x_j\ \partial\ x_i}$ there exist on an open neighborhood V of the point $a \in V$, $V \subset A$, and they are continuous at the point a, then:

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \, \partial x_i}(a).$$

The definition of partial derivatives of order greater than two is obvious. E.g, if there exists $\frac{\partial}{\partial x_k}\left(\frac{\partial^2 f}{\partial x_i \, \partial x_j}\right)$, then it is denoted by $\frac{\partial^3 f}{\partial x_k \, \partial x_i \, \partial x_j}$.

Definition 6.6.3 Any element $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ is called **multi-index**. We will denote by $|k| = k_1 + \ldots + k_n$ and by $D^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$, where $f : A \subset \mathbb{R}^n \to \mathbb{R}$, and $A \subset \mathbb{R}^n$ is an open subset.The function f is said to be of C^p- **class** on$C^p - A$, if there exist $D^k f$ and are continuous on A, for any multi-index k with $|k| \leq p$.We will use the notation $f \in C^p(A)$.

Example 6.6.3 If $k = (2, 1, 0, 3)$, then $|k| = 6$ and:

$$D^k f = \frac{\partial^6 f}{\partial x_1^2 \, \partial x_2 \, \partial x_4^3}.$$

The function $f \in C^5(A)$, where $A \subset \mathbb{R}^4$ is an open subset, if there exist $D^k f$ and are continuous on A, for any multi-index $k \in \mathbb{N}^4$ with $|k| \leq 5$.

For mixed derivatives of order greater than 2, the possibility of permuting the derivation order is proved by applying Theorem 6.6.1 several times.

Example 6.6.4 Let $A \subset \mathbb{R}^3$ be an open subset and $f \in C^4(A)$.Then we have:

$$\begin{aligned}
\frac{\partial^4 f}{\partial z \, \partial x \, \partial z \, \partial y} &= \frac{\partial^2}{\partial z \, \partial x}\left(\frac{\partial^2 f}{\partial z \, \partial y}\right) = \frac{\partial^2}{\partial x \, \partial z}\left(\frac{\partial^2 f}{\partial y \, \partial z}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial z \, \partial y}\left(\frac{\partial f}{\partial z}\right)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y \, \partial z}\left(\frac{\partial f}{\partial z}\right)\right) = \frac{\partial^4 f}{\partial x \, \partial y \, \partial z^2}.
\end{aligned}$$

Definition 6.6.4 Let $A \subset \mathbb{R}^n$ be an open subset, $a \in A$ and $f \in C^2(A)$. Is called the **second-order differential of function** f **at the point** a**,** and is denoted by $d^2 f(a)$ the following quadratic form on $\mathbb{R}^n$:

$$d^2 f(a)(h) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j}(a) \cdot h_i \cdot h_j, \; \forall \; h = (h_1, \ldots, h_n) \in \mathbb{R}^n.$$

Notation is also used:

$$d^2 f(a) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j}(a) \cdot d x_i \, d x_j.$$

The symmetrical matrix $H_f(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ is called **the Hessian matrix attached to the function f at the point a.**

Remark 6.6.4 Consider the particular $n = 2$.

Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of C^2- class on the open subset A and $(a, b) \in A$. The second-order differential of the function f at the point $(a, b) \in A$ is the following quadratic form on $\mathbb{R}^2$:

$$\begin{aligned} d^2 f(a,b)(h,k) = \frac{\partial^2 f}{\partial x^2}(a,b) \cdot h^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(a,b) \cdot h \cdot k \\ + \frac{\partial^2 f}{\partial y^2}(a,b) \cdot k^2, \\ \forall (h,k) \in \mathbb{R}^2. \end{aligned}$$

or

$$\begin{aligned} d^2 f(a,b) = \frac{\partial^2 f}{\partial x^2}(a,b) \cdot dx^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(a,b) \cdot dx \cdot dy \\ + \frac{\partial^2 f}{\partial y^2}(a,b) \cdot dy^2. \end{aligned}$$

Example 6.6.5 Let $f : \left\{(x, y) \in \mathbb{R}^2 ;\ x > 0,\ y > 0\right\} \subset \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x, y) = x^2 \cdot y^3 - 2 \cdot y \cdot \ln x + y^x$. Compute $d^2 f(1, 1)$ and $d^2 f(1, 1)(2, -1)$.

From Example 6.1.1 we deduce:

$$\frac{\partial^2 f}{\partial x^2}(1, 1) = 4; \quad \frac{\partial^2 f}{\partial y^2}(1, 1) = 6; \quad \frac{\partial^2 f}{\partial y \partial x}(1, 1) = \frac{\partial^2 f}{\partial y \partial x}(1, 1) = 5.$$

Therefore we have:

$d^2 f(1, 1) = 4 \cdot dx^2 + 10 \cdot dx \cdot dy + 6 \cdot dy^2$ and $d^2 f(1, 1)(2, -1) = 2$.

Remark 6.6.5 Let $A \subset \mathbb{R}^3$ be an open subset and let $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ be a function of three variables, $f \in C^2(A)$. Then the differential of second-order of f at the point $(a, b, c) \in A$ has the following form:

$$\begin{aligned} d^2 f(a,b,c) = \frac{\partial^2 f}{\partial x^2}(a,b,c) \cdot dx^2 + \frac{\partial^2 f}{\partial y^2}(a,b,c) \cdot dy^2 \\ + \frac{\partial^2 f}{\partial z^2}(a,b,c) \cdot dz^2 \\ + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(a,b,c) \cdot dx \cdot dy \end{aligned}$$

$$+2 \cdot \frac{\partial^2 f}{\partial x \partial z}(a, b, c) \cdot dx \cdot dz$$
$$+2 \cdot \frac{\partial^2 f}{\partial y \partial z}(a, b, c) \cdot dy \cdot dz.$$

Returning to the general case of real functions of n variables and of C^2- class, we find that the second order differential of funtion f at the point a is the following quadratic form on $\mathbb{R}^n$:

$$d^2 f(a)(h) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \cdot h_i \cdot h_j , \ \forall\, h = (h_1, \ldots, h_n) \in \mathbb{R}^n .$$

If we denote by $a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$, then $a_{ij} = a_{ji}$, $\forall\, i, \ j = \overline{1, n}$, and:

$$d^2 f(a)(h) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \cdot h_i \cdot h_j , \ \forall\, h = (h_1, \ldots, h_n) \in \mathbb{R}^n .$$

We recall the following definitions and results of linear algebra: the second - order differential $d^2 f(a)$ is called ***positive*** (***negative***) ***definite*** if $d^2 f(a)(h) > 0$ $\left(d^2 f(a)(h) < 0\right)$ for any $h \in \mathbb{R}^n$, $h \neq 0_{\mathbb{R}^n}$. If there exist $h \neq k$, such that $d^2 f(a)(h) \cdot d^2 f(a)(k) < 0$ then, the quadratic form $d^2 f(a)$ is called ***alternating.***

Further we will use the notations:

$$\Delta_1 = a_{11}; \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}; \ \ldots; \ \Delta_n = \det H_f(a).$$

Theorem 6.6.2 (Sylvester). *The necessary and sufficient condition that $d^2 f(a)$ to be positive (negative) definite is:*

$\Delta_i > 0$, $\forall\, i = \overline{1, n}$ (respectively $(-1)^i \Delta_i > 0$, $\forall\, i = \overline{1, n}$).

Next, we will introduce the following notation convention:

$\left(\frac{\partial f}{\partial x}(a, b)\right)^{(2)}$ instead of $\frac{\partial^2 f}{\partial x^2}(a, b)$.

$\frac{\partial f}{\partial x}(a, b) \cdot \frac{\partial f}{\partial y}(a, b)$ instead of $\frac{\partial^2 f}{\partial x \partial y}(a, b)$.

$\left(\frac{\partial f}{\partial y}(a, b)\right)^{(2)}$ instead of $\frac{\partial^2 f}{\partial y^2}(a, b)$.

With this convention, for functions of two variables we have:

$$d^2 f(a, b) = \frac{\partial^2 f}{\partial x^2}(a, b) \cdot dx^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(a, b) \cdot dx \cdot dy$$
$$+ \frac{\partial^2 f}{\partial y^2}(a, b) \cdot dy^2$$

$$= \left(\frac{\partial f}{\partial x}(a\,,\ b)\ \cdot\ d\,x\ +\ \frac{\partial f}{\partial y}(a\,,\ b)\ \cdot\ d\,y\right)^{(2)}\ .$$

To define the differential of the $m-$ order, we will naturally extend the previous convention, namely, we will note with:

$\left(\frac{\partial f}{\partial x}(a\,,\ b)\right)^{(m)}$ instead of $\frac{\partial^m f}{\partial x^m}(a\,,\ b)$

$\frac{\partial^{m-k} f}{\partial x^{m-k}}(a\,,\ b)\ \cdot\ \frac{\partial^k f}{\partial y^k}(a\,,\ b)$ instead of $\frac{\partial^m f}{\partial x^{m-k}\,\partial y^k}\ (a\,,\ b)$ and so on.

Definition 6.6.5 Let $A \subset \mathbb{R}^2$ be an open subset, $f\,:\ A \subset \mathbb{R}^2 \to \mathbb{R}$ and $(a\,,\ b) \in A$. If $f\ \in\ C^m(A)$, then **the** $mth-$ **order differential of the function** f **at the point** $(a\,,\ b)$ is defined as follows:

$$d^m f(a\,,\ b)\ =\ \left(\frac{\partial f}{\partial x}(a\,,\ b)\ \cdot\ d\,x\ +\ \frac{\partial f}{\partial y}(a\,,\ b)\cdot\ d\,y\right)^{(m)}$$

$$=\ \sum_{k\,=\,0}^{m} C_m^k\ \cdot\ \frac{\partial^m f}{\partial\,x^{m-k}\,\partial\,y^k}(a\,,\ b)\cdot\ d\,x^{m-k}\cdot d\,y^k$$

$$=\ C_m^0\ \cdot \frac{\partial^m f}{\partial\,x^m}(a\,,\ b)\cdot d\,x^m + C_m^1 \cdot \frac{\partial^m f}{\partial\,x^{m-1}\,\partial\,y}(a\,,\ b)\cdot d\,x^{m-1}\cdot d\,y + \cdots +$$

$$+ C_m^m \cdot \frac{\partial^m f}{\partial\,y^m}(a\,,\ b)\cdot d\,y^m.$$

The following formula that generalizes Newton's binomial can be proved by mathematical induction:

$(a_1\ +\ \ldots\ +\ a_n)^m\ =\ \sum\limits_{\substack{k_1\,+\ldots+\,k_n\,=\,m\\ k_i\in\mathbb{N}}} \frac{m!}{k_1!\ \ldots\ k_n!}\ a_1^{k_1}\ldots\ a_n^{k_n}.$

Using this formula and the previous conventions, we can define the $mth-$ order differential of the function f at the point a as follows:

Let $A \subset \mathbb{R}^n$ *be an open subset,* $f\,:\ A \subset \mathbb{R}^n \to \mathbb{R}$ *and* $a = (a_1\,,\ \ldots\,,\ a_n) \in A.$

If $f\ \in\ C^m(A)$*, then* ***the*** $mth-$ ***order differential of the function*** f ***at the point*** $a = (a_1\,,\ \ldots\,,\ a_n) \in A$ is***:***

$$d^m f(a)\ =\ \left(\frac{\partial f}{\partial\,x_1}(a)\ \cdot\ d\,x_1\ +\ \ldots\ +\ \frac{\partial f}{\partial\,x_n}(a)\ \cdot\ d\,x_n\right)^{(m)}$$

$$=\ \sum_{\substack{k_1\,+\ldots+\,k_n\,=\,m\\ k_i\in\mathbb{N}}} \frac{m\ !}{k_1!\ \ldots\ k_n!}\ \cdot\ \frac{\partial^m f}{\partial\,x_1^{k_1}\ \ldots\ \partial\,x_n^{k_n}}(a)\cdot\ d\,x_1^{k_1}\ \cdot\ \ldots\ \cdot\ d\,x_n^{k_n}\ .$$

6.7 Second-Order Partial Derivatives of Functions Composed of Two Variables

Theorem 6.7.1 *Let $A\,,\ \ B\subset\mathbb{R}^2$ be two open subsets,*

$$F=(u,v):A\subset\mathbb{R}^2\to B\subset\mathbb{R}^2,\ \ f:B\subset\mathbb{R}^2\to\mathbb{R}\text{ and}$$
$$h=f\circ F:A\subset\mathbb{R}^2\to\mathbb{R},$$
$$h(x,y)=f(u(x,y),v(x,y)),x,y\in A.$$

If $F\in C^2(A)$ and $f\in C^2(B)$, then $h\in C^2(A)$ and we have the following formulas:

$$\frac{\partial^2 h}{\partial x^2}=\frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2+2\cdot\frac{\partial^2 f}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}$$
$$+\frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}.$$
$$\frac{\partial^2 h}{\partial x\partial y}=\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$
$$+\frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}$$
$$+\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial x\partial y}.$$

$$\frac{\partial^2 h}{\partial y^2}=\frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2$$
$$+2\cdot\frac{\partial^2 f}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial y}\right)^2$$
$$+\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}.$$

Proof According to (6.10) we have:

$$\begin{aligned}\frac{\partial h}{\partial x}&=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}\\ \frac{\partial h}{\partial y}&=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\,.\end{aligned}\tag{6.19}$$

We will examine the existence of the derivative $\frac{\partial^2 h}{\partial x^2}$ and its computation method. From Theorem 6.3.1 it results that the function $(x,\ y)\ \to \frac{\partial f}{\partial u}(u(x,\ y)\,,\ v(x,\ y))$ has the first order partial derivaties on A(with respect to x respectively to y), namely:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial v}{\partial x}$$
$$= \frac{\partial^2 f}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v\, \partial u} \cdot \frac{\partial v}{\partial x} \tag{6.20}$$

Similarly we have:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial^2 f}{\partial u\, \partial v} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2} \cdot \frac{\partial v}{\partial x}. \tag{6.21}$$

From (6.19), (6.20) and (6.21) it results:

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$
$$= \frac{\partial^2 f}{\partial u^2} \cdot \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 f}{\partial v\, \partial u} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} +$$
$$+ \frac{\partial^2 f}{\partial u\, \partial v} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 f}{\partial v^2} \cdot \left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial f}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}.$$

Because in our conditions $\frac{\partial^2 f}{\partial u\, \partial v} = \frac{\partial^2 f}{\partial v\, \partial u}$, finally we get:

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} \cdot \left(\frac{\partial u}{\partial x}\right)^2 + 2 \cdot \frac{\partial^2 f}{\partial u\, \partial v} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 f}{\partial v^2} \cdot \left(\frac{\partial v}{\partial x}\right)^2$$
$$+ \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial f}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}.$$

Similarly, the other two formulas in the theorem statement are proved.

Example 6.7.1 Let $f \in C^2(\mathbb{R}^2)$, $h(x, y) = f(2 \cdot x + y^2, 1 - y \cdot \cos x)$. $\forall (x, y) \in \mathbb{R}^2$. Compute $\frac{\partial^2 h}{\partial x^2}(x, y)$, $\frac{\partial^2 h}{\partial x\, \partial y}(x, y)$ and $\frac{\partial^2 h}{\partial y^2}(x, y)$.

If we denote by $u(x, y) = 2 \cdot x + y^2$, $v(x, y) = 1 - y \cdot \cos x$, then we have:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} \cdot 2 + \frac{\partial f}{\partial v} \cdot y \cdot \sin x.$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} \cdot 2 \cdot y + \frac{\partial f}{\partial v} \cdot (-\cos x).$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} \cdot 4 + 4 \cdot \frac{\partial^2 f}{\partial u\, \partial v} \cdot y \cdot \sin x + \frac{\partial^2 f}{\partial v^2} \cdot y^2 \cdot \sin^2 x$$

$$+\frac{\partial f}{\partial v} \cdot y \cdot \cos x.$$

$$\frac{\partial^2 h}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial u^2} \cdot 4 \cdot y + \frac{\partial^2 f}{\partial u\,\partial v} \cdot \left(-2 \cdot \cos x + 2\,y^2 \cdot \sin x\right)$$
$$+\frac{\partial^2 f}{\partial v^2} \cdot (-\,y \cdot \sin x \cdot \cos x) + \frac{\partial f}{\partial v} \cdot \sin x.$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} \cdot 4 \cdot y^2 + 4 \cdot \frac{\partial^2 f}{\partial u\,\partial v} \cdot (-\,y \cdot \cos x)$$
$$+\frac{\partial^2 f}{\partial v^2} \cdot \cos^2 x + \frac{\partial f}{\partial u} \cdot 2\,.$$

Remark 6.7.1 If we will use the following formal notations:

$\left(\frac{\partial\, f}{\partial\, u}\right)^{(2)}$ instead of $\frac{\partial^2\, f}{\partial\, u^2}$.

$\frac{\partial\, f}{\partial\, u} \cdot \frac{\partial\, f}{\partial\, v}$ instead of $\frac{\partial^2 f}{\partial\, u\,\partial\, v}$.

$\left(\frac{\partial\, f}{\partial\, v}\right)^{(2)}$ instead of $\frac{\partial^2\, f}{\partial\, v^2}$.

then the expressions of the second order partial derivatives of the composed functions are more easily retained in the form:

$$\frac{\partial^2\, h}{\partial\, x^2} = \left(\frac{\partial\, h}{\partial\, x}\right)^{(2)} + \frac{\partial\, f}{\partial\, u} \cdot \frac{\partial^2\, u}{\partial\, x^2} + \frac{\partial\, f}{\partial\, v} \cdot \frac{\partial^2\, v}{\partial\, x^2},$$
$$\frac{\partial^2 h}{\partial\, x\,\partial\, y} = \left(\frac{\partial\, h}{\partial\, x}\right) \cdot \left(\frac{\partial\, h}{\partial\, y}\right) + \frac{\partial\, f}{\partial\, u} \cdot \frac{\partial^2 u}{\partial\, x\,\partial\, y} + \frac{\partial\, f}{\partial\, v} \cdot \frac{\partial^2 v}{\partial\, x\,\partial\, y},$$
$$\frac{\partial^2\, h}{\partial\, y^2} = \left(\frac{\partial\, h}{\partial\, y}\right)^{(2)} + \frac{\partial\, f}{\partial\, u} \cdot \frac{\partial^2\, u}{\partial\, y^2} + \frac{\partial\, f}{\partial\, v} \cdot \frac{\partial^2\, v}{\partial\, y^2}.$$

where by $\left(\frac{\partial h}{\partial x}\right)^{(2)}$ we understand:

$$\left(\frac{\partial\, h}{\partial\, x}\right)^{(2)} = \left(\frac{\partial\, f}{\partial\, u} \cdot \frac{\partial\, u}{\partial\, x} + \frac{\partial\, f}{\partial\, v} \cdot \frac{\partial\, v}{\partial\, x}\right)^{(2)}$$
$$= \frac{\partial^2\, f}{\partial\, u^2} \cdot \left(\frac{\partial\, u}{\partial\, x}\right)^2 + 2\,\frac{\partial^2 f}{\partial\, u\,\partial\, v} \cdot \frac{\partial\, u}{\partial\, x} \cdot \frac{\partial\, v}{\partial\, x} + \frac{\partial^2\, f}{\partial\, v^2} \cdot \left(\frac{\partial\, v}{\partial\, x}\right)^2, \text{ and so on.}$$

In the case of Example 6.7.1 we have:

$$\left(\frac{\partial\, h}{\partial\, x}\right)^{(2)} = \left(\frac{\partial\, f}{\partial\, u} \cdot 2 + \frac{\partial\, f}{\partial\, v} \cdot y\,\sin x\right)^{(2)} =$$
$$= \frac{\partial^2\, f}{\partial\, u^2} \cdot 4 + 4\,\frac{\partial^2 f}{\partial\, u\,\partial\, v} \cdot y \cdot \sin x + \frac{\partial^2\, f}{\partial\, v^2} \cdot y^2 \cdot \sin^2 x \text{ , and so on.}$$

6.8 Change of Variables

Let I and J two open intervals of $\mathbb{R}$ and $u : I \to J$ with the properties:

(i) $u \in C^k(I)\,,\; k > 1$;
(ii) $u'(x) \neq 0\,,\; \forall\, x \in I$;
(iii) u is bijection.

If $y : I \to \mathbb{R}$ is such that $y \in C^2(I)$, then the composed function:

$$\overline{y} = y \circ u^{-1} : J \to \mathbb{R}$$

is also a function of C^2- class on J.

As $y(x) = \overline{y}\,(u(x))\,,\; \forall x \in I$, it results:

$$y'(x) = \overline{y}\,'(u(x)) \cdot u'(x). \tag{6.22}$$

If we denote by $t = u(x) \in J$ then, the formula (6.22) becomes:

$$\frac{d\,y}{d\,x} = \frac{d\,\overline{y}}{d\,t} \cdot \frac{d\,t}{d\,x}. \tag{6.23}$$

Formula (6.23) shows the connection between the derivation operator with respect to x and the derivation operator with respect to t, namely:

$$\frac{d}{d\,x} = \frac{d\,t}{d\,x} \cdot \frac{d}{d\,t} = u'(x) \cdot \frac{d}{d\,t}. \tag{6.24}$$

Further we have:

$$\begin{aligned} y''(x) &= \frac{d}{d\,x}\big(y'(x)\big) = \frac{d}{d\,x}\left(\frac{d\,\overline{y}}{d\,t} \cdot \frac{d\,t}{d\,x}\right) = \frac{d}{d\,x}\left(\frac{d\,\overline{y}}{d\,t}\right) \cdot \frac{d\,t}{d\,x} + \frac{d\,\overline{y}}{d\,t} \cdot \frac{d^2\,t}{d\,x^2} \\ &= \frac{d}{d\,t}\left(\frac{d\,\overline{y}}{d\,t}\right) \cdot \frac{d\,t}{d\,x} \cdot \frac{d\,t}{d\,x} + \frac{d\,\overline{y}}{d\,t} \cdot \frac{d^2\,t}{d\,x^2} = \frac{d^2\,\overline{y}}{d\,t^2} \cdot \left(\frac{d\,t}{d\,x}\right)^2 + \frac{d\,\overline{y}}{d\,t} \cdot \frac{d^2\,t}{d\,x^2}. \end{aligned} \tag{6.25}$$

Similarly,if we assume that u and y are functions of C^3- class, we get the formula:

$$y'''(x) = \frac{d^3\,\overline{y}}{d\,t^3} \cdot \left(\frac{d\,t}{d\,x}\right)^3 + 3\,\frac{d^2\,\overline{y}}{d\,t^2} \cdot \frac{d\,t}{d\,x} \cdot \frac{d^2\,t}{d\,x^2} + \frac{d\,\overline{y}}{d\,t} \cdot \frac{d^3\,t}{d\,x^3}. \tag{6.26}$$

Following the change of the independent variable $x \to t : I \to J$, any form expression $E = E\big(x\,,\; y\,,\; y'\,,\; y''\,,\; \ldots\big)$ becomes $\overline{E} = \overline{E}\left(t\,,\; \overline{y}\,,\; \frac{d\,\overline{y}}{d\,t}\,,\; \frac{d^2\,\overline{y}}{d\,t^2}\,,\; \ldots\right)$.

Example 6.8.1 What becomes the following differential equation $\left(1 - x^2\right) \cdot y'' - x \cdot y' + y = 0$, if we make the change of variable:

$$x = \cos t \,,\ t \in \left(0\,,\ \frac{\pi}{2}\right)?$$

By inversing the function, we get $t = \cos^{-1} x \,,\ x \in (0\,,\ 1)$ and further:

$$\begin{aligned}\frac{d\,t}{d\,x} &= -\frac{1}{\sqrt{1 - x^2}} = -\frac{1}{\sin t}\\ \frac{d^2\,t}{d\,x^2} &= \frac{d}{d\,x}\left(\frac{d\,t}{d\,x}\right) = \frac{d}{d\,t}\left(-\frac{1}{\sin t}\right) \cdot \frac{d\,t}{d\,x}\\ &= -\frac{1}{\sin t} \cdot \frac{d}{d\,t}\left(-\frac{1}{\sin t}\right) = -\frac{\cos t}{\sin^3 t}\,.\end{aligned}$$

From (6.23) and (6.25) we deduce:

$$y' = -\frac{1}{\sin t} \cdot \frac{d\,y}{d\,t} \text{ and } y'' = \frac{1}{\sin^2 t} \cdot \frac{d^2\,y}{d\,t^2} - \frac{\cos t}{\sin^3 t} \cdot \frac{d\,y}{d\,t}\,.$$

Substituting in the initial differential equation we obtain:

$$\frac{d^2\,y}{d\,t^2} - \frac{\cos t}{\sin t} \cdot \frac{d\,y}{d\,t} + \frac{\cos t}{\sin t} \cdot \frac{d\,y}{d\,t} + y = 0,$$

or:

$$\frac{d^2\,y}{d\,t^2} + y = 0.$$

Therefore, following the change of variable $x = \cos t\,,\ t \in \left(0\,,\ \frac{\pi}{2}\right)$, the differential equation $\left(1 - x^2\right) y'' - x\,y' + y = 0$, becomes $\frac{d^2\,y}{d\,t^2} + y = 0$.

In formulas (6.23) and (6.25) we noted the new function also with y instead of $\overline{y}$, because in an equation it does not matter how the unknown function is noted.

Let $x \to y(x) : I \to J$ be a bijective function, $y \in C^k(I), k > 1$. Further we will show how the derivatives of the inverse function $y \to x(y) : J \to I$ are computed. According to inverse function therem (Teorema 6.11.1 below), it follows:

$$x'(y) = \frac{d\,x}{d\,y} = \frac{1}{y'(x)}$$

from which we deduce that:

$$y'(x) = \frac{1}{x'(y)}. \tag{6.27}$$

Next we have:

$$y''(x) = \frac{d}{dx}\left(y'(x)\right) = \frac{\frac{d}{dy}\left(\frac{1}{x'(y)}\right)}{\frac{dx}{dy}}$$

$$= \frac{-\frac{x''(y)}{(x'(y))^2}}{x'(y)} = -\frac{x''(y)}{(x'(y))^3}.$$

Example. 6.8.2 What becomes the differential equation $y \cdot \left(y'\right)^3 + y'' = 0$ following the inversion of the variables $y(x) \leftrightarrow x(y)$?

Taking into account (6.27) and (6.28) the differential equation becomes:

$$y \cdot \frac{1}{(x'(y))^3} - \frac{x''(y)}{(x'(y))^3} = 0$$

or, after simplification:

$x''(y) - y = 0.$

Further we consider the case of the functions of two variables.

Let $\Omega\,,\ D \subset \mathbb{R}^2$ be two open and connected subsets. The function:

$$F = (u\,,\ v) : \ \Omega \to D,$$

$(F(x,\ y) = (u(x,\ y)\,,\ v(x,\ y))\,, \forall\,(x,\ y) \in \Omega)$, is said to be ***a change of variables*** if has the properties:

(i) $F \in C^1(\Omega)$;
(ii) F is bijection;
(iii) The Jacobian of this transformation i.e.:

$$\frac{D(u,\ v)}{D(x\,,\ y)}(x\,,\ y) \neq 0\,,\ \ \forall\ (x\,,\ y) \in \Omega.$$

Written on components the change of variables is given in the form:

$$\begin{cases} u = u(x\,,\ y) \\ v = v(x\,,\ y) \end{cases},\ \ \forall\,(x\,,\ y) \in \Omega.$$

Let $z\ :\ \Omega \ \rightarrow\ \mathbb{R}$ be such that $f \in C^2(\Omega)$ and let be the composed function: $\overline{z} = z \circ F^{-1} : D \to \mathbb{R}$.

Further, if we assume that $F \in C^2(\Omega)$, it results that $\overline{z} \in C^2(D)$ and $z = \overline{z} \circ F : \Omega \to \mathbb{R}$.

As $z(x,\ y) = \overline{z}\,(u(x,\ y)\,,\ v(x,\ y))\,, \forall\,(x,\ y) \in \Omega$, and taking into account the derivation rules (6.10) it results:

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial \overline{z}}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \overline{z}}{\partial v} \cdot \frac{\partial v}{\partial x}. \\ \frac{\partial z}{\partial y} &= \frac{\partial \overline{z}}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \overline{z}}{\partial v} \cdot \frac{\partial v}{\partial y}. \end{aligned} \tag{6.28}$$

And:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial^2 \overline{z}}{\partial u^2} \cdot \left(\frac{\partial u}{\partial x}\right)^2 + 2\, \frac{\partial^2 \overline{z}}{\partial u\, \partial v} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \\ &+ \frac{\partial^2 \overline{z}}{\partial v^2} \cdot \left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial \overline{z}}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial \overline{z}}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2} \\ \frac{\partial^2 z}{\partial x\, \partial y} &= \frac{\partial^2 \overline{z}}{\partial u^2} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \\ &+ \frac{\partial^2 \overline{z}}{\partial u\, \partial v} \cdot \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}\right) + \frac{\partial^2 \overline{z}}{\partial v^2} \cdot \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y} + \\ &+ \frac{\partial \overline{z}}{\partial u} \cdot \frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial \overline{z}}{\partial v} \cdot \frac{\partial^2 v}{\partial x\, \partial y} \end{aligned} \tag{6.29}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 \overline{z}}{\partial u^2} \cdot \left(\frac{\partial u}{\partial y}\right)^2 + 2\, \frac{\partial^2 \overline{z}}{\partial u\, \partial v} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 \overline{z}}{\partial v^2} \cdot \left(\frac{\partial v}{\partial y}\right)^2 \\ &+ \frac{\partial \overline{z}}{\partial u} \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial \overline{z}}{\partial v} \cdot \frac{\partial^2 v}{\partial y^2} \end{aligned}$$

Let us consider the expression:

$$E = E\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\, \partial y}, \frac{\partial^2 z}{\partial y^2}\right)(x, y) \in \Omega.$$

Following the change of variables F, the expression E becomes:

$$\overline{E} = \overline{E}\left(u, v, \overline{z}, \frac{\partial \overline{z}}{\partial u}, \frac{\partial \overline{z}}{\partial v}, \frac{\partial^2 \overline{z}}{\partial u^2}, \frac{\partial^2 \overline{z}}{\partial u\, \partial v}, \frac{\partial^2 \overline{z}}{\partial v^2}\right), \text{ where } (u, v) \in D$$

Example 6.8.3

1. What becomes the equation:

$x^2 \frac{\partial^2 z}{\partial x^2} - 2\,x\,y\,\frac{\partial^2 z}{\partial x\, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + 2\,x\,\frac{\partial z}{\partial x} = 0$ following the change of variables:

$$\begin{cases} u = x \cdot y \\ v = y \end{cases}, \quad y \neq 0\,?$$

Taking into account the formulas (6.29), (6.30) we get successively:

$$\frac{\partial z}{\partial x} = \frac{\partial \overline{z}}{\partial u} \cdot y$$

$$\frac{\partial z}{\partial y} = \frac{\partial \overline{z}}{\partial u} \cdot x + \frac{\partial \overline{z}}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 \overline{z}}{\partial u^2} \cdot y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 \overline{z}}{\partial u^2} \cdot x y + \frac{\partial^2 \overline{z}}{\partial u \partial v} \cdot y + \frac{\partial \overline{z}}{\partial u}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 \overline{z}}{\partial u^2} \cdot x^2 + 2 \frac{\partial^2 \overline{z}}{\partial u \partial v} \cdot x + \frac{\partial^2 \overline{z}}{\partial v^2}.$$

Substituting these partial derivatives into the given equation is found:

$$v^2 \cdot \frac{\partial^2 \overline{z}}{\partial v^2} = 0 \text{ or } \frac{\partial^2 \overline{z}}{\partial v^2} = 0.$$

2. Find the expression of the Laplacian operator in polar coordinates. In other words, what becomes the expression $\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ after the change of variables:

$$\begin{cases} x = \rho \cdot \cos\theta \\ y = \rho \cdot \sin\theta \end{cases}, \quad \rho > 0 \tag{6.30}$$

To begin with we inverse the transformation (6.31) to obtain the new coordinates (ρ, θ) according to the old coordinates (x, y), as the theory is presented. We immediately notice that:

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \frac{y}{x} \end{cases}, \quad x \cdot y \neq 0. \tag{6.31}$$

We have successively:

$$\frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{y^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}, \quad \frac{\partial^2 \rho}{\partial y^2} = \frac{x^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}.$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}.$$

$$\frac{\partial^2\theta}{\partial\, x^2} = \frac{2\,x\,y}{\left(x^2 + y^2\right)^2}, \frac{\partial^2\theta}{\partial\, y^2} = \frac{-\,2\,x\,y}{\left(x^2 + y^2\right)^2}.$$

$$\frac{\partial\, z}{\partial\, x} = \frac{\partial\, \overline{z}}{\partial\, \rho} \cdot \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial\, \overline{z}}{\partial\, \theta} \cdot \frac{-\,y}{x^2 + y^2}.$$

$$\frac{\partial\, z}{\partial\, y} = \frac{\partial\, \overline{z}}{\partial\, \rho} \cdot \frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial\, \overline{z}}{\partial\, \theta} \cdot \frac{x}{x^2 + y^2}.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2\overline{z}}{\partial\rho^2} \cdot \frac{x^2}{x^2 + y^2} + 2 \cdot \frac{\partial^2\overline{z}}{\partial\rho\,\partial\theta} \cdot \frac{-\,x \cdot y}{\left(x^2 + y^2\right)^{\frac{3}{2}}}$$

$$+ \frac{\partial^2\overline{z}}{\partial\theta^2} \cdot \frac{y^2}{\left(x^2 + y^2\right)^2} + \frac{\partial\overline{z}}{\partial\rho} \cdot \frac{y^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}} + \frac{\partial\overline{z}}{\partial\theta} \cdot \frac{2\,x \cdot y}{\left(x^2 + y^2\right)^2}.$$

$$\frac{\partial^2\, z}{\partial\, y^2} = \frac{\partial^2\, \overline{z}}{\partial\, \rho^2} \cdot \frac{y^2}{x^2 + y^2} + 2 \cdot \frac{\partial^2\overline{z}}{\partial\, \rho\, \partial\, \theta} \cdot \frac{x \cdot y}{\left(x^2 + y^2\right)^{\frac{3}{2}}} + \frac{\partial^2\, \overline{z}}{\partial\, \theta^2} \cdot \frac{x^2}{\left(x^2 + y^2\right)^2}$$

$$+ \frac{\partial\, \overline{z}}{\partial\, \rho} \cdot \frac{x^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}} + \frac{\partial\, \overline{z}}{\partial\, \theta} \cdot \frac{-\,2 \cdot x \cdot y}{\left(x^2 + y^2\right)^2}.$$

$$\Delta\, z = \frac{\partial^2\, \overline{z}}{\partial\, \rho^2} + \frac{\partial^2\overline{z}}{\partial\, \theta^2} \cdot \frac{1}{x^2 + y^2} + \frac{\partial\, \overline{z}}{\partial\, \rho} \cdot \frac{1}{\sqrt{x^2 + y^2}}.$$

Finally, taking into account (6.32) we obtain the expression of the Laplacian in polar coordinates:

$$\Delta\, z = \frac{\partial^2\, \overline{z}}{\partial\, \rho^2} + \frac{\partial^2\overline{z}}{\partial\, \theta^2} \cdot \frac{1}{\rho^2} + \frac{\partial\, \overline{z}}{\partial\, \rho} \cdot \frac{1}{\rho}.$$

6.9 Taylor's Formula for Functions of Several Variables

For$n$ any $a\,,\ b \in \mathbb{R}^n$, we denote by $[a\,,\ b] = \{(1 - t) \cdot a + t \cdot b;\ t \in [0,\ 1]\}$.

The subset $[a\,,\ b] \subset \mathbb{R}^n$ is called ***the closed segment of heads** a **and** b.*

The open segment of heads** a **and** b **is denoted by.

$(a\,,\ b) = \{(1 - t) \cdot a + t \cdot b;\ t \in (0,\ 1)\}$.

Definition 6.9.1 *A subset $A \subset \mathbb{R}^n$ is called* **convex** *if for any $a\,,\ b \in A$ it results that $[a\,,\ b] \subset A$.*

Remark 6.9.1 Any open (closed) n-dimentional ball in $\mathbb{R}^n$ is a convex subset.

Indeed, if $x\,,\ y \in B(a,\ r)$,then $\|x - a\| < r$ and $\|y - a\| < r$.

For any $t \in [0,\ 1]$ we have:

$$\|(1-t)\cdot x + t\cdot y - a\| = \|(1-t)\cdot(x-a) + t\cdot(y-a)\| \leq$$
$$\leq (1-t)\cdot \|x-a\| + t\cdot\|y-a\| < (1-t)\cdot r + t\cdot r = r,$$

hence $[x, y] \subset B(a, r)$.

From Remark 6.9.1, we deduce that any interval in $\mathbb{R}$, any disk (square)in $\mathbb{R}^2$, any sphere (cube) in $\mathbb{R}^3$ are convex subsets.

Theorem 6.9.1 (Taylor's formula for functions of n variables). *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$,$a \in A$ be an interior point and let $r > 0$ be such that.*

$V = B(a, r) \subset A$.

If $f \in C^{m+1}(V)$ then, for any $x \in V$ there exists $\xi \in (a, x)$ such that:

$$f(a+h) = f(a) + \frac{1}{1!}\,df(a)(h) + \frac{1}{2!}\,d^2 f(a)(h) + \cdots +$$
$$+ \frac{1}{m!}\,d^m f(a)(h) + \frac{1}{(m+1)!}\cdot d^{m+1} f(\xi)(h),$$

where $h = x - a$.

Proof To simplify the writing, we prove the theorem in the particular case $n = 2$..Let $a = (a_1, a_2) \in A$ be an interior point, $x = (x_1, x_2) \in V$ a fix point and $h = (h_1, h_2) == (x_1 - a_1, x_2 - a_2) = x - a$. For any $t \in [0, 1]$ we denote by:

$$u_1(t) = a_1 + t\cdot h_1 = a_1 + t\cdot(x_1 - a_1)$$
$$u_2(t) = a_2 + t\cdot h_2 = a_2 + t\cdot(x_2 - a_2)$$

and we consider the composed function:

$g : [0, 1] \to \mathbb{R}, \; g(t) = f(u_1(t), u_2(t))$.

Obvious, g is a function of $C^{m+1}-$ class on $[0, 1]$, and according to Maclaurin's formula for function of one variable with Lagrange's remainder (Cap. 4, Sect. 4.4), there exists $0 < \theta < 1$ such that:

$$g(1) = g(0) + \frac{g'(0)}{1!} + \frac{g''(0)}{2!} + ... + \frac{g^{(m)}(0)}{m!} + \\ + \frac{g^{(m+1)}(\theta)}{(m+1)!}. \tag{6.32}$$

We remark that:

$$g(1) = f(x) = f(a+h) \text{ and } g(0) = f(a). \tag{6.33}$$

On the other hand, we have:

$$g'(t) = \frac{\partial f}{\partial x_1}(u_1(t),\ u_2(t)) \cdot u_1'(t) + \frac{\partial f}{\partial x_2}(u_1(t),\ u_2(t)) \cdot u_2'(t)$$

or:

$$g'(t) = \frac{\partial f}{\partial x_1}(u_1(t),\ u_2(t)) \cdot h_1 + \frac{\partial f}{\partial x_2}(u_1(t),\ u_2(t)) \cdot h_2,$$

hence:

$$g'(0) = \frac{\partial f}{\partial x_1}(a) \cdot h_1 + \frac{\partial f}{\partial x_2}(a) \cdot h_2 = d f(a)(h) \tag{6.34}$$

Also,from Theorem 6.7.1, it results:

$$\begin{aligned} g''(t) = &\frac{\partial^2 f}{\partial x_1^2}(u_1(t),\ u_2(t)) \cdot \left(u_1'(t)\right)^2 \\ &+ 2\,\frac{\partial^2 f}{\partial x_1\,\partial x_2}(u_1(t),\ u_2(t)) \cdot u_1'(t) \cdot u_2'(t) \\ &+ \frac{\partial^2 f}{\partial x_2^2}(u_1(t),\ u_2(t)) \cdot \left(u_2'(t)\right)^2 + \frac{\partial f}{\partial x_1}(u_1(t),\ u_2(t)) \cdot u_1''(t) \\ &+ \frac{\partial f}{\partial x_2}(u_1(t),\ u_2(t)) \cdot u_2''(t)\,. \end{aligned}$$

Since $u_1''(t) = u_2''(t) = 0$, it follows:

$$\begin{aligned} g''(0) &= \frac{\partial^2 f}{\partial x_1^2}(a) \cdot h_1^2 + 2 \cdot \frac{\partial^2 f}{\partial x_1\,\partial x_2}(a) \cdot h_1.\, h_2 + \frac{\partial^2 f}{\partial x_2^2}(a) \cdot h_2^2 \\ &= d^2 f(a)(h). \end{aligned} \tag{6.35}$$

It can be shown that:

$$g^{(k)}(0) = d^k f(a)(h)\,,\ \forall k \in \mathbb{N}^* \tag{6.36}$$

If we denote by $\xi = a + h \cdot \theta$ and taking into account (6.34)–(6.37) in (6.33) we get:

$$\begin{aligned} f(x) = f(a) + \frac{1}{1!} df(a)(h) + \frac{1}{2!} d^2 f(a)(h) + \cdots + \\ + \frac{1}{m!} d^m f(a)(h) + + \frac{1}{(m+1)!} d^{m+1} f(\xi)(h). \end{aligned}$$

where $\xi \in (a\ ,\ x)$, that is, we obtained the formula from the statement of the theorem.

Remark 6.9.2 Let us consider the particular case $n = 2$.

Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, $(a\,,\ b) \in A$ an interior point and let V be a neighborhood of the point $(a\,,\ b)$. If $f \in C^{m\,+\,1}(V)$, then for any $(x\,,\ y) \in V$, there exists a point $(\xi\,,\ \eta) \in V$, with $\xi \in (a\,,\ x)$ and $\eta \in (b\,,\ y)$, such that:

$$f(x\,,\ y) = f(a\,,\ b) + \frac{1}{1!}\left[\frac{\partial f}{\partial x}(a\,,\ b)\cdot(x - a) + \frac{\partial f}{\partial y}(a\,,\ b)\cdot(y - b)\right]$$
$$+\cdots+ \frac{1}{m\,!}\left[\frac{\partial f}{\partial x}(a\,,\ b)\cdot(x - a) + \frac{\partial f}{\partial y}(a\,,\ b)\cdot(y - b)\right]^{(m)}$$
$$+ \frac{1}{(m + 1)!}\left[\frac{\partial f}{\partial x}(\xi\,,\ \eta)\cdot(x - a) + \frac{\partial f}{\partial y}(\xi\,,\ \eta)\cdot(y - b)\right]^{(m+1)}.$$

Remark 6.9.3 In the hypotheses of Theorem 6.9.1 and Remark 6.9.2, if $a = 0$, then the Taylor's formula is called the **Maclaurin's formula**.

Example 6.9.1 Write the third degree Maclaurin's formula for the function:

$$f : \mathbb{R}^2 \to \mathbb{R}\,,\ f(x\,,\ y) = e^y \cdot \cos x$$

We have successive:

$$f'_x = -\,e^y\cdot\sin x\,,\ \ f'_x(0\,,\ 0) = 0\,,\ \ f'_y = e^y\cdot\cos x\,,\ \ f'_y(0\,,\ 0) = 1$$
$$f''_{x^2} = -\,e^y\cdot\cos x\,,\ \ f''_{x^2}(0\,,\ 0) = -1\,,\ \ f''_{y^2} = e^y\cdot\cos x\,,\ \ f''_{y^2}(0\,,\ 0) = 1$$
$$f''_{x\,y} = -\,e^y\cdot\sin x\,,\ \ f''_{x\,y}(0\,,\ 0) = 0\,,\ \ f(0\,,\ 0) = 1$$
$$f'''_{x^3} = e^y\cdot\sin x\,,\ \ f'''_{x^3}(\xi\,,\ \eta) = e^\eta\cdot\sin\xi\,,$$
$$f'''_{x^2y} = -\,e^y\cdot\cos x\,,\ \ f'''_{x^2y}(\xi\,,\ \eta) = -\,e^\eta\cdot\cos\xi$$
$$f'''_{y^3} = e^y\cdot\cos x\,,\ \ f'''_{y^3}(\xi\,,\ \eta) = e^\eta\cdot\cos\xi\,,$$
$$f'''_{x\,y^2} = -\,e^y\cdot\sin x\,,\ \ f'''_{x\,y^2}(\xi\,,\ \eta) = -\,e^\eta\cdot\sin\xi\,.$$

The third degree Maclaurin's formula is:

$$f(x\,,\ y) = f(0\,,\ 0) + \frac{1}{1!}\cdot\left[f'_x(0\,,\ 0)\cdot x + f'_y(0\,,\ 0)\cdot y\right]$$
$$+ \frac{1}{2!}\cdot\left[f''_{x^2}(0\,,\ 0)\cdot x^2 + 2\cdot f''_{x\,y}(0\,,\ 0)\cdot x\cdot y + f''_{y^2}(0\,,\ 0)\cdot y^2\right]$$
$$+ \frac{1}{3!}\cdot\left[\begin{array}{l} f'''_{x^3}(\xi\,,\ \eta)\cdot x^3 + 3\cdot f'''_{x^2y}(\xi\,,\ \eta)\cdot x^2\cdot y + 3\cdot f'''_{x\,y^2}(\xi\,,\ \eta)\cdot x\cdot y^2 + \\ + f'''_{y^3}(\xi\,,\ \eta)\cdot y^3 \end{array}\right]$$

or:

$$f(x\,,\ y) = 1 + y - \frac{1}{2}\cdot x^2 + \frac{1}{2}\cdot y^2 + \frac{1}{6}\cdot(e^\eta\cdot\sin\xi\cdot x^3$$
$$-3\cdot e^\eta\cdot\cos\xi\cdot x^2\cdot y - 3\cdot e^\eta\cdot\sin\xi\cdot x\cdot y^2 + e^\eta\cdot\cos\xi\cdot y^3)$$

Theorem 6.9.2 (Lagrange's Theorem). *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, $a = (a_1, ..., a_n) \in A$ an interior point, and V an open and convex neighborhood of a, $V \subset A$. If $f \in C^1(V)$, then for any $x = (x_1, ..., x_n) \in V$ there exists $\xi \in (a, x)$ such that:*

$$f(x) - f(a) = \frac{\partial f}{\partial x_1}(\xi) \cdot (x_1 - a_1) + ... + \frac{\partial f}{\partial x_n}(\xi) \cdot (x_n - a_n).$$

Proof The proof follows immediately from Theorem 6.9.1 for the particular case $m = 0$. If $n = 2$ we get:

$$\begin{aligned} f(x_1, x_2) - f(a_1, a_2) = &\frac{\partial f}{\partial x_1}(\xi_1, \xi_2) \cdot (x_1 - a_1) \\ &+ \frac{\partial f}{\partial x_2}(\xi_1, \xi_2) \cdot (x_2 - a_2). \end{aligned}$$

6.10 Local Extrema of a Function of Several Variables

Definition 6.10.1 We shall say that $a \in A$ is **a local maximum (minimum) point** of the function $f : A \subset \mathbb{R}^n \to \mathbb{R}$, if there exists a neighborhood V of the point a, such that:

$f(x) \leq f(a)$ (respectively, $f(x) \geq f(a)$), for any $x \in V \cap A$.

As with the functions of one variable, a local maximum (minimum) point is called **a local extreme point.**

Definition 6.10.2 Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, differentiable at the interior point $a \in A$. If $df(a) = 0$, then a is called **critical (stationary) point for the function** f.

Observatia 6.10.1. The critical points of a differentiable function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ are determined by solving the following system:

$$\begin{cases} \frac{\partial f}{\partial x_1}(x_1, ..., x_n) = 0 \\ .... \\ \frac{\partial f}{\partial x_n}(x_1, ..., x_n) = 0. \end{cases}$$

Theorem 6.10.1 (Fermat's Theorem). *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, differentiable at the interior point $a \in A$. If the point $x = a$ is a local extreme point for f, then $x = a$ is a critical point for f, i.e. $df(a) = 0$.*

Proof Let us denote by $A_i = \{t \in \mathbb{R};\ (a_1, ..., a_{i-1}, t, a_{i+1}, ..., a_n) \in A\}$ and let us consider the functions of one variable:

$$\varphi_i : A_i \to \mathbb{R},\ \varphi_i(t) = f\big(a_1,\ \ldots,\ a_{i-1},\ t,\ a_{i+1},\ \ldots,\ a_n\big),\ t \in A_i,\ i = \overline{1,\ n}$$

Obvious, φ_i is differentiable at the point $t = a_i$, and this point is a local extreme point for φ_i. According to Fermat's Theorem for functions of one variable it results that $\varphi_i'(a_i) = 0$, hence $\frac{\partial f}{\partial x_i}(a) = 0$, for any $i = \overline{1,\ n}$.Therefore $d f(a) = 0$, and so $x = a$ is a critical point for f.

As with the functions of one variable, not every critical point is an extreme local point. The following theorem establishes sufficient conditions for a critical point to be an extreme local point.

Theorem 6.10.2 *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$, $a \in A$ an interior point, and let $r > 0$ be such that $V = B(a,\ r) \subset A$.We assume also that $f \in C^2(V)$ and that $x = a$ is a critical point for f.Then we have:*

(1) *If $d^2 f(a)$ is positive definite, then $x = a$ is a local minimum point for f.*
(2) *If $d^2 f(a)$ is negative definite, then $x = a$ is a local maximum point for f.*
(3) *If $d^2 f(a)$ is alternating on any neighborhood V of the point a, then the point a does not is a local extreme point of f.*

Proof Let $x \in V$ be an arbitrary fixed point and $h = x - a$. From Theorem 6.9.1, it follows that there exists $\xi \in (a\ ,\ x)$ such that:

$$f(x) = f(a) + \frac{1}{1!}\, d f(a)(h) + \frac{1}{2!}\, d^2 f(\xi)(h),$$

and so: $f(x) - f(a) = \frac{1}{2} \cdot d^2 f(\xi)(h) = \frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi) \cdot h_i \cdot h_j$.

If we denote by $\alpha_i = \frac{h_i}{\| h \|_2}$, then $\sum_{i=1}^{n} \alpha_i^2 = 1$, and:

$$f(x) - f(a) = \frac{\| h \|^2}{2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\xi) \cdot \alpha_i \cdot \alpha_j. \tag{6.37}$$

If we denote also by:

$$\omega(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\xi) - \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right] \cdot \alpha_i \cdot \alpha_j$$

and taking into account that $f \in C^2(V)$, it results that $\lim_{x \to a} \omega(x) = 0$. The formula (6.38) becomes:

$$f(x) - f(a) = \frac{\| h \|^2}{2}\big(d^2 f(a)(\alpha) + \omega(x)\big).$$

Let:

$$S = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ \sum_{i=1}^{n} x_i^2 = 1 \right\} \subset \mathbb{R}^n.$$

Obvious S is a closed and bounded subset, hence a compact subset.

The application $\alpha \to d^2 f(a)(\alpha) : S \to \mathbb{R}$ is continuous (being a polynomial function), so it is bounded and attains its bounds on S, according to Theorem 5.6.2.

(1) Let us suppose that $d^2 f(a)$ is positive definite, and let $\alpha_0 \in S$ be such that:

$$m = \inf_{\alpha \in S} d^2 f(a)(\alpha) = d^2 f(a)(\alpha_0) > 0.$$

As $\lim\limits_{x \to a} \omega(x) = 0$, it follows that there exists a neighborhood V_1 of the point a, $V_1 \subset V$, such that $|\omega(x)| < \frac{m}{2}$, $\forall\, x \in V_1$. Therefore, for any $x \in V_1$ we have:

$f(x) - f(a) = \frac{\|h\|^2}{2} \cdot \left(d^2 f(a)(\alpha) + \omega(x)\right) > \frac{\|h\|^2}{2} \cdot \left(m - \frac{m}{2}\right) \geq 0.$

It results that $f(x) \geq f(a)$, $\forall\, x \in V_1$, hence a is a local minimum point for f.

(2) If $d^2 f(a)$ is negative definite, it is proved as in (1) that there is a neighborhood $V_2 \subset V$ of the point a, such that $f(x) \leq f(a)$, $\forall\, x \in V_2$, where results that the point a is a local maximum point for f.

(3) If $d^2 f(a)$ is alternating, the difference $f(x) - f(a)$ does not keep a constant sign on any neighborhood of the point a, so the point a is not an extreme local point for f. In the particular case of real functions of two variables, the nature of the local extreme points can be determined with the following theorem:

Theorem 6.10.3 *Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, and $(a, b) \in A$ an interior point. We assume also that (a, b) is a critical point for f and that there exists a neighborhood V of the point (a, b) such that $f \in C^2(V)$. Then:*

(1) *If $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$ and*

$$\frac{\partial^2 f}{\partial x^2}(a, b) \cdot \frac{\partial^2 f}{\partial y^2}(a, b) - \left[\frac{\partial^2 f}{\partial x \partial y}(a, b)\right]^2 > 0,$$

then (a, b) is a local minimum point for f.

(2) *If $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$ and*

$$\frac{\partial^2 f}{\partial x^2}(a, b) \cdot \frac{\partial^2 f}{\partial y^2}(a, b) - \left[\frac{\partial^2 f}{\partial x \partial y}(a, b)\right]^2 > 0,$$

then (a, b) is a local maximum point for f.

(3) *If $\frac{\partial^2 f}{\partial x^2}(a, b) \cdot \frac{\partial^2 f}{\partial y^2}(a, b) - \left[\frac{\partial^2 f}{\partial x \partial y}(a, b)\right]^2 < 0$,*

then $(a\ ,\ b)$ *does not is a local extreme point for* f.

Remark 6.10.2 In the hypotheses of Theorem 6.10.3, if:

$$\frac{\partial^2 f}{\partial x^2}(a\ ,\ b)\ \cdot\ \frac{\partial^2 f}{\partial y^2}(a\ ,\ b)\ -\ \left[\frac{\partial^2 f}{\partial x\,\partial y}(a\ ,\ b)\right]^2 = 0,$$

we cannot say anything about the point $(a\ ,\ b)$; there are cases where it is the extreme point of the function, there are cases when it is not.

In the particular case of real functions of three real variables, the nature of the extreme points can be determined with the following theorem:

Theorem 6.10.4 *Let* $f : A \subset \mathbb{R}^3 \to \mathbb{R}$, *and* $a \in A$ *an interior point. We assume also that* a *is a critical point for* f *and that there exists a neighborhood* V *of this point,,* $V \subset A$ *such that* $f \in C^2(V)$. *If we denote by:*

$$\Delta_1 =, \Delta_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}, \Delta_3 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix}$$

(where the all partial derivatives of the function f *are computed at the point* a).

then:

(1) *If* $\Delta_1\ > 0\ ,\ \Delta_2\ >\ 0\ ,\ \Delta_3\ >\ 0$, *then* a *is a local minimum point for* f.
(2) *If* $\Delta_1\ < 0\ ,\ \Delta_2\ >\ 0\ ,\ \Delta_3\ <\ 0$, *then* a *is a local maximum point for* f.

Example 6.10.1 Find the local extreme points of the follloing functions**:**

1. $f : \mathbb{R}^2 \to \mathbb{R}\ ,\ f(x\ ,\ y)\ =\ x^3 + y^3 +\ 21\ x\,y\ +\ 36\ x\ +\ 36\,y\ -\ 3.$

   By solving the system:

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,\ y) = x^2\ +\ 7\cdot y\ +\ 12 = 0 \\ \dfrac{\partial f}{\partial y}(x,\ y) = y^2\ +\ 7\cdot x\ +\ 12 = 0 \end{cases}$$

we obtain two critical points i.e., $M_1(-4,\ -4)$ and $M_2(-3,\ -3)$.

Further we have:

$$d^2 f(x\ ,\ y)\ =\ 6\cdot x\cdot d\,x^2\ +\ 42\cdot d\,x\cdot d\,y\ +\ 6\cdot\ y\cdot d\,y^2.$$

$$d^2 f(-4,\ -4)\ =\ -24\cdot d\,x^2\ +\ 42\cdot d\,x\cdot\ d\,y\ -\ 24\cdot\ y\cdot d\,y^2.$$

$$\Delta_1 = -24 < 0, \ \Delta_2 = \begin{vmatrix} -24 & 21 \\ 21 & -24 \end{vmatrix} = 135 > 0.$$

As $d^2 f(-4, -4)$ is negative definite we deduce that the point $M_1(-4, -4)$ is a local maximun point for f.

On the other hand we have:

$$d^2 f(-3, -3) = -18 \cdot dx^2 + 42 \cdot dx \cdot dy - 18 \cdot y \cdot dy^2.$$

$$\Delta_1 = -18 < 0, \ \Delta_2 = \begin{vmatrix} -18 & 21 \\ 21 & -18 \end{vmatrix} = -117 < 0.$$

As $d^2 f(-3, -3)$ is altrenating it results that the point $M_2(-3, -3)$ does not is a local extreme point for f.

2. $f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = x^4 + y^4 - 2 \cdot x^2 - 2 \cdot y^2 + 4 \cdot x \cdot y.$

We have successively:

$$\frac{\partial f}{\partial x} = 4 \cdot x^3 - 4 \cdot x + 4 \cdot y; \ \frac{\partial f}{\partial y} = 4 \cdot y^3 + 4 \cdot x - 4 \cdot y,$$
$$\frac{\partial^2 f}{\partial x^2} = 12 \cdot x^2 - 4; \ \frac{\partial^2 f}{\partial y^2} = 12 \cdot y^2 - 4; \ \frac{\partial^2 f}{\partial x \, \partial y} = 4.$$

The critical point are $M_1\left(\sqrt{2}, -\sqrt{2}\right)$; $M_2\left(-\sqrt{2}, \sqrt{2}\right)$ and $M_3(0, 0)$.

$$d^2 f\left(\sqrt{2}, -\sqrt{2}\right) = 20 \cdot dx^2 + 8 \cdot dx \cdot dy + 20 \cdot dy^2.$$

$\Delta_1 = 20 > 0, \ \Delta_2 = \begin{vmatrix} 20 & 4 \\ 4 & 20 \end{vmatrix} = 384 > 0$, so $M_1\left(\sqrt{2}, -\sqrt{2}\right)$ is a local minimum point.

$$d^2 f\left(-\sqrt{2}, \sqrt{2}\right) = 20 \cdot dx^2 + 8 \cdot dx \cdot dy + 20 \cdot dy^2 = d^2 f\left(\sqrt{2}, -\sqrt{2}\right),$$

It results that $M_2\left(-\sqrt{2}, \sqrt{2}\right)$ is also a local minimum point.

$$d^2 f(0, 0) = -4 \cdot dx^2 + 8 \cdot dx \cdot dy - 4 \cdot dy^2.$$

$$\Delta_1 = -4 < 0, \quad \Delta_2 = \begin{vmatrix} -4 & 4 \\ 4 & -4 \end{vmatrix} = 0.$$

Because, $\Delta_2 = 0$, we cannot apply Theorem 6.10.3 to the point $M_3(0, 0)$. We can figure out whether or not the point is an extreme point starting from the definition. Indeed, we notice that $f(0, 0) = 0$, $f(x, 0) = x^2 \cdot \left(x^2 - 2\right)$ and $f(x, x) = 2 \cdot x^4$. It results that in any neighborhood of the point $(0, 0)$ there are points where $f > 0$, and points where $f < 0$, while $f(0, 0) = 0$. Therefore, $M_3(0, 0)$ does not is a local extreme point for f.

3. $f : \mathbb{R}^3 \to \mathbb{R}, \ f(x, y, z) = x^2 + y^2 + z^2 - x \cdot y + x - z.$

By solving the system:

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we find only a critical point, namely $M\left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}\right)$.

Further we have:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = -1, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial y \partial z} = 0.$$

$$d^2 f(-1, -2, 2) = 2 \cdot \left(d x^2 + d y^2 + d z^2 - d x \, d y\right).$$

$$\Delta_1 = 2 > 0, \quad \Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0, \quad \Delta_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 > 0.$$

As, $d^2 f\left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}\right)$ is positive definite, it results that:
$M\left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}\right)$ is a local minimum point.

6.11 Local Inversion Theorem

Definition 6.11.1 Let $A, B \subset \mathbb{R}^n$ be two open subsets. A function $F : A \to B$ is called **diffeomorphism** if it has the properties:

(a) F is bijective,
(b) $F \in C^1(A)$,
(c) $F^{-1} \in C^1(B)$.

Such a definition is necessary because there are functions that satisfy conditions (a) and (b) but do not meet condition (c).

Example 6.11.1 The function $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^3$ does not is diffeomorphism.

Obvious, f is bijective and $f \in C^1(\mathbb{R})$, but f is not diffeomorphism because its inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$, $f^{-1}(y) = \sqrt[3]{y}$, is not derivable at
$y = 0$, therefore $f^{-1} \notin C^1(\mathbb{R})$.

Remark 6.11.1 Let A, B and D three open subsets of $\mathbb{R}^n$ and $F : A \to B$, $G : B \to D$ two diffeomorphisms.

Then the composed function $H = G \circ F : A \to D$ is also diffeomorphism.

The statement results from the fact that a composition of C^1- class functions is also a function of C^1- class, and by the ramark that:

$$(G \circ F)^{-1} = F^{-1} \circ G^{-1}.$$

Example 6.11.2 The simplest example of diffeomorphism is the translation operator. Let $y \in \mathbb{R}^n$ be an arbitraty fixed point, and let $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$ defined by:
$\tau_y(x) = x + y$, $\forall x \in \mathbb{R}^n$.

Obvious $\tau_y \in C^1(\mathbb{R}^n)$, τ_y is bijective and $\tau_y^{-1} = \tau_{-y} \in C^1(\mathbb{R}^n)$.

The following result is known as the local inversion theorem for functions of one variable.

Teorema 6.11.1
Let $A \subset \mathbb{R}$ be an open subset, $a \in A$ and $f : A \subset \mathbb{R} \to \mathbb{R}$. If $f \in C^1(A)$ and $f'(a) \neq 0$, then there exists an open neighborhood U of the point a, $U \subset A$, such that $f'(x) \neq 0$, $\forall x \in U$, $f : U \to V = f(U)$ is a diffeomorphism and:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}, \quad \forall\, y = f(x) \in V.$$

Proof Let us suppose that $f'(a) > 0$. As f' is continuous at the point a, it results that there exists an open interval U containing the point a, $U \subset A$ such that $f'(x) > 0$, $\forall\, x \in U$(See Theorem 5.5.4).

If we denote by $V = f(U)$, then V is an open interval and $f : U \to V$ is bijective because is strictly increasing. On the other hand it is known that if $f' \neq 0$ on U, then $f^{-1} : V \to U$ is derivable on V,and $\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}$, $\forall\, y \in V$, $y = f(x)$.Obvious $f^{-1} \in C^1(V)$, therefore f is a diffeomorphism.

Next we present without proof the local inversion theorem for vector functions.

Theorem 6.11.2 (Inverse function theorem). *Let $A \subset \mathbb{R}^n$ be an open subset, $a \in A$ and $F = (f_1, \ldots, f_n) : A \subset \mathbb{R}^n \to \mathbb{R}^n$.*

If $F \in C^1(A)$ and the jacobian $\det J_F(a) = \frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(a) \neq 0$, *then there exists an open neighborhood U of the point a with the properties: $U \subset A$, $\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(x) \neq 0$, $\forall\, x \in U$ and $F : U \to V = F(U)$ is diffeomorphism.*

More, if $F^{-1} = G = (g_1, \ldots, g_n) : V \to U$, then:

$$\frac{D(g_1, \ldots, g_n)}{D(y_1, \ldots, y_n)}(y) = \frac{1}{\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(x)}, \quad \forall\, y \in V, \; y = F(x).$$

6.12 Regular Transformations

Definition 6.12.1 Let $F = (f_1, \ldots, f_n) : A \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector function. We say that F is a **regular transformation** at the point $a \in A$,if there exists an open neighborhood U of the point a, $U \subset A$ with the properties:

$F \in C^1(U)$ and

$$\det J_F(a) = \frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(a) \neq 0.$$

We say that F is a regular transformation on the set A if F is a regular transformation at any point of A.

Proposition 6.12.1 *If $A \subset \mathbb{R}^n$ is an open subset and $F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ is a regular transformation at the point $a \in A$, then F is continuous at the point a.*

Proof If F is a regular transformation at the point a, then, according to Theorem 6.1.3, F is differentiable at a, hence F is continuous at a, as it results from Proposition 6.1.4.

Proposition 6.12.2 *Let A, $B \subset \mathbb{R}^n$ be two open subsets.*

If $F : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^n$ is a regular transformation at the point $a \in A$ and $G : B \subset \mathbb{R}^n \to \mathbb{R}^n$ is a regular transformation at the point $b = F(a) \in B$, then the composed function $H = G \circ F : A \to \mathbb{R}^n$ is a regular transformation at the point $a \in A$.

Proof From the hypothesis it results that $\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(a) \neq 0$, where $f_1, \ldots, f_n$ are the scalar components of the vector function F, and that there is an open neighborhood U of the point a,$U \subset A$, such that $F \in C^1(U)$. Also there exists an open neighborhood V of the point $b = F(a) \in B$, $V \subset B$ such that $G \in C^1(V)$ and $\frac{D(g_1, \ldots, g_n)}{D(y_1, \ldots, y_n)}(b) \neq 0$, where $g_1, \ldots, g_n$ are the scalar components of G.

Since F is continuous at the point a, it follows that there is an open neighborhood U_1 of a, $U_1 \subset U$ such that $F(U_1) \subset V$. Obvious $H = G \circ F$ is of C^1- class on U_1. On the other hand, we have:

$$\det J_H(a) = \det J_G(b) \cdot \det J_F(a) \neq 0,$$

hence H is a regular transformation at the point $a \in A$.

If we denote by $h_1, \ldots, h_n$ the scalar components of the vector function H, then we get the equality:

$$\frac{D(h_1, \ldots, h_n)}{D(x_1, \ldots, x_n)}(a) = \frac{D(g_1, \ldots, g_n)}{D(y_1, \ldots, y_n)}(b) \cdot \frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)}(a).$$

The following theorem highlights a remarkable property of regular transformations, namely that the direct image of an open set by a regular transformation is also an open set.

Theorem 6.12.1 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a regular transformation on $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is an open subset then the subset $F(A) \subset \mathbb{R}^n$ is also an open subset.*

Proof Let $b \in F(A)$ arbitrary, and $a \in A$ such that $b = F(a)$. From Theorem 6.11.2 it results that there exists an open neighborhood U of the point a, $U \subset A$ and an open neighborhood $V = F(U)$ of the point b such that $F : U \to V$ is diffeomorphism. Obvious, $V \subset F(A)$, hence b is an interior point of the set $F(A)$. As the point b was arbitrary it results that $F(A)$ is an open subset.

Definition 6.12.2 An open and connected subset of $\mathbb{R}^n$ is called **domain**$\mathbb{R}^n$**.**

Proposition 6.12.3 *Let $F = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a regular transformation on $\mathbb{R}^n$. If $D \subset \mathbb{R}^n$ is a domain, then $F(D)$ is also a domain and the jacobian $\frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)}$ keeps constant sign on D.*

Proof The fact that $F(D)$ is an open subset of $\mathbb{R}^n$ follows from Theorem 6.12.1. On the other hand, from Proposition 6.12.1 it results that F is continuous on $\mathbb{R}^n$. As a continuous function transforms a connected set into a connected set (Theorem 5.6.5), it follows that $F(D)$ is connected, so $F(D)$ is a domain.

Since the function $\frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)} : \mathbb{R}^n \to \mathbb{R}$ is continuous and D is a connected subset it results that the set $\frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)}(D)$ is a connected subset of $\mathbb{R}$, so an interval (Theorem 5.6.2). If we suppose that there are two points $u, v \in D$ such that $\frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)}(u) < 0$ and $\frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)}(v) > 0$ then, from Corollary 5.6.2. it results that there is a point $w \in D$ such that such that:

$$\frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)}(w) = 0,$$

which contradicts the hypothesis that F is regular.

6.13 Implicit Functions

Let us consider the rectangle $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and the equation:

$$F(x, y) = 0, \ (x, y) \in D. \tag{6.38}$$

We wonder if for any $x \in [a, b]$ there is only one value $y \in [c, d]$ so that the pair (x, y) verifies the Eq. (6.39).

If this happens, we will note by $y(x)$ the value y corresponding to value x. The function $x \to y(x) : [a, b] \to [c, d]$ it is said to be ***implicitly defined by*** Eq. (6.39). Obviously we have:

$$F(x, y(x)) = 0, \forall x \in [a, b].$$

Example 6.13.1 Let us consider the equation:

$$x^2 + y^2 - 1 = 0 \tag{6.39}$$

The set of points in the plane that verifies Eq. (6.40) represents, the circle $C(0;\ 1)$, with the center in origin and radius 1 (Fig. 6.3).

Let $D_1 = [a_1,\ b_1] \times [c_1,\ d_1]$. We remark that for any $x \in [a_1,\ b_1]$, there is a unique value $y = y(x) \in [c_1,\ d_1]$, namely $y(x) = \sqrt{1-x^2}$ such that the pair $(x,\ y)$ verifies the Eq. (6.40), i.e.,the point $(x,\ y)$ belongs to circle $C(0;\ 1)$. It results that on the rectangle D_1 the Eq. (6.40) defines an implicit function.

On the other hand, we notice that on the rectangle D_2, Eq. (6.40) does not define any implicit function of the form $y = y(x)$, because for any $x \in [a_2,\ -1)$, there is no value y such that the pair $(x,\ y) \in C(0;\ 1)$, and for any $x \in [-1,\ b_2]$ there are two values, namely $y_1 = -\sqrt{1-x^2}$ and $y_2 = \sqrt{1-x^2}$ such that the points $(x,\ y_1)$ and $(x,\ y_2)$ belong to the circle $C(0;\ 1)$.

The following theorem establishes sufficient conditions for the existence of implicit functions.

Theorem 6.13.1 (Implicit functions defined of equations of two variables). *Let $A \subset \mathbb{R}^2$ be an open subset, $(a,\ b) \in A$ and $F : A \subset \mathbb{R}^2 \to \mathbb{R}$ with the properties:*

(a) $F \in C^1(A)$;
(b) $F(a,\ b) = 0$;
(c) $\frac{\partial F}{\partial y}(a,\ b) \neq 0$.

Then there exists an open neighborhood U of the point a, and an open neighborhood V of the point b such that $U \times V \subset A$ and an unique function $f : U \to V$ such that:

(1) $F(x,\ f(x)) = 0,\ \forall\, x \in U$;

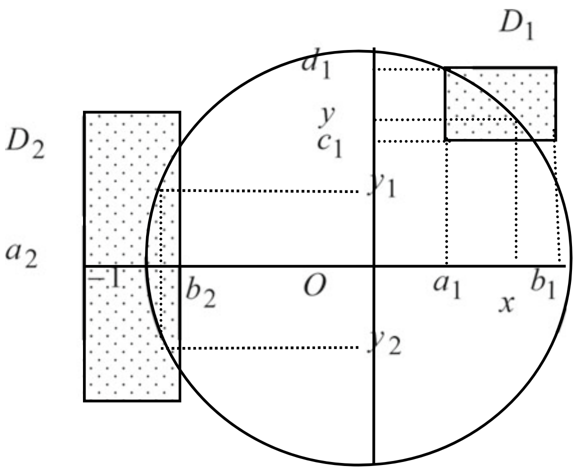


Fig. 6.3 The circle with center in origin and radious one $= C(O; 1)$

(2) $f(a) = b$;

(3) $f \in C^1(U)$ *and* $f'(x) = -\dfrac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}, \quad \forall x \in U$.

More, if $F \in C^p(A)$, *then* $f \in C^p(U)$.

Proof Let us consider the vector function $\Phi = (\varphi_1, \varphi_2) : A \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$\Phi(x, y) = (x, F(x, y)), \ (x, y) \in A.$$

Obvious we have:
$\varphi_1(x, y) = x$ and $\varphi_2(x, y) = F(x, y)$, $(x, y) \in A$, $\Phi(a, b) = (a, 0)$.
and

$$\det J_\Phi(a, b) = \frac{D(\varphi_1, \varphi_2)}{D(x, y)}(a, b) = \begin{vmatrix} 1 & 0 \\ \frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \end{vmatrix} = \frac{\partial F}{\partial y}(a, b) \neq 0.$$

According to Theorem 6.11.2, there exists an open neighborhood $U \times V$ of the point (a, b) and an open neighborhood $U \times W$ of the point $(a, 0)$ such that $\Phi : U \times V \to U \times W$ is diffeomorphism. More,

$$\det J_\Phi(x, y) = \frac{D(\varphi_1, \varphi_2)}{D(x, y)}(x, y) = \frac{\partial F}{\partial y}(x, y) \neq 0, \ \forall (x, y) \in U \times V.$$

If we denote by $\Psi = (\psi_1, \psi_2) = \Phi^{-1} : U \times W \to U \times V$, then $\Psi \in C^1(U \times W)$.

Now we define $f(x) = \psi_2(x, 0)$, $\forall x \in U$. Obvious, $f : U \to V$, $f \in C^1(U)$ and $f(a) == \psi_2(a, 0) = b$. Further, for any $x \in U$ we have:

$$\begin{aligned}(x, 0) = \Phi(\Psi(x, 0)) &= \Phi(\psi_1(x, 0), \psi_2(x, 0)) \\ = \Phi(x, f(x)) &= (x, F(x, f(x)))\end{aligned}$$

whence it results:

$$F(x, f(x)) = 0, \ (\forall) \ x \in U. \tag{6.40}$$

By deriving the relationship (6.41) we get:

$$\frac{\partial F}{\partial x}(x, f(x)) + \frac{\partial F}{\partial y}(x, f(x)) \cdot f'(x) = 0$$

whence it results:

$$f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}, \forall x \in U.$$

Example 6.13.2

1. Show that the equation $x^4 + y^4 - 2 \cdot x^2 \cdot y - 2 \cdot x^2 - y = 0$ defines in a neighborhood of the point $(2, 1)$ the implicit function $y = y(x)$, and compute $y'(2)$ and $y''(2)$.

We denote by $F(x, y) = x^4 + y^4 - 2 \cdot x^2 \cdot y - 2 \cdot x^2 - y = 0$ and we check the conditions in Theorem 6.13.1:

(a) F is a polynomial function, so indefinitely derivable on $\mathbb{R}^2$;
(b) $F(2, 1) = 0$;
(c) $\frac{\partial F}{\partial y}(x, y) = 4 \cdot y^3 - 2 \cdot x^2 - 1$ and $\frac{\partial F}{\partial y}(2, 1) = -5 \neq 0$.

From Teorema 6.13.1, it results that there exists an open neighborhood U of the point 2, an open neighborhood V of the point 1 and an unique implicit function $x \to y(x) : U \to V$.

with the properties:

1. $F(x, y(x)) = 0, \forall x \in U$;
2. $y(2) = 1$;
3. $y \in C^1(U)$ and $y'(x) = -\frac{\frac{\partial F}{\partial x}(x,y(x))}{\frac{\partial F}{\partial y}(x,y(x))} = -\frac{4\cdot x^3 - 4\cdot x\cdot y - 4\cdot x}{4\cdot y^3 - 2\cdot x^2 - 1}, \forall x \in U$,

where we deduce that $y'(2) = \frac{16}{5}$.
More,

$$y''(x) = -\frac{\left(12 \cdot x^2 - 4 \cdot y - 4 - 4 \cdot x \cdot y'\right)\left(4 \cdot y^3 - 2 \cdot x^2 - 1\right)}{\left(4 \cdot y^3 - 2 \cdot x^2 - 1\right)^2}$$
$$-\frac{\left(4 \cdot x^3 - 4 \cdot x \cdot y - 4 \cdot x\right)\left(12 \cdot y^2 \cdot y' - 4 \cdot x\right)}{\left(4 \cdot y^3 - 2 \cdot x^2 - 1\right)^2}$$

hence:

$$y''(2) = \frac{2792}{125}.$$

2. Parametric representation of the ellipse.

We recall that the most commonly used parametric equations of the circle with the center at origin and radius R of the equation $x^2 + y^2 = R^2$ are:

$$\begin{cases} x = R \cdot \cos t \\ y = R \cdot \sin t \end{cases}, \quad t \in [0, 2\pi] \tag{6.41}$$

Note that in this representation the parameter t represents the angle at the center between the position vector $\overrightarrow{OM}$ and the axis Ox, where $M(x, y)$ is a current point on the circle. By analogy with this representation, the parametric representation is used for the ellipse of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:

$$\begin{cases} x = a \cdot \cos t \\ y = b \cdot \sin t \end{cases}, \quad t \in [0, 2\pi] \tag{6.42}$$

There is a fundamental difference between the two representations. In the case of the ellipse, the parameter t does not have the same geometric significance as in the case of the circle, t is no more the angle at the center formed by the position vector $\overrightarrow{OM}$ with the axis Ox. We realize this, for example, when the point M on the ellipse is on the first bisector. Indeed, then $x = y$, whence it follows that $a \cdot \cos t = b \cdot \sin t$, and further $t = \operatorname{arctg} \frac{b}{a} \neq \frac{\pi}{4}$, which contradicts the fact that M is on the first bisector.

In the following we present another parametric representation of the ellipse used especially in Geodesy.

Let $M(x, y)$, $y \neq 0$ be a current point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is chosen as a parameter the angle B which the normal in M at the ellipse does with the axis Ox(Fig. 6.4).

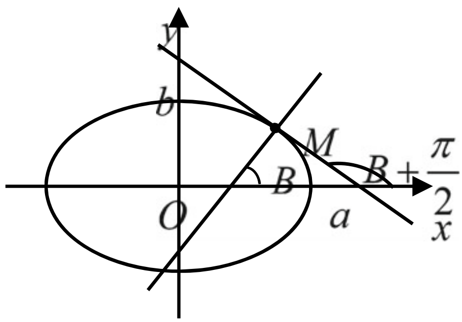
Since the angle that the tangent in M to the ellipse makes with the axis Ox is $B + \frac{\pi}{2}$ it results that the slope of the tangent is $m = \operatorname{tg}\left(B + \frac{\pi}{2}\right) = -\operatorname{ctg} B$.

On the other hand, from the theorem of implicit functions, it results that locally, in a neighborhood of the point M, the ellipse is the graph of the implicit function $y = y(x)$ defined by the equation $F(x, y) = 0$, where:

$$F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

It follows that the slope of the tangent is equal to:

Fig. 6.4 The geometric representation of the ellipse



$$y'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{x \cdot b^2}{y \cdot a^2}.$$

By equalizing those two expressions of the slope of the tangent we obtain:

$$\frac{x \cdot b^2}{y \cdot a^2} = \frac{\cos B}{\sin B}$$

. and further:

$$y = \frac{b^2}{a^2} \cdot \frac{\sin B}{\cos B} \cdot x. \tag{6.43}$$

Substituting (6.44) in the equation of the ellipse results:

$$x^2 \cdot \left(1 + \frac{b^2}{a^2} \cdot \frac{\sin^2 B}{\cos^2 B}\right) = a^2.$$

Taking into account that $\sqrt{1 - \frac{b^2}{a^2}} = e \in (0, 1)$.
represents the first eccentricity of the ellipse, we find:

$$x^2 \cdot \left(1 + \left(1 - e^2\right) \cdot \frac{\sin^2 B}{\cos^2 B}\right) = a^2$$

or:

$$x^2 \cdot \frac{1 - e^2 \sin^2 B}{\cos^2 B} = a^2,$$

from which we deduce that:

$$x = \frac{a \cdot \cos B}{\sqrt{1 - e^2 \cdot \sin^2 B}}.$$

Taking into account (6.44), it results that $y = \frac{a \cdot \left(1 - e^2\right) \cdot \sin B}{\sqrt{1 - e^2 \cdot \sin^2 B}}$.
Finally, if we note with $w = \sqrt{1 - e^2 \cdot \sin^2 B}$,
the parametric representation of the ellipse used in Geodesy is obtained:

$$\begin{cases} x = \frac{a \cdot \cos B}{w} \\ y = \frac{a \cdot \left(1 - e^2\right) \cdot \sin B}{w} \end{cases} \tag{6.44}$$

where B is the ***geodesic latitude*** (the angle between the normal at M to the ellipse and axis $O\,x$).

Remark 6.13.1 In the particular case of the circle with center in origin and radius $R = a$, the eccentricity $e = 0$ and if we replace in Eqs. (6.45) we obtain the parametric Eqs. (6.42) of the circle, where parameter B is the angle at the center between the position vector $\overrightarrow{OM}$ and the axis Ox, because in this case the normal passes through origin.

Next we present, without proof, two important generalizations of Theorem 6.13.1.

Theorem 6.13.2 (Implicit functions defined of equations of $n+1$ variables).

Let $A \subset \mathbb{R}^{n+1}$ be an open subset, $n \geq 2$, $(a, b) = (a_1, \ldots, a_n, b) \in A$ and $F : A \to \mathbb{R}$.

with the properties:

(a) $F \in C^1(A)$;
(b) $F(a_1, \ldots, a_n, b) = 0$;
(c) $\frac{\partial F}{\partial y}(a_1, \ldots, a_n, b) \neq 0$;

Then there are an open neighborhood U of the point $a = (a_1, \ldots, a_n)$, an open neighborhood V of the point b such that $U \times V \subset A$ and an unique function $f : U \to V$ with the properties:

1. $F(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0, \ \forall x = (x_1, \ldots, x_n) \in U$;
2. $f(a_1, \ldots, a_n) = b$;
3. $f \in C^1(U)$;

and $\forall\ x = (x_1, \ldots, x_n) \in U, \ \forall i = \overline{1, n}$ we have:

$$\frac{\partial f}{\partial x_i}(x) = -\frac{\frac{\partial F}{\partial x_i}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}.$$

More, if $F \in C^p(A)$, then $f \in C^p(U)$.

Example 6.13.3 Show that the equation $x + y + z = e^z$ defines on the some neighborhood of the point $(-1, e, 1)$ an implicit function $z = f(x, y)$, and compute $df(-1, e)$.

If we denote by $F(x, y, z) = x + y + z - e^z = 0$, then $F \in C^1(\mathbb{R}^3)$, $F(-1, e, 1) = 0$ and $\frac{\partial F}{\partial z}(-1, e, 1) = 1 - e \neq 0$

We remark that the conditions of Theorem 6.13.2 are met, so there is an open neighborhood U of the point $(-1, e)$, an open neighborhood V of the point 1 and an unique function $z = f(x, y) : U \to V$ with the properties (1), (2) and (3) from this theorem:

We have:

$$\frac{\partial f}{\partial x}(-1, e) = -\frac{\frac{\partial F}{\partial x}(-1, e, 1)}{\frac{\partial F}{\partial z}(-1, e, 1)} = \frac{1}{e - 1}.$$

and

$$\frac{\partial f}{\partial y}(-1,\ e) = -\frac{\frac{\partial F}{\partial y}(-1,\ e,\ 1)}{\frac{\partial F}{\partial z}(-1,\ e,\ 1)} = \frac{1}{e-1}.$$

Therefore:

$$d f(-1,\ e) = \frac{1}{e-1}(d x + d y).$$

Theorem 6.13.3 (Implicit functions defined by a system of equations).
Let $A \subset \mathbb{R}^{n+m}$ be an open subset, $m \geq 2$, $(a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_m) \in A$ and the vector function $F = (F_1, \ldots, F_m) : A \subset \mathbb{R}^{n+m} \to \mathbb{R}^m$ with the properties:

(a) $F \in C^1(A)$;

(b) $\begin{cases} F_1(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ;\\ F_m(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0 \end{cases}$

(c) $\frac{D(F_1, \ldots, F_m)}{D(y_1, \ldots, y_m)}(a, b) \neq 0.$

Then there exists an open neighborhood U of the point $a = (a_1, \ldots, a_n)$ and an open neighborhood $V = V_1 \times \ldots \times V_m$ of the point $b = (b_1, \ldots, b_m)$ such that $U \times V \subset A$ and m uniquely determined functions $f_i : U \to V_i$, $i = \overline{1, m}$, with the properties:

(1) $\begin{cases} F_1(x_1, \ldots, x_n, f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ;\\ F_m(x_1, \ldots, x_n, f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) = 0 \end{cases}$
$\forall\, x = (x_1, \ldots, x_n) \in U$;

(2) $f_1(a_1, \ldots, a_n) = b_1, \ldots, f_m(a_1, \ldots, a_n) = b_m$;

(3) $f_i \in C^1(U)$, $\forall\, i = \overline{1, m}$

and for $\forall\, j = \overline{1, n}$, $\forall\, x = (x_1, \ldots, x_n) \in U$, we have:

$$\frac{\partial f_1}{\partial x_j}(x) = -\frac{\frac{D(F_1, F_2, \ldots, F_m)}{D(x_j, y_2, \ldots, y_m)}(x, f_1(x), \ldots, f_m(x))}{\frac{D(F_1, F_2, \ldots, F_m)}{D(y_1, y_2, \ldots, y_m)}(x, f_1(x), \ldots, f_m(x))}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial f_m}{\partial x_j}(x) = -\frac{\frac{D(F_1, \ldots, F_{m-1}, F_m)}{D(y_1, \ldots, y_{m-1}, x_j)}(x, f_1(x), \ldots, f_m(x))}{\frac{D(F_1, \ldots, F_{m-1}, F_m)}{D(y_1, \ldots, y_{m-1}, y_m)}(x, f_1(x), \ldots, f_m(x))}.$$

Example 6.13.4 Show that the nonlinear system:

$$\begin{cases} x^3 + 3 \cdot y^2 - z^2 + x - y - 8 = 0 \\ x^2 - y^2 - 3 \cdot z - 3 = 0 \end{cases}$$

defines on a some neighborhood of the point $(1\,,\ 2\,,\ -2)$ two implicit functions $y \ = \ f(x)$ and $z \ = \ g(x)$ and compute $f'(1)$ and $g'(1)$.

We have met the conditions of Theorem 6.13.3,i.e.:

$$\begin{cases} F(x\,,\ y\,,\ z) = x^3 + \ 3\,y^2 - \ z^2 + \ x \ - \ y \ - \ 8 \ = \ 0 \\ G(x\,,\ y\,,\ z) = x^2 - \ y^2 \ - \ 3\,z \ - \ 3 \ = \ 0 \end{cases},\ F\,,\ G \in C^1\left(\mathbb{R}^3\right)$$

$$F(1\,,\ 2\,,\ -2) = 0,\ G(1\,,\ 2\,,\ -2) = 0$$

$$\frac{D(F\,,\ G)}{D(y\,,\ z)}\ (1\,,\ 2\,,\ -2) = \begin{vmatrix} \frac{\partial\,F}{\partial\,y} & \frac{\partial\,F}{\partial\,z} \\ \frac{\partial\,G}{\partial\,y} & \frac{\partial\,G}{\partial\,z} \end{vmatrix} (1\,,\ 2\,,\ -2) \ = \ \begin{vmatrix} 11 & 4 \\ -4 & -3 \end{vmatrix} \ = \ -\ 17 \ \neq 0.$$

From Teoremei 6.13.3, it follows that there are an open neighborhood U of the point 1, an open neighborhood $V \ \times \ W$ of the point $(2\,,\ -2)$ and two functions $y \ = \ f(x) \ : \ U \to V, z \ = \ g(x) \ : \ U \to W$ with the properties (1), (2), (3) of this theorem.

Further we have:

$$\frac{D(F\,,\ G)}{D(x\,,\ z)}\ (1\,,\ 2\,,\ -2) \ = \ \begin{vmatrix} 4 & 4 \\ 2 & -3 \end{vmatrix} \ = \ -\ 20$$

and:

$$\frac{D(F\,,\ G)}{D(y\,,\ x)}\ (1\,,\ 2\,,\ -2) \ = \ \begin{vmatrix} 11 & 4 \\ -4 & 2 \end{vmatrix} \ = \ 38.$$

whence it results:

$$f'(1) \ = \ - \ \frac{-\ 20}{-\ 17} = - \ \frac{20}{17}$$

and

$$g'(1) \ = \ - \ \frac{38}{-\,17} \ = \ \frac{38}{17}.$$

Example 6.13.5 Find the locally extreme points of the function $y \ = \ y(x)$ defined implicitly by the equation $x^3 \ + \ y^3 - \ 3\,x^2 \cdot y \ - \ 3 \ = \ 0$.

We shall denote by $F(x\,,\ y) = x^3 \ + \ y^3 - \ 3\,x^2 \cdot y \ - \ 3$.

The critical points of the implicit function $y \ = \ y(x)$ belong to the curve, i.e. verifies the equation $F(x\,,\ y) = 0$, and satisfy the equation $y'(x) \ = \ 0$, so they are the solutions of the system:

$$\begin{cases} \frac{\partial\,F}{\partial\,x}(x\,,\ y) = 3\cdot x^2 - \ 6 \cdot x \cdot y & = 0 \\ F(x\,,\ y) \ = x^3 \ + \ y^3 - \ 3\cdot x^2 \cdot y \ - \ 3 & = 0\,. \end{cases}$$

Solving this system we obtain the two critical points:
$M_1\left(0\,,\ \sqrt[3]{3}\right)$ and $M_2(-2\,,\ -1)$.
Now we check if the condition $\frac{\partial F}{\partial y} \neq 0$ is fulfilled at the points M_1 and M_2.
As $\frac{\partial F}{\partial y}(x\,,\ y) = 3\,y^2 - 3\,x^2$, it results that:
$\frac{\partial F}{\partial y}\left(0\,,\ \sqrt[3]{3}\right) = 3\,\sqrt[3]{9} \neq 0$ and $\frac{\partial F}{\partial y}(-2\,,\ -1) = -9 \neq 0$.
To decide whether M_1 and M_2 are locally extreme points, as well as their nature, we shall compute y''.We have successively:

$$y' = \frac{x^2 - 2\,x\,y}{x^2 - y^2}$$

and

$$y'' = \frac{\left(2\,x - 2\,y - 2\,x\,y'\right)\left(x^2 - y^2\right) - \left(x^2 - 2\,x\,y\right)\left(2\,x - 2\,y\,y'\right)}{\left(x^2 - y^2\right)^2}.$$

Since $y'\left(0\,,\ \sqrt[3]{3}\right) = 0$, it results $y''\left(0\,,\ \sqrt[3]{3}\right) = \frac{2}{\sqrt[3]{3}} > 0$, so $M_1\left(0\,,\ \sqrt[3]{3}\right)$ is a locally minimum point for the implicit function $y = y(x)$ and since $y'(-2\,,\ -1) = 0$ and $y''(-2\,,\ -1) = -\frac{2}{3} < 0$, we deduce that $M_2(-2\,,\ -1)$ is a locally maximum point for this function.

6.14 Local Conditional Extremum

In applications, the problem of determining the extreme values of a function of several variables often occurs in situations where the variables are subject to certain restrictions (satisfy certain connection relations).

Example 6.14.1 Find the extreme values of the function:

$$f : \mathbb{R}^2 \to \mathbb{R}\,,\ f(x\,,\ y) = x^2 + y^2$$

with the constraint:

$$x + y - 2 = 0$$

The case being very simple, the problem is immediately reduced to an extremely free problem. Indeed, by substituting $y = 2 - x$ in the expression of the function f we obtain the function $g : \mathbb{R} \to \mathbb{R}\,,\ g(x) = 2 \cdot x^2 - 4 \cdot x + 4, x \in \mathbb{R}$.

Since $g'(x) = 4 \cdot x - 4$ and $g''(x) = 4 > 0$, it results that the function g has a minimum equal with 2 at the point $x = 1$, hence the function $f(x\,,\ y) = x^2 + y^2$, with the constraint $x + y - 2 = 0$, has a minimum equal with 2 at the point $(1,\ 1)$.

Note that the function $f(x\,,\ y) \ = \ x^2 + \ y^2$ admits the minimum value 0 at the point $(0\,,\ 0)$ if the variables are not subject to any constraint.

Let $A \ \subset \mathbb{R}^n$ be an open subset and let $f\,,\ F_1\,,\ \dots\,,\ F_m\ :\ A\ \subset \mathbb{R}^n \to \mathbb{R}$ be $m \ + \ 1$ functions, $1 \ \leq \ m \ < \ n$.

Let $S \ \subset \mathbb{R}^n$ be the set of all system solutions:

$$\begin{cases} F_1(x_1\,,\ \dots\,,\ x_n) &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \dots\ \dots \\ F_m(x_1\,,\ \dots\,,\ x_n) &= 0\,. \end{cases} \tag{6.45}$$

Definition 6.14.1 *We say that the point $a \in \ A \cap S$ is* ***a local conditional maximum (minimum) point for the function f with the constraints*** (6.46) *if there exists an open neighborhood U of the point $x = a$ such that $f(x) \ \leq \ f(a)$.*

$(f(x) \ \geq \ f(a))$, for any $x \in \ A \ \cap \ S \ \cap \ U$.

For a better understanding, consider the particular case $n \ = \ 3\,,\ m \ = \ 1$.

Let $f\ :\ A \ \subset \mathbb{R}^3 \to \mathbb{R}$ be a funtion of C^1- class and let the constraint:

$$F(x_1\,,\ x_2\,,\ x_3) \ = \ 0. \tag{6.46}$$

From a geometric point of view, the set of solutions of Eq. (6.47) represents a surface. We will denote by S this surface.The problem consists on determining the maximum and minimum value of the function f on the surface S. The following theorem establishes sufficient conditions for the existence of the local onditional extreme points of a real function with three variable and with one constraint.

Theorem 6.14.1 *Let $A \ \subset \mathbb{R}^3$ be an open subset, $f\,,\ F \in C^1(A)$ and let $a \in \ A \cap S$ be a local conditional point for f with the constraint* (6.47)*. If $\frac{\partial F}{\partial x_3}(a) \ \neq \ 0$,then there exists an unique number $\lambda \in \mathbb{R}$ such that:*

$$\begin{cases} \frac{\partial f}{\partial x_1}(a) + \lambda \ \frac{\partial F}{\partial x_1}(a) &= 0 \\ \frac{\partial f}{\partial x_2}(a) + \lambda \ \frac{\partial F}{\partial x_2}(a) &= 0 \\ \frac{\partial f}{\partial x_3}(a) + \lambda \ \frac{\partial F}{\partial x_3}(a) &= 0\,. \end{cases} \tag{6.47}$$

(λ *is called* ***Lagrange's multiplier***).

Proof To fix the ideas, suppose that $a \ = \ (a_1,\ a_2,\ a_3)$ is a local conditional minimum point for f. Then there is an open neighborhood U of the point a, such that:

$$f(a) \ \leq \ f(x)\,,\ \forall\, x = (x_1,\ x_2,\ x_3) \in \ A \ \cap \ S \ \cap \ U.$$

As a is an interior point of the subset A we can assume that $U \subset A$. Therefore:

$$f(a) \ \leq \ f(x)\,,\ \ \forall\, x \in \ \ S \ \cap \ U. \tag{6.48}$$

Since $\frac{\partial F}{\partial x_3}(a) \neq 0$, from Theorem 6.13.2, it result that there exist $\varepsilon > 0$ and an implicit function $\varphi : U_1 \times U_2 \to U_3$, where $U_1 = (a_1 - \varepsilon, a_1 + \varepsilon)$,$U_2 = (a_2 - \varepsilon, a_2 + \varepsilon)$,$U_3 = (a_3 - \varepsilon, a_3 + \varepsilon)$,

with the properties:

(i) $F(x_1, x_2, \varphi(x_1, x_2)) = 0, \forall (x_1, x_2) \in U_1 \times U_2$;
(ii) $\varphi(a_1, a_2) = a_3$;
(iii) $\varphi \in C^1(U_1 \times U_2)$ and:

$$\frac{\partial \varphi}{\partial x_1}(a_1, a_2) = -\frac{\frac{\partial F}{\partial x_1}(a)}{\frac{\partial F}{\partial x_3}(a)}; \quad \frac{\partial \varphi}{\partial x_2}(a_1, a_2) = -\frac{\frac{\partial F}{\partial x_2}(a)}{\frac{\partial F}{\partial x_3}(a)}.$$

Obviously we can choose $\varepsilon > 0$ so that the cartesian product $U_1 \times U_2 \times U_3 \subset U$.
We remark that if $(x_1, x_2) \in U_1 \times U_2$,then:

$$(x_1, x_2, \varphi(x_1, x_2)) \in U_1 \times U_2 \times U_3 \subset U.$$

On the other hand, if $(x_1, x_2) \in U_1 \times U_2$,then, from the property (i) it results:

$$(x_1, x_2, \varphi(x_1, x_2)) \in S.$$

Therefore, if $(x_1, x_2) \in U_1 \times U_2$,then:

$$(x_1, x_2, \varphi(x_1, x_2)) \in S \cap U \tag{6.49}$$

Let us denote by $g(x_1, x_2) = f(x_1, x_2, \varphi(x_1, x_2)), \quad (x_1, x_2) \in U_1 \times U_2$.
Taking into account (ii), (6.49) and (6.50) it results:

$$g(a_1, a_2) = f(a_1, a_2, a_3) = f(a) \leq f(x_1, x_2, \varphi(x_1, x_2)) = g(x_1, x_2),$$

$$\forall (x_1, x_2) \in U_1 \times U_2.$$

Therefore, (a_1, a_2) is a free local minimum point for the function g.
From Fermat's Theorem (6.10.1) we deduce that:

$$\frac{\partial g}{\partial x_1}(a_1, a_2) = 0 \text{ and } \frac{\partial g}{\partial x_2}(a_1, a_2) = 0.$$

According to the rule of derivation of composed functions we obtain:

$$\frac{\partial g}{\partial x_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_1};$$

$$\frac{\partial g}{\partial x_2} = \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} \cdot \frac{\partial \varphi}{\partial x_2}.$$

Taking into account (iii), further we have:

$$\frac{\partial g}{\partial x_1}(a_1, a_2) = \frac{\partial f}{\partial x_1}(a) + \frac{\partial f}{\partial x_3}(a) \cdot \frac{-\frac{\partial F}{\partial x_1}(a)}{\frac{\partial F}{\partial x_3}(a)} = 0 \tag{6.50}$$

$$\frac{\partial g}{\partial x_2}(a_1, a_2) = \frac{\partial f}{\partial x_2}(a) + \frac{\partial f}{\partial x_3}(a) \cdot \frac{-\frac{\partial F}{\partial x_2}(a)}{\frac{\partial F}{\partial x_3}(a)} = 0. \tag{6.51}$$

We will analyze the following cases:

$$\textbf{Case 1.}\ \frac{\partial F}{\partial x_1}(a) \neq 0$$

and $\frac{\partial F}{\partial x_2}(a) \neq 0$.

Then, from (6.51)and (6.52) it results:

$$\frac{\frac{\partial f}{\partial x_1}(a)}{\frac{\partial F}{\partial x_1}(a)} = \frac{\frac{\partial f}{\partial x_2}(a)}{\frac{\partial F}{\partial x_2}(a)} = \frac{\frac{\partial f}{\partial x_3}(a)}{\frac{\partial F}{\partial x_3}(a)} = -\lambda,$$

where we immediately obtain the relations (6.48) from the theorem statement.

$$\textbf{Case 2.}\ \frac{\partial F}{\partial x_1}(a) = 0.$$

and $\frac{\partial F}{\partial x_2}(a) \neq 0$.

From (6.51) and (6.52) we deduce that:

$$\frac{\partial f}{\partial x_1}(a) = 0. \tag{6.52}$$

$$\frac{\frac{\partial f}{\partial x_2}(a)}{\frac{\partial F}{\partial x_2}(a)} = \frac{\frac{\partial f}{\partial x_3}(a)}{\frac{\partial F}{\partial x_3}(a)} = -\lambda. \tag{6.53}$$

It follows from (6.53) that the first relation in (6.48) is verified for any λ, and from (6.54) that the last two relations in (6.48) are also true.

$$\textbf{Case 3.}\ \frac{\partial F}{\partial x_1}(a) = 0$$

and $\frac{\partial F}{\partial x_2}(a) = 0$.

In this case, from (6.51) and (6.52) we deduce that $\frac{\partial f}{\partial x_1}(a) = \frac{\partial f}{\partial x_2}(a) = 0$ and it is obvious that the first two relations in (6.46) are true for any $\lambda \in \mathbb{R}$.

If we denote by $\frac{\frac{\partial f}{\partial x_3}(a)}{\frac{\partial F}{\partial x_3}(a)} = -\lambda$, we also obtain the third relation in (6.48) and with this theorem is proved.

Next, consider the following auxiliary function:

$$\phi(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda \cdot F(x_1, x_2, x_3). \tag{6.54}$$

We remark that:

$$\begin{cases} \frac{\partial \phi}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda \cdot \frac{\partial F}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda \cdot \frac{\partial F}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} = \frac{\partial f}{\partial x_3} + \lambda \cdot \frac{\partial F}{\partial x_3} \\ \frac{\partial \phi}{\partial \lambda} = F. \end{cases} \tag{6.55}$$

From Theorem 6.14.1 and from (6.56), it results that if $a = (a_1, a_2, a_3)$ is a local conditional extreme point for f,with the constraint (6.47), and $\overline{\lambda}$ verifies the system (6.48), then $\left(a_1, a_2, a_3, \overline{\lambda}\right)$ is a critical point for the auxiliary function ϕ defined in (6.55).

It follows that the local conditional points of the function f with the constraint (6.47), are find among the points (x_1, x_2, x_3) with the property that (x_1, x_2, x_3, λ) are critical points of the auxiliary function given by (6.55).

Let $\left(a_1, a_2, a_3, \overline{\lambda}\right)$ be a critical point for ϕ and let:

$$\overline{\phi}(x) = f(x) + \overline{\lambda}\, F(x), \ \forall\, x = (x_1, x_2, x_3) \in A.$$

If $x \in U \cap S$, then $F(x) = 0$ and we have:

$$f(x) - f(a) = \overline{\phi}(x) - \overline{\phi}(a) = d\,\overline{\phi}(a)(x - a) + \frac{1}{2!} d^2\overline{\phi}(\xi)(x - a)$$

As, $d\,\overline{\phi}(a) = 0$, it results that:

$$f(x) - f(a) = \frac{1}{2!} d^2\overline{\phi}(\xi)(x - a) = \frac{1}{2!} d^2\overline{\phi}(a)(x - a) + \omega(x) \tag{6.56}$$

where ω is an infinitely small function i.e. $\omega = 0(x - a)$.

By differentiating the constraint (6.47) we get:

$$\frac{\partial F}{\partial x_1}(a) \cdot d\,x_1 + \frac{\partial F}{\partial x_2}(a) \cdot d\,x_2 + \frac{\partial F}{\partial x_3}(a) \cdot d\,x_3 = 0 \tag{6.57}$$

From the relation (6.58) we can solving, for example, $d\,x_3$ in relation to $d\,x_1$ and $d\,x_2$ and replace it in $d^2\overline{\phi}(a)$. So, a quadratic form is obtained in two variables.

If this quadratic form is positive (negative) definite, then the point a is a local conditional minimum (maximum) point for f with the constraint (6.47).

Example 6.14.2 Find the local conditional points of the function:

$$f : \mathbb{R}^3 \backslash \{(0, 0, 0)\} \to \mathbb{R}, \ f(x, y, z) = x^2 \cdot y^3 \cdot z^4$$

with the constraint:

$$2 \cdot x + 3 \cdot y + 4 \cdot z = 9.$$

We consider the auxiliary function:

$$\phi(x, y, z) = x^2 \cdot y^3 \cdot z^4 + \lambda \cdot (2 \cdot x + 3 \cdot y + 4 \cdot z - 9).$$

The critical points of this function are obtained by solving the system:

$$\begin{cases} \frac{\partial \phi}{\partial x} = 2 \cdot x \cdot y^3 \cdot z^4 + 2 \cdot \lambda & = 0 \\ \frac{\partial \phi}{\partial y} = 3 \cdot x^2 \cdot y^2 \cdot z^4 + 3 \cdot \lambda & = 0 \\ \frac{\partial \phi}{\partial z} = 4 \cdot x^2 \cdot y^3 \cdot z^3 + 4 \cdot \lambda & = 0 \\ \frac{\partial \phi}{\partial \lambda} = 2 \cdot x + 3 \cdot y + 4 \cdot z - 9 & = 0 \, . \end{cases}$$

The system has only one solution, namely:

$$x = y = z = 1, \ \lambda = -1.$$

If we denote by:

$$\overline{\phi}(x, y, z) = x^2 \cdot y^3 \cdot z^4 - (2 \cdot x + 3 \cdot y + 4 \cdot z - 9)$$

then we have:

$$\begin{aligned} d^2\overline{\phi}(1, 1, 1) = {} & 2 \cdot dx^2 + 6 \cdot dy^2 + 12 \cdot dz^2 + 12 \cdot dx \cdot dy \\ & + 16 \cdot dx \cdot dz + 24 \cdot dy \cdot dz \end{aligned} .$$

By differentiating the the constraint $F(x, y, z) = 2 \cdot x + 3 \cdot y + 4 \cdot z - 9 = 0$ we get:

$$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy + \frac{\partial F}{\partial z} \cdot dz = 2 \cdot dx + 3 \cdot dy + 4 \cdot dz = 0,$$

whence it results:

$$\begin{aligned} (2 \cdot dx + 3 \cdot dy + 4 \cdot dz)^2 = {} & 4 \cdot dx^2 + 9 \cdot dy^2 + 16 \cdot dz^2 \\ & + 12 \cdot dx \cdot dy + 16 \cdot dx \cdot dz + 24 \cdot dy \cdot dz = 0 \end{aligned}$$

and further:

$$12 \cdot dx \cdot dy + 16 \cdot dx \cdot dz + 24 \cdot dy \cdot dz$$
$$= -4 \cdot dx^2 - 9 \cdot dy^2 - 16 \cdot dz^2$$

If we take into account the last relation in the expression of $d^2\,\overline{\phi}(1, 1, 1)$ we get:

$$d^2\,\overline{\phi}(1, 1, 1) = -\left(2\,dx^2 + 3\,dy^2 + 4\,dz^2\right).$$

This quadratic form is negative definite because:

$$\Delta_1 = -2 < 0; \quad \Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} = 6 > 0;$$

$$\Delta_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{vmatrix} = -24 < 0$$

Therefore, the point $(1, 1, 1)$ is a local conditional maximum point for f with the constraint $F(x, y, z) = 2 \cdot x + 3 \cdot y + 4 \cdot z - 9 = 0$.

Example 6.14.3 Find the local conditional points of the function:

$$f(x, y, z) = x \cdot y + y \cdot z + z \cdot x$$

with the constraint:

$$x \cdot y \cdot z = 1.$$

We consider the auxiliary function:

$$\phi(x, y, z) = x \cdot y + y \cdot z + z \cdot x + \lambda \cdot (x \cdot y \cdot z - 1)$$

The critical points of this function are obtained by solving the system:

$$\begin{cases} \frac{\partial \phi}{\partial x} = y + z + \lambda \cdot y \cdot z = 0 \\ \frac{\partial \phi}{\partial y} = x + z + \lambda \cdot x \cdot z = 0 \\ \frac{\partial \phi}{\partial z} = y + x + \lambda \cdot x \cdot y = 0 \\ \frac{\partial \phi}{\partial \lambda} = x \cdot y \cdot z - 1 = 0 \end{cases}$$

The system has the unique solution:

$$x = y = z = 1, \ \lambda = -2.$$

If we denote by:

$$\overline{\phi}(x\,,\ y\,,\ z)\ =\ x\cdot y+y\cdot z+z\cdot x-2\cdot(x\cdot y\cdot z-1)$$

then we have:

$$\mathrm{d}^2\overline{\phi}(x,y,z)=2\cdot[(1-2\cdot z)\cdot \mathrm{d}x\,\mathrm{d}y+(1-2y)\cdot \mathrm{d}x\,\mathrm{d}z+(1-2x)\cdot \mathrm{d}y\,\mathrm{d}z]$$

and:

$$\mathrm{d}^2\overline{\phi}(1,1,1)=-2\cdot(\mathrm{d}x\,\mathrm{d}y+\mathrm{d}x\,\mathrm{d}z+\mathrm{d}y\,\mathrm{d}z).$$

By differentiating the the constraint $F(x,\ y,\ z)=\ x\cdot y\cdot z-1\ =\ 0$ we get:

$$\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial y}\mathrm{d}y+\frac{\partial F}{\partial z}\mathrm{d}z=y\cdot z\,\mathrm{d}x+x\cdot z\,\mathrm{d}y+x\cdot y\,\mathrm{d}z=0,$$

For $x=z=1$ we have $\mathrm{d}x+\mathrm{d}y+\mathrm{d}z=0$, whence it results $\mathrm{d}z=-(\mathrm{d}x+\mathrm{d}y)$. Replacing in $\mathrm{d}^2\overline{\phi}(1,1,1)$ we get:

$$\mathrm{d}^2\overline{\phi}(1,1,1)=2\cdot\left(\mathrm{d}x^2-\mathrm{d}x\,\mathrm{d}y+\mathrm{d}y^2\right),$$

This quadratic form is positive definite because:

$$\Delta_1=a_{11}=1>0\ \text{ and }\Delta_2=\begin{vmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{vmatrix}=\begin{vmatrix}1&-\frac{1}{2}\\-\frac{1}{2}&1\end{vmatrix}=\frac{3}{4}>0.$$

Therefore, the point $(1,1,1)$ is a conditional minimum point for the function $f(x,y,z)==x\cdot y+y\cdot z+z\cdot x$ with the constraint $x\cdot y\cdot z=1$.

Theorem 6.14.1 admits the following generalization.

Theorem 6.14.2 Let $A\ \subset\mathbb{R}^n$ *be an open subset,* $f\ ,\ \ F_1\ ,\ \ldots,\ \ F_m\in C^1(A)$ *and let* $a\in\ \ A\ \cap\ S$ *be a local extreme point for* f *with the constraints* (6.46).

If $\frac{D(F_1,\ \ldots,\ F_m)}{D(x_1\,,\ \ldots,\ x_m)}(a)\ \neq\ 0$*, then there are* $\lambda_1\,,\ \ldots,\ \lambda_m\in\mathbb{R}$*, uniquely determined, such that:*

$$\begin{cases}\frac{\partial\ f}{\partial\ x_1}(a)\ +\ \lambda_1\cdot\ \frac{\partial\ F_1}{\partial\ x_1}(a)\ +\ \ldots\ +\ \lambda_m\cdot\ \frac{\partial\ F_m}{\partial\ x_1}(a)\ =\ \ 0\\ \quad\ldots\ldots\qquad\qquad\ldots\ldots\ldots\ldots\qquad\qquad\ldots\ldots\ldots\\ \frac{\partial\ f}{\partial\ x_n}(a)\ +\ \lambda_1\cdot\ \frac{\partial\ F_1}{\partial\ x_n}(a)\ +\ \ldots\ +\ \lambda_m\cdot\ \frac{\partial\ F_m}{\partial\ x_n}(a)\ =0\,.\end{cases}\tag{6.58}$$

($\lambda_1\,,\ \ldots,\ \lambda_m$ *are called* ***Lagrange's multipliers****).*

Consider the auxiliary $\phi:\ A\ \times\ \mathbb{R}^m\rightarrow\mathbb{R}$ defined by:

$$\phi(x_1\,,\ \ldots,\ x_n\,,\ \lambda_1\,,\ \ldots,\ \lambda_m)\ =\ f(x)\ +\ \lambda_1\cdot F_1(x)\ +\ \ldots\ +\ \lambda_m\cdot\ F_m(x)\,,\quad x\in A\tag{6.59}$$

and be the following system of $n + m$ equations with the $n + m$ unknowns. $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)$. Further we have:

$$\begin{cases} \frac{\partial \phi}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1 \cdot \frac{\partial F_1}{\partial x_1} + \ldots + \lambda_m \cdot \frac{\partial F_m}{\partial x_1} = 0 \\ \ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots \\ \frac{\partial \phi}{\partial x_n} = \frac{\partial f}{\partial x_n} + \lambda_1 \cdot \frac{\partial F_1}{\partial x_n} + \ldots + \lambda_m \cdot \frac{\partial F_m}{\partial x_n} = 0 \\ \frac{\partial \phi}{\partial \lambda_1} = F_1 = 0 \\ \ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots \\ \frac{\partial \phi}{\partial \lambda_m} = F_m = 0 . \end{cases} \tag{6.60}$$

From Theorem 6.14.2 it results that, if $a = (a_1, \ldots, a_n)$ is a local conditional extreme point for f with the constrains (6.46), and $\overline{\lambda}_1, \ldots, \overline{\lambda}_m$ satisfies the system (6.60), then the point $\left(a_1, \ldots, a_n, \overline{\lambda}_1, \ldots, \overline{\lambda}_m\right)$ is a critical point for the auxiliary function ϕ. It follows that the local conditional extreme points of f with the constraints (6.46) are find among the points $(x_1, \ldots, x_n)$ with the property that $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)$ are critical points of the auxiliary function ϕ.

Let $\left(a_1, \ldots, a_n, \overline{\lambda}_1, \ldots, \overline{\lambda}_m\right)$ be a critical point for the auxiliary function ϕ and let:

$$\overline{\phi}(x) = f(x) + \overline{\lambda}_1 F_1(x) + \ldots + \overline{\lambda}_m F_m(x), \ \forall\, x = (x_1, \ldots, x_n) \in A.$$

If $x \in U \cap S$, then $F_i(x) = 0,\ i = \overline{1, m}$ and we have:

$$f(x) - f(a) = \overline{\phi}(x) - \overline{\phi}(a) = d\,\overline{\phi}(a)(x - a) + \frac{1}{2!} d^2\overline{\phi}(\xi)(x - a)$$

As $d\,\overline{\phi}(a) = 0$,it results:

$$f(x) - f(a) = \frac{1}{2!} d^2\overline{\phi}(\xi)(x - a) = \frac{1}{2!} d^2\overline{\phi}(a)(x - a) + \omega(x)$$

where $\omega = 0(x - a)$.

By differentiating the constraints (6.46) we get:

$$\begin{cases} \frac{\partial F_1}{\partial x_1} \cdot d\,x_1 + \ldots + \frac{\partial F_1}{\partial x_n} \cdot d\,x_n = 0 \\ \ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots \\ \frac{\partial F_m}{\partial x_1} \cdot d\,x_1 + \ldots + \frac{\partial F_m}{\partial x_n} \cdot d\,x_n = 0 . \end{cases} \tag{6.61}$$

From the system (6.62) we can solving, for example, $d\,x_1, \ldots, d\,x_m$ in relation to $d\,x_{m+1}, \ldots, d\,x_n$ and replace in the expression of $d^2\overline{\phi}(a)$. So, a quadratic form in $n - m$ variables is thus obtained.

If this quadratic form is positive (negative) define, then the point a is a local conditional minimum (maximum) point for f with the constraint (6.46).

As with the free local extreme points, if $d^2\overline{\phi}(a)$ is positive (negative) define, then a is a local conditional minimum (maximum) point for f with the constraints (6.46).

6.15 Dependent and Independent Functions

Let $A \subset \mathbb{R}^n$ be an open subset and $f_1, \ldots, f_m \in C^1(A), m \leq n$.

Definition 6.15.1 *The functions $f_1, \ldots, f_m$ are said to be* ***dependent in the domain*** *A if at last one of the functions, say f_m, is espressed in terms of the other functions i.e.,there exists a $\varphi \in C^1\left(\mathbb{R}^{m-1}\right)$ such that:*

$$f_m(x) = \varphi(f_1(x), \ldots, f_{m-1}(x)), \quad \forall x \in A.$$

Example 6.15.1 Consider the functions $f_i : \mathbb{R}^3 \to \mathbb{R}, \ i = \overline{1, 3}$ defined by:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 \\ f_2(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\ f_3(x_1, x_2, x_3) &= 2 \cdot x_1 \cdot x_2 + 2 \cdot x_1 \cdot x_3 + 2 \cdot x_2 \cdot x_3 . \end{aligned}$$

It is immediately noticeable that:

$$f_1(x_1, x_2, x_3) = \varphi(f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))$$

where:

$$\varphi(u, v) = u^2 - v, \ \forall (u, v) \in \mathbb{R}^2.$$

Therefore, the functions f_1, f_2, f_3 are dependent in $\mathbb{R}^3$.

Remark 6.15.1 We recall that the functions $f_1, \ldots, f_m$ are ***linearly dependent*** in the subset A if there are m real numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, not all null, such that:

$$\lambda_1 \cdot f_1(x) + \ldots + \lambda_m \cdot f_m(x) = 0, \ \forall x \in A.$$

If we suppose, for example, that $\lambda_m \neq 0$, it results that:

$$f_m(x) = \left(-\frac{\lambda_1}{\lambda_m}\right) \cdot f_1(x) + \ldots + \left(-\frac{\lambda_{m-1}}{\lambda_m}\right) \cdot f_{m-1}(x), \ \forall x \in A.$$

Therefore, the notion of linearly dependent functions is a particular case of Definition 6.15.1, namely the case when the function φ is linear.

Theorem 6.15.1 *Let $A \subset \mathbb{R}^n$ be an open subset, $m \leq n$, and let $F \in C^1(A)$ be the vector function $F = (f_1, \ldots, f_m) : A \to \mathbb{R}^m$. If $f_1, \ldots, f_m$ are dependent in A, then:*

$$\operatorname{rang} J_F(x) < m, \quad \forall x \in A.$$

Proof let $\varphi \in C^1\left(\mathbb{R}^{m-1}\right)$ be such that:

$$f_m(x) = \varphi(f_1(x), \ldots, f_{m-1}(x)), \forall x = (x_1, \ldots, x_n) \in A. \tag{6.62}$$

By differentiating the relation (6.63) we get:

$$\frac{\partial f_m}{\partial x_j} = \frac{\partial \varphi}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_j} + \ldots + \frac{\partial \varphi}{\partial f_{m-1}} \cdot \frac{\partial f_{m-1}}{\partial x_j}, \; j = \overline{1, n}. \tag{6.63}$$

Since the Jacobian matrix is:

$$J_F(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_m}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \ldots & & \ldots \ldots & & \ldots \ldots \\ \frac{\partial f_m}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_m}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix},$$

from (6.64) it follows that the last line of the matrix $J_F(x)$ is a linear combination of the other lines, so any minor of the order m of the matrix $J_F(x)$ is null.

Definition 6.15.2 *Let $A \subset \mathbb{R}^n$ be an open subset. We say that the functions $f_1, \ldots, f_m$ are **independent in the point** $a \in A$, if they are not dependent in any neighborhood V of the point a, $V \subset A$. We say that $f_1, \ldots, f_m$ are **independent in** A if they are independent in every point of A.*

With this definition, from Theorem 6.15.1 results:.

Corollary 6.15.1 *If $F = (f_1, \ldots, f_m) : A \to \mathbb{R}^m, F \in C^1(A)$ and.*
rang $J_F(x) = m$, $\forall x \in A$, *then the functions $f_1, \ldots, f_m$ are independent in A.*

Corollary 6.15.2 *If the Jacobian $\frac{D(f_1, \ldots, f_m)}{D(x_1, \ldots, x_m)}(x) \neq 0$, $\forall x \in A$, then the functions $f_1, \ldots, f_m$ are independent in A.*

In the following we present, without proof, the general theorem of functional dependence and independence:

Theorem 6.15.2 *Let $A \subset \mathbb{R}^n$ be an open subset, $m \leq n$, $a \in A$ and the vector function $F = (f_1, \ldots, f_m) : A \subset \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1(A)$.*

If rang $J_F(a) = s < m \leq n$, *then there exists an open neighborhood U of the point a, $U \subset A$ such that s functions between the functions $f_1, \ldots, f_m$ are independent in U, and the other $m - s$ functions depend on them in U.*

Example 6.15.2 If we return to Example 6.15.1 we find that:

$$J_F(x) = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \\ 1 & 1 & 1 \\ 2(x_2 + x_3) & 2(x_1 + x_3) & 2(x_1 + x_2) \end{pmatrix}.$$

It is easy to verify that $\det J_F(x) = 0$, $\forall x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Let us denote by $M = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3;\; x_1 = x_2 = x_3\right\}$.

and by $A = \mathbb{R}^3 \backslash M$.
If $(x_1\,,\ x_2\,,\ x_3) \in A$, then at least one of the minors of order 2:

$$\begin{vmatrix} 2\,x_1 & 2\,x_2 \\ 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2\,x_1 & 2\,x_3 \\ 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2\,x_2 & 2\,x_3 \\ 1 & 1 \end{vmatrix}$$

is different from zero. It results that the functions $f_1\,,\ f_2$ are independent in A, and f_3 depend on f_1 and f_2 in A.

References

1. Ilyn, V.A., and E.G. Pozniak. 1982. *Fundamentals of Mathematical Analysis.* Moscow: Mir Publishers, Part 1 and Part 2, 1982.
2. Krasnov, M., A. Kiselev, G. Makarenko, and E. Shikin. 1990. *Mathematical Analysis for Engineers.* Moscow: Mir Publishers, Vol 1, 1989 and Vol 2, 1990.
3. Lange, S. 1987. *Calculus of Several Variables,* Third Edition. New York Inc.: Spinger.
4. Nikolsky, S.M. 1977. *A Course of Mathematical Analysis.* Moscow: Mir Publishers, Vol 1 and Vol 2, 1977.
5. Paltineanu, G., and M. Zamfir. 2015. *Higher Mathematics 1 (Differential Calculus).* Bucuresti: Conspress (In Romanian).
6. Rudin, W. 1964. *Principles of Mathematical Analysis,* McGraw-Hill, Inc.
7. Rudin, W. 1987. *Real and Complex Analysis*, Third Edition, McGraw-Hill, Inc.
8. Stanasila, O. 1981. *Mathematical Analysis,* E.D.P.Bucuresti (In Romanian).
9. Trench, W.F. 2003. *Introduction to Real Analysis,* Library of Congress Cataloging-in-Publication Data.

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