NUMERICAL INTEGRATION

Aim: To find $\int_a^b f(x)\,dx$ where $f(x)$ is actually difficult to integrate using normal techniques.

Let $f(x) \approx \phi(x)$

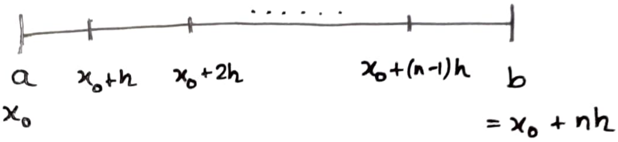
$$\Rightarrow \int_a^b f(x)\,dx - \int_a^b \phi(x)\,dx = \text{Error Term}$$

Methods of Numerical ~~Interp~~ Integration:

1. **Trapezoidal rule:** (Assume $f(x)$ is continuous)

Let $a = x_0$

$b = x_0 + nh$



$$\Rightarrow h = \frac{b - x_0}{h} = \frac{b-a}{h}$$

To find $\phi(x)$, we use Newton's Forward interpolation polynomial: (Here $y(x) \approx \phi(x)$)

$$\Rightarrow \phi(x) = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{\lfloor 2} + \dots u(u-1)\cdots(u-\overline{n-1})\frac{\Delta^n y_0}{\lfloor n}$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx \approx \int_{x_0}^{x_0+nh} \phi(x)\,dx.$$

$$\Rightarrow \int_{x_0}^{x_0+nh} \phi(x)\,dx = \int_{x_0}^{x_0+nh} \left[y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{\lfloor 2} + \dots \right] dx$$

If $u = \dfrac{x - x_0}{h}$

$$du = \frac{dx}{h} \Rightarrow dx = h\,du \Rightarrow \left[\begin{array}{l} \text{If} \\ x = x_0, \quad u = 0 \\ x = x_0 + nh, \quad \cancel{u} = n \end{array}\right]$$

So,

$$\phi(x) = \int_0^n \left[y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{\lfloor 2} + \cdots \right] h\,du$$

Put $n = 1$. (only one interval)

$$\int_{x_0}^{x_0+h} f(x)\,dx \approx \int_0^1 [y_0 + u\Delta y_0]\, h\,du$$

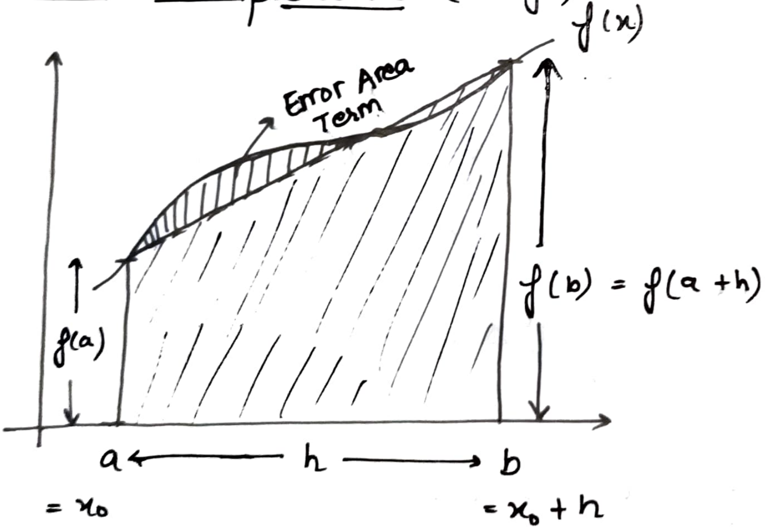
$$= h\left[y_0 + \frac{\Delta y_0}{2} \right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}[y_0 + y_1] = \frac{h}{2}[f(x_0) + f(x_1)]$$

$$= \frac{h}{2}[f(x_0) + f(x_0+h)]$$

<u>Geometrical</u> <u>Interpretation</u> (Single)



So,

Area of trapezoid = $\frac{1}{2}$ (sum of $\parallel$ sides) $\times$ $\perp$ distance

$$= \frac{1}{2}(f(a) + f(a+h)) \times h$$

$$= \frac{1}{2}(f(x_0) + f(x_0+h)) \times h.$$

Now, from a to b with n sub intervals.

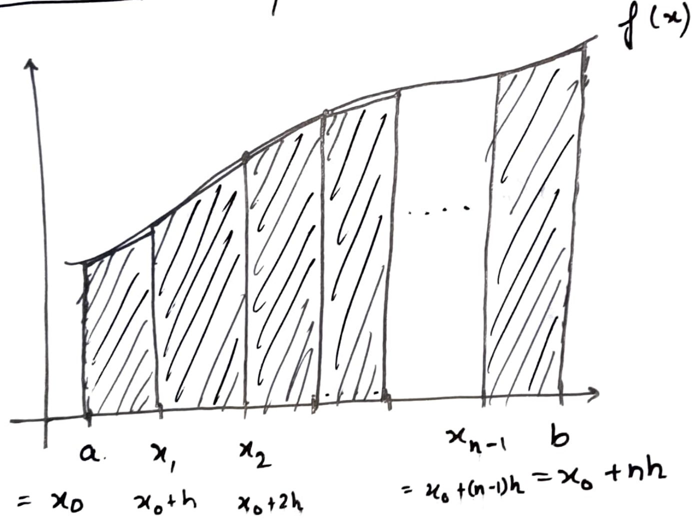
$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots$$

$$+ \cdots \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx.$$

$$= \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \frac{h}{2}[y_2 + y_3] + \cdots + \frac{h}{2}[y_{n-1} + y_n]$$

$$= \left\{ \frac{h}{2}\left[(y_0 + y_n) + 2[y_1 + y_2 + \cdots + y_{n-1}]\right] \right.$$
(Composite trapezoidal rule) }

<u>Geometrical Interpretation</u> (composite)



$$[a, x_1] = A_1 = \frac{h}{2}[f(x_0) + f(x_0+h)] = \frac{h}{2}[y_0 + y_1]$$

$$= A_2 = \frac{h}{2}[f(x_1) + f(x_2)] = \frac{h}{2}[y_1 + y_2]$$

$$\vdots$$

$$A_n = \frac{h}{2}[f(x_0+(n-1)h) + f(x_0+nh)] = \frac{h}{2}[y_{n-1} + y_n]$$

$\therefore$ Total approx area $= A_1 + A_2 + \cdots + A_n$.

So, if n increases, the solution will be more accurate and very close to exact solution

Error:

If $f(x)$ is continuous in $[x_0, x_0+h]$ and possesses continuous derivatives of all orders, then $\exists$ a function $F(x)$ in (x_0, x_0+h) such that $F'(x) = f(x)$ [Fundamental theorem of integration]

$$\int_{x_0}^{x_0+h} f(x)\,dx = \cancel{F(x_0+h) - F(x_0+h)}$$
$$F(x_0+h) - F(x_0)$$

Exact value

$$* \int_{x_0}^{x_0+h} f(x)\,dx = F(x_0+h) - F(x_0)$$

Using Taylor's theorem,

$$= \left[\cancel{F(x_0)} + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \cdots\right] - \cancel{F(x_0)}$$

$$= hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \cdots \quad -(1)$$

Apprx. value of $\int_{x_0}^{x_0+h} f(x)\,dx \approx \frac{h}{2}[f(x_0) + f(x_0+h)]$

$$= \frac{h}{2}\left[f(x_0) + f(x_0) + \frac{hf'(x_0)}{1!} + \frac{h^2}{2!}f''(x_0) + \cdots\right] \quad -(2)$$

$= I$ (Approx)

In (1) $F'(x) = f(x)$

So,

$$(1) \Rightarrow hf(x_0) + \frac{h^2}{2!}f'(x_0) + \frac{h^3}{3!}f''(x_0) + \cdots$$

$= I$ (Exact)

$\therefore$ Error: (For one interval)

$$I_E - I_A$$

$$= \left(h f(x_0) + \frac{h^2}{\underline{|2}} f'(x_0) + \frac{h^3}{\underline{|3}} f''(x_0) + \cdots\right) - \left(h f(x_0) + \frac{h^2 f'(x_0)}{\underline{|2}} + \frac{h^3}{2 \cdot \underline{|3}} f + \cdots\right)$$

$$= \left(\frac{1}{6} - \frac{1}{4}\right) f''(x_0) \cdot h^3 + \cdots$$

$$= -\frac{h^3}{12} f''(x_0) + \cdots$$

$$\approx -\frac{h^3}{12} f''(x_0).$$

$$E = -\frac{h^3}{12} f''(\xi) \qquad \text{where } x_0 < \xi < x_0 + h.$$

- If h is very small, then the corresponding error will also be very small.

- If we take any first degree polynomial of x then always

Approx = Exact value

$$\Rightarrow \text{Error} = -\frac{h^3}{12} f''(x)$$

$$= -\frac{h^3}{12}(0) = 0 \qquad \left[\begin{array}{l} \because f(x) = a_0 + a_1 x \\ \therefore f''(x) = 0 \end{array}\right]$$

* For the composite trapezoidal rule:

$$\text{Error} = -\frac{h^3 . n}{12} f''(\xi) \qquad \text{where } x_0 < \xi < x_0 + h.$$

and n = No. of sub intervals.

Algorithm:

1. Input : $f(x)$, a, b, n
2. Output : Approx. value of $\int_a^b f(x)\,dx$.

Steps : 1. $h = \dfrac{b-a}{n}$

2. Compute $S_0 \cancel{=} \cancel{y_0} + y_r$

   for $i = 0, 1, 2 \ldots, n$

   $y_i = f(x_i)$ where $x_i = a + ih$

3. $S_1 = y_0 + y_n$

4. $S_2 = \sum_{i=1}^{n-1} y_i$

5. Result : $\dfrac{h}{2}[S_1 + 2S_2]$

Ex. Evaluate $\int_0^1 (4x - 3x^2)\,dx$ taking 10 intervals by trapezoidal rule. Compute the exact value. Find the absolute and relative error.

Ans.

Exact : $\int_0^1 (4x - 3x^2)\,dx$

$= [2x^2 - x^3]_0^1 = 2 - 1 = 1.$

Approx : $n = 10$, $a = 0$, $b = 1$

$f(x) = 4x - 3x^2$

$h = \dfrac{b-a}{n} = \dfrac{1-0}{10} = 0.1$

$\ldots 06236)] = 0.15\ldots \approx 0.15$

	x_i	$f(x)$
0	0	0
1	0.1	0.37
2	0.2	0.68
3	0.3	0.93
4	0.4	1.12
5	0.5	1.25
6	0.6	1.32
7	0.7	1.33
8	0.8	1.28
9	0.9	1.17
10	1.0	1

$$\text{Approx} = \frac{h}{2}[1 + 2(9.45)] = \cancel{5.225}\ 0.9950.$$

$$\text{Absolute error} = |1 - 0.9950| = 0.0050$$

$$\text{Relative error} = \frac{0.0050}{1} = 0.0050.$$

Ex. Evaluate $\int_0^5 \frac{1}{1+x}\,dx$ taking 10 intervals by Trapezoidal rule.

Ans. $n = 10$, $a = 0$, $b = 5$

$f(x) = \frac{1}{1+x}$, $h = \frac{5-0}{10} = 0.5$

$$\text{Exact} = \int_0^5 \frac{1}{1+x}\,dx = [\ln(1+x)]_0^5$$

$$= \ln 6.$$

Approx:

i	x_i	$f(x_i)$
0	0	1.000
1	0.5	0.6667
2	1.0	0.5000
3	1.5	0.4000
4	2.0	0.3333
5	2.5	0.2857
6	3.0	0.2500
7	3.5	0.2222
8	4.0	0.2000
9	4.5	0.1818
10	5.0	0.1667

Approx value $= \frac{0.5}{2}[1.1667 + 2(3.0397)]$

$= 1.8115$

<u>Simpson's $\frac{1}{3}$ rule</u> :

$$\int_{x_0}^{x_0+nh} f(x)\,dx \approx \int_0^n \left(y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{\lfloor 2} + \cdots\right) h \cdot du$$

If $n = 2$, (The interval is divided into 2 sub-intervals)

$$\int_{x_0}^{x_0+2h} f(x)\,dx \approx \int_0^2 \left(y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{\lfloor 2}\right) h\,du.$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\frac{\Delta^2 y_0}{\lfloor 2}\right]_0^2$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{\cancel{2}}{3}\frac{\Delta^2 y_0}{\cancel{\lfloor 2}}\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[\cancel{6y_0} + 6y_1 - \cancel{6y_0} + y_2 - 2y_1 + y_0\right]$$

$$= \boxed{\frac{h}{3}\left[y_0 + 4y_1 + y_2\right]} = \text{Approx integral.}$$

- <u>Composite rule</u>

$[a, b]$ is divided into n - intervals

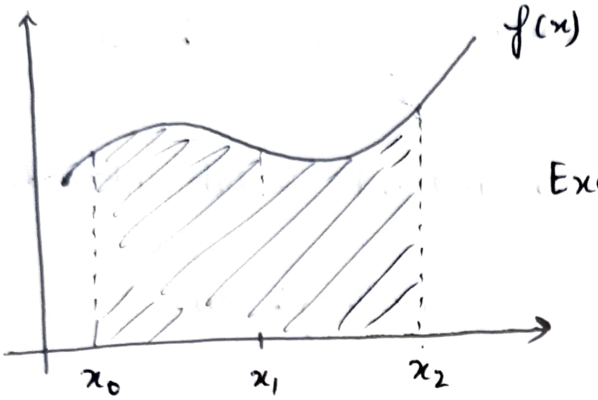
where $n = 2m$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \cdots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx.$$

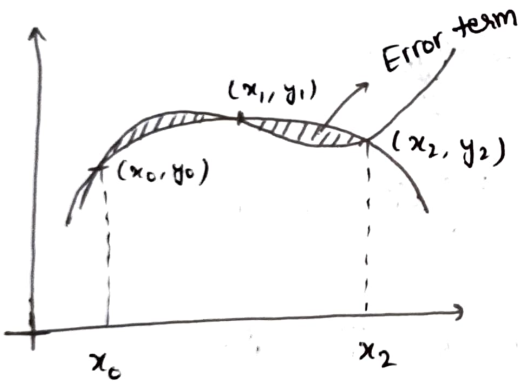
$$= \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \dots + \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right]$$

Geometrical Intepprretation:



Exact integral.



Approx. integral

Error:

$$I_{exact} = \int_{x_0}^{x_0+2h} f(x)\,dx = F(x_0+2h) - F(x_0)$$

Using Taylor's Theorem,

$$I_{exact} = \cancel{F(x_0)} + 2h\,F'(x_0) + \frac{(2h)^2}{\underline{|2}} F''(x_0) + \dots - \cancel{F(x_0)}$$

$$= 2h\,F'(x_0) + 2h^2\,F''(x_0) + \frac{8h^3}{\underline{|3}} F'''(x_0) + \dots$$

$$= 2hf(x_0) + 2h^2 f'(x_0) + \frac{8h^3}{3!} f''(x_0) + \cdots$$

$$I_{approx} = \frac{h}{3}[f(x_0) + 4f(x_0 + h) + f(x_0 + 2h)]$$

$$= \frac{h}{3}\left[f(x_0) + 4\left[f(x_0) + hf'(x_0) + \frac{h^2}{\lfloor 2} f''(x_0) + \cdots\right]\right.$$

$$\left. + \left[f(x_0) + 2hf'(x_0) + \frac{4h^2}{\lfloor 2} f''(x_0) + \cdots\right]\right]$$

$$= 2hf(x_0) + 2h^2 f'(x_0) + \frac{4}{3}h^3 f''(x_0) + \frac{2}{3}h^4 f'''(x_0) + \cdots$$

$\therefore$ Error: $= -\dfrac{h^5}{90} f^{iv}(\xi)$ where $a < \xi < b$

For composite rule:

Error: $= \dfrac{n}{2}\left(-\dfrac{h^5}{90} f^{iv}(\xi)\right)$

$$= -\frac{nh^5}{180} f^{iv}(\xi).$$

(or atmost 3)

- If the function is of the third order, then the error is always zero.

Algorithm:

Input: $f(x)$, a, b, n

Output: Approx value of $\int_a^b f(x)\,dx$

Steps: 1. Compute $h = \dfrac{b-a}{n}$

2. Compute $x_i = a + ih$
$y_i = f(x_i)$
for $i = 0, 1, 2, \ldots n$.

3. $S_1 = [y_0 + y_n]$

4. $S_2 = \sum_{j=1}^{m} y_{2j-1}$

5. $S_3 = \sum_{j=1}^{m-1} y_{2j}$

6. $S = \frac{h}{3} [S_1 + 4S_2 + 2S_3]$

<u>Ex</u>: Evaluate $\int_0^1 \frac{x}{1+x} dx$ correct upto 3 decimal places taking six intervals by Simpson's 1/3 rule.

<u>Ans</u>: $f(x) = \frac{x}{1+x}$ $a = 0$, $b = 1$, $n = 6$.

$h = \frac{b-a}{n} = \frac{1}{6}$.

i	x_j	$f(x_i)$
0	0	0
1	1/6	0.1429
2	2/6	0.2500
3	3/6	0.3333
4	4/6	0.4000
5	5/6	0.4545
6	6/6 = 1	~~1.0000~~ 0.500

$S_1 = 0 + 0.5000 = 0.5000$

$S_2 = 0.1429 + 0.3333 + 0.4545 = 0.9307$

$S_3 = 0.6500$

Approx value $= \frac{h}{3}[S_1 + 4S_2 + 2S_3]$

$$= \frac{1}{18}[\overset{0.5000}{\cancel{2.0000}} + 0.9307 \times 4 + 2 \times 0.6500]$$

$$= \frac{1}{18}[5.5228] = 0.3068 \approx 0.307.$$

Ex. Evaluate $\int_0^{\pi/2} \sin x \, dx$ correct upto 3 decimal places using Simpson 1/3 rule taking 10 intervals.

Ans. $f(x) = \sin x$, $a = 0$, $b = \frac{\pi}{2}$, $n = 10$,

$$h = \frac{\frac{\pi}{2} - 0}{10} = \frac{\pi}{20}.$$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{\pi}{20}$	0.1564
2	$\frac{2\pi}{20}$	0.3090
3	$\frac{3\pi}{20}$	0.4540
4	$\frac{4\pi}{20}$	0.5878
5	$\frac{5\pi}{20}$	0.7071
6	$\frac{6\pi}{20}$	0.8090
7	$\frac{7\pi}{20}$	0.8910
8	$\frac{8\pi}{20}$	0.9511
9	$\frac{9\pi}{20}$	0.9877
10	$\frac{\pi}{2}$	1

$S_1 = 1$

$S_2 = 3.1962$

$S_3 = 2.6569$

Approx value $= \frac{h}{3}[1 + 4(3.1962) + 2(2.6569)]$

$= \frac{\pi}{60}[19.0986] = 0.9999 \sim 1.000$ upto 3 decimal places.

$$\int_{x_0}^{x_0+nh} f(x)\,dx \approx \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \cdots\right] h\cdot du$$

$n = 1$ Trapezoidal rule

$n = 2$ Simpson's $1/3$ rule

$n = 3$ Simpson's $3/8$ rule.

<u>Simpson's 3/8 rule</u>:

$$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Error : $-\frac{3h^5}{80} f^{iv}(\xi)$, $a < \xi < b$.

<u>Composite rule</u>:

$n = 3m$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

Total m integrations

$$\frac{3h}{8}[S_1 + 3S_2 + 2S_3]$$

$$= \frac{3(0.1)}{8}[0.47000 + 3(1.01957) + 2(0.26236)] = 0.15200 \approx 0.$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6] +$$

$$\frac{3h}{8}[y_6 + 3y_7 + 3y_8 + y_9] + \cdots + \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

$$= \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})]$$

Error: (In composite rule):

$$= \frac{n}{\cancel{3}} \cdot \left(-\frac{\cancel{3}h^5}{80} f^{iv}(\xi)\right) \qquad a < \xi < b$$

$$= -\frac{nh^5}{80} f^{iv}(\xi) \qquad a < \xi < b.$$

Ex: Evaluate $\int_0^2 (4x - 3x^2)\,dx$ taking 9 intervals by Simpson's $\frac{3}{8}$ rule correct upto 3 decimal places.

Ans. $f(x) = 4x - 3x^2$, $a = 0$, $b = 2$, $n = 9$.

$h = \frac{2}{9}$.

i	x_i	$f(x_i)$
0	0	0
1	2/9	0.7407
2	4/9	1.1852
3	6/9	1.3333
4	8/9	1.1852
5	10/9	0.7407
6	12/9	0.0000
7	14/9	−1.0370
8	16/9	−2.3704
9	18/9 = 2	−4.0000

$S_1 = -4.0000$

$S_2 = 0.4444$

$S_3 = 1.3333$

Ans Approx value $= \frac{2}{9} \cdot \frac{3}{8} [-4.0000 + 3(0.4444) + 2(1.3333)]$

$= \frac{6}{72} [-0.0002]$

$\approx 0.0000 = 0.000$ upto 3 decimal place

Weddle's Rule:

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \cdots \right] h \cdot du.$$

If $n = 6$, we get Weddle's rule of integration

$$\int_{x_0}^{x_0+6h} f(x)\, dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$\boxed{n = 6m}$ Criteria for using Weddle's rule.

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \int_{x_0}^{x_0+6h} f(x) dx + \int_{x_0+6h}^{x_0+12h} f(x) dx + \int_{x_0+12h}^{x_0+18h} f(x)\, dx + \cdots$$

$$+ \cdots \int_{x_0+(n-6)h}^{x_0+nh} f(x)\, dx$$

$$\frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] + \frac{3h}{10} [y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$+ \cdots \frac{3h}{10} [y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n].$$

$$= \frac{3h}{10}\Big[y_0 + 5(y_1 + y_5 + y_7 + y_{11} + \cdots y_{n-5} + y_{n-1}) + (y_2 + y_4 + y_8 + \cdots y_{n-4} + y_{n-2}) + 6(y_3 + y_9 + \cdots y_{n-3}) + 2(y_6 + y_{12} + \cdots + y_{n-6})] + y_n\Big]$$

This is the composite Weddle rule!

Error: $E_W = -\frac{h^7}{140} f^{vi}(\xi) \qquad a < \xi < b.$

$$E_W^{C} = -\frac{nh^7}{(140)(6)} f^{vi}(\xi) \qquad a < \xi_c < b$$

Ex: $\int_0^4 \frac{\tan^{-1}x}{1+x^2}\,dx$. Use Weddle's rule using $n=12$ correct upto 4 decimal places.

Ans. $f(x) = \frac{\tan^{-1}x}{1+x^2}$, $a=0$, $b=4$, $h = \frac{4}{12} = \frac{1}{3}$.

i	x_i	$f(x_i)$
0	0	0
1	1/3	0.2896
2	2/3	0.4071
3	3/3 = 1	0.3927
4	4/3	0.3338
5	5/3	0.2727
6	6/3 = 2	0.2214
7	7/3	0.1809
8	8/3	0.1494

9	$9/3 = 3$	0.1249
10	$10/3$	0.1056
11	$11/3$	0.0903
12	$12/3 = 4$	0.0780

$S_1 = 0.0780$

$S_2 = 0.8335$

$S_3 = 0.9954$

$S_4 = 0.5176$

$S_5 = 0.2214$

$$\text{Ans} = \frac{\cancel{3} \times \frac{1}{\cancel{3}}}{10} \left[0.0780 + 5(0.8335) + 0.9954 + 6(0.5176) + 2 \cdot (0.2214) \right]$$

$$= 0.8789.$$

Using Simpson's 1/3 rule.

$S_0 = 0.0780$

$S_1 = 1.3511$

$S_2 = 1.2099$

$$\text{Ans} = \frac{1}{3 \times 3} \left[0.0780 + 4(1.3511) + 2(1.2099) \right]$$

$$= 0.8780$$