DIAGONALIZATION OF A MATRIX

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Example. *$A = \begin{bmatrix} 4 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 3, \lambda_2 = 5$ and eigenvectors $\vec{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{x_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$*

$$A\vec{x_1} = \lambda_1 \vec{x_1} \tag{0.1}$$

$$A\vec{x_2} = \lambda_2 \vec{x_2} \tag{0.2}$$

Let $X = \begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix}$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

We can rewrite equations (0.1) as

$$A \begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{x_1} & \lambda_2 \vec{x_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{(\lambda_1 \vec{x_1} = \vec{x_1} \lambda_1)}$$

$$AX = X\Lambda$$

Note that $\vec{x_1}$ and $\vec{x_1}$ are linearly independent, hence $X = \begin{bmatrix} \vec{x_1} & \vec{x_2} \end{bmatrix}$

$$AX = X\Lambda$$
$$X^{-1}AX = X^{-1}X\Lambda$$
$$X^{-1}AX = \Lambda$$

Date: May 31, 2021.

$\Rightarrow$ Transforms A into a diagonal matrix. A is diagonlized by its eigenvectors.

1. Eigendecomposition

$$\begin{aligned} AX &= X\Lambda \\ AXX^{-1} &= X\Lambda X^{-1} \\ A &= X\Lambda X^{-1} \end{aligned}$$

Matrix decomposition of A into eigenvector/values
Eigendecomposition of A

2. Conditions for validity

Both $A = X\Lambda X^{-1}, X^{-1}AX = \Lambda$ hold true for general nxn matrices under the following conditions.

(1) Eigenvector matrix X is invertible, which is true when the eigenvectors $\vec{x_1}, \vec{x_2}, \cdots, \vec{x_n}$ are linearly independent.

Theorem 1. *Eigenvectors $\vec{x_1}, \vec{x_2}, \cdots, \vec{x_j}$ that correspond to distinct eigenvalues (all different) are linearly independent. Moreover, an nxn matrix that has n different eigenvalues (no repeated numbers) must be diagonalizable.*

Proof. Assume $c_1\vec{x_1} + c_2\vec{x_2} = \vec{0}$ (*) Multiply equation (*) by A:

$$Ac_1\vec{x_1} + Ac_2\vec{x_2} = \vec{0}$$

$$c_1A\vec{x_1} + c_2A\vec{x_2} = \vec{0}$$

$$c_1\lambda_1\vec{x_1} + c_2\lambda_2\vec{x_2} = \vec{0}(**)$$

Multiply eq(*) by λ_2

$$\lambda_2c_1\vec{x_1} + \lambda_1c_2\vec{x_2} = \vec{0}$$

$$c_1\lambda_2\vec{x_1} + c_2\lambda_1\vec{x_2} = \vec{0}(***)$$

Now subtract eq(***) from eq(**)

$$c_1\lambda_1\vec{x_1} + c_2\lambda_2\vec{x_2} - (c_1\lambda_2\vec{x_1} + c_2\lambda_1\vec{x_2}) = \vec{0}$$

$$c_1(\lambda_1 - \lambda_2)\vec{x_1} = \vec{0}$$

Since $\lambda_1 \neq \lambda_2$ (by assumption)
$\Rightarrow c_1 = 0$
Repeating steps with λ_1

$$\Rightarrow c_2(\lambda_1 - \lambda_2)\vec{x_2} = \vec{0}$$
$$\Rightarrow c_2 = 0$$

$\Longrightarrow \vec{x_1}$ and $\vec{x_2}$ are linearly independent.
When all $\vec{x_1}, \vec{x_2}, \cdots, \vec{x_j}$, $X = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \cdots & \vec{x_j} \end{bmatrix}$ is invertible. $\square$

3. Powers of diagonalizable matrix

Let A be diagonalizable $A = X\Lambda X^{-1}$
Then $A^2 = X\Lambda \underbrace{X^{-1}X}_{I} \Lambda X^{-1} = X\Lambda^2 X^{-1}$
Generally $A^k = X\Lambda^k X^{-1}$
If A is invertible, then Λ, X are invertible and
$A^{-1} = \underbrace{X\Lambda X^{-1}}_{A} = (X^{-1})^{-1}\Lambda^{-1}X^{-1} = X\Lambda^{-1}X^{-1}$

$$A^{-1} = X\Lambda^{-1}X^{-1}$$