SYMMETRIC MATRICES

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Recall: S is a symmetric matrix if $S = S^T$

Theorem 1. *If S is a symmetric matrix, then all its eigenvalues are real values.*

Proof. Let $S\vec{x} = \lambda\vec{x}$. Λ could be complex-valued: $\Lambda = a + ib$, $i = \sqrt{-1}$. We want to show that Λ is real-valued, i.e. $b = 0$.

Let $\bar{\Lambda} = a - ib$ be the conjugate of Λ and $\bar{\vec{x}}$ be the conjugate of $\vec{x}$.

Know: $\bar{\Lambda\vec{x}} = \bar{\Lambda}\bar{X}$

Also know: $S = \bar{S}$, since S is real-valued.

$$\begin{aligned}
S\vec{x} &= \lambda\vec{x}, \text{ Now take conjugate on both sides} \\
\bar{S\vec{x}} &= \bar{\lambda\vec{x}} \\
\bar{S}\bar{\vec{x}} &= \bar{\lambda}\bar{\vec{x}}, \text{S is real valued, so } \bar{S} = S \\
S\bar{\vec{x}} &= \bar{\lambda}\bar{\vec{x}}, \text{ Now take the transpose on both sides} \\
(S\bar{\vec{x}})^T &= (\bar{\lambda}\bar{\vec{x}})^T \\
\bar{\vec{x}}^T S^T &= \bar{\vec{x}}^T \bar{\lambda}, \text{ S is symmetric, so } S = S^T \\
\bar{\vec{x}}^T S &= \bar{\vec{x}}^T \bar{\lambda}, \text{ Now take product with } \vec{x}
\end{aligned}$$

$$\bar{\vec{x}}^T S x = \bar{\vec{x}}^T \bar{\lambda} x \tag{1.1}$$

Now consider $S\vec{x} = \lambda\vec{x}$ take dot product with $\vec{x}$

$$\bar{\vec{x}}^T S \vec{x} = \bar{\vec{x}}^T \lambda \vec{x} \tag{1.2}$$

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Left sides of (1.1) and (1.2) are equal $\Rightarrow$ right sides are equal.

$$\Rightarrow \bar{\vec{x}}^T \bar{\lambda} \vec{x} = \bar{\vec{x}} \lambda \vec{x}$$

$$\bar{\lambda} \underbrace{\bar{\vec{x}}^T \vec{x}} = \lambda \underbrace{\bar{\vec{x}}^T \vec{x}}$$

$$\text{These terms are equal: } \bar{\vec{x}}^T = \sum_{k=1}^{n} \bar{x}_k x_k = \sum_{k=1}^{n} |x_k^2|$$

$$\Rightarrow \bar{\vec{x}}^T \vec{X} \neq 0$$

$$\Rightarrow \bar{\lambda} = \lambda$$

$$\Rightarrow a - ib = a + ib \Rightarrow b = 0 \Rightarrow \lambda \text{ is real}$$

$\square$