EIGENVECTORS AND EIGENVALUES

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Contents

Definition 1. *Let A be an nxn matrix. An* ***eigenvector*** *of A is a nonzero vector $\vec{v}$ in $\mathbb{R}^n$ such that $Av = \lambda v$, for some scalar λ. An* ***eigenvalue*** *of A is a scalar λ such that the equation $Av = \lambda v$ has a nontrivial solution.*

$$A\vec{x} = \lambda\vec{x} \qquad \lambda \text{ is an } \textit{eigenvalue} \text{ of A} \tag{0.1}$$
$$\vec{x} \text{ is an } \textit{eigenvector} \text{ of A} \tag{0.2}$$
$$\text{if equation is true} \tag{0.3}$$

$\Rightarrow$ "eigen" = "self" (from German)

λ tells us by how much we shrink (if $\|\lambda\| < 1$) or stretch (if $\|\lambda\| > 1$)$\vec{x}$ and reverse direction (if $\lambda < 0$)

Theorem 1. *if $\vec{x}$ is an eigenvector of A, then so is $\lambda\vec{x}$ for any* $\alpha \neq 0$

Proof.

$$A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha\lambda\vec{x} = \lambda(\alpha\vec{x})$$

$\alpha \& \lambda$ are both scalars, so we can move them around

□

We have infinitely many eigenvectors, so we typically normalize them

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Example 1. *Let*

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow A\vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, A\vec{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$A\vec{v} = 3\vec{v} \Rightarrow \lambda = 3$$

Step: Normalize $\vec{v}$

$$\|\vec{v}\| = \sqrt{2}$$

Eigenvector is

$$\vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \|\vec{x}\| = 1$$

Then

$$A\vec{x} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} * \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{3}_{\lambda} * \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}}$$

$\Rightarrow \vec{x}$ *is an* eigenvector *of A with an associated* eigenvalue *of 3.*

Example 2. *Is* $\lambda = 5$ *true with eigenvector* $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$*?*

$$\underbrace{\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \underbrace{5}_{\lambda} * \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}}$$

Two eigenvectors $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and eigenvalues* $\lambda_1 = 3, \lambda_2 = 5$
Are there more eigenvalues in this case? No, how do we find that?

1. How we find eigenvalues and eigenvectors

$$\begin{aligned} &A\vec{x} = \lambda\vec{x} \\ &\Leftrightarrow A\vec{x} - \lambda\vec{x} = \vec{0} \\ &\Leftrightarrow (A - \lambda I)\vec{x} = \vec{0} \end{aligned}$$

This means $\vec{x}$ is an eigenvector of A if and only if $\vec{x}$ is in the *nullspace* of $(A - \lambda I)$

$$\vec{x} \in \mathcal{N}(A - \lambda I) \tag{1.1}$$

But we do not know $\lambda \Rightarrow$ we do not know $A - \lambda I$ yet.

- $(A - \lambda I)$ has a non trivial nullspace (= nullspace contains not just the $\vec{v}$)
  - This is true if A is invertible, because invertible matrices have only $\vec{0}$ in the nullspace.
  - if $det(A - \lambda I) = 0$
- $\Rightarrow$ Find those λ for which $det(A - \lambda I) = 0$

Example 3.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$det(A - \lambda I) = det \underbrace{\begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}}_{A-\lambda I}$$

$$det \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (4-\lambda)^2 - 1^2 = \underbrace{0}_{Nullspace}$$

$$\Rightarrow \lambda^2 - 8\lambda + 15 = 0$$

$$\Rightarrow \underbrace{\lambda_1 = 3, \lambda_2 = 5}_{Eigenvalues\ of\ A}$$

Example 4. ***Finding eigenvectors***

Condition: equation (1.1)

$$\lambda = 3 : A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathcal{N}(A - \lambda I) : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = t, x_1 = t$$

$$\vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} \textit{for some } t \neq 0$$

We pick arbitrary t, want unit vector

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\lambda = 5 : A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = t, x_1 = -t$$
$$\Rightarrow \vec{x} = \begin{bmatrix} -t \\ t \end{bmatrix}$$
$$\textit{Normalize } \vec{x}$$
$$\boxed{\vec{x_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

1.1. Summary.

Summary

$$\Rightarrow \lambda_1 = 3, \quad \vec{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \lambda_2 = 5, \quad \vec{x_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1.2. Steps.

For an mxn matrix A

(1) Find eigenvalues
  (a) Find λ such that $(A - \lambda I) = 0$
  (b) $(A - \lambda I) = 0$ returns a polynomial of degree n.
  (c) Find the roots of this polynomial
(2) Find eigenvectors
  (a) For each λ solve $(A - \lambda I)\vec{x} = 0$ to find the associated eigenvectors $\vec{x_1}, \vec{x_2}, \cdot$
  (b) Ensure these vectors are normalized
(3) Note: $\lambda_1 * \lambda_2 = 3 * 5 = det(A) = 15$
(4) and $\lambda_1 + \lambda_2 =$ Sum of diagonal entries of A. (called trace of A)

2. Eigen Stuffs

$$A^p \vec{x} = \lambda^p \vec{x} \tag{1.2}$$

$$A^{-1}\vec{x} = \lambda^{-1}\vec{x} \text{ if A is invertible} \tag{1.3}$$

A, A^p, A^{-1} have some eigenvector with associated eigenvalues $\lambda, \lambda^p, \lambda^{-1}$

Let A be a tridiagonal matrix, then the eigenvalues of A are its diagonal entries

Ex: $A = \begin{bmatrix} 4 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$
$\Rightarrow$ eigenvalues are $\lambda_1 = 4, \lambda_2 = -1, \lambda_3 = -1$
We also say that eigenvalue $\lambda_2 = -1$ has a multiplicity of 2

2.1. **Eigenvalues and Eigenvectors of A+B, A*B.** If λ is an eigenvalue of A, and μ is an eigenvalue of B, then it is generally not true that $\lambda + \mu$ is an eigenvalue of A+B, $\lambda * \mu$ is an eigenvalue of A*B.