CIRCULANT MATRICES

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Definition 1. *An nxn matrix C is called circulant if C =*

$$\begin{bmatrix} c_1 & c_n & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_n & \cdots & \cdots \\ c_3 & c_2 & c_1 & \cdots & \cdots \\ \cdots & c_3 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & \cdots & c_3 & \cdots & c_n \\ c_n & c_{n-1} & c_{n-1} & \cdots c_2 & c_1 \end{bmatrix}$$

Example.

$$C = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

1. Discrete Fourier Transform

Definition 2. *The nxn DFT matrix is given by*

$$F_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & omega^9 & \cdots & \omega^{3(n-1)} \\ & & & \cdots & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

where $\omega = e^{2\pi i/n}$ *and* $i = \sqrt{-1}$

Theorem 1. *A circulant matrix is diagonalized by the DFT matrix. Then* $F_n^{-1} C F_n = \Lambda$ *where* $\Lambda =$ *diagonal matrix, containing eigenvalues of C.*

Remark. $F_n^{-1} = F_n^H$ *and* $C^{-1} = F_n \Lambda^{-1} F_n^H$

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Multiplication by the DFT matrix F_N can be done in $n \log n$ computations instead of n^2 computations. Done via Fast Fourier Transforms.