DETERMINANTS

DYLAN ANG

Contents

Definition 1. *The determinant of a square matrix is a single number which carries important info about the matrix. We write det A.*

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ Then } det(A) = ad - bc \tag{0.1}$$

$$\text{Recall that } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow A \text{ is invertible} \Leftrightarrow det(A) \neq 0$$

$$\text{Can be written } det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

We can define det(A) for any $n * n$ matrix recursively

(0.2)

For 3 by 3 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{0.3}$$

$$det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{0.4}$$

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1. Rules for Determinant

(1) $det(A)$ changes signs when we exchange two rows.
(2) $det(A)$ is unchanged if a multiple of one row is subtracted from another row.
  - $\Rightarrow A$ and U have the same determinant.

(3) For a triangular matrix, it's determinant is the product of the diagonal entries.
(4) A matrix with a row of zeros has a $det(A) = 0$.
  - $\Rightarrow$ if A has two identical rows $\Rightarrow det(A) = 0$.

(5) $det(A) =$ product of it's pivots.
(6) $det(A) \neq 0$ if and only if A is invertible.
(7) $det(A) = det(A^T)$.

Theorem 1.

$$det(AB) = det(A) * det(B)$$
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

Proof.

$$\begin{aligned} det(A) * det(B) &= (ad - bc) * (ps - qr) \\ &= adps - bcps - adqr + bcqr \\ &= (ad + br)(cq + ds) - (aq + bs)(cp + dr) \\ AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp * dr & cq + ds \end{bmatrix} \\ det(AB) &= (ap + br)(cq + ds) - (cp + dr)(aq + bs)\square \end{aligned}$$

$\square$

2. Linearity of Determinant

$$\begin{vmatrix} xa + yA & xb + yB \\ c & d \end{vmatrix} = x(ad - bc) + y(Ad - Bc)$$

$$\Rightarrow \text{det is linear in row 1}$$

Example 1.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 1*0*1+4*1*3+(-2)*2*1 \\ &= (-2*0*3-1*1*1-4*2*1) \\ &= 0+12-4-(0-3-8) \\ &= 8+11=19 \end{aligned}$$

Example 2. ***Identity.***

$$det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

Example 3. ***Row Exchanges.*** *By Rule 1 for determinants, two row exchanges $\rightarrow$ determinant flips sign*

$$det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1$$

$$det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = -(-1) = 1$$

This happens because the $det(I^4) = 1$

3. Example: Finding Determinant

Example 4. ***Finding Determinant.*** *Compute*

$$\begin{aligned} det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 4 & 4 & 1 & 1 \end{bmatrix} &= \Leftarrow \textit{lower-triangular matrix} \\ &= 1*1*0*1 = 1 \end{aligned}$$

Why?

- *Lower-triangular matrices are invertible if and only if diagonal entries are not 0.*
- *Therefore, this matrix is not invertible.a*
- *determinant of 0 means non-invertible.*
- *Therefore the determinant is 0.*

Theorem 2. *If Q is an orthogonal square matrix, then $det(Q) = \pm 1$ Prove with $Q^T Q = I$, $Q^T = Q^{-1}$, and $det(A)det(A^{-1}) = det(AA^{-1})$*

4. Parallelogram Theorem

Theorem 3. ***Parallelogram*** *Let P be the parallelogram spanned by the rows of A. Then*

$$vol(P) = \lfloor det(A) \rfloor \tag{3.1}$$

Example 5.

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$
$$\lfloor det(A) \rfloor = \lfloor 4(3) - 1(2) \rfloor$$
$$area\ of\ parallelogram = 12 - 2 = 10$$

5. With LU Factorization

$$det(AA) = det(LU) = det(L) * det(U) \tag{3.2}$$

Example 6.

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$
$$A = LU$$
$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}$$
$$U = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}$$
$$\begin{aligned} det(A) &= det(L)det(U) \\ &= (1 * 1 * 1) * (4 * 3 * \frac{8}{3}) = 32 \end{aligned}$$