SIMILAR MATRICES

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Definition 1. *We call two matrices A,B similar if there exists an invertible matrix C such that* $A = C^{-1}BC$ *(or* $B = CAC^{-1}$*)*

Theorem 1. *Similar matrices have the same eigenvalues.*

Proof. Assume A,B are similar $\rightarrow A = C^{-1}BC$. Consider $A\vec{x} = \lambda\vec{x}$. Then

$$\underbrace{(C^{-1}BC)}_{A}\vec{x} = \lambda\vec{x}$$

$$\underbrace{CC^{-1}}_{I}BC\vec{x} = C\lambda x$$

$$B\underbrace{C\vec{x}}_{\vec{v}} = \lambda C\vec{x}, \text{ denote } \vec{v} = C\vec{x}$$

$$B\vec{v} = \lambda\vec{v}$$

$$\Rightarrow \lambda \text{ is an eigenvalue of B with eigenvector } \vec{v}(= C\vec{x})$$

□

Theorem 2. *Suppose X diagonalizes both A and B. Then* $AB = BA$ *(i.e., A and B commute).*

Proof.

$$X^{-1}AX = \Lambda_1, X^{-1}BX = \Lambda_2 \quad \text{, (A,B have the same eigenvectors)}$$

$$\Rightarrow A = X\Lambda_1X^{-1}, B = X\Lambda_2X^{-1}$$

$$AB = X\Lambda_1\underbrace{X^{-1}X}_{I}\Lambda_2X^{-1}$$

$$= X\Lambda_1\Lambda_2X^{-1}$$

$$= X\Lambda_2\Lambda_1X^{-1} \quad \text{, } (\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1) \text{ since diagonal matrices commute}$$

$$= \underbrace{X\Lambda_2}_{B}\underbrace{X^{-1}X}_{I}\underbrace{Lambda_1X^{-1}}_{A}$$

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$\square$

Example.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\textit{eigenvectors of A:} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\textit{eigenvectors of B:}$$

$$det(B - \lambda I) = 0$$

$$det \begin{bmatrix} 3 - \lambda 1 & \\ 1 & 3 - \lambda I \end{bmatrix} = 0 \Rightarrow \cdots \mu_1 = 2, \mu_2 = 4$$

$$\Rightarrow \textit{B has same eigenvalues as A}$$

$$\Rightarrow \textit{B,A commute}$$

$$AB = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 7 \\ 7 & 13 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 13 & 7 \\ 7 & 13 \end{bmatrix}$$