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Answer 1

a) First, we should check the basis step. For n = 1 :
$6^2 - 1 = 35 = 5 * 7$ Hence, it is divisible by both 5 and 7.

b.1) For $n = k$:
Suppose k is a positive integer. If $6^{2k} - 1$ is divisible by 5, then there exists an integer x such that
$6^{2k} - 1 = 5x$
$6^{2k} = 5x + 1$

b.2) For $n = k$:
Suppose k is a positive integer. If $6^{2k} - 1$ is divisible by 7, then there exists an integer y such that
$6^{2k} - 1 = 7y$
$6^{2k} = 7y + 1$

c.1) For $n = k + 1$:

- $6^{2(k+1)} - 1 = 6^{2k} * 6^2 - 1$
- $6^2 * 6^{2k} - 1 = 6^2 * (5x + 1) - 1$
- $= 180x + 36 - 1 = 180x + 35$
- $= 5(36x + 7)$
- Hence, $6^{2(k+1)} - 1 = 5z$
- $z = 36x + 7$

c.2) For $n = k + 1$:

- $6^{2(k+1)} - 1 = 6^{2k} * 6^2 - 1$
- $6^2 * 6^{2k} - 1 = 6^2 * (7y + 1) - 1$
- $= 252y + 36 - 1 = 252y + 35$
- $= 7(36y + 5)$
- Hence, $6^{2(k+1)} - 1 = 7t$

- $t = 36y + 5$

Clearly, $6^{2(k+1)} - 1$is divisible by both 5 and 7.
Therefore, the statement is true for all positive integers by the principle of mathematical induction.

Answer 2

- For the base cases, let's consider $n = 3$. We have that $H_3 = 8 * H_2 + 8 * H_1 + 9H_0 = 105$. We have $H_3 = 105 \leq 9^3 = 729$. So, the result is true when $n = 3$.

- For the strong inductive hypothesis, suppose that for some $n \geq 2$ we have that $H_k \leq 9^k$ for all $0 \leq k \leq n$. Consider our values $H_0 = 1 \leq 9^0, H_1 = 5 \leq 9^1, H_2 = 7 \leq 9^2$. Since these given values are coherent to our assumption we can now consider H_{n+1}. We have :

$$H_{n+1} = 8H_n + 8H_{n-1} + 9H_{n-2}$$

(Because $n + 1 \geq 3$)

$$H_{n+1} = 8H_n + 8H_{n-1} + 9H_{n-2} \leq 8 * 9^n + 8 * 9^{n-1} + 9 * 9^{n-2}$$

$$H_{n+1} = 8H_n + 8H_{n-1} + 9H_{n-2} \leq 8 * 9^2 * 9^{n-2} + 8 * 9 * 9^{n-2} + 9 * 9^{n-2}$$

$$H_{n+1} = 8H_n + 8H_{n-1} + 9H_{n-2} \leq 9^{n+1}$$

- Hence, $H_{n+1} \leq 9^{n+1}$

- Therefore, the result holds for $n + 1$ as well. Hence, by the Principle of Strong Induction we have $H_n \leq 9^n$ for all $n \in \mathbb{N}$.

Answer 3

Let's start with possible positions for bit strings of length 8 with 4 consecutive 1's. Totally 5 possible such positions exist.

1111xxxx (4 slots $\rightarrow 2^4$)
01111xxx (3 slots $\rightarrow 2^3$)
x01111xx (3 slots $\rightarrow 2^3$)
xx01111x (3 slots $\rightarrow 2^3$)
xxx01111 (3 slots $\rightarrow 2^3$)

A) 48 bit strings of length 8 contain 4 consecutive ones.

In the first position there are 4 slots which can be either 0 or 1. i.e. 2^4 bit strings. In each of the next positions, there are 3 slots which can be either 0 or 1 which means there are in total $2^4 + 2^3 * 4$ bit strings $= 48$

Note: Extra 0 is added from the second position to avoid counting duplicate bit strings twice.

B) 48 bit strings of length 8 contain 4 consecutive zeros.

It's just a mirror of the above problem so this is 48 bit strings too.

94 bit strings of length 8 contain either four consecutive 0s or four consecutive 1s.

A = bit strings of length 8 contain either four consecutive 1s.

B = bit strings of length 8 contain either four consecutive 0s.

By set theory, we know that:

$$\mathbf{A} \cup \mathbf{B} = \mathbf{A} + \mathbf{B} - (\mathbf{A} \cap \mathbf{B})$$

Here A $\cap$ B means that the bit string contains 4 consecutive ones and 4 consecutive zeroes. There are only two such possibilities:

11110000
00001111

Therefore, the answer is 48 + 48 - 2 = 94

Answer 4

Let's start with choosing our star, nonhabitable planets, and habitable planets.

- For choosing the star, we have 10 options. So it is $\binom{10}{1}$.
- For habitable planets, we have 20 options. So it is $\binom{20}{2}$.
- For nonhabitable planets, we have 80 options. So it is $\binom{80}{8}$.

$\rightarrow$ Since our problem says at least 6 nonhabitable planets between the 2 habitable ones, first consider the case of 6 nonhabitable planets:

HXXXXXXHXX
H: Habitable planets

- Our 6 nonhabitable planets can be inserted in 6! ways between two habitable planets.(6!)
- Our 2 habitable planets can interchange places. So we have 2! ways.(2!)
- Our other 2 nonhabitable planets that are not between habitable ones can interchange places. So we have 2!.
- Lastly, our galaxy can be formed in 2 other ways as follows(Total = 3 with the beginning form):

XHXXXXXXHX
XXHXXXXXXH

Hence, with 6 nonhabitable planets we have $\binom{10}{1} * \binom{20}{2} * \binom{80}{8} * \binom{8}{6} * 6! * 2! * 3$

$\rightarrow$ Let's consider the case of 7 nonhabitable planets:

HXXXXXXXHX
H: Habitable planets

- Our 7 nonhabitable planets can be inserted in 7! ways between two habitable planets.(7!)
- Our 2 habitable planets can interchange places. So we have 2! ways.(2!)
- Lastly, our galaxy can be formed in 1 other way as follows(Total = 2 with the beginning form):

XHXXXXXXXH

Hence, with 7 nonhabitable planets we have $\binom{10}{1} * \binom{20}{2} * \binom{80}{8} * \binom{8}{7} * 7! * 2! * 2$

$\rightarrow$ Finally let's consider the case of 8 nonhabitable planets:

HXXXXXXXXH
H: Habitable planets

- Our 8 nonhabitable planets can be inserted in 8! ways between two habitable planets.(8!)
- Our 2 habitable planets can interchange places. So we have 2! ways.(2!)

Hence, with 8 nonhabitable planets we have $\binom{10}{1} * \binom{20}{2} * \binom{80}{8} * \binom{8}{8} * 8! * 2$

If we add all of our solutions, our answer is:

$$\binom{10}{1} * \binom{20}{2} * \binom{80}{8} * 8! * 12$$

Answer 5

a) **Case-1** Let's consider the case where our robot lands one cell away:
There is only 1 way to go one cell.

(1)

Case-2 Consider the case where our robot lands two cells away:
There are 2 ways of doing it.

(1-1 or 2)

Case-3 Now consider the case where our robot lands three cells away:
There are 4 ways of doing it.

(1-1-1 , 2-1 , 1-2 , 3)

Case-4 Lastly, consider the case where our robot lands four cells away:
There are 7 ways of doing it.

(1-1-1-1 , 1-2-1 , 1-1-2 , 2-1-1 , 2-2 , 1-3 , 3-1)

As it can be seen clearly, our recurrence relation is $a_n = a_{n-3} + a_{n-2} + a_{n-1}$ for $n \geq 4$.

b) **Case-1** Let's consider the case where our robot lands one cell away:
There is only 1 way to go one cell.

(1)

Case-2 Consider the case where our robot lands two cells away:
There are 2 ways of doing it.

(1-1 or 2)

Case-3 Now consider the case where our robot lands three cells away:
There are 4 ways of doing it.

(1-1-1 , 2-1 , 1-2 , 3)

So, $a_1 = 1, a_2 = 2, a_3 = 4$ are initial conditions.

c) Since our recurrence relation is $a_n = a_{n-3} + a_{n-2} + a_{n-1}$, we can find a_9 easily.

$$\begin{gathered}
a_4 = a_1 + a_2 + a_3 = 7 \\
a_5 = a_2 + a_3 + a_4 = 13 \\
a_6 = a_3 + a_4 + a_5 = 24 \\
a_7 = a_4 + a_5 + a_6 = 44 \\
a_8 = a_5 + a_6 + a_7 = 81 \\
a_9 = a_6 + a_7 + a_8 = 149
\end{gathered}$$

Hence, the answer is 149.