CENG 223

Discrete Computational Structures

Fall 2022-2023

Take Home Exam 3 - Sample Solution

Answer 1

- **Basis Step:** For $n = 1$, $6^2 - 1 = 35$ which is clearly divisible by both 5 and 7.

- **Inductive Step:** For $k \geq 1$, suppose that $6^{2k} - 1$ is divisible by both 5 and 7. Under this assumption, it must be shown that the proposition is true for $k+1$, that is $6^{2(k+1)} - 1$, i.e. $6^{2k+2} - 1$ is divisible by both 5 and 7. Because of the assumption, we know that

$$6^{2k} - 1 = 5 \cdot 7 \cdot l$$
$$6^{2k} = 5 \cdot 7 \cdot l + 1,$$

where l is a positive integer. Multiplying both sides with 6^2, we get:

$$6^{2k+2} = 6^2 \cdot (5 \cdot 7 \cdot l + 1)$$
$$6^{2k+2} = 36 \cdot 5 \cdot 7 \cdot l + 36.$$

Subtracting 1 from both sides, we get:

$$6^{2k+2} - 1 = 36 \cdot 5 \cdot 7 \cdot l + 35$$
$$6^{2k+2} - 1 = 5 \cdot 7 \cdot (36l + 1).$$

Since l is an integer, we know that $(36l + 1)$ is also an integer. Therefore, we show that $6^{2k+2} - 1$ is divisible by both 5 and 7. This proves the given proposition.

Answer 2

- **Basis Step:** We have to show that given proposition holds for $k = 0, 1, 2$ to be able to construct the inductive step for any $k \geq 2$. We can easily observe that $H_0 \leq 9^0 = 1$, where $H_0 = 1$. In the same manner, $H_1 \leq 9^1 = 9$ for $H_1 = 5$ and $H_2 \leq 9^2 = 81$ for $H_2 = 7$. This concludes the basis step.

- **Inductive Step:** Assume that $H_j \leq 9^j$ for $0 \leq j \leq k$, where $k \geq 2$. Then we must show that $H_{k+1} \leq 9^{k+1}$. Since we suppose that this proposition holds for each H_j where $0 \leq j \leq k$, we are assuming the following inequalities:

$$H_k \leq 9^k,\ H_{k-1} \leq 9^{k-1},\ H_{k-2} \leq 9^{k-2}.$$

To show that $H_{k+1} \leq 9^{k+1}$ using these assumptions, we can replace H_{k+1} by $8H_k + 8H_{k-1} + 9H_{k-2}$. So, we have to show that

$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 9^{k+1}.$$

Since each term is smaller or equal to the corresponding right hand side of the inequalities above, we get:

$$\begin{aligned}
8H_k + 8H_{k-1} + 9H_{k-2} &\leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9 \cdot 9^{k-2} \\
8H_k + 8H_{k-1} + 9H_{k-2} &\leq 9^{k-2}(8 \cdot 9^2 + 8 \cdot 9 + 9) \\
8H_k + 8H_{k-1} + 9H_{k-2} &\leq 9^{k-2}(648 + 72 + 9) \\
8H_k + 8H_{k-1} + 9H_{k-2} &\leq 9^{k-2} \cdot 729
\end{aligned}$$

Since $729 = 9^3$,

$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 9^{k-2} \cdot 9^3 = 9^{k+1}$$

Therefore,

$$H_{k+1} \leq 9^{k+1}.$$

Hence, the proof is completed.

Answer 3

There are 5 different cases where the consecutive 0s can start in the bit strings of length 8:

- **0000xxxx** $\rightarrow 2^4$ strings
- **10000xxx** $\rightarrow 2^3$ strings
- **x10000xx** $\rightarrow 2^3$ strings
- **xx10000x** $\rightarrow 2^3$ strings
- **xxx10000** $\rightarrow 2^3$ strings

Note that we fixed the bit before consecutive 0s as 1 (except the first case) to avoid repetition in the produced strings. By using the sum rule, we get 48 different bit strings having 4 consecutive 0s. With the same process, we get the same number for the bit strings having 4 consecutive 1s. Now we have to exclude intersection of both type of bit-strings. There are 2 such strings (00001111 and 11110000). Hence, the result is:

$$48 + 48 - 2 = 94.$$

Answer 4

Let us select the star and the planets before ordering them according to given constraints. Since we have to choose a star from 10 stars, 2 habitable and 8 nonhabitable planets among 20 habitable ones and 80 nonhabitable ones respectively. Hence, we have

$$\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8}$$

different selections. Now we can form the galaxies in 3 different manner:

- The galaxy may have 6 nonhabitable planets between 2 habitable ones. In this case, we should pick the nonhabitable ones that reside outside of the habitable ones. So, we have $\binom{8}{2}$ different selections for them. Since the planets are distinct and the order matters, these selected planets can create 2! different formations. 2! and 6! many different ordering can be done for the habitable and nonhabitable ones in the middle respectively. Finally, we have 3 different ways to place the outsiders (*NNHNNNNNNH*, *HNNNNNNHNN*, *NHNNNNNNHN*). Hence, factors of this case become as:

$$\binom{8}{2} \cdot 2! \cdot 2! \cdot 6! \cdot 3.$$

- For the second case, we may have 7 nonhabitable planets in the middle of 2 habitable ones. In this case, we should select the only nonhabitable one that resides outside of the habitable planets as $\binom{8}{1}$. 2! and 7! many different ordering can be done for the habitable and nonhabitable ones in the middle respectively. Finally, we have 2 different ways to place the outsider (*NHNNNNNNNH*, *HNNNNNNNHN*). Hence, factors of this case become as:

$$\binom{8}{1} \cdot 2! \cdot 7! \cdot 2.$$

- For the third and last case, we may have 8 nonhabitable planets in the middle of 2 habitable ones. In this case there is no outsiders. Hence, only difference between the galaxies comes from the ordering of the habitable and nonhabitable ones internally. 2! and 8! many different ordering can be done for the habitable and nonhabitable ones in the middle respectively. Hence, factors of this case become as:

$$2! \cdot 8!.$$

Therefore, the final calculation becomes:

$$\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8} \left(\binom{8}{2} \cdot 2! \cdot 2! \cdot 6! \cdot 3 + \binom{8}{1} \cdot 2! \cdot 7! \cdot 2 + 2! \cdot 8! \right).$$

To avoid repetition of the same term, we can rewrite this multiplication as:

$$\binom{10}{1} \cdot \binom{20}{2} \cdot \binom{80}{8} \cdot 2! \cdot \left(\binom{8}{2} \cdot 2! \cdot 6! \cdot 3 + \binom{8}{1} \cdot 7! \cdot 2 + 8! \right).$$

Answer 5

a. In 3 ways the robot can begin to jump:

- To one cell away with a jump, then a_{n-1} different ways to complete rest.
- To two cells away with a jump, then a_{n-2} different ways to complete rest.
- To three cells away with a jump, then a_{n-3} different ways to complete rest.

Hence, the recurrence relation can be defined as:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

b.
- $a_1 = 1$
- $a_2 = 2$ $(1 + 1$ or $2)$
- $a_3 = 4$ $(1 + 1 + 1,\ 1 + 2,\ 2 + 1,\ 3)$

c. $a_4 = a_3 + a_2 + a_1 = 7$
$a_5 = a_4 + a_3 + a_2 = 13$
$a_6 = a_5 + a_4 + a_3 = 24$
$a_7 = a_6 + a_5 + a_4 = 44$
$a_8 = a_7 + a_6 + a_5 = 81$
$a_9 = a_8 + a_7 + a_6 = 149$