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Answer 1

a) The function $f(x) = x^2$ from the set of real numbers to the set of real numbers is not one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Since our domain is set of real numbers x might be both positive and negative. Hence, it is not injective.
For example, $f(-1) = f(1) = 1$
= **Not Injective**
Furthermore, $f(x) = x^2$ is not onto when the codomain is the set of all real numbers, because negative real numbers don't have a square root and also we cannot find a real number whose square is negative.
= **Not Surjective**

b) The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x+y)(x-y) = 0$. This means that $x+y = 0$ or $x-y = 0$, so $x = -y$ or $x = y$. Because both x and $y$are nonnegative, we must have $x = y$. So, this function is one-to-one.
= **Injective**
Furthermore, $f(x) = x^2$ is not onto when the codomain is the set of all real numbers, because negative real numbers don't have a square root.
= **Not Surjective**

c) The function $f(x) = x^2$ from the set of real numbers to the set of nonnegative real numbers is not one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x+y)(x-y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Since our domain is set of real numbers x might be both positive and negative. Hence, it is not injective.
For example, $f(-1) = f(1) = 1$
= **Not Injective**
Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of real numbers to the set of nonnegative real numbers is onto.
= **Surjective**

d) The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both x and $y$are nonnegative, we must have $x = y$. So, this function is one-to-one.
= **Injective**
Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto.
= **Surjective**

Answer 2

a) Let $D \subseteq$R A function $f : D \rightarrow$R is continuous at some $x_0 \in D$ for each $\epsilon > 0$ there should be some $\delta > 0$ for any $x \in D$ with $|x - x_0| < \delta$. So, we can construct $|f(x) - f(x_0)| < \epsilon$. Let's choose a function f :Z$\rightarrow$R pick $x_0 \in Z$ and choose $\epsilon > 0$. After this point, we need to find δ such that it satisfies the statement of continuity. Let's choose $\delta = 1/7$
Let's say $x \in Z$ and $|x - x_0| < \delta = 1/7$. Because the only integer within 1/7 distance of x_0 is itself, we have $x = x_0$. Therefore, $f(x) = f(x_0)$, accordingly $|f(x) - f(x_0)| = 0$. We can easily see that it is lesser than ϵ. Now that we have proven that f is continuous at x_0. Because we have chosen x_0 arbitrarily, it shows us that f is continuous everywhere in its domain.

b) Because if f were not a constant function, it would be creating discontinuity. Since the image set is all Z, if f were not a constant function, there would be some points which are not in the image set's domain. For example, let's imagine that $f(1) = 1$ and $f(2) = 2$. What about the values between 1 and 2? Since our codomain consist of integers, it won't be taking the values between 1 and 2. So, it won't be continuous. That's why our function f should be a constant function. Because, otherwise f is not continuous.

Answer 3

a)

- For example, if both A_1, A_2 and ... A_n are finite with $|A_1| = m$ and let's say $|A_2| = n$, we can show that $|A_1$ x $A_2| = mn$. Hence A_1 x A_2 is finite. So it is countable.
- If A and B are both countable then there exists bijective functions such that $f : A \rightarrow Z$ and $g : B \rightarrow Z$ and defining $h : A \times B \rightarrow Z^2$ gives us an injective function, since the set of positive integers Z^+ , Z and $Z \times Z$ have the same cardinality. So $A \times B$ is countable.

b) $\{En\}_{n\in N}$ is a sequence of countable sets and $S = E_1 \times ... \times E_n \times ...$
By the definition of Cartesian product of sets,

$$S = \Pi_{n\in N}\{f : \mathrm{N} \to \bigcup_{n\in N} E_n | \forall n, f(n) \in E_n\}$$

If $E_n = \{0, 1\}$, then

$$S_{01} = \Pi_{n\in N}\{0, 1\} = E^N$$

,where $E = \{0, 1\}$. By a theorem, $\bigcup_{n\in N} E_n$ is countable since the sequence is countable. By using Cantor's diagonalization, we can suppose S is countable. Let $(F_n : n \in N)$ be an enumeration of S. For each n, I picked two points which are $a_n, b_n \in E_n$.Defining a new function $F \in S$ as follows:

$$F(m) = b_m \text{ if } F_m(m) = a_m$$
$$F(m) = a_m \text{ otherwise}$$

It follows that $F \in S$ but it is different of all F_n's which is a contradiction.

Answer 4

$$(\log n)^2 \ , \ \sqrt{n}\mathrm{log}n \ , \ n^{50} \ , \ n^{51} + n^{49}, \ 2^n, \ 5^n \ , \ (n!)^2$$

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a) $\lim_{x\to\infty} \frac{\sqrt{n}logn}{(logn)^2} = \lim_{x\to\infty} \frac{\sqrt{n}}{logn} = \lim_{x\to\infty} \frac{\sqrt{n}(\ln 10)}{2}$ (By L'Hospital Rule) $= \infty$
Since it goes to infinity, we can conclude that $\sqrt{n}\mathrm{log}n > (\log n)^2$, and $(\mathrm{log n})^2 = O(\sqrt{n}\mathrm{log\ n})$.

b) $\lim_{x\to\infty} \frac{n^{50}}{\sqrt{n}logn} = \frac{99}{2} * n^{\frac{99}{2}} * \ln 10$ (By L'Hospital Rule) $= \infty$
Since it goes to infinity, we can conclude that $n^{50} > \sqrt{n}\mathrm{log}n$, and $\sqrt{n}\mathrm{log\ n} = O(n^{50})$.

c) $\lim_{x\to\infty} \frac{n^{51}+n^{49}}{n^{50}} = \lim_{x\to\infty} n + \frac{1}{n} = \infty$
Since it goes to infinity, we can conclude that $n^{51} + n^{49} > n^{50}$, and $n^{50} = O(n^{51} + n^{49})$.

d) $\lim_{x\to\infty} \frac{2^n}{n^{51}+n^{49}} = \lim_{x\to\infty} \frac{2^n * \ln 2}{51n^{50}+49n^{48}}$ (By L'Hospital Rule) $= \lim_{x\to\infty} \frac{2^n * (\ln 2)^{51}}{51!}$ (By using L'Hospital Rule several times) $= \infty$
Since it goes to infinity, we can conclude that $2^n > n^{51} + n^{49}$,and $n^{51} + n^{49} = O(2^n)$.

e) $\lim_{x\to\infty} \frac{5^n}{2^n} = \lim_{x\to\infty} (\frac{5}{2})^n = \infty$
Hence, since it goes to infinity, we can conclude that $5^n > 2^n$,and $2^n = O(5^n)$.

f) For the comparison of 5^n and $(n!)^2$, we know that for $n = 5$ we have $5^n = 3125 < (n!)^2 = 14400$. Using this for $n \geq 5$ we can establish that $(n!)^2$ grows faster:

$$5^n = 5^5 * 5^{n-5} \leq (5!)^2 * 5^{n-5} \leq (5!)^2 * (\frac{n!}{5!})^2 = n!$$

Here, we made use of the fact that 5^{n-5} has $n - 5$ elements, all of which are lesser than those in $(\frac{n!}{5!})^2$ which also has $n - 5$ elements. Hence, we can conclude that $(n!)^2 > 5^n$,and $5^n = O((n!)^2)$.

Answer 5

a)

$$\begin{array}{rcll} & & \text{Since for } a \geq b \geq 0 \text{ ,} & \\ \text{gcd(a , b)} & \equiv & \text{gcd(b,a mod b)} & \text{by Euclidean Algorithm} \\ & & & \\ \text{gcd(94, 134)} & \equiv & \text{gcd(134,94)} & \\ & \equiv & \text{gcd(94, 40)} & \\ & \equiv & \text{gcd(40, 14)} & \\ & \equiv & \text{gcd(14, 12)} & \\ & \equiv & \text{gcd(12, 2)} & \\ & \equiv & \text{gcd(2,0)} & \\ & \equiv & 2 & \end{array}$$

b) I assumed that every even integer greater than 2 is the sum of two primes according to Goldbach's conjecture, and I have choosen n which is an integer greater than 5. If n is odd, then we can write n = 3 + (n - 3), decompose n - 3 = p + q into the sum of two primes (since n - 3 is an even integer greater than 2), and therefore have written n = 3 + p + q as the sum of three primes. If n is even, we can write n = 2 + (n - 2), decompose n - 2 = p + q into the sum of two primes (since n - 2 is an even integer greater than 2), and therefore have written n = 2 + p + q as the sum of three primes.

For the converse, I assumed that every integer greater than 5 is the sum of three primes, and again I have chosen n which is an even integer greater than 2. By our assumption we can write n + 2 as the sum of three primes. Since n + 2 is even, these three primes cannot all be odd, so we have n + 2 = 2 + p + q, where p and q are primes, hence n = p + q, as desired.