CENG 223

Discrete Computational Structures

Fall 2022-2023
Take Home Exam 4

Solution

Question 1 (30 pts)

Use generating functions to solve the following recurrence relation;

$$a_n = 3a_{n-1} + 4a_{n-2}, n \geq 2$$

where $a_0 = a_1 = 1$. Any solution that does not use generating functions would not gain partial credits.

$$G(x) = a_0 + a_1 x + \sum_{k=2}^{\infty} a_k x^k$$

Substituting $a_n = 3a_{n-1} + 4a_{n-2}$;

$$G(x) = a_0 + a_1 x + 3\sum_{k=2}^{\infty} a_{k-1} x^k + 4\sum_{k=2}^{\infty} a_{k-2} x^k$$

$$G(x) = a_0 + a_1 x + 3x\sum_{k=1}^{\infty} a_k x^k + 4x^2\sum_{k=0}^{\infty} a_k x^k$$

Since we know that $\sum_{k=1}^{\infty} a_k x^k = G(x) - a_0$;

$$G(x) = a_0 + a_1x + 3x(G(x) - a_0) + 4x^2G(x)$$

Rewriting to group all terms with $G(x)$;

$$G(x) - 3xG(x) - 4x^2G(x) = a_0 + a_1x - 3xa_0$$

Substituting $a_0 = a_1 = 1$ and solving for $G(x)$;

$$G(x) = \frac{1 - 2x}{1 - 3x - 4x^2}$$

At this point we need to find roots of the equation for $G(x)$. Rewriting the equation;

$$\frac{1 - 2x}{(1 - 4x)(1 + x)} = \frac{A}{1 - 4x} + \frac{B}{1 + x}$$

$$A + Ax + B - 4Bx = 1 - 2x$$

$$\text{This makes } A = \frac{2}{5} \text{ and } B = \frac{3}{5}$$

The equation is;

$$G(x) = \frac{2}{5}(\frac{1}{1 - 4x}) + \frac{3}{5}(\frac{1}{1 + x})$$

$$G(x) = \frac{2}{5}(\sum_{k=0}^{\infty}(4x)^n) + \frac{3}{5}(\sum_{k=0}^{\infty}(-x)^n)$$

Finally, $a_n = \frac{2}{5}4^n + \frac{3}{5}(-1)^n$.

Question 2 (30 pts)

a)

Find the generating function (in closed form) for the sequence $< 2, 5, 11, 29, 83, 245, \cdots >$. Show all the steps clearly. *(15 pts)*

Firstly observe that the given sequence can be written as the sum of two different sequences;
$< 0, 3, 9, 27, 81, 243, \cdots >$ and $< 2, 2, 2, 2, 2, \cdots >$

For the first sequence;

Start with; $< 1, 1, 1, 1, 1, \cdots > \leftrightarrow \dfrac{1}{1-x}$

Substitute $3x$ for x; $< 1, 3, 9, 27, 81, \cdots > \leftrightarrow \dfrac{1}{1-3x}$

Multiply with 3; $< 3, 9, 27, 81, 243, \cdots > \leftrightarrow \dfrac{3}{1-3x}$

Shift right; $< 0, 3, 9, 27, 81, 243, \cdots > \leftrightarrow \dfrac{3x}{1-3x}$

For the first sequence;

Start with; $< 1, 1, 1, 1, 1, \cdots > \leftrightarrow \dfrac{1}{1-x}$

Multiply with 2; $< 2, 2, 2, 2, 2, \cdots > \leftrightarrow \dfrac{2}{1-x}$

Their sum is $\dfrac{3x}{1-3x} + \dfrac{2}{1-x} = \dfrac{2-3x-3x^2}{1-4x+3x^2}$

b)

Find the sequence corresponding to the generating function; *(15 pts)*

$$G(x) = \frac{7 - 9x}{1 - 3x + 2x^2}$$

Initially we must factor out $G(x)$. Note that the denominator is $(1 - x)(1 - 2x)$.

$$G(x) = \frac{A}{1 - x} + \frac{B}{1 - 2x}$$

When solved for A and B, we get $A + B = 7$ and $2A + B = 9$, making $A = 2$ and $B = 5$. The resulting function is;

$$G(x) = \frac{2}{1 - x} + \frac{5}{1 - 2x}$$

First part can be written as $2(1 + x + x^2 + x^3 + \ ... \ + x^n \ ... \ ) = 2$

Second part can be written as $5(1 + 2x + 4x^2 + 8x^3 \ ... \ 2^n x^n \ ... \ ) = 5.2^n$

$G(x) = 5.2^n + 2$ which can also be written as $< 7, 22, 42, 82, 162, \cdots >$

Question 3 (20 pts)

a)

The relation R is defined on as $\mathbb{Z}$ follows;

$$aRb \text{ iff there exists a right triangle that has the edges } a, b, n \text{ where } n \in \mathbb{Z}$$

Is R an equivalence relation? If it is an equivalence relation what is the equivalence class of 3? *(5 pts)*

For a relation to be an equivalence relation, it must satisfy three conditions; reflexivity, symmetry, transitivity.

Reflexivity; The relation is not reflexive, $\exists x(\sqrt{2x^2} \notin \mathbb{Z})$ where $x \in \mathbb{Z}$

Symmetry; The relation is symmetric. First assume n is the hypotenuse, $n^2 = a^2 + b^2 = b^2 + a^2$.
Then, assume a is the hypotenuse, $a^2 = n^2 + b^2$. In the symmetric case, same n can be used as another edge along with a where b is the hypotenus and $b^2 = n^2 + a^2$.

Transitivity; It is not transitive, prove it with a counter-example.
$3R5$ is true, because there is a right triangle with edges $3 - 4 - 5$.
$5R12$ is also true, because there is a right triangle with edges $5 - 12 - 13$.
However, $3R12$ is not true because neither $\sqrt{9 + 144}$ or $\sqrt{144 - 9}$ is in $\mathbb{Z}$.

The relation is not an equivalence relation, so equivalence classes do not exist. Proof of either reflexivity or transitivity is enough here.

b)

The relation R is defined on as $\mathbb{R}$ follows;

$$(x_1, y_1)R(x_2, y_2) \text{ iff } 2x_1 + y_1 = 2x_2 + y_2$$

Is R an equivalence relation? If it is an equivalence relation what is the equivalence class of $(1, -2)$? What does it represent in the Cartesian coordinate system? *(15 pts)*

Reflexivity; For $\forall x_1, y_1 \in \mathbb{R}$, $2x_1 + y_1 = 2x_1 + y_1$ and the relation is reflexive.

Symmetry; If $(x_1, y_1)R(x_2, y_2)$ is true, then it must be proven that $(x_2, y_2)R(x_1, y_1)$ is also true for $\forall x_1, y_1, x_2, y_2 \in \mathbb{R}$. If the equation $2x_1 + y_1 = 2x_2 + y_2$ holds, then the equation $2x_2 + y_2 = 2x_1 + y_1$ must also hold because of the symmetric property of the algebraic symbol of "=".

Transitivity; If $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$, then $(x_1, y_1)R(x_3, y_3)$. Given that $2x_1+y_1 = 2x_2+y_2$ and $2x_2+y_2 = 2x_3+y_3$ it can be inferred that $2x_1+y_1 = 2x_3+y_3$, making the relation transitive.

Hence the relation is an equivalence relation.

The equivalence class of $(1, -2)$ is all real pairs (x, y) where $2x + y = 0$. This refers to a line in Cartesian coordinate system.

Question 4 (20 pts)

$R = \{(a, b) | a \text{ divides } b\}$ is a relation defined on $A = \{2, 5, 10, 18, 60\}$.

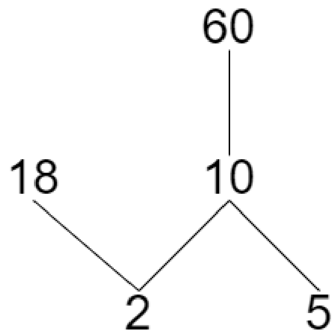
a) Draw the Hasse diagram of R.

b) What is the matrix representation for R?

c) What is the matrix representation for R_s, where R_s is the symmetric closure of R. List all pairs (x, y) where $(x, y) \in R_s \land (x, y) \notin R$.

d) You are allowed to remove a single element in A and add another element. Is it possible to create a total ordering that includes all elements of A. What if you are allowed to remove two elements and add one? Which elements would you remove and add to create such total ordering?

Each item is worth *5 pts*. Note that partial points may not be given to the items.

a) 

b) $R = \begin{array}{c|c} & \begin{array}{ccccc} 2 & 5 & 10 & 18 & 60 \end{array} \\ \begin{array}{c} 2 \\ 5 \\ 10 \\ 18 \\ 60 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

c) The pairs (x, y) where $(x, y) \in R_s \wedge (x, y) \notin R$ refers to the pairs that needs to be added to create a symmetric closure of R. These are $(10, 2), (10, 5), (18, 2), (60, 2), (60, 5), (60, 10)$.

$$R_s = \begin{array}{c|c} & \begin{array}{ccccc} 2 & 5 & 10 & 18 & 60 \end{array} \\ \begin{array}{c} 2 \\ 5 \\ 10 \\ 18 \\ 60 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

d) Since there are two partial ordering between different elements $(2 - 5$ and $10 - 18)$ removing a single element and adding a new one would not create a total ordering with all items in A.
However if we are allowed to remove two elements, we can remove 5 and 18 and add any n where n is divisible by $2, 10, 60$ or n divides 60 while being divisible by 2 and 10. Some examples for n could be $20, 30$ or 120.