THE 5 Solutions

Answer 1

a)

(3 pts) According to the Handshaking Theorem (p.653) the answer is $2m$ which is $2 \times 7 = 14$, where m is the number of edges.

b)

(3 pts) Since it is undirected, again the result is 14.

c)

(3 pts) There are 35 entries in total in the incidence matrix of G (5 nodes and 7 edges). Since it is undirected, 14 of the entries will be non-zero. Therefore $35 - 14 = 21$ entries will be zero entries.

d)

(3 pts) **Complete graph:** exactly one edge between each pair of distinct vertices. If G had a complete graph with at least four vertices, it would contain vertices of degree 3. Such vertices are a, b, c and e. But there is no edge between c and e. Therefore, G does not have a complete graph with at least four vertices as a subgraph.

e)

(3 pts) A **bipartite graph** is a graph that does not contain any odd-length cycles. Therefore, G is not a bipartite graph.

f)

(3 pts) For every edge, there exist two possible direction options and we have 7 edges, so the answer is $2^7 = 128$.

g)

(3 pts) It shouldn't contain the same edge more than once. Such a path is a, b, c, d, e, a, c and its length is 6.

h)

(3 pts) There is 1 connected component which is G itself. The reason is that there is a simple path between every pair of distinct vertices of G. So it is already a connected component.

i)

(3 pts) An Euler circuit in a graph is a simple circuit containing every edge of the graph. In order to have an Euler circuit, every vertex must have even degree. We see that $deg(e) = 3$ so there is not an Euler circuit in G.

j)

(3 pts) An Euler path in a graph is a simple path containing every edge of the graph. If G had exactly two vertices of odd degree, then it would have an Euler path. But it has four vertices of odd degree. Therefore, it does not have an Euler path.

k)

(3 pts) A simple circuit in a graph that passes through every vertex exactly once is called a Hamilton circuit. There exists such a circuit in G: a, b, c, d, e, a.

l)

(3 pts) A simple path in a graph that passes through every vertex exactly once is called a Hamilton path. There exists such a path in G: a, b, c, d, e.

Answer 2

(7 pts) First, let's check the invariants: the number of vertices, the number of edges and the number of vertices of each degree. They all must be the same.
G has 5 vertices, 5 edges and the degree of each vertex in G is 2. H also has 5 vertices, 5 edges and the degree of each vertex in H is 2, too. However, that is not enough to conclude that they are isomorphic.
(13 pts) We now will define a function f and then determine whether it is an isomorphism. By examining the adjacent vertices, we can define the following one-to-one and onto function: $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, $f(d) = d'$, $f(e) = e'$.
We now have a one-to-one correspondence between the vertex set of G and the vertex set of H. To see whether f preserves edges, we examine the adjacency matrix of G,

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

and the adjacency matrix of H with the rows and columns labeled by the images of the corresponding vertices in G,

$$A_H = \begin{array}{c} \\ a' \\ b' \\ c' \\ d' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} a' & b' & c' & d' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Because $A_G = A_H$, it follows that f preserves edges. We conclude that f is an isomorphism, so G and H are isomorphic.

Answer 3

(10 pts) The iterations of Dijkstra's algorithm are described in the following table.

Step	Visited	s	u	v	w	x	y	z	t
0	$-$	0	∞	∞	∞	∞	∞	∞	∞
1	s	0	4_s	5_s	3_s	∞	∞	∞	∞
2	w	0	4_s	5_s	3_s	11_w	∞	15_w	∞
3	u	0	4_s	5_s	3_s	11_w	15_u	15_w	∞
4	v	0	4_s	5_s	3_s	7_v	11_v	15_w	∞
5	x	0	4_s	5_s	3_s	7_v	8_x	13_x	∞
6	y	0	4_s	5_s	3_s	7_v	8_x	12_y	17_y
7	z	0	4_s	5_s	3_s	7_v	8_x	12_y	15_z
8	t	0	4_s	5_s	3_s	7_v	8_x	12_y	15_z

(10 pts)

Step 0: Initialize the value for s as 0. The other lengths are ∞ for now.

Step 1: The closest vertex to s is w and the length is 3. Update the values for x and z.

Step 2: The second closest vertex to s is u and the length is 4. Update the value for y.

Step 3: The third closest vertex to s is v and the length is 5. Update the values for x and y.

Step 4: The next one is x. Update the values for y and z.

Step 5: The next one is y. Update the values for z and t.

Step 6: The next one is z. Update the value for t.

Step 7: The next one is t. There is no update.

As we reached the target node, we now terminate the algorithm. We find that a shortest path from s to t is s, v, x, y, z, t with length 15.

Answer 4

a)

(8 pts)
If we choose Prim's algorithm, we start by selecting an initial edge of minimum weight and continue by successively adding edges of minimum weight that are incident to a vertex in the tree and that do not form simple circuits. We stop when $n-1$ edges have been added.
According to the table below, the order in which the edges are added to the tree is $\{b, c\}$,$\{c, f\}$, $\{c, d\}$, $\{d, k\}$, $\{f, j\}$, $\{a, b\}$, $\{e, f\}$, $\{f, g\}$, $\{f, i\}$ and $\{i, h\}$.

Choice	Edge	Cost
1	$\{b, c\}$	2
2	$\{c, f\}$	2
3	$\{c, d\}$	3
4	$\{d, k\}$	2
5	$\{f, j\}$	3
6	$\{a, b\}$	3
7	$\{e, f\}$	4
8	$\{f, g\}$	4
9	$\{f, i\}$	4
10	$\{i, h\}$	2

If we choose Kruskal's algorithm, we start by choosing an edge in the graph with minimum weight and continue by successively adding edges with minimum weight that do not form a simple circuit with those edges already chosen. We stop after $n - 1$ edges have been selected.

Choice	Edge	Cost
1	$\{b, c\}$	2
2	$\{c, f\}$	2
3	$\{d, k\}$	2
4	$\{h, i\}$	2
5	$\{a, b\}$	3
6	$\{c, d\}$	3
7	$\{f, j\}$	3
8	$\{e, f\}$	4
9	$\{f, i\}$	4
10	$\{g, j\}$	4

According to the table above, the order in which the edges are added to the tree is $\{b, c\}$,$\{c, f\}$, $\{d, k\}$, $\{h, i\}$, $\{a, b\}$, $\{c, d\}$, $\{f, j\}$, $\{e, f\}$, $\{f, i\}$ and $\{g, j\}$.

b)

(8 pts)
Prim's:

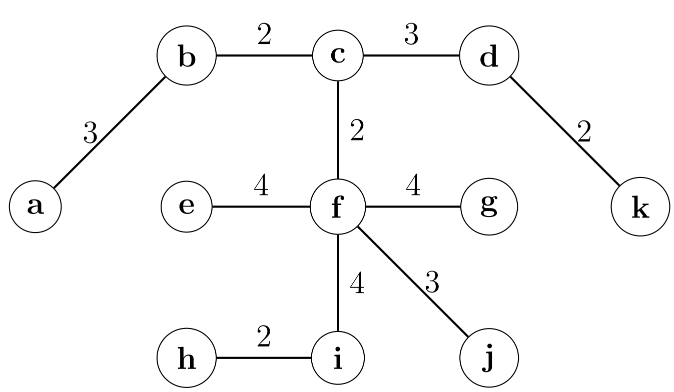

Figure 1: Minimum Spanning Tree for G.

Kruskal's:

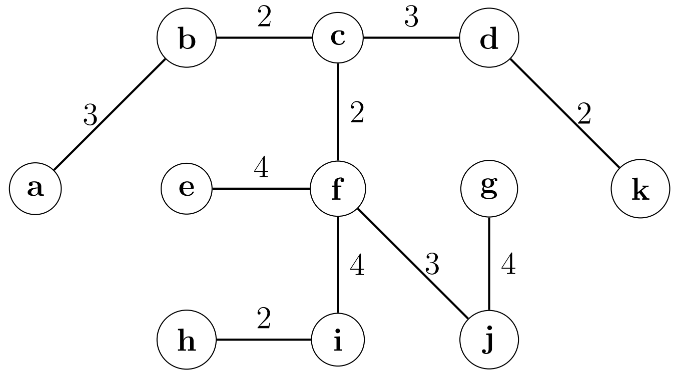

Figure 2: Minimum Spanning Tree for G.

c)

(8 pts)
Prim's:
No, it is not unique. We could choose another option at Choice 8, for example, edge $\{g, j\}$ because its cost is same with the edge we chose. Therefore, we would get another minimum spanning tree.
Kruskal's:
No, it is not unique. We could choose another option at Choice 10, for example, edge $\{f, g\}$ because its cost is same with the edge we chose. Therefore, we would get another minimum spanning tree.
As we can see, we found different minimum spanning trees with different algorithms because of the different choices we made during the execution of the algorithms.