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Answer 1

$$\sum_{n\geq 2}^{\infty} = a_n * x^n = \sum_{n\geq 2}^{\infty} 3a_{n-1} * x^n + \sum_{n\geq 2}^{\infty} 4a_{n-2} * x^n$$

$i)$ Let's assume that

$$A(x) = \sum_{n\geq 0}^{\infty} = a_n * x^n$$

In this case,

$$\sum_{n\geq 2}^{\infty} = a_n * x^n = A(x) - a_0 - a_1 x = A(x) - 1 - x$$

$ii)$ Now let's consider

$$\sum_{n\geq 2}^{\infty} 3a_{n-1} * x^n$$

We know that,

$$\sum_{n\geq 2}^{\infty} 3a_{n-1} * x^n = \sum_{n\geq 1}^{\infty} 3a_n * x^{n+1} = 3x \sum_{n\geq 1}^{\infty} a_n * x^n$$

That's why we can say that:

$$\sum_{n\geq 1}^{\infty} a_n * x^n = A(x) - a_0 = A(x) - 1$$

$$3x \sum_{n\geq 1}^{\infty} a_n * x^n = 3x(A(x) - 1)$$

$iii)$ Lastly, we should realize that

$$\sum_{n\geq 2}^{\infty} 4a_{n-2} * x^n = \sum_{n\geq 0}^{\infty} 4a_n * x^{n+2} = 4x^2 \sum_{n\geq 0}^{\infty} a_n * x^n = 4x^2(A(x))$$

When we substitute these values, we have the equation:

$$A(x) - 1 - x = 3x(A(x) - 1) + 4x^2(A(x))$$

$$A(x) = \frac{1 - 2x}{(1 - 4x)(1 + x)} = \frac{A}{(1 - 4x)} + \frac{B}{(1 + x)}$$

$$A + Ax + B + -4Bx = 1 - 2x$$

$$A = \frac{2}{5}, B = \frac{3}{5}$$

$$A(x) = \frac{2}{5}\frac{1}{(1-4x)} + \frac{3}{5}\frac{1}{(1+x)}$$

$$A(x) = \frac{2}{5}\sum_{n\geq 0}^{\infty}(4x)^n + \frac{3}{5}\sum_{n\geq 0}^{\infty}(-x)^n = \sum_{n\geq 0}^{\infty} x^n \left(\frac{2}{5}(4)^n + \frac{3}{5}(-1)^n\right)$$

We have come to the conclusion

$$a_n = \frac{2}{5}(4)^n + \frac{3}{5}(-1)^n$$

Answer 2

a)

Let's choose a function $A(x)$ which is:

$$A(x) = 2 + 5x + 11x^2 + 29x^3 + 83x^4 + 245x^5 + ...$$

$$= (2 + 2x + 2x^2 + 2x^3 + 2x^4 + ...) + (3x + 9x^2 + 27x^3 + 81x^4 + ...)$$

$$= 2(1 + x + x^2 + x^3 + x^4 + ...) + 3x(1 + 3x + 9x^2 + 27x^3 + ...)$$

It can be observed that:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty}(3x)^n = 1 + 3x + 9x^2 + ... = \frac{1}{1-3x}$$

Hence, we have found our generating function as:

$$A(x) = 2\left(\frac{1}{1-x}\right) + 3x\left(\frac{1}{1-3x}\right) = \frac{-3x^2 - 3x + 2}{(1-x)(1-3x)} = \frac{-3x^2 - 3x + 2}{(1 - 4x + 3x^2)}$$

b)

$$G(x) = \frac{7-9x}{(1-3x+2x^2)} = \frac{7-9x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$$

$$A(1-x) + B(1-2x) = 7 - 9x \rightarrow A = 5, B = 2$$

$$G(x) = \frac{5}{1-2x} + \frac{2}{1-x} = 5\left(\frac{1}{1-2x}\right) + 2\left(\frac{1}{1-x}\right)$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}(x)^n$$

$$G(x) = \sum_{n=0}^{\infty}5(2x)^n + \sum_{n=0}^{\infty}2(x)^n$$

$$G(x) = \sum_{n=0}^{\infty}(x)^n(5 * 2^n + 2)$$

Hence $a_n = 5 * 2^n + 2$ and the sequence is $< 7, 12, 22, 42, 82, 162, ... >$.

Answer 3

a)

aRb iff there exists a right triangle that has the edges a, b, n where $n \in Z$ represents Pythagoras' Theorem.
Pythagoras' Theorem says that there exist three positive integers a, b, and n such that $\sqrt{a^2 + b^2} = n$. a, b and n can be swapped, such as $\sqrt{a^2 + n^2} = b$ or $\sqrt{b^2 + n^2} = a$
In line with this information, we can check whether R is an equivalence relation, or not.

In order to be an equivalence relation, R must meet these conditions:

1. It should be reflexive.

2. It should be symmetric.

3. It should be transitive.

- Let's first check if R is reflexive:
  If R is reflexive, then we should be able to construct a relationship like aRa which satisfies $\sqrt{a^2 + a^2} = n$, or $\sqrt{a^2 + n^2} = a$, or $\sqrt{n^2 + a^2} = a$.

  In the first equation ($\sqrt{a^2 + a^2} = n$), we find a solution such that $n = a\sqrt{2}$

However, our question says that $n \in Z$ in the beginning, and according to the theorem a must be a positive integer, which creates a contradiction with the solution $n = a\sqrt{2}$.

In the second and third equation ($\sqrt{a^2+n^2} = a$, $\sqrt{n^2+a^2} = a$) , we find a solution such that $n = 0$, which is impossible to create a right triangle.

That is why R is not an equivalence relation.

Since R is not an equivalence relation, there is no equivalence class of 3.

b)

In order to be an equivalence relation, R must meet these conditions:
1. It should be reflexive.
2. It should be symmetric.
3. It should be transitive.

- Let's check first if it is reflexive.

$$x_2 = x_1, y_2 = y_1 \rightarrow (x_1, y_1)R(x_1, y_1) \iff 2x_1 + y_1 = 2x_1 + y_1$$

  So, it is reflexive.

- Second, we should check if it is symmetric.

$$(x_1, y_1)R(x_2, y_2) \iff 2x_1 + y_1 = 2x_2 + y_2$$

$$(x_2, y_2)R(x_1, y_1) \iff 2x_2 + y_2 = 2x_1 + y_1$$

  So, R is symmetric.

- Lastly, we need to find if it is transitive. We already know that

$$(x_1, y_1)R(x_2, y_2) \iff 2x_1 + y_1 = 2x_2 + y_2$$

  Let's change left side of R as $x_1 = x_2, y_1 = y_2$ and right side of it as $x_2 = x_3, y_2 = y_3$ So we have:

$$(x_2, y_2)R(x_3, y_3) \iff 2x_2 + y_2 = 2x_3 + y_3$$

  Since $2x_2 + y_2 = 2x_3 + y_3$ we can substitute $2x_2 + y_2$ with $2x_3 + y_3$ in the first equation accordingly. As a result we have $2x_1 + y_1 = 2x_3 + y_3$

  Hence we have proven that R is transitive.

- Finally, since R is both reflexive, symmetric, and transitive, we can conclude that R is an equivalence relation.

- Since R is an equivalence relation, now we can find an equivalence class of (1, -2).
  For the equivalence class of (1 , -2) we should find (x , y) pairs satisfying the condition of $2 * (1) + (-2) = 2x + y$
  After calculations, we find 2x + y = 0 which represents the y = -2x line in the Cartesian coordinate system.

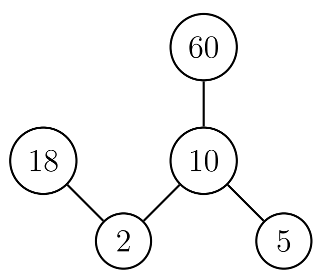
Answer 4

a)

We should first find a's and b's. To do this we should check the values in A.

- If a = 2, then b can be 2, 10, 18, and 60. Connections are (2,2), (2,10), (2,18), and (2,60).
- If a = 5, then b can be both 10 and 60. Connections are (5,5), (5,10), (5,60).
- If a = 10, then b can be 10 and 60. Connections are (10,10), (10, 60).
- If a = 18, then b can be only 18. Connection is (18,18).
- If a = 60, then b can be only 60. Connection is (60,60).

To draw Hasse Diagram, we should remove all loops, and also the connections that can be made by transitive property.
So the rest of the connections are (2,10), (5,10), (10,18), and (10,60).



b)

$$A = \{2, 5, 10, 18, 60\}$$

$$R = \{(2,2), (2,10), (2,18), (2,60), (5,5), (5,10), (5,60), (10,10), (10,60), (18,18), (60,60)\}$$

$$\begin{array}{c|c} & \begin{matrix} 2 & 5 & 10 & 18 & 60 \end{matrix} \\ \hline \begin{matrix} 2 \\ 5 \\ 10 \\ 18 \\ 60 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

c)

For the symmetric closure of R (R_S), we should also take the values which b divides a.
Because of that,
$R_S = \{(2,2),(2,10),(2,18),(2,60),(5,5),(5,10),(5,60),(10,10),(10,60),(18,18),(60,60),(10,2),$
$(10,5),(18,2),(60,2),(60,5),(60,10)\}$.

That is why our matrix representation is:

$$\left[\begin{array}{c|ccccc} & 2 & 5 & 10 & 18 & 60 \\ \hline 2 & 1 & 0 & 1 & 1 & 1 \\ 5 & 0 & 1 & 1 & 0 & 1 \\ 10 & 1 & 1 & 1 & 0 & 1 \\ 18 & 1 & 0 & 0 & 1 & 0 \\ 60 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

The question asks for the list of all pairs (x,y) where $(x,y) \in R_S \wedge (x,y) \notin R$.
So, the list is:

$$[(10,2),(10,5),(18,2),(60,2),(60,5),(60,10)]$$

d)

- Total ordering can be described as follows. If every two elements of A are comparable and related, then we can create total ordering.
- Let's look at our incomparable relations which are:

  $$(2,5),(5,18),(10,18),(18,60)$$

  It is clear that 5 and 18 are the troublemaker elements since they are incomparable with more than 1 element.

- In line with this information, let's remove element 5 and add another one. If we remove 5, element 18 will be still a troublemaker. Because it will have an incomparable relation with 10 and 60 again.

  Hence, it is not possible to create a total ordering that includes all elements of A, when we remove a single element in A and add another element.

- Now, let's remove 5 and 18 since they are the troublemakers. Moreover, let's add 20 since it has a comparable relation with both 2, 10, and 60 ($20 \mod 2 = 60 \mod 20 = 20 \mod 10 = 0$).
  It is clear that all of these elements are comparable with each other and there is no troublemaker anymore. Hence, it is possible to create a total ordering that includes all elements of A.