CENG 223

Discrete Computational Structures

Fall 2022-2023

Take Home Exam 2 - Sample Solutions

Question 1

f_1 is not surjective as the negative reals are not mapped to. It neither is injective as $-x$ and x both map to x^2 for $x \in \mathbb{R}$.

f_2 is not surjective as the negative reals are not mapped to. It is injective since the domain is only composed of nonnegative real numbers.

f_3 is surjective: for all values in the codomain there exists a square root in the domain. It is not injective as $-x$ and x both map to x^2 for $x \in \mathbb{R}$.

f_4 is surjective: for all values in the codomain there exists a square root in the domain. It is also injective since the domain is only composed of nonnegative real numbers.

Question 2

a) For all values of ε we can choose $\delta = 1/2$ which will then give $\{x : \|x - x_0\| < 1/2\} = \{x_0\}$. Clearly for each member of this set $\|f(x_0) - f(x)\| < \varepsilon$ holds.

b) First we show that $f : \mathbb{R} \to \mathbb{Z}$ is continuous if it is constant. We can choose any δ since for any pair x and x_0 we have $\|f(x) - f(x_0)\| = 0 < \varepsilon$.

Next to show: $f : \mathbb{R} \to \mathbb{Z}$ is not continuous if it is not constant. Assuming f is not constant, there exists some $x_0 \in \mathbb{R}$ such that for all $\delta \in \mathbb{R}$ we have $|f([x_0 - \delta, x_0 + \delta])| \neq 1$. That is to say, all intervals around a point of change are multi-valued. Since integers are separated by at least 1, at the point x_0 for any $\varepsilon < 1$ we cannot find a suitable δ.

Question 3

a) Let's start by considering two countable sets A and B and their product. If any of these sets is empty, so is their product. Hence the countability. If not, there exists surjective functions $f : \mathbb{Z} \to A$ and $g : \mathbb{Z} \to B$. Using these we can construct a surjective function $h : \mathbb{Z} \times \mathbb{Z} \to A \times B$ defined by $h(n, m) = (f(n), g(m))$. Since $\mathbb{Z} \times \mathbb{Z}$ is countable then so is $A \times B$.

For the finite product $A_1 \times A_2 \times \cdots \times A_n$ we apply the above logic $n-1$ times to establish the final product's countability: we first consider $B = A_1 \times A_2$ which is countable and then proceed similarly by treating the product as $B \times A_3 \times \ldots A_n$.

b) Let's denote this set by X^ω. Then we will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

For a such defined function g, we have $g(n) = (x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots)$ where each x_{ij} are either 0 or 1. Then we consider $y = (y_1, y_2, \ldots) \in X^\omega$ given by

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1 \\ 1 & \text{if } x_{nn} = 0\,. \end{cases}$$

Such defined y is not mapped to by g: it differs from each $g(n)$ by at least one coordinate. Hence g cannot be surjective.

Question 4

$$(\log n)^2, \quad \sqrt{n}\log n, \quad n^{50}, \quad n^{51} + n^{49}, \quad 2^n, \quad 5^n, \quad (n!)^2$$

Most of the following solutions make use of the limits, and in particular, l'Hospital's rule. Note that it is also possible to find witnesses (see Definition 1 from §3.2 and the paragraph following it) to get the same results.

a)

$$\lim_{n\to\infty} \frac{(\log n)^2}{\sqrt{n}\log n} = \lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

b)

$$\lim_{n\to\infty} \frac{\sqrt{n}\log n}{n^{50}} = \lim_{n\to\infty} \frac{\log n}{n^{49.5}} = \lim_{n\to\infty} \frac{1/n}{49.5\, n^{48.5}} = \lim_{n\to\infty} \frac{1}{49.5\, n^{49.5}} = 0$$

c)

$$\lim_{n\to\infty} \frac{n^{50}}{n^{51} + n^{49}} = \lim_{n\to\infty} \frac{1}{n + 1/n} = 0$$

d) We can apply L'Hopital's rule multiple times to establish that

$$\lim_{n\to\infty} \frac{n^{51} + n^{49}}{2^n} = 0$$

e)

$$\lim_{n\to\infty}\frac{2^n}{5^n}=\lim_{n\to\infty}\left(\frac{2}{5}\right)^n=0$$

f) We first note that $5^5=3125<(5!)^2=14400$. Choosing $k=5$ and $C=1$ we have for $n>k$

$$5^n=5^5\times\underbrace{5\times\cdots\times 5}_{n-5\text{ terms}}\qquad\text{and}\qquad(n!)^2=(5!)^2\underbrace{\times 6^2\times\cdots\times n^2}_{n-5\text{ terms}}$$

Each term of the remaining $n-5$ terms in 5^n is lesser than each term in the remaining terms in the expression $(n!)^2$. Hence $|f(n)|=|5^n|\leq C|g(n)|=C|(n!)^2|$ for $n>k$.

Question 5

a)

$$\mathbf{gcd}(94,134)=\mathbf{gcd}(40,94)=\mathbf{gcd}(14,40)=\mathbf{gcd}(12,14)=\mathbf{gcd}(2,12)=2$$

b) $(\rightarrow)$ Let $n>5$ be even. Then $m=n-2>2$ is also even and we can apply Goldbach's conjecture to it to write it as $m=p+q$ for some prime numbers p and q. Consequently, $n=p+q+2$, a sum of three primes. Next assume that $n>5$ is odd. Then $m=n-3>2$ is even. Analogously to the even case we have $n=p+q+3$.

$(\leftarrow)$ Let $2n>2$ for some integer n. Then it holds that $m=2n+2>5$ and m is also even. As such, we can write $m=p+q+r$ where at least one of p,q,r must be 2 (here, we have used the facts that three odd numbers sum to another odd number, and 2 is the only even prime). Without loss of generality, let $p=2$. Then $2n=q+r$; Goldbach's conjecture.