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Notes on Perturbing Period Floor

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1. INTRO AND EQUATIONS

The TTV super-period equation, as described in (Agol & Fabrycky 2018), is

$$P_{sup} = \frac{1}{|j/P_t - k/P_p|}. \tag{1}$$

Here j and k are integers which represent the ratio of the MMR and P_t and P_p are the two exoplanet periods (P_t stands for period of transiting exoplanet and P_p stands for period of perturbing exoplanet). The super-period for near MMR planets tends to have both a longer period and larger amplitude than conjunction induced TTVs. The distance from MMR is directly related to both the super-period and the amplitude of the TTV signal (Agol & Fabrycky 2018).

We can re-write this equation, scaled by the period of the transiting as:

$$P'_{sup} = \frac{1}{|j - k/P'_p|}. \tag{2}$$

TTVs caused by the conjunctions have a period equal to the synodic period, which equals

$$P_{syn} = \frac{1}{|1/P_t - 1/P_p|}. \tag{3}$$

Again, we can re-write this equation, scaled by the period of the transiting as:

$$P'_{syn} = \frac{1}{|1/ - 1/P'_p|}. \tag{4}$$

The conjunction TTVs normally act as a second order harmonic, on top of the primary near MMR super-period TTV. The second order harmonic is called the "chopping effect" and can be used to break the mass-eccentricity degeneracy present in near MMR super-period TTV equations (Lithwick et al. 2012; Nesvorný & Vokrouhlický 2014; Schmitt et al. 2014; Deck & Agol 2015).

One minor caveat here is that the second order harmonic, or chopping signal, can also be caused by distant MMRs, if the planet orbits are far distant enough that during conjunctions there would not be a significant gravitational effect. For example, in Yahalomi et al. (2023), they found that Kepler-1513 b's TTVs could be explained by a near 5:1 exterior planet, Kepler-1513'c, with the long term TTV trend from the 5:1 super-period and the chopping effect caused by the distant 2:1 aliased super-period.

In order to determine the observable aliased period, we follow the same derivation as presented in McClellan et al. (1998), and then adopted in Dawson & Fabrycky (2010) and subsequently in Kipping (2021). We find that the observed aliased TTV frequency peaks, ν, in terms of the non-aliased physical TTV period, P_{TTV}, and the period of the transiting exoplanet, P_{trans}, occur at

$$\nu = |\frac{1}{P_{TTV}} \pm m\frac{1}{P_t}| \tag{5}$$

where m is a positive real integer. Or, in terms of observed aliased TTV periods, P'_{TTV}, we have

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$$\bar{P}_{TTV} = \frac{1}{\nu} = \frac{1}{|\frac{1}{P_{TTV}} \pm m\frac{1}{P_t}|} \tag{6}$$

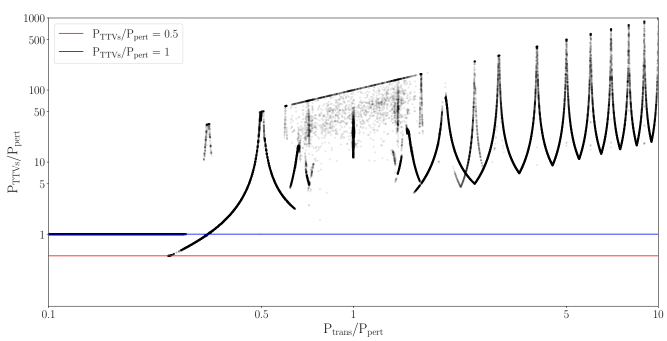
Again, scaling by the transiting period, we get

$$\bar{P}'_{TTV} = \frac{1}{\nu} = \frac{1}{|\frac{1}{P'_{TTV}} \pm m|} \tag{7}$$

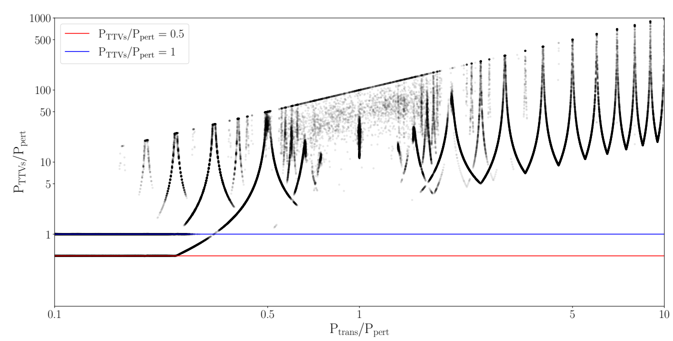
In order to understand these observations, we must also consider that for TTVs induced via the conjunction effect (ie the chopping signal) we expect the amplitude of the TTVs to scale with e^0, were e is a combined vector of free eccentricity for the two planets in the system. Additionally, for TTVs induced via the near MMR, we expect the amplitude of the signal to scale with e^n, where n is the order of the resonance (or $|j-k|$). Therefore, for perfectly circular orbits, none of the TTVs induced via near MMR would survive. And at small eccentricities, we expect the first order resonant terms to dominate.

2. OBSERVATIONS FROM SIMULATIONS

We found through simulations with TTVFast that the TTV period is never less than half the perturbing period. Here we try to explain this observation.



(a) circular simulated systems



(b) non-circular ($0 < e < 0.1$) systems

Figure 1

3. ANALYTIC SOLUTION: CIRCULAR SYSTEMS

Let's start by looking at circular systems. As previously stated, for circular orbits, we expect the conjunction effect to be the driving TTV effect. Therefore, we expect all TTV periods to be observed at the synodic period or aliases of the synodic period. Therefore, to understand why no TTV period is observed below the perturbing period, we can plot the aliased synodic period for external perturbers (see Figure 2).

For the range $2/3 < P_p/P_t < 2$, the synodic signal will not be aliased, as ita TTV period is greater than $2P_t$. However, outside of this range, the TTV period of the synodic period will be aliased. We find that the aliased synodic period in this range ($2 < P_p/P_t$) is equal to the perturbing period.

To see this mathematically, we used the alias equation. We find that only m=-1 solves the aliased equation (for $2 < P_p/P_t$) I should probably prove this more rigorously, but I tested a bunch of values and this is what I found... and we can re-write the aliased equation as:

$$\bar{P}'_{TTV} = \frac{1}{|\frac{1}{P_{syn}} - 1|} \tag{8}$$

Plugging in P_{syn} into this equation, we get:

$$\bar{P}'_{TTV} = \frac{1}{|\frac{1}{\frac{1}{|1/-1/P'_p|}} - 1|} \tag{9}$$

Plugging in any $P'_p > 2$, you find that $P_{TTV} = P_p$.

We plot this equation over the span of $2/3 < P_p/P_t < 10$ – note that for $P_p/P_t < 2/3$, other aliases (ie. m values) dominate – in Figure 2.

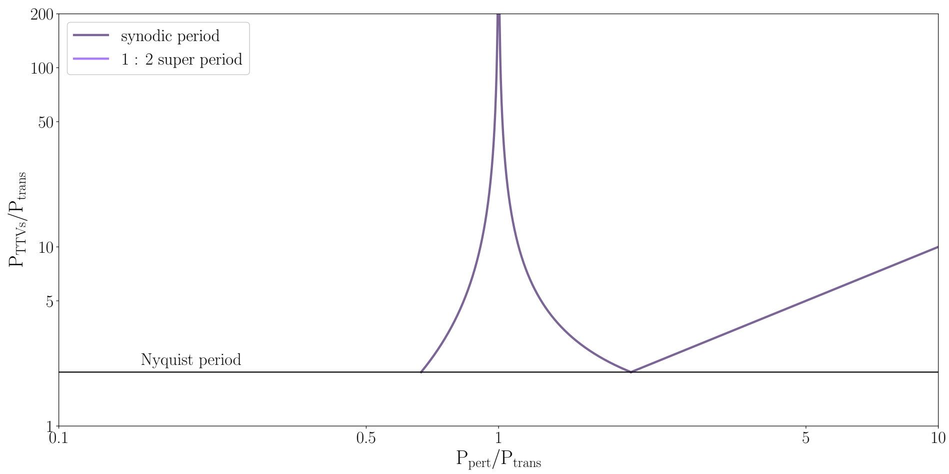


Figure 2: synodic period

Therefore, we find that in circular orbits, you expect the minimum recoverable TTV period to always equal the perturbing period.

As the Nyquist period is equal to $2P_t$, and we've shown that for all $P_p >= 2P_t$, the minimum recoverable TTV period from the aliased perturber affect is equal to the perturber period, we cover all of the relevant parameter space as for any $P_p < 2P_t$, the Nyquist period alias affect would kick in.

4. ANALYTIC SOLUTION: LOW ECCENTRICITY SYSTEMS

We can follow this same procedure now for systems with small eccentricities, where we expect first order near MMR TTVs and conjunction TTVs to dominate. We won't reproduce the math presented above for the conjunction effect, but we need to for first order near MMRs.

For an external perturber, the first order resonance with the most distant perturber is the j:k = 1:2 MMR, so we will focus on this MMR to start.

For the range $4/3 < P_p/P_t < 4$, the 1:2 MMR TTV signal will not be aliased, as its TTV period is greater than $2P_t$. However, outside of this range, the TTV period of the synodic period will be aliased. We find that the aliased synodic period in the range $(4 < P_p/P_t)$ is equal to the half the perturbing period.

To see this mathematically, we find that only m=-1 solves the aliased equation (for $4 < P_p/P_t$) I should probably prove this more rigorously, but I tested a bunch of values and this is what I found... and we can re-write the aliased equation as:

$$P'_{TTV} = \frac{1}{|\frac{1}{P_{syn}} - 1|} \tag{10}$$

Plugging in P_{syn} into this equation, we get:

$$\bar{P}'_{TTV} = \frac{1}{|\frac{1}{\frac{1}{|1/-2/P'_p|}} - 1|} \tag{11}$$

Plugging in any $P'_p > 4$, you find that $P_{TTV} = 1/2\, P_p$.

We plot this equation over the span of $4/3 < P_p/P_t < 10$ – note that for $P_p/P_t < 4/3$, other alias (m values) dominate – in Figure 3. We also plot the conjunction effect TTV periods, as we expect that to also play a role at low eccentricities.

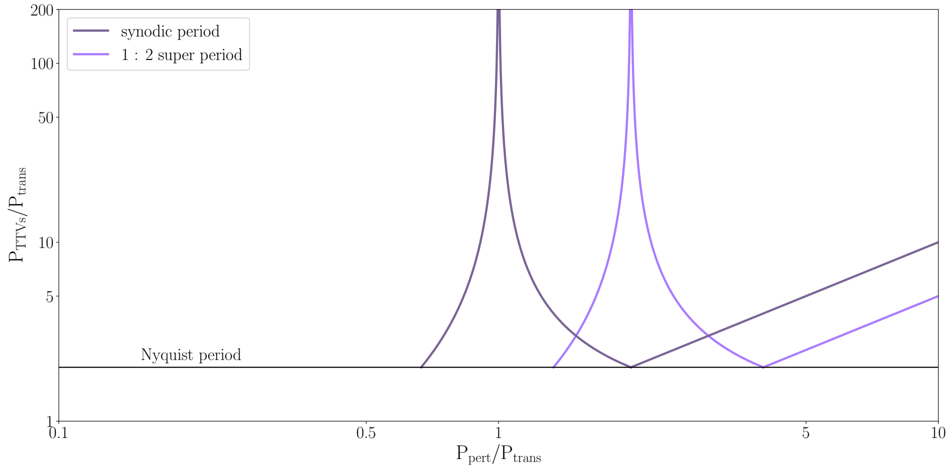


Figure 3: synodic period and 1:2 super period

Therefore, we find that in low eccentricity orbits, you expect the minimum recoverable TTV period to equal half the perturbing period.

As the Nyquist period is equal to $2P_t$, and we've shown that for all $P_p >= 4P_t$, the minimum recoverable TTV period from the aliased perturber affect is equal to the half the perturber period, we cover all of the relevant parameter space as for any $P_p < 2P_t$, the Nyquist period alias affect would kick in.

5. ANALYTIC SOLUTION: HIGH ECCENTRICITY SYSTEMS

Naively, one might think that once we allow for high eccentricity orbits, we would simply reproduce the same process, but now allowing for higher order MMR terms. However, once we allow for higher order MMR terms, there are now many MMRs for external perturbers (ie 1:3, 1:4, 1:5, etc.). If we reproduce the process for 1:3 MMR that we did for 1:2 and synodic periods, we would find that 1:3 signal would not be aliased until it reaches a P_p value equal to $6P_t$. At P_p near $6P_t$, we would not expect the 1:3 signal to be dominant over the much closer MMR terms such as 1:6. The difference is in the high eccentricity regime, higher order MMR are expected to contributed significantly to the TTV signal.

This continues for higher order MMRs, such as 1:4, where the signal would not be aliased until it reaches a P_p value equal to $8P_t$.

I suppose that the more relevant period value to consider is not when the 1:3 super period would be aliased due to transiting Nyquist frequency, but instead when P_{sup} drops below $1/2P_p$. I am fairly certain that this is also far enough away from 1:3 MMR, where we would expect other MMR super-period to dominate, but worth proving more rigourously.

6. CONCLUSION

Going back to the original simulated data in Figure 1, we see that both the aliased synodic TTV signal, with $P_{TTV} = P_p$, and the aliased 1:2 MMR TTV signal, with $P_{TTV} = 1/2P_p$.

For circular orbits, we see the aliased synodic period, with $P_{TTV} = P_p$, dominate. We note that we didn't correctly model circular orbits as we assumed an time-instantaneous eccentricity of 0 rather than a free-eccentricity of 0. Therefore, we think that is why the 1:2 signal is still present in the data.

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