Homework 3

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1. **Prove that if $e \leq 0$ and $v \leq 1$ are integers with $e \leq \frac{v(v-1)}{2}$, then there is a graph with v vertices, e edges, and at most e^3/v^2 crossings.**

   Solution. We define an explicit construction as follows. Space out the v vertices equally along the circumference of a unit circle, and order them a_1 to a_v by clockwise order. We define the i^{th}, for $i \geq 1$, generation of edges in our construction to be the edges $\overline{a_j a_{(j+i)\,\%v}}$ for each j from 1 to v. For example, the first generation of edges constitutes the boundary of an n-gon.

   If our graph G has e edges, then we require at most $\left\lfloor \frac{e}{n} \right\rfloor + 1$[1] generations in our construction. We will bound the crossing number of G by bounding the number of crosses for each of the e edges in our construction.

   Consider any of the e edges in our graph. Since we have completed at most $\left\lfloor \frac{e}{n} \right\rfloor + 1$ generations in our construction, each edge must have at most $\left(\left\lfloor \frac{e}{n} \right\rfloor + 1\right) - 1 = \left\lfloor \frac{e}{n} \right\rfloor$ vertices between them. Each of these vertices also has at most $\left\lfloor \frac{e}{n} \right\rfloor$ interior (not counting the boundary) edges with it as an endpoint in our n-gon since each generation creates one interior edge from a given vertex. Thus, the crossing number is bounded by the number of edges multiplied by the maximum number of crossings per edge:

   $$\mathrm{Cr}(G) \leq e\left(\left\lfloor \frac{e}{n} \right\rfloor\right)\left(\left\lfloor \frac{e}{n} \right\rfloor\right) = e\left(\left\lfloor \frac{e}{n} \right\rfloor\right)^2.$$

   Clearly, since $\left\lfloor \frac{e}{n} \right\rfloor \leq \frac{e}{n}$, we find that

   $$\mathrm{Cr}(G) \leq e\left(\left\lfloor \frac{e}{n} \right\rfloor\right)^2 \leq e\left(\frac{e}{n}\right)^2 = \frac{e^3}{n^2}$$

   as desired. ■

[1] every generation consists of n edges, besides generation n/2 when n is even — this case is still handled by the +1 term in the expression

2. **The Sierpinksi right triangle is defined recursively as follows: at stage 0, make a solid right triangle of height 1 and base length 1. At stage n, for each positive integer n, slice each solid triangle of height 2^{1-n} into four congruent triangles of height 2^{-n} and remove the interior of the middle one.**

   Prove that there is some $\log_2 3$-gale d with the following property: for all points p in the right Sierpinski triangle, there is an infinite sequence $q_0 \supset q_1 \ldots$ of nested dyadic squares such that $p \in \bigcap_{n \in \mathbb{N}} q_n$ and $d(q_n) = \Omega(1)$.

   Solution. We claim that the gale d with $d\left(q_i^{\ulcorner}\right) = d(q_i^{\llcorner}) = d(q_i^{\lrcorner}) = 1$ and $d\left(q_i^{\urcorner}\right) = 0$ where each of $q_i^{\ulcorner}, q_i^{\llcorner}, q_i^{\lrcorner}$, and $q_i^{\urcorner}$ represent the four dyadic subsquares of q_i is a $\log_2 3$-gale with the desired properties. We will prove this inductively, on the generation number n.

   For our base case(s), note that $d(q_0) = 1$ holds by definition. For the inductive step, assume that $d(q_k) = \Omega(1)$ for some $k \in \mathbb{N}$. We want to show that $d(q_{k+1}) = \Omega(1)$.

   By definition,

$$\begin{aligned} d(q_{k+1}) &= 2^{-\log_2 3}\left(d\left(q_k^{\ulcorner}\right) + d(q_k^{\llcorner}) + d(q_k^{\lrcorner}) + d\left(q_k^{\urcorner}\right)\right) \\ &= \frac{1}{3}(3\Omega(1) + 0) = \Omega(1) \end{aligned}$$

   and so $d(q_{k+1}) = \Omega(1)$, as desired. Furthermore, this is indeed a $\log_2 3$-gale.

   Thus, by induction, we know that the above gale is a $\log_2 3$-gale d satisfying the property that for all points p in the right Sierpinski triangle, there is an infinite sequence $q_0 \supset q_1 \ldots$ of nested dyadic squares such that $p \in \bigcap_{n \in \mathbb{N}} q_n$ and $d(q_n) = \Omega(1)$. ■

3. **Use the Szemeredi-Trotter theorem to prove that if P is any set of n points in the plane, then there are $O(n^{4/3})$ pairs of points in P that are at distance exactly 1 from each other:**

$$|\{(p,q) \in P^2 : ||p-q|| = 1\}| = O(n^{4/3})$$

Solution. Let us begin by creating a set of circles, C, each of radius 1, centered at each point in P. Note that a pair of points in P separated by a unit distance corresponds directly to two point-circle incidences between P and C. Consequently, to bound the number of pairs of points in P that are exactly a unit distance away from each other, it suffices to bound the number of point-circle incidences in P and C.

It remains to show that there are $O(n^{4/3})$ point-circle incidences between P and C, which we will prove by appealing to a modified version of the proof for the Szemeredi-Trotter Theorem.

Let I be the number of point-circle incidences between P and C, and let m_i represent the number of points of P on the boundary of circle C_i, which is centered at p_i. By definition, we have that

$$I = \sum_{i=1}^{n} m_i.$$

We will now create a graph G. We remove circles with $m_i \leq 2$, i.e. circles that have two or less points on their boundary. This removes at most $2n$ incidences. Now, each individual circle is partitioned into ≥ 3 arcs between consecutive points on that circle. We define an edge between each consecutive point; more formally, for points $q_1, q_2, \ldots, q_n$ (ordered in clockwise order) around circle P_i, define the edges $\overline{q_1 q_2}, \ldots, \overline{q_n q_1}$, and do this for each of the n circles. If there are multi-edges corresponding to edges that belong to multiple circles, remove one of their copies. We know that the number of edges, e, in G satisfies

$$e \geq \frac{\sum\limits_{i=1}^{n} m_i}{2} - 2n$$

where the $2n$ term comes from the fact that our first step removes at most $2n$ incidences and consequently, at most $2n$ edges. Since we know $I = \sum\limits_{i=1}^{n} m_i$ by definition, we substitute to get $e \geq \frac{I}{2} - 2n$, meaning $I \leq 2e + 4n$.

We define the vertices of G as the points in P. We now consider two cases separately to appeal to the Crossing Number Inequality.

Note that if $e \leq 4v$, then

$$I \leq 2e + 4n \leq 2(4n) + 4n$$

and so $I = O(n) = O(n^{4/3})$ as desired.

On the other hand, if $e \geq 4v$, then we know by the Crossing Number Inequality that

$$\mathrm{Cr}(G) \geq \frac{e^3}{n^2}.$$

Since $e \geq \frac{I}{2} - 2n$, we have the bound $e \geq \frac{I}{2}$. Similarly, since each pair of circles intersects at most twice, we know that $\mathrm{Cr}(G) \leq 2\binom{n}{2} \leq n^2$. Combining these facts together, we find that

$$n^2 \geq \mathrm{Cr}(G) \geq \frac{\left(\frac{I}{2}\right)^3}{n^2}.$$

Solving for I, we find that $I = O(n^{4/3})$. Thus, we conclude that there are $O(n^{4/3})$ pairs of points in P that are at distance exactly 1 from each other, as desired. $\blacksquare$