Homework 6

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1. **Let E2S be the problem of determining, given a graph $G = (V, E)$ and a positive integer k, whether there is a set $S \subset V$ such that $|S| = k$ and for all distinct $u, v \in S$, the length of the shortest path between u and v is exactly two. Prove that IS $\leq_p$ E2S.**

   Solution. Consider an instance of IS, $(G(V, E), k)$. Define a new graph G' with all vertices in V and an additional vertex v^*. Furthermore, define the edges of G' to be the edges E, and edges between each vertex in V to v^*. Formally, G' is a graph with

   - vertex set $V' = V \cup v^*$
   - edge set $E' = E \cup \{\overline{vv^*} \mid v \in V\}$

   Note that this is a transformation that can clearly be implemented in polynomial time in the number of edges and vertices.

   We claim that (G, k) is in IS if and only if (G', k) is in E2S.

   For the forward implication, note that for any two vertices u, v in an independent set of G, the minimum length between u and v is by definition ≥ 2. Furthermore, with the construction of v^*, there is a path $u - v^* - v$ of length 2 in G', so the length between any two vertices u, v in the same set of vertices in G' is exactly 2. Thus, if (G, k) is in IS, then (G', k) is in E2S.

   For the reverse implication, consider some set $S \subset V'$ in G' with $|S| = k$ satisfying E2S. Consider any vertices u, v in S. By definition, the minimum length between u and v is two, and so the length between u and v in the graph G must be greater than or equal to 2. Consequently, any vertices in some set S satisfying E2S will also be vertices in some independent set in G.[1] Thus, if (G', k) is in E2S then (G, k) is in IS. ■

[1] this also follows since v^* cannot be in this E2S set, since the distance between v^* and every other vertex in G' is one by construction.

2. **Consider the two following decision problems.**

PARTITION: Given a set X of positive integers, can X can be partitioned into two sets L and R (meaning $L \cap R = \emptyset$ and $L \cup R = X$) such that

$$\sum_{\ell \in L} \ell = \sum_{r \in R} r?$$

KNAPSACK: Given a list of n positive weights $w_1, \ldots, w_n$, n positive values $v_1, \ldots, v_n$, a positive capacity W, and a target value V , is there a subset $S \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} v_i \geq V?$$

Prove that PARTITION $\leq_p$ KNAPSACK.

Solution. Consider an instance of PARTITION defined by a set X of positive integers $X = \{x_1, \ldots, x_n\}$ which has sum $T = \sum_{i=1}^{n} x_i$. Define an instance of KNAPSACK with a list of n positive weights and values as follows:

$$\{w_1 = v_1 = x_1, w_2 = v_2 = x_2, \ldots, w_n = v_n = x_n\}$$

and $W = V = \frac{T}{2}$, which is a transformation that can be done in $O(n)$ time.

We claim that X is in PARTITION if and only if the designed instance is in KNAPSACK.

For the forward implication, note that if X is in PARTITION, X can be partitioned into two sets L, R with equal sum. By definition, this sum must be equal to half the total sum of the elements of X, which is simply $\frac{T}{2}$. Since the weights and values are defined as the elements in X, $S = L$ is a subset satisfying KNAPSACK with $W = V = \frac{T}{2}$ and the defined list.

For the reverse implication, note that if there is a subset $S \subset \{1, \ldots, n\}$ satisfying

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} v_i \geq V$$

with $W = V = \frac{T}{2}$, then the subsets S and $\{1, \ldots, n\} \setminus S$ are the corresponding indices for the sets L and R satisfying PARTITION, with the original set X. ■

3. **We define a geometric knapsack problem GEOKNAP as follows. The input consists of a set of n small polygons $\mathcal{P}_1, \dots, \mathcal{P}_n$ each given by clockwise lists of their vertices; a value v_i for each polygon $\mathcal{P}_i$; a big polygon $\mathcal{Q}$; and a target value V. The question is whether there is some set of small polygons with total value $\geq V$ that can fit inside $\mathcal{Q}$ without overlapping, assuming that we are allowed to translate or rotate the polygons. Prove that KNAPSACK $\leq_p$ GEOKNAP.**

Solution. Consider an instance K of KNAPSACK defined by a list of n positive weights and values $w_1, \dots, w_n$ and $v_1, \dots, v_n$ with a positive capacity W and target value V. Define an instance G of GEOKNAP defined by

- a set of n rectangles $\mathcal{P}_1, \dots, \mathcal{P}_n$ each of which has dimension
$$\left(\min_{i \in \{1,\dots,n\}} w_i\right) \times w_i.$$
- a big polygon $\mathcal{Q}$ of size $1 \times W$
- a target value V

where W, V are the same values as defined in the instance of KNAPSACK. This is a transformation that can be done in $O(n)$ time.

We claim that K is in KNAPSACK if and only if G is in GEOKNAP.

For the forward implication, note that K being in KNAPSACK implies a subset $S \subseteq \{1, \dots, n\}$ with
$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} v_i \geq V.$$

By construction, then, the set $SP = \{\mathcal{P}_i \mid i \in S\}$ is a set of small polygons that can fit inside $\mathcal{Q}$ (since the sum of widths of elements in SP is less than W) that also has total sum $\geq V$ by definition; thus, G is in GEOKNAP.

For the reverse implication, consider some set SP of rectangles $\mathcal{P}_i$ (with the previously defined W, V and list of weights and values) satisfying GEOKNAP. Note that rotations of these rectangles force them out of $\mathcal{Q}$, so we know that the axes of each rectangle lie parallel to the axes of $\mathcal{Q}$. Consequently, since the rectangles in SP fit in $\mathcal{Q}$, we must have that
$$\sum_{i \in SP} w_i \leq W.$$

Furthermore, since their values sum to a total value $\geq V$,
$$\sum_{i \in SP} v_i \geq V$$

is also satisfied. Thus, by definition, the set S of the indices of rectangles is part of an instance K in KNAPSACK. ■

4. **Define the problem TRIHIT as follows. The input consists of a set T of m triangles, each given by its vertices, a set P of n points, and a parameter k. The question is whether there is a subset $S \subseteq P$ of cardinality k such that each triangle in T has some point from S in its interior. Prove that IS $\leq_p$ TRIHIT.**

   Solution. First, note that IS $\leq_p$ VC. We claim that $S \subset V$ be a vertex cover in G if and only if $V \setminus S$ is an independent set in G. This follows directly from the definitions of vertex cover and independent set. A vertex cover S must "hit" every edge, in the sense that every edge has an endpoint in S. Consequently, $V \setminus S$ is a set containing at most one endpoint of every edge, which is an independent set of G by definition.

   Since IS $\leq_p$ VC, to show IS $\leq_p$ TRIHIT, it suffices to show that VC $\leq_p$ TRIHIT. Consider an instance of VC, defined by a graph G with vertex set V and edge set E, and the parameter k representing the size of a potential vertex set.

   For each edge $e_i \in E$, create a triangle defined by three vertices $v_{i_1}, v_{i_2}, v_{i_3}$, and two points p_{i_1}, p_{i_2} such that $\overline{p_{i_1} p_{i_2}}$ is enclosed in the interior of this triangle. This transformation can be implemented in $O(|E|)$ time, and yields an instance of TRIHIT with input set T consisting of $|E|$ triangles, a set P of $2|E|$ points, and the same parameter k as the instance of VC.

   We claim that (G, k) is in VC if and only if this new (T, P, k) is in TRIHIT.

   For the forward implication, note that if (G, k) is in VC, there is some vertex cover of size k in G, which includes at least one endpoint of every edge in G; by construction, this means that each of our constructed triangles has at least one point (representing the endpoints of an edge) in its interior. Thus, if (G, k) is in VC, then (T, P, k) is in TRIHIT.

   For the reverse implication, note that if (T, P, k) is in TRIHIT, then is some subset $S \subseteq P$ of cardinality k such that each triangle in T contains a point from S in its interior. By construction, then, this means that every edge in E has one of its endpoints represented in S, and thus, G must have a valid vertex cover of size k, i.e. (G, k) is in VC. ∎