Circle Packings from Tilings of the Plane

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Philadelphia Undergraduate Mathematics Conference

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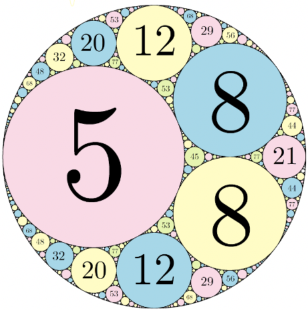
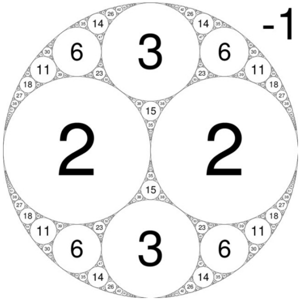
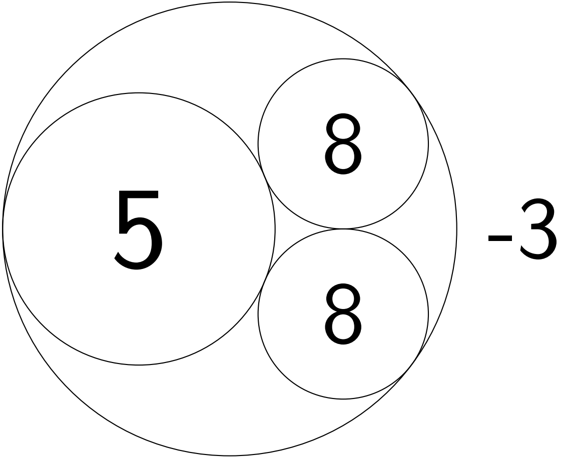
Advised by Professor Ian Whitehead
Collaborated with Phil Rehwinkel

Apollonian Circle Packings





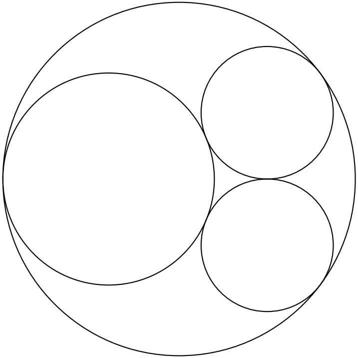
Constructing a Packing: Start



Constructing a Packing: Dual Circles

Definition (Dual Circles)

A *dual circle* of the Apollonian packing is a circle orthogonal to a ring of mutually tangent circles at their points of tangencies. Inversions about dual circles generate the packing.



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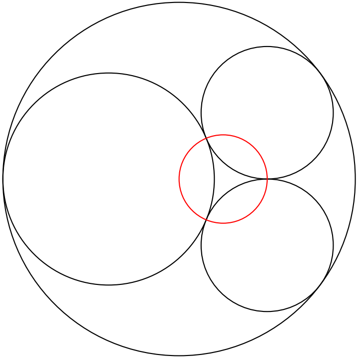

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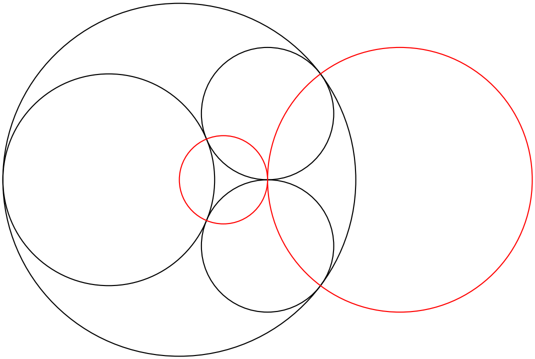

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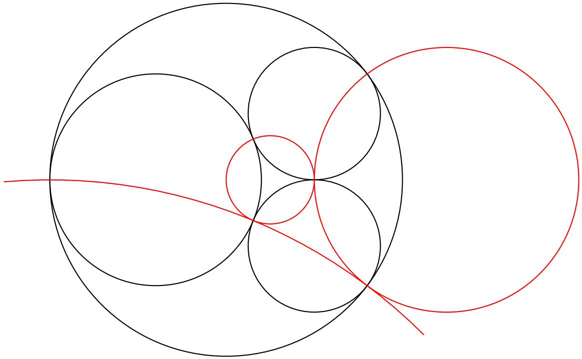

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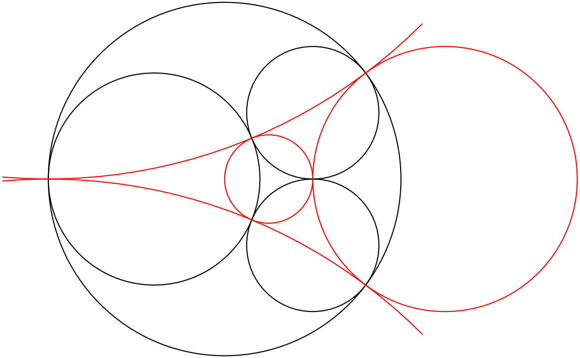
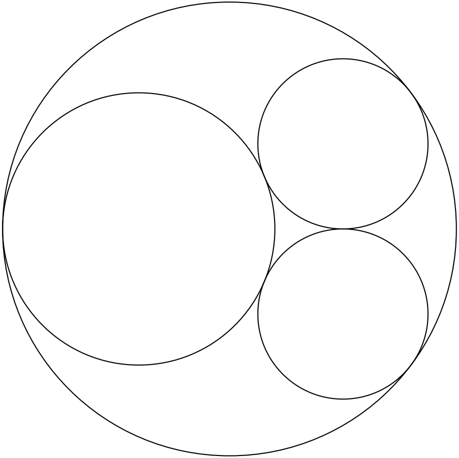


Constructing a Packing: Inversions



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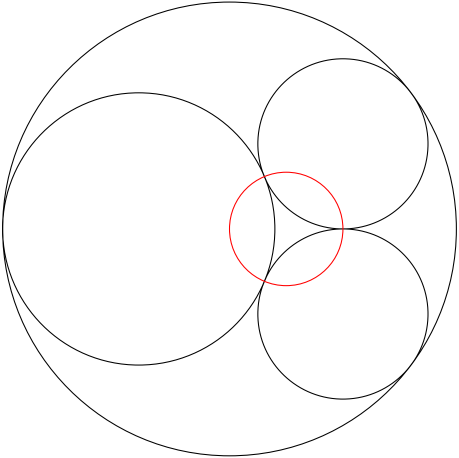

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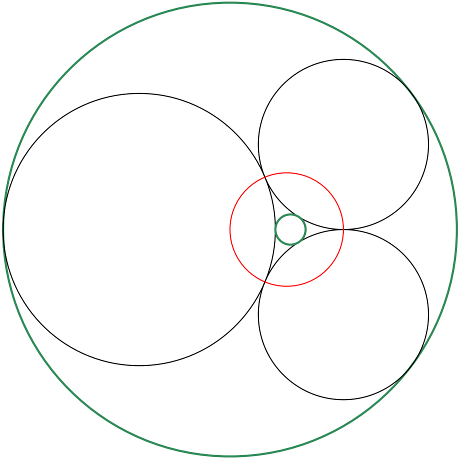

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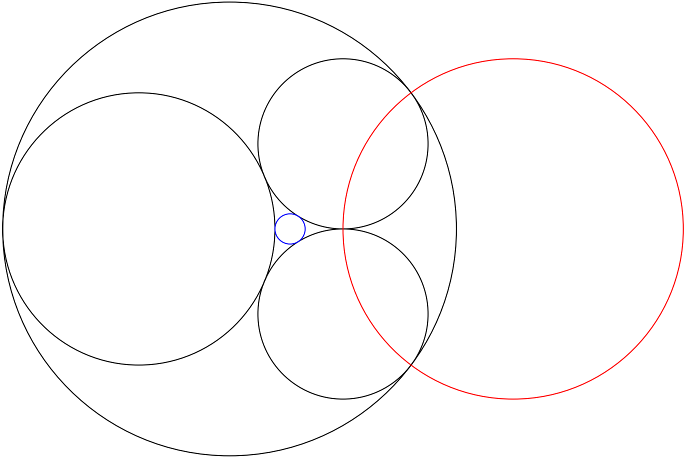

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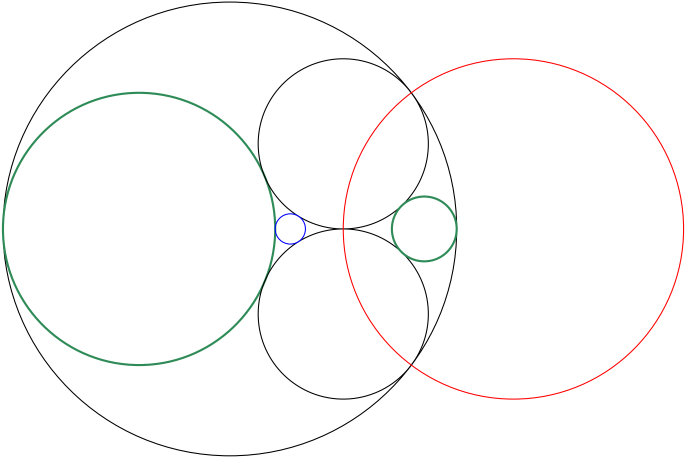

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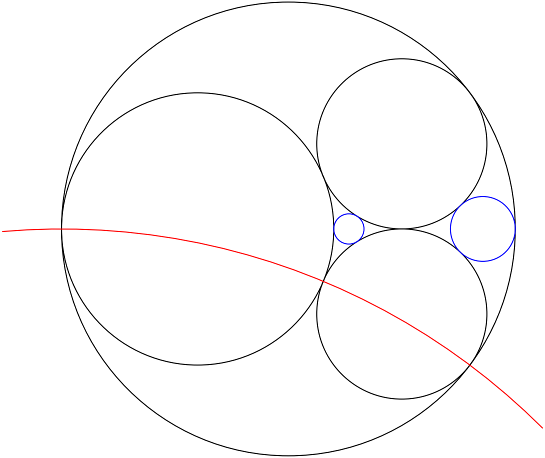

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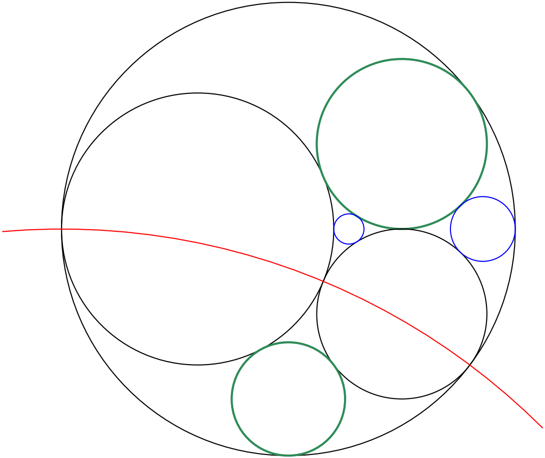

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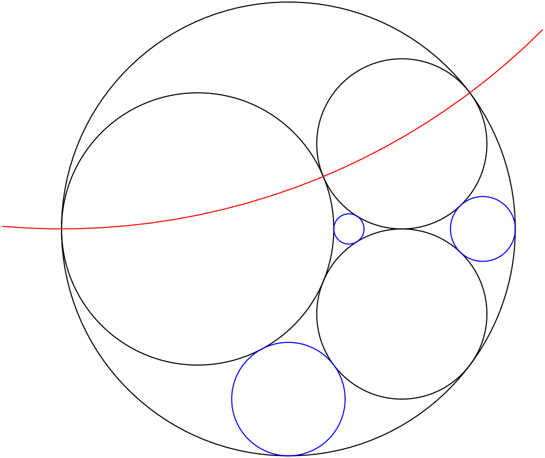

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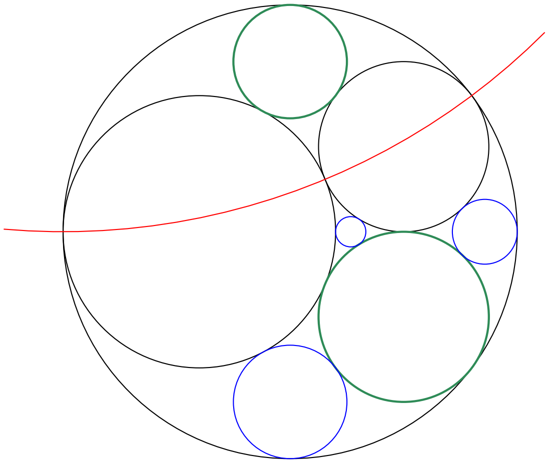
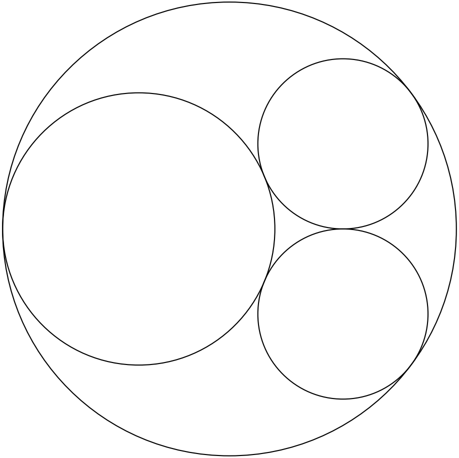


Apollonian Circle Packing



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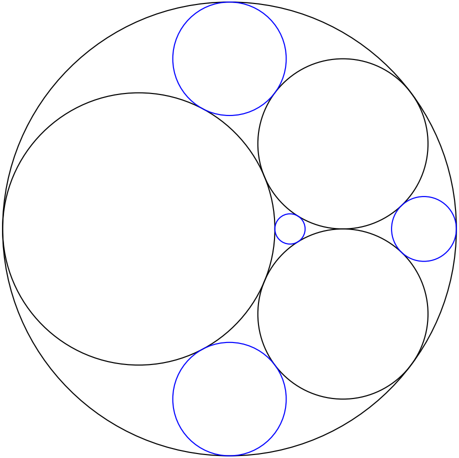

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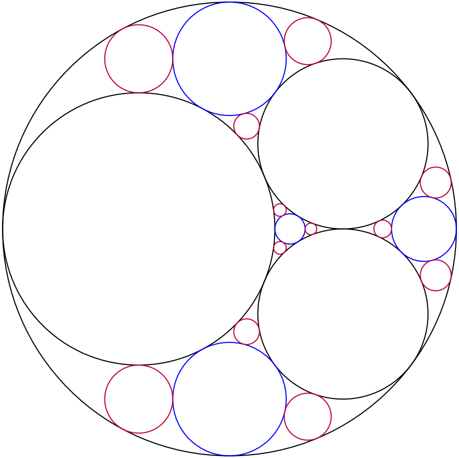
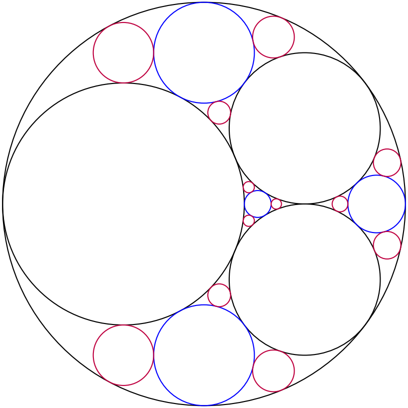
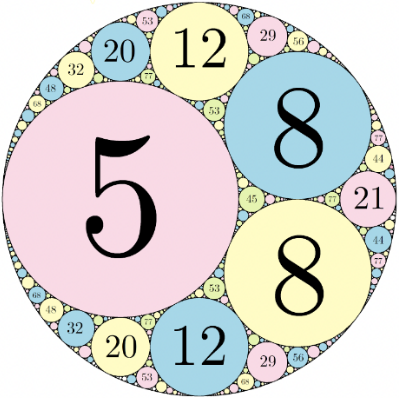
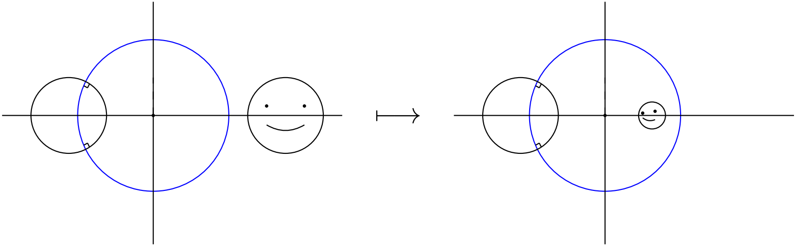


Apollonian Circle Packing: Complete





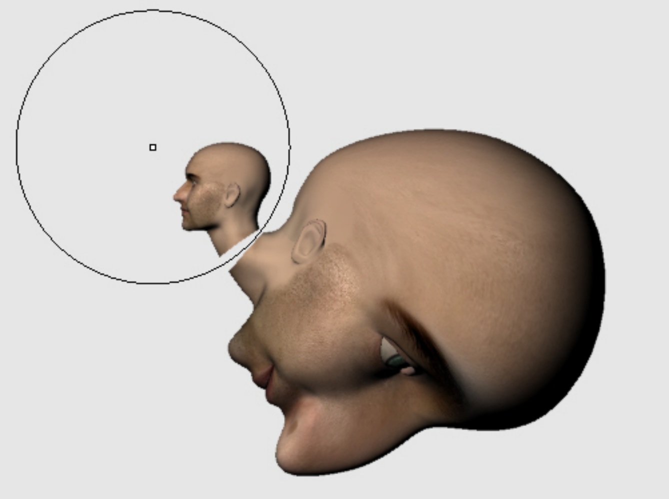
Circular Inversion Example



- Circles map to circles (though they can also map to lines, and vice versa)
- Orthogonal circles to circle of inversion map to themselves
- Inversion preserves tangency relations

Möbius Transformations

More Inversion Examples



Images from Space Symmetry Structure Wordpress

More Inversion Examples

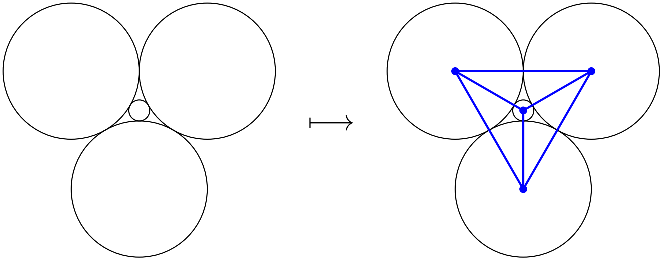
Images from Space Symmetry Structure Wordpress

Tangency Graphs

Definition

The *tangency graph* of a circle packing is a graph with

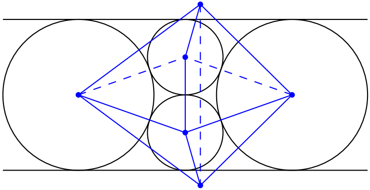
1. a vertex at the center of each circle
2. an edge between a pair of vertices if the corresponding circles are tangent



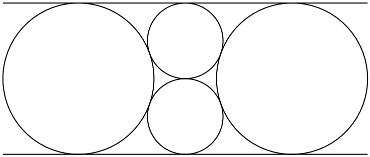
Polyhedral Packings

Definition

A packing is *polyhedral* if the tangency graph is a polyhedron.



Octahedral Packing: Dual Circles



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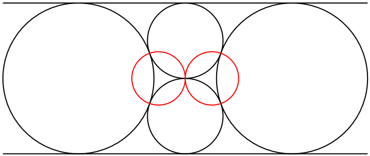

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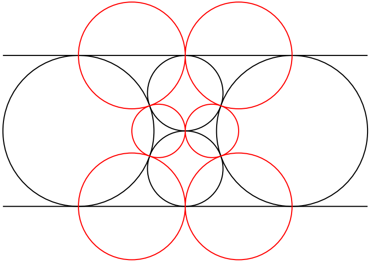

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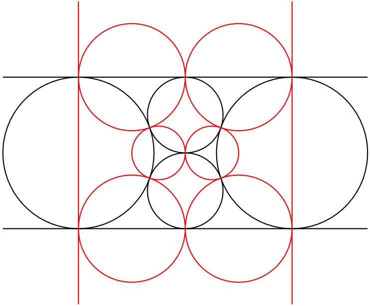
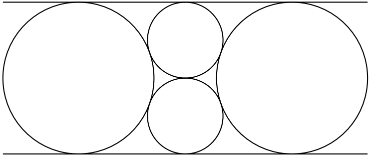


Generating the Octahedral Packing



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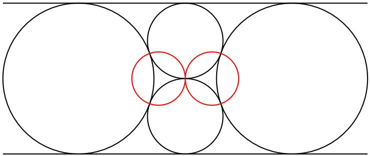

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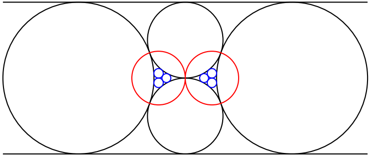

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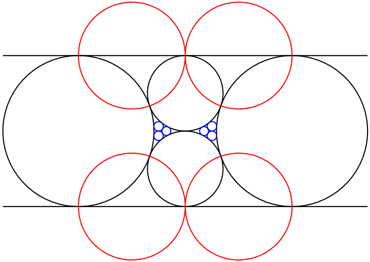

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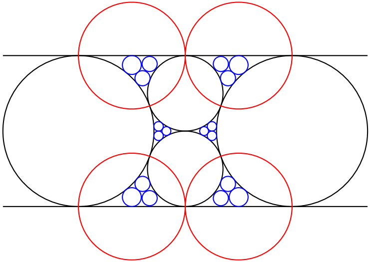

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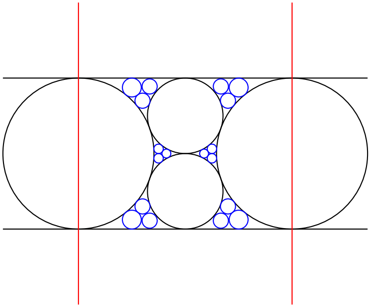
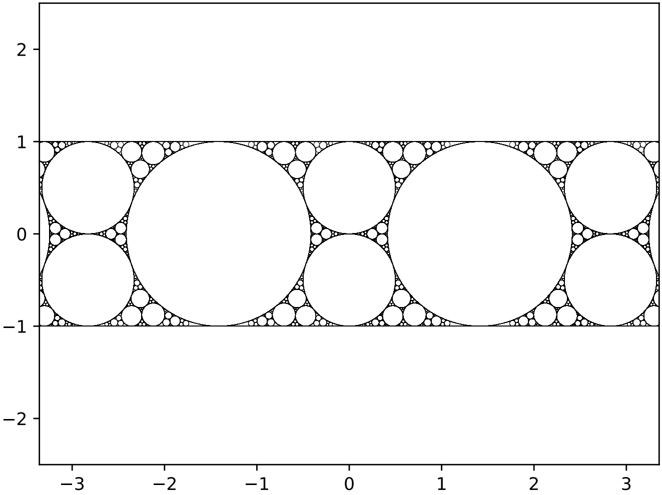


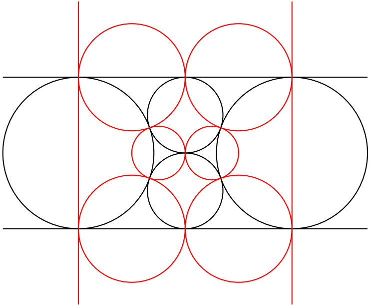
Generated Octahedral Circle Packing

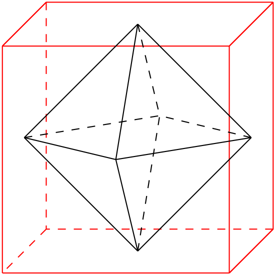


Dual Packings and Dual Polyhedra

Definition

The faces of the polyhedron correspond to *dual circles*, which are circles that intersect the circles corresponding to vertices in the tangency graph orthogonally.





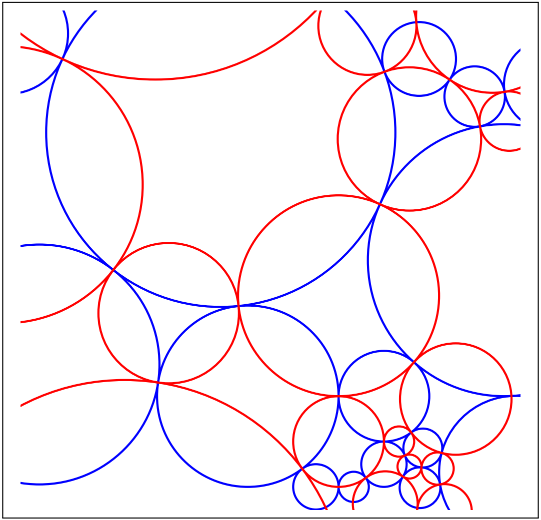
Base and Dual Configuration of Infinite Packings

Definition

Let B and $\hat{B}$ be two collections of oriented circles with the tangency graphs Γ_B and $\Gamma_{\hat{B}}$, respectively. Then B is called a base configuration and $\hat{B}$ is called a dual configuration if the following properties hold:

1. the interiors of the circles in B are pairwise disjoint and the interiors of the circles in $\hat{B}$ are pairwise disjoint.
2. the tangency graphs Γ_B and $\Gamma_{\hat{B}}$ are each nontrivial, connected, and are duals of each other.
3. if circles in B and $\hat{B}$ intersect, they do so orthogonally and they correspond to a face-vertex pair in the tangency graph.
4. $B \cup \hat{B}$ have at most one accumulation point.

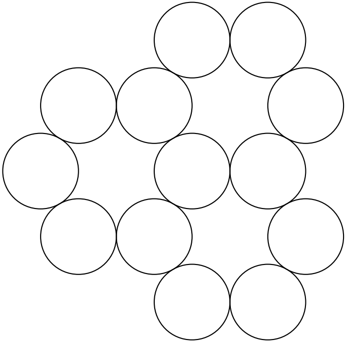
Base and Dual Configurations



Infinite Packings

Definition

The *packing* $\mathscr{P}$ is the orbit of B under the group generated by reflections across circles in $\hat{B}$.



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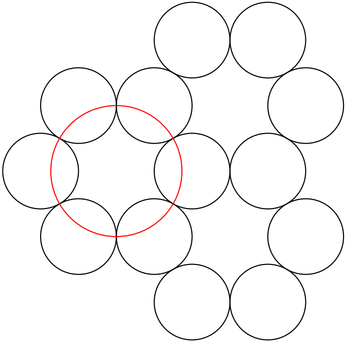

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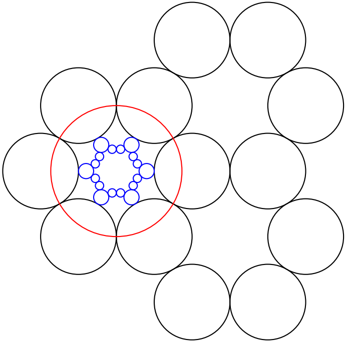

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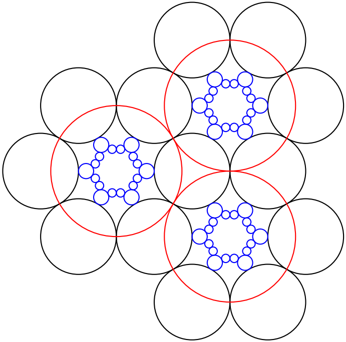
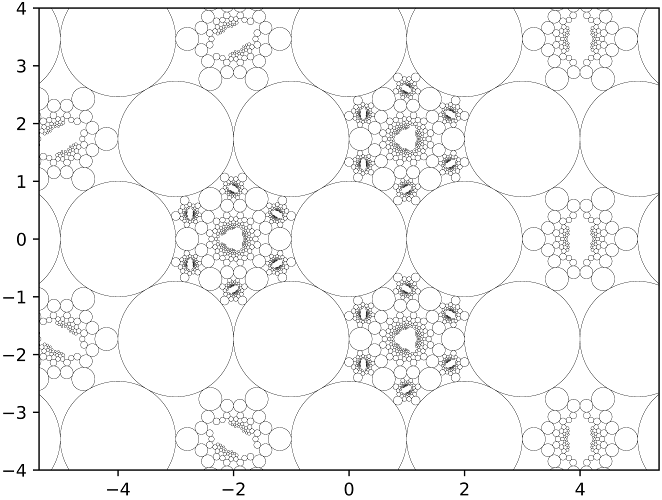


Generated Infinite Packing



Dual Packings

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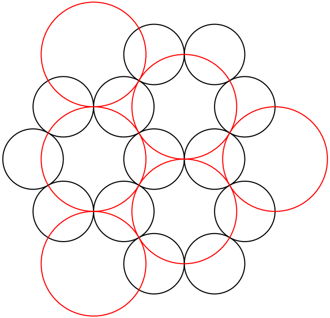

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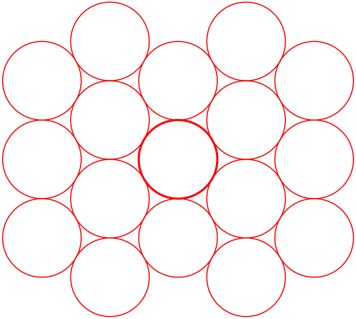

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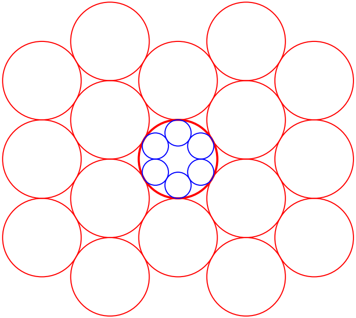

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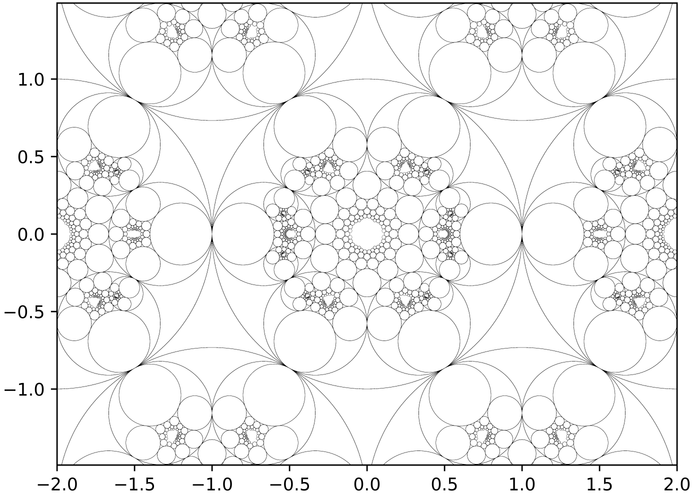
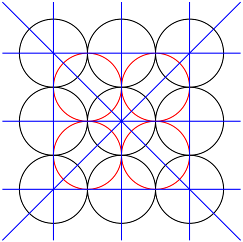


Symmetry Groups of a Packing

Definition

For each packing $\mathscr{P}$, we define the following symmetry groups:

1. $\Gamma = \mathrm{SYM}(\mathscr{P}, \hat{\mathscr{P}})$: the group of Möbius transformations that preserve both packing and dual packing,
2. $\Gamma_1 = \langle \hat{B} \rangle$: the group generated by reflections through the dual circles,
3. $\Gamma_2 = \mathrm{SYM}(B, \hat{B})$: the group of Möbius transformations that preserve both base and dual configuration.



Group Structure Theorems

Proposition

Γ_1 is a free Coxeter group.

Proposition

If B is finite, then Γ_2 is the group of symmetries of a polyhedron. If B is infinite, then Γ_2 is conjugate to a discrete group of isometries of the plane: a cyclic group, dihedral group, frieze group, or wallpaper group.

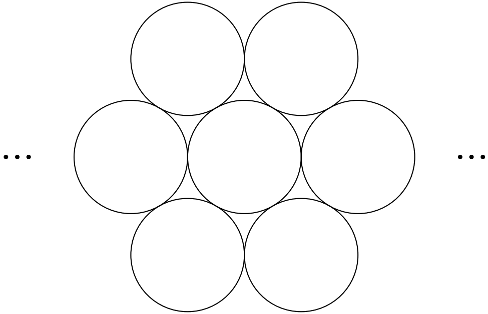
Theorem

$\Gamma \cong \Gamma_1 \rtimes \Gamma_2$.

Wallpaper Groups

Theorem (Refined Circle Packings)

Any wallpaper group is the symmetry group of the base configuration (Γ_2) *of some circle packing.*



Wallpaper Group 17 (p6m)

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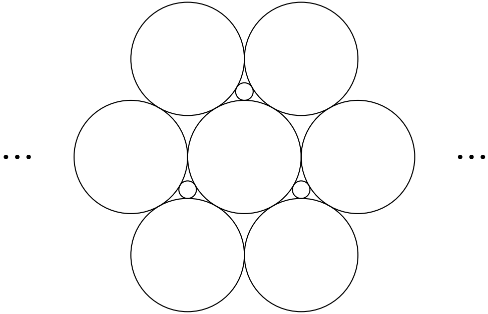

Wallpaper Group 15 (p3m1)

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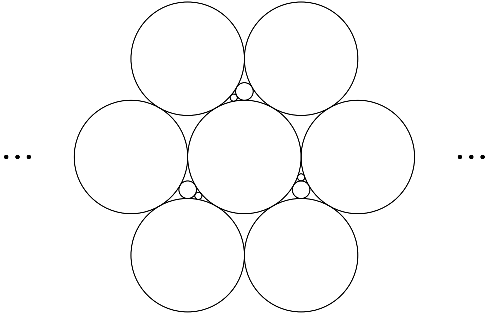
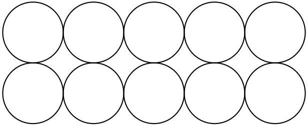
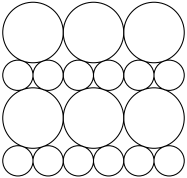


Wallpaper Group 13 (p3)

Limit Packings

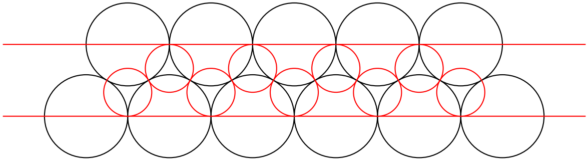


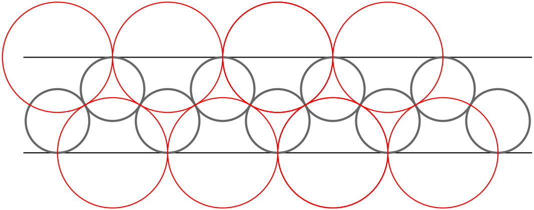
admits integrality and superintegrality
related to Apollonian packing



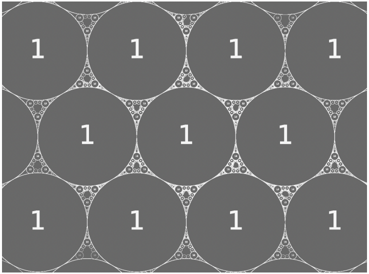
admits integrality and superintegrality
related to Octahedral packing

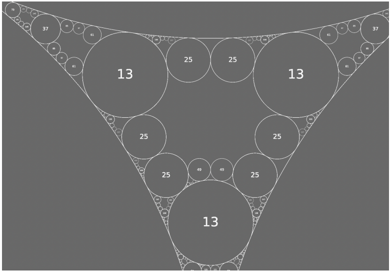
Limit Packings cont.





Integral* Packings





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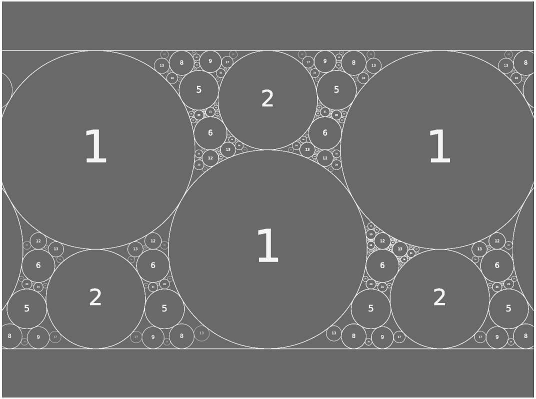

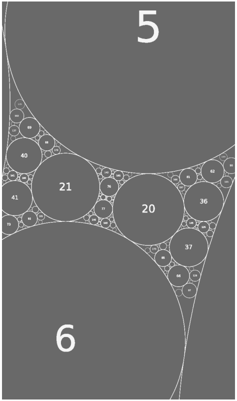

Acknowledgments

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