MATH103: Complex Analysis **Fall 2023**

Homework 5

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Chapter IV (Complex Integration and Analyticity) Problems.

Section IV.4 (The Cauchy Integral Formula), Problem 4

Let D be a bounded domain with smooth boundary ∂D, and let $z_0 \in D$. Using the Cauchy integral formula, show that there is a constant C such that

$$|f(z_0)| \leq C \sup\{|f(z)| : z \in \partial D\}$$

for any function $f(z)$ analytic on $D \cup \partial D$. By applying this estimate to $f(z)^n$, taking nth roots, and letting $n \to \infty$, show that the estimate holds with $C = 1$. *Remark.* This provides an alternative proof of the maximum principle for analytic functions.

Solution. Since $f(z)$ is analytic on $D \cap \partial D$ and D is a bounded domain with smooth boundary, we know by Cauchy's Integral Formula that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0}\, dw$$

and so consequently,

$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0}\, dw \right|$$

Furthermore, since $\left|\frac{1}{2\pi i}\right| = \frac{1}{2\pi}$ and $|f(w)| \leq \sup\{|f(z)| : z \in \partial D\}$ for $w \in \partial D$, we know that

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0}\, dw \right| \\ &= \frac{1}{2\pi} \left| \int_{\partial D} \frac{f(w)}{w - z_0}\, dw \right| \\ &\leq \frac{1}{2\pi} \sup\{|f(z)| : z \in \partial D\} \left| \int_{\partial D} \frac{1}{w - z_0}\, dw \right|. \end{aligned}$$

However, note that $\frac{1}{w - z_0}$ is bounded, since $w \in \partial D$ and $z_0 \in D$. Consequently, the term $\left| \int_{\partial D} \frac{1}{w - z_0}\, dw \right|$ is bounded, say by a constant M. Thus, we have that

$$|f(z_0)| \leq C \sup\{|f(z)| : z \in \partial D\}$$

for a constant $C = \frac{M}{2\pi}$, where M is defined above.

We can now apply this estimate to the function $f(z)^n$, which is analytic since $f(z)$ is analytic. We get that

$$|f(z_0)^n| \leq C \sup\{|f(z)^n| : z \in \partial D\}$$

Note that $|f(z_0)^n| = |f(z_0)|^n$ and $\sup\{|f(z)^n| : z \in \partial D\} \leq (\sup\{|f(z)| : z \in \partial D\}^n)$, and so we know that our estimate is equivalent to

$$|f(z_0)|^n \leq C \left(\sup\{|f(z)| : z \in \partial D\}\right)^n .$$

Taking the nth root of both sides of our inequality, we find that

$$|f(z_0)| \leq C^{\frac{1}{n}} \sup\{|f(z)| : z \in \partial D\}.$$

Note that $\lim_{n\to\infty} C^{\frac{1}{n}} = 1$ for any nonzero constant C. Thus, we find that our estimate, as $n \to \infty$, becomes

$$|f(z_0)| \leq \sup\{|f(z)| : z \in \partial D\}.$$

Thus, our estimate holds for $C = 1$ and we also have an alternative proof of the maximum principle for analytic functions. ∎

Section IV.5 (Liouville's Theorem), Problem 4

Suppose that $f(z)$ is an entire function such that $f(z)/z^n$ is bounded for $|z| \geq R$. Show that $f(z)$ is a polynomial of degree at most n. What can be said if $f(z)/z^n$ is bounded on the entire complex plane?

Solution. Let D be a disk of radius R centered at the origin. By definition, $\frac{f(z)}{z^n}$ is bounded on D, and so we know that

$$\left|\frac{f(z_0)}{z_0^n}\right| \leq M$$

for all $z_0 \in D$.
Equivalently, we can multiply both sides by $|z_0|^n$ to get that

$$|f(z_0)| \leq M|z_0|^n \leq MR^n$$

for all $z_0 \in D$ (where the last step follows from the fact that $|z_0| \leq R$). ■