MATH103: Complex Analysis **Fall 2023**

Homework 12

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Chapter IX (The Schwarz Problems and Hyperbolic Geometry) Problems.

Section IX.2 (Conformal Self-Maps of the Unit Disk), Problem 8

Show that every conformal self-map of the Riemann sphere $\mathbb{C}^*$ is given by a fractional linear transformation.

Solution. Let $f(z)$ be a conformal self-map of the Riemann sphere $\mathbb{C}^*$. We know that $f(\infty) = \infty$ or $f(\infty) \neq \infty$, and we will consider these cases separately.

In the former case, $f(\infty) = \infty$. Consequently, since f is a conformal self-map of $\mathbb{C}^*$ with a fixed point at ∞, f must also be a conformal self-map of $\mathbb{C}$. By Exercise IX.2.7, we know that

$$f(z) = az + b,$$

with $a \neq 0$. Since $f(z) = az + b$ is a fractional linear transformation with $c = 0$ and $d = 1$, any conformal self-map $f(z)$ satisfying $f(\infty) = \infty$ is given by a fractional linear transformation.

In the latter case, $f(\infty) = c$ for some $c \neq \infty$. Consider the fractional linear transformation

$$g(z) = \frac{1}{z - c}.$$

Since $g(z)$ is a fractional linear transformation, it is a conformal self-map of $\mathbb{C}^*$.

Consider the function $g \circ f$. Since g and f are themselves conformal self-maps of $\mathbb{C}^*$ and the composition of conformal self-maps of $\mathbb{C}^*$ is also a conformal self-map of $\mathbb{C}^*$, we know $g \circ f$ is a conformal self-map of $\mathbb{C}^*$. Furthermore, note that

$$(g \circ f)(\infty) = g(f(\infty)) = g(c) = \infty$$

so the function $g \circ f$ is a conformal self-map of C^* with a fixed point at ∞. By our work above, we know that $g \circ f$ must be a fractional linear transformation of the form $az + b$.

Since $(g \circ f)(z) = \frac{1}{f(z) - c}$, we have that

$$(g \circ f)(z) = \frac{1}{f(z) - c} = az + b$$

with $a \neq 0$. Solving for $f(z)$ by taking the reciprocal of both sides and simplifying, we get that

$$\begin{aligned} f(z) - c &= \frac{1}{az + b} \\ \Rightarrow f(z) &= c + \frac{1}{az + b} \\ \Rightarrow f(z) &= \frac{c(az + b) + 1}{az + b}. \end{aligned}$$

Simplifying, we get that

$$f(z) = \frac{(ac)z + (bc+1)}{az+b}.$$

Note that $b(ac) - a(bc+1) = -a \neq 0$, and so $f(z)$ is a fractional linear transformation.

In both cases, we find that a conformal self-map $f(z)$ of $\mathbb{C}^*$ is a fractional linear transformation. Thus, every conformal self-map of the Riemann sphere $\mathbb{C}^*$ is given by a fractional linear transformation, as desired. $\blacksquare$

Section IX.2 (Conformal Self-Maps of the Unit Disk), Problem 13

Suppose $f(z)$ is an analytic function from the open unit disk $\mathbb{D}$ to itself that is not the identity map z. Show that $f(z)$ has at most one fixed point in $\mathbb{D}$. *Hint.* Make a change of variable with a conformal self-map of $\mathbb{D}$ to place the fixed point at 0.

Solution. We will prove the contrapositive; namely, that if $f(z)$ has at least two distinct fixed points, which we can denote z_0 and z_1 in $\mathbb{D}$, then $f(z)$ is the identity map.

Let $g(z)$ be the conformal self-map of $\mathbb{D}$ mapping z_0 to 0 and z_1 to some nonzero value c:

$$g(z) = \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Consider $h(z) = (g \circ f \circ g^{-1})(z)$. By definition, since g and g^{-1} are both conformal self-maps, they are analytic maps from $\mathbb{D}$ to $\mathbb{D}$. Similarly, f is analytic from $\mathbb{D}$ to $\mathbb{D}$. Thus, h, the composition of these functions, is also an analytic function from $\mathbb{D}$ to $\mathbb{D}$.

Furthermore, note that

$$h(0) = g(f(g^{-1}(0))) = g(f(z_0)) = g(z_0) = 0$$

by construction, as $g(z_0) = 0$, $g^{-1}(0) = z_0$, and $f(z_0) = z_0$ as z_0 is a fixed point of f by assumption.

Similarly, consider the image of the nonzero value $c = g(z_1)$ under h: since by construction $g^{-1}(c) = z_1, g(z_1) = c$, and $f(z_1) = z_1$ as z_1 is a fixed point of f, we have

$$h(c) = g(f(g^{-1}(c))) = g(f(z_1)) = g(z_1) = c,$$

giving us a nonzero fixed point for h.

By Schwarz Lemma, since h is an analytic function from $\mathbb{D}$ to itself, $|h(z)| \leq 1$ (as h is bounded by the unit disk) for $|z| < 1$, and $h(0) = 0$, we know that

$$|h(z)| \leq |z|.$$

Since equality holds at $z = c \neq 0$, we know that by Schwarz Lemma,

$$h(z) = \lambda z$$

for some λ of unit modulus. Even more, we must have that $\lambda = 1$, since $h(c) = c$. Thus, $h(z) = z$. Consequently, we know that

$$h(z) = (g \circ f \circ g^{-1})(z) = z$$

for $z \in \mathbb{D}$. This tells us that

$$(f \circ g^{-1})(z) = g^{-1}(z),$$

Treating $g^{-1}(z)$ as z_1 for some $z_1 \in D$, we must have that $f(z_1) = z_1$ for all $z_1 \in \mathbb{D}$. Consequently, f is the identity function.

Thus, by proving the contrapositive statement, we know that $f(z)$ must have at most one fixed point in $\mathbb{D}$, as desired. ■