MATH103: Complex Analysis **Fall 2023**

Homework 7

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Chapter VI (Laurent Series and Isolated Singularities) and Chapter VII (The Residue Calculus) Problems.

Section VI.2 (Isolated Singularities of an Analytic Function), Problem 12

Show that if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity of $e^{f(z)}$.

Solution. First, note that if z_0 is an isolated singularity of $f(z)$, it must also be an isolated singularity of $e^{f(z)}$.[1] Since z_0 is an isolated singularity of $f(z)$ that is not removable, we know that z_0 is either an essential singularity of $f(z)$ or z_0 is a pole of $f(z)$. We will consider these two cases separately.

First, suppose that z_0 is an essential isolated singularity of $f(z)$. Then by the Casorati-Weierstrass Theorem, we know that for every complex number w_0, there exists a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$.

Consider two sequence $a_n \to z_0$ and $b_n \to z_0$ such that $f(a_n) \to 0$ and $f(b_n) \to x$, for any complex x. Then we must have that

$$\left|e^{f(a_n)}\right| \to e^0 = 1 \text{ and } \left|e^{f(b_n)}\right| \to e^x.$$

Since $\left|e^{f(a_n)}\right| \to e^0 = 1$ for a sequence $a_n \to z_0$, we know that z_0 cannot be a pole for $e^{f(z)}$ (since the magnitude of $e^{f(z)}$ does not approach ∞ as $a_n \to z_0$). Similarly, since $\left|e^{f(b_n)}\right| \to e^x$ and the choice of x is arbitrary, we know that $e^{f(z)}$ is not bounded near z_0, and thus, z_0 cannot be a removable singularity of $e^{f(z)}$.

On the other hand, suppose that z_0 is a pole of $f(z)$, say of order N at z_0. By definition, then, we know that

$$f(z) = \frac{g(z)}{(z-z_0)^N} = \frac{h(z)}{(z-z_0)^N} + r(z),$$

where the first statement follows from the definition of a pole of $f(z)$ at order N at z_0, with $g(z)$ analytic at z_0 and $g(z_0) \neq 0$. The second statement follows from the fact that we can rewrite $f(z)$ as its Laurent decomposition (where the first term represents the principal part at the pole z_0, and the second term $r(z)$ is analytic). Furthermore, note that under this definition, $h(z)$ is a polynomial of degree $< N$ with $h(z_0) \neq 0$.

Now, consider

$$e^{f(z)} = e^{\frac{h(z)}{(z-z_0)^N} + r(z)} = e^{r(z)} e^{\frac{h(z)}{(z-z_0)^N}}.$$

[1] this follows from the definition of an isolated singularity – we can take the same radius r that $f(z)$ is analytic in the punctured disk with radius r and center z_0 in and the same property will hold for $e^{f(z)}$.

Note that the Laurent expansion of $e^{r(z)}$, about z_0, includes no negative powers of $(z - z_0)$, since $r(z)$ is analytic at z_0. On the other hand, the Laurent expansion of $e^{\frac{h(z)}{(z-z_0)^N}}$ about z_0 is

$$e^{\frac{h(z)}{(z-z_0)^N}} = \sum_{n=0}^{\infty} \frac{\left(\frac{h(z)}{(z-z_0)^N}\right)^n}{n!}.$$

Since $h(z_0) \neq 0$ by construction and $h(z)$ is a polynomial of degree $< N$, we know that for each fixed n, there is a term with a negative power. Since this is an infinite sum, and these terms will not cancel with the resulting terms for larger values of n, there must be infinitely many negative power terms in the Laurent expansion about z_0. Thus, by definition, $e^{f(z)}$ has an essential singularity at z_0.

In both cases, we find that $e^{f(z)}$ has an essential singularity at z_0, and thus, if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity of $e^{f(z)}$, as desired. $\blacksquare$

Section VII.1 (The Residue Theorme), Problem 2

Calculate the residue at each isolated singularity in the complex plane of the following functions.

a) $e^{1/z}$

Solution. Note that the isolated singularity of $e^{1/z}$ is at $z=0$. The Laurent Series of $e^{1/z}$ at $z=0$ is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \dots$$

By definition, the residue of $e^{1/z}$ at the isolated singularity $z=0$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion at $z=0$, so

$$\operatorname{Res}\left[e^{1/z}, 0\right] = \boxed{1}.$$

■

b) $\tan z$

Solution. Note that the isolated singularities of $\tan z = \frac{\sin z}{\cos z}$ occur when $\cos z = 0$, so $z = \frac{\pi}{2} + n\pi$ for any $n \in \mathbb{Z}$.

Since $\cos z$ has a simple zero at each of these singularities (they are simple since its derivative, $-\sin z$, is nonzero at the singularities), and both $\cos z$ and $\sin z$ are analytic at the singularities, then we know by Rule 3 that

$$\operatorname{Res}\left[\frac{\sin z}{\cos z}, s_n\right] = \frac{\sin s_n}{-\sin s_n} = \boxed{-1}$$

for each singularity $s_n = \frac{\pi}{2} + n\pi$. ■

c) $\frac{z}{(z^2+1)^2}$

Solution. Note that the isolated singularities of $\frac{z}{(z^2+1)^2}$ occur when $z^2+1=0$, so $z = \pm i$. Furthermore, since

$$\frac{1}{\left(\frac{z}{(z^2+1)^2}\right)} = \frac{(z^2+1)^2}{z} = \frac{(z+i)^2(z-i)^2}{z}$$

has zeros of order 2 at the singularities $z = \pm i$, we know that $\frac{z}{(z^2+1)^2}$ has double poles at $\pm i$.

By Rule 2, we have that

$$\begin{aligned}
\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, i\right] &= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z}{z^2+1}\right] \\
&= \lim_{z\to i} \frac{d}{dz}\left[\frac{z}{(z+i)^2}\right] \\
&= \lim_{z\to i} \frac{-z+i}{(z+i)^3} \\
&= 0.
\end{aligned}$$

Similarly, by Rule 2, we have that

$$\begin{aligned}
\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, -i\right] &= \lim_{z\to -i} \frac{d}{dz}\left[(z+i)^2\frac{z}{z^2+1}\right] \\
&= \lim_{z\to -i} \frac{d}{dz}\left[\frac{z}{(z-i)^2}\right] \\
&= \lim_{z\to -i} -\frac{z+i}{(z-i)^3} \\
&= 0.
\end{aligned}$$

Thus, we conclude that $\boxed{\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, i\right] = 0 \text{ and } \operatorname{Res}\left[\frac{z}{(z^2+1)^2}, -i\right] = 0}$. ■

d) $\frac{1}{z^2+z}$

Solution. Note that the isolated singularities of $\frac{1}{z^2+z} = \frac{1}{z(z+1)}$ occur at $z = 0, -1$. Furthermore, note that

$$\frac{1}{z^2+z} = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}.$$

Note that the Laurent expansion of this expression at $z = 0$ is

$$\frac{1}{z} - \frac{1}{z+1} = \frac{1}{z} - \frac{1}{1-(-z)} = \frac{1}{z} + \sum_{n=0}^{\infty}(-z)^k.$$

By definition, the residue at the isolated singularity $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion at $z = 0$, so

$$\operatorname{Res}\left[\frac{1}{z^2+z}, 0\right] = 1.$$

By a similar line of reasoning, note that the Laurent expansion of this expression at $z = -1$ is

$$\frac{1}{z} - \frac{1}{z+1} = \frac{-1}{1-(z+1)} - \frac{1}{z+1} = -\frac{1}{z+1} - \sum_{n=0}^{\infty}(z+1)^k.$$

By definition, the residue at the isolated singularity $z = -1$ is the coefficient of $\frac{1}{z+1}$ in the Laurent expansion at $z = -1$, so

$$\operatorname{Res}\left[\frac{1}{z^2+z}, -1\right] = -1.$$

Thus, we find that $\boxed{\operatorname{Res}\left[\frac{1}{z^2+z}, 0\right] = 1 \text{ and } \operatorname{Res}\left[\frac{1}{z^2+z}, -1\right] = -1}$. ■