MATH103: Complex Analysis **Fall 2023**

Homework 4

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Chapter III (Line Integrals and Harmonic Functions) Problems.

Section III.3 (Harmonic Conjugates), Problem 3

Let $D = \{a < |z| < b\} \setminus (-b, -a)$, an annulus slit along the neagtive real axis. Show that any harmonic function on D has a harmonic conjugate on D. *Suggestion.* Fix c between a and b, and define $v(z)$ explicitly as a line integral along the path consisting of the straight line from c to $|z|$ followed by the circular arc from $|z|$ to z. Or map the slit annulus to a rectangle by $w = \operatorname{Log} z$.

Solution. Fix A in D and let B be any point in D. We define $v(B)$ explicitly as

$$v = \int_\gamma -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

where γ consists of the straight line segment from point A to $|B|e^{i\operatorname{Arg}(A)}$ and the circular arc from $|B|e^{i\operatorname{Arg}(A)}$ to B. Note that in polar coordinates, we know that

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Consequently, we know that

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta \text{ and } dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

Substituting these equations into our integral, we get that

$$\begin{aligned} v &= \int_\gamma -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \\ &= \int_\gamma -\frac{\partial u}{\partial y} \left(\cos\theta\, dr - r\sin\theta\, d\theta\right) + \frac{\partial u}{\partial x}\left(\sin\theta\, dr + r\cos\theta\, d\theta\right). \end{aligned}$$

Grouping the dr and $d\theta$ terms together, this integral becomes

$$\int_\gamma \left(-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right) dr + \left(\frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta\right) d\theta.$$

Note that $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$, $\frac{\partial x}{\partial r} = \cos\theta$, and $\frac{\partial y}{\partial r} = \sin\theta$. Substituting these values into our integral, we get

$$\int_\gamma \left(-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right) dr + \left(\frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta\right) d\theta$$

$$= \int_\gamma \left(-\frac{1}{r}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} - \frac{1}{r}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} \right) dr + \left(r\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + r\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} \right) d\theta$$

$$= \int_\gamma -\frac{1}{r}\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} \right) dr + r\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} \right) d\theta$$

However, we also know that the expressions inside the parantheses are simply

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} \text{ and } \frac{\partial u}{\partial r} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}.$$

Thus, substituting once more, we find that the integral simplifies to

$$v = \int_\gamma -\frac{1}{r}\frac{\partial u}{\partial \theta}\, dr + r\frac{\partial u}{\partial r}\, d\theta.$$

To show that v is the harmonic conjugate of u, it suffices to show $u + iv$ is analytic, which we will do by appealing to the Cauchy-Riemann equations. Observe that

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

Thus, we find that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r},$$

matching the polar form of the Cauchy-Riemann equations. Thus, we know that the function $u+iv$ is analytic and since u is harmonic on D, v is the harmonic conjugate of u, as desired. ∎

Section III.4 (The Mean Value Property), Problem 4

Formulate the mean value property for a function on a domain in $\mathbb{R}^3$, and show that any harmonic function has the mean value property. *Hint.* For $A \in \mathbb{R}^3$ amd $r > 0$, let B_r be the ball of radius r centered at A, with volume element $d\tau$, and let ∂B_r be its boundary sphere, with area element $d\sigma$ and unit outward normal vector n. Apply the Gauss divergence theorem

$$\int\int_{\partial B_r} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int\int\int_{B_r} \nabla \cdot \mathbf{F} \, d\tau$$

to $\mathbf{F} = \triangle u$.

Solution. We say that a continuous function $h(z)$ on a domain D in $\mathbb{R}^3$ has the mean value propety if for each point $z_0 \in D$, $h(z_0)$ is the average of its values over the boundary of any ball centered at z_0.

We will now show that any harmonic function u defined on this domain has the mean value property. Let us first define $g(r)$ to be a function of r representing the average of u on ∂B_r. By definition, for center $A \in \mathbb{R}^3$ and radius $r > 0$, we have

$$g(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(A)} u \, dS = \frac{1}{|\partial B_r|} \int_{\partial B_r(A)} u(y) \, d\sigma(y).$$

We can now change the domain of integration to one that is independent of r; to do so, we will note that $y = A + rn$ and apply a change of variables to get $\frac{\partial\sigma(y)}{\partial\sigma(n)} = r^2$ (as we scale down from a ball of radius r to a ball of radius 1, the surface area scales down by a factor of r^2).

Thus, rewriting $g(r)$ in our new domain of integration, we get

$$\begin{aligned} g(r) &= \frac{1}{4\pi r^2} \int\int_{\partial B_1(0)} u(A + nr) r^2 \, d\sigma(n) \\ &= \frac{1}{4\pi} \int\int_{\partial B_1(0)} u(A + nr) \, d\sigma(n). \end{aligned}$$

To show that the mean value property holds, we will first show that $g'(r) = 0$. Taking the derivative with respect to r of both sides of our above equation, we get that

$$\begin{aligned} g'(r) &= \frac{\partial}{\partial r} \left(\frac{1}{4\pi} \int\int_{\partial B_1(0)} u(A + nr) \, d\sigma(n) \right) \\ &= \frac{1}{4\pi} \int\int_{\partial B_1(0)} \nabla u(A + nr) \cdot n \, d\sigma(n). \end{aligned}$$

By the Divergence Theorem, we know that

$$\int\int_{\partial B_1(0)} \nabla u(A + nr) \cdot n \, d\sigma(n) = \int\int\int_{B_1} \nabla \cdot \triangle u \, d\tau,$$

and the right-hand side simplifies to 0 since u is harmonic. Thus, we get that

$$\begin{aligned} g'(r) &= \frac{1}{4\pi} \int\int_{\partial B_1(0)} \nabla u(A + nr) \cdot n \, d\sigma(n) \\ &= \int\int\int_{B_1} \nabla \cdot \triangle u \, d\tau = 0. \end{aligned}$$

To show that the mean value property holds, it remains to show that $\lim_{r \to 0} g(r) = u(A)$, i.e. the average of all values over the boundary of the ball is the value of the harmonic function at its center.

Since $g(r)$ is constant with respect to r, it is also continuous, and so as r approaches 0, $g(r)$ approaches the value of u at the center, namely $u(A)$. Thus, we conclude that any harmonic function on a domain in $\mathbb{R}^3$ also has the mean value property, as desired. $\blacksquare$

Section III.5 (The Maximum Principle), Problem 3

Use the maximum principle to prove the fundamental theorem of algebra, that any polynomial $p(z)$ of degree $n \geq 1$ has a zero, by applying the maximum principle to $1/p(z)$ on a disk of a large radius.

Solution. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 1$. Let us assume for the sake of contradiction that p has no zeros, so $p(z) \neq 0$ for all $z \in \mathbb{C}$, or equivalently, $a_0 \neq 0$. Consider the function $f(z) = \frac{1}{p(z)}$. Note that $f(z)$ is continuously differentiable for all $z \in \mathbb{C}$ and

$$f'(z) = -\frac{1}{p(z)^2}$$

is continuous since $p(z) \neq 0$ for all z. Thus, $f(z)$ is analytic and it is also harmonic.

Consider any large disk $D = \{|z| < r\}$ for large r. Since $p(z) \neq 0$, we know that $f(z)$ extends continuously to the boundary ∂D. Let z_0 be the point on ∂D where f is maximized, and use M to denote $f(z_0)$. By the Maximum Principle, we know that

$$|f(z)| \leq |f(z_0)| = M$$

for all $z \in D$.

We claim that as the radius of the disk increases, i.e. $r \to \infty$, $|f(z_0)| \to 0$ for any point z_0 on the boundary of D. Note that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right)$$

Thus,

$$|p(z)| = |z^n||a_n|.$$

as we can write $z = re^{i\theta}$ for some θ and observe that the other terms tend to 0 when $r \to \infty$. Thus, we find that as $r \to \infty$, $p(z_0) \to \infty$ and so

$$\lim_{r\to\infty} |f(z_0)| = \lim_{r\to\infty} \left| \frac{1}{p(z_0)} \right| \to 0.$$

By the Maximum Principle, we also know that $\lim_{r\to\infty} |f(z)| = 0$. By definition, we now know that for large r and all $\epsilon > 0$, $|f(z)| < \epsilon$ for all z such that $|z| < r$.

Consider $\epsilon = \frac{1}{a_0}$. Clearly, we know that $|0| < r$ but

$$|f(0)| = \left| \frac{1}{a_0} \right| \not< \epsilon.$$

Thus, we have reached a contradiction and so we know that any polynomial of degree ≥ 1 has a zero, as desired. ■