MATH103: Complex Analysis **Fall 2023**

Homework 2

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Chapter II (Analytic Functions) Problems.

Section II.1 (Review of Basic Analysis), II.1.14

Let $h(t)$ be a continuous complex-valued function on the unit interval $[0,1]$, and consider

$$H(z) = \int_0^1 \frac{h(t)}{t-z}\,dt.$$

Where is $H(z)$ defined? Where is $H(z)$ continuous? Justify your asnwer. *Hint.* Use the fact that if $|f(t)-g(t)| < \epsilon$ for $0 \le t \le 1$, then $\int_0^1 |f(t)-g(t)|\,dt < \epsilon$.

Solution. $H(z) = \int_0^1 \frac{h(t)}{t-z}$ is defined only when the integrand is defined; this happens only when the denominator of the fraction $\frac{h(t)}{t-z}$ is nonzero. Put simply, we need $t - z \neq 0$ or $z \neq t$. Since by definition $t \in [0,1]$, $H(z)$ is defined for $z \in \mathbb{C} \setminus [0,1]$.

We claim that $H(z)$ is continuous for all $z \in \mathbb{C} \setminus [0,1]$ (by definition, it can only be continuous where it is defined, and so we aim to show that $H(z)$ is continuous at all points where it is defined). To do so, we will appeal to the limit definition of continuity, that $H(z)$ is continuous at z_0 if

$$\lim_{z \to z_o} H(z) = H(z_0),$$

To make use of the hint, let us define $f(t) = \frac{h(t)}{t-z}$ and $g(t) = \frac{h(t)}{t-z_0}$ for any $z, z_0 \in \mathbb{C} \setminus [0,1]$ and $t \in [0,1]$. Then

$$|f(t) - g(t)| = \left| \frac{h(t)}{t-z} - \frac{h(t)}{t-z_0} \right| = \left| \frac{h(t)(z-z_0)}{(t-z)(t-z_0)} \right|$$

Since $h(t)$ is defined on the compact interval $[0,1]$, it has a bounded maximum value, which we will denote M. Equivalently, $h(t) \le M$ for all $t \in [0,1]$. Thus, substituting this back into our above equation and using the fact that absolute values are multiplicative, we get

$$\begin{aligned} |f(t) - g(t)| = \left| \frac{h(t)(z-z_0)}{(t-z)(t-z_0)} \right| &\le \left| \frac{M(z-z_0)}{(t-z)(t-z_0)} \right|. \\ &= \left| \frac{(z-z_0)}{(t-z)(t-z_0)} \right| |M| \end{aligned}$$

We claim that

$$\lim_{z \to z_0} \left(\left| \frac{(z-z_0)}{(t-z)(t-z_0)} \right| |M| \right) = 0.$$

To see this, note that as $z \to z_0$, the denominator $(t-z)(t-z_0)$ approaches $(t-z_0)(t-z_0) = (t-z_0)^2$. Thus, rewriting the above limit, we have

$$\lim_{z\to z_0}\left(\left|\frac{(z-z_0)}{(t-z)(t-z_0)}\right||M|\right)=\lim_{z\to z_0}\left(\left|\frac{(z-z_0)}{(t-z_0)^2}\right||M|\right).$$

Note that since by definition, $z_0 \notin [0,1]$, z_0 cannot get arbitrarily close to $t \in [0,1]$. On the other hand, the numerator $z - z_0$ tends towards 0 as z approaches z_0. Thus,

$$\lim_{z\to z_0}\left(\left|\frac{(z-z_0)}{(t-z_0)^2}\right||M|\right)=0.$$

By the hint, we know that since $|f(t)-g(t)| < \epsilon$ for $0 \le t \le 1$, then $\int_0^1 |f(t)-g(t)|\,dt < \epsilon$. Equivalently, we know that since $\lim\limits_{z\to z_0}\left|\frac{h(t)}{t-z}-\frac{h(t)}{t-z_0}\right|=0$,

$$\lim_{z\to z_0}\int_0^1\left|\frac{h(t)}{t-z}-\frac{h(t)}{t-z_0}\right|=0.$$

Furthermore, note that by an absolute value property of integrals, we know that

$$\begin{aligned}\int_0^1\left|\frac{h(t)}{t-z}-\frac{h(t)}{t-z_0}\right| &\ge \left|\int_0^1\frac{h(t)}{t-z}-\frac{h(t)}{t-z_0}\right|\\ &=\left|\int_0^1\frac{h(t)}{t-z}-\int_0^1\frac{h(t)}{t-z_0}\right|\\ &=|H(z)-H(z_0)|\end{aligned}$$

Put succinctly, we know that

$$|H(z)-H(z_0)|\le\int_0^1\left|\frac{h(t)}{t-z}-\frac{h(t)}{t-z_0}\right|.$$

Thus, since $\lim\limits_{z\to z_0}\int_0^1\left|\frac{h(t)}{t-z}-\frac{h(t)}{t-z_0}\right|=0$, we know that

$$\lim_{z\to z_0}|H(z)-H(z_0)|=0.$$

for any $z_0 \in \mathbb{C}\setminus[0,1]$ (where H is defined). Thus, by the limit definition of continuity, $H(z)$ is continuous everywhere it is defined. ■

Section II.3 (The Cauchy-Riemann Equations), II.3.4

Show that if f is analytic on a domain D, and if $|f|$ is constant, then f is constant. *Hint.* Write $\overline{f} = |f|^2/f$.

Solution. We will split our work into two cases: if either f is 0 anywhere in the domain or if $f \neq 0$ everywhere in the domain. Note that by construction, these represent all possible cases for f.

First, if f is zero anywhere in the domain, then $|f| = 0$ at that point. Since $|f|$ is constant, this means that $|f|$ is zero for every point in the domain, which only occurs when f is zero everywhere. When f is zero everywhere in this domain, f is constant, as desired.

On the other hand, if $f \neq 0$ everywhere in the domain, then we consider $\overline{f}$. By the hint, we know

$$\overline{f} = \frac{|f|^2}{f} = \frac{C}{f}$$

for some constant C, since $|f|$ is constant. Furthermore, f is analytic on D. Since the quotient of an analytic function is also analytic (when the denominator does not vanish – which it does not, since f is 0 nowhere in the domain), we know $\overline{f}$ is also analytic on D.

By Exercise II.3.3, since both f and $\overline{f}$ are analytic on D, f is constant.

Thus, by our two cases, we know that if f is analytic on a domain D, and if $|f|$ is constant, then f is constant. ■

Section II.4 (Inverse Mappings and the Jacobian), II.4.3

Consider the branch of $f(z) = \sqrt{z(1-z)}$ on $\mathbb{C} \setminus [0,1]$ that has positive imaginary part at $z = 2$. What is $f'(z)$? Be sure to specify the branch of the expression for $f'(z)$.

Solution. We can calculate $f'(z)$ using the Chain Rule: note that $f(z) = (z(1-z))^{\frac{1}{2}}$ so

$$\begin{aligned} f'(z) &= \frac{1}{2}(z(1-z))^{-\frac{1}{2}} \cdot (z(1-z))' \\ &= \frac{1}{2}(z(1-z))^{-\frac{1}{2}} \cdot (1-2z). \end{aligned}$$

Simplifying, we get that

$$f'(z) = \frac{1-2z}{2\sqrt{z(1-z)}}.$$

Multiplying both the numerator and denominator of $f'(z)$ by $\sqrt{z(1-z)}$, we get that

$$\begin{aligned} f'(z) &= \frac{1-2z}{2\sqrt{z(1-z)}} \cdot \frac{\sqrt{z(1-z)}}{\sqrt{z(1-z)}} \\ &= \frac{(1-2z)\sqrt{z(1-z)}}{2z(1-z)} \end{aligned}$$

To determine the branch of the expression for $f'(z)$, we can first analyze the branch of $f(z)$ at $z = 2$. By definition, we know

$$\begin{aligned} f(z) = z\sqrt{1-z} &= z(1-z)^{\frac{1}{2}} \\ &= ze^{\frac{1}{2}(\log|1-z| + i\operatorname{Arg}(1-z) + i\cdot 2\pi m)}. \end{aligned}$$

At $z = 2$, $\log|1-z| = \log|-1| = 0$ and $\operatorname{Arg}(1-z) = \operatorname{Arg}(-1) = \pi$, so we have

$$z\sqrt{1-z} = 2e^{\frac{1}{2}(i\pi + i2\pi m)} = 2e^{i\frac{\pi}{2}}e^{i\pi m}.$$

Since we are considering the branch of $f(z)$ on $\mathbb{C} \setminus [0,1]$ that has positive imaginary part at $z = 2$, we are considering the principal branch where $m = 0$.

For $f'(z)$, notice that the expression $\sqrt{z(1-z)} = f(z)$ appears in the numerator and the other terms in the remaining expression $\left(\frac{1-2z}{2z(1-z)}\right)$ do not affect the branch of $f'(z)$; consequently, the branch of the expression for $f'(z)$ is simply the same principal branch of $f(z)$, which has positive imaginary part at $z = 2$. ∎