Homework 10

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Chapter VIII (The Logarithmic Integral) Problems.

Section VIII.4 (Open Mapping and Inverse Function Theorems), Problem 1

Suppose D is a bounded domain with piecewise smooth boundary. Let $f(z)$ be meromorphic and $g(z)$ analytic on D. Suppose that both $f(z)$ and $g(z)$ extend analytically across the boundary of D, and that $f(z) \neq 0$ on ∂D. Show that

$$\frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} m_j g(z_j)$$

where $z_1, \ldots, z_n$ are the zeros and poles of $f(z)$ and m_j is the order of $f(z)$ at z_j.

Solution. Note that $g(z)\frac{f'(z)}{f(z)}$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities at $z_1, \ldots, z_n$. Consequently, by the Residue Theorem, we have that

$$\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = 2\pi i\sum_{j=1}^{n}\text{Res}\left[g(z)\frac{f'(z)}{f(z)}, z_j\right].$$

Consider a given singularity z_j, which is either a zero or pole of order m_j at $f(z)$. By definition, we have that

$$f(z) = (z - z_j)^{m_j} h(z)$$

for a function $h(z)$ satisfying $h(z_j) \neq 0$ and $h(z)$ analytic at z_j. By the Chain Rule, we also find that

$$f'(z) = m_j(z - z_j)^{m_j - 1}h(z) + (z - z_j)^{m_j}h'(z)$$

and so

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m_j(z - z_j)^{m_j-1}h(z) + (z - z_j)^{m_j}h'(z)}{(z - z_j)^{m_j}h(z)} \\ &= \frac{m_j}{z - z_j} + \frac{h'(z)}{h(z)}.\end{aligned}$$

Since $h(z_j) \neq 0$ and $h(z)$ is analytic at z_j, $\frac{h'(z)}{h(z)}$ is also analytic at z_j. Thus, the residue of $\frac{f'(z)}{f(z)}$ at z_j, which is the coefficient of the $\frac{1}{z - z_j}$ term in the Laurent expansion about z_j, is simply m_j.

Plugging this residue into our result from the Residue Theorem, we find that

$$\begin{aligned}\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz &= 2\pi i\sum_{j=1}^{n}\text{Res}\left[g(z)\frac{f'(z)}{f(z)}, z_j\right] \\ &= 2\pi i\sum_{j=1}^{n} m_j g(z_j)\end{aligned}$$

where the factor of $g(z_j)$ follows from the fact that $g(z)$ is analytic at z_j.

Dividing both sides of our equation by $2\pi i$, we arrive at the desired result,

$$\frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} m_j g(z_j)$$

where $z_1, \ldots, z_n$ are the zeros and poles of $f(z)$ and m_j is the order of $f(z)$ at z_j. ■

Section VIII.6 (Winding Numbers), Problem 6

Let γ be a closed path in a domain D such that $W(,\gamma,\xi) = 0$ for all $\xi \notin D$. Suppose that $f(z)$ is analytic on D except possibly at finite number of isolated singularities $z_1, \ldots, z_m \in D \setminus \Gamma$. Show that

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} W(\gamma, z_k)\text{Res}[f, z_k].$$

Solution. Note that $f(z)$ is analytic on D except at its isolated singularities z_1 to z_m. Consequently, the function

$$g(z) = f(z) - \sum_{k=1}^{m} \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n$$

obtained by subtracting the principal parts of $f(z)$ at each singularity from $f(z)$ is analytic both at each singularity and everywhere else on D. Equivalently, $g(z)$ is analytic everywhere on D.

Since $g(z) = f(z) - \sum_{k=1}^{m} \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n$ is analytic on D, and for the closed path γ in D, $W(\gamma, \xi) = 0$ for all $\xi \notin D$, we know by the Theorem on page 243 that

$$\int_\gamma g(z)\,dz = \int_\gamma \left(f(z) - \sum_{k=1}^{m} \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n \right) dz = 0.$$

Rearranging the equation $\int_\gamma \left(f(z) - \sum_{k=1}^{m} \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n \right) dz = 0$, we find that

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma \sum_{k=1}^{m} \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n\,dz \\
&= \sum_{k=1}^{m} \int_\gamma \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n\,dz
\end{aligned}$$

By VIII.6 Problem 5, we know that for each $n \leq -2$ and each singularity z_k (since no singularity is on the trace of γ), $\int_\gamma (z - z_k)^n\,dz = 0$. Thus,

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \sum_{k=1}^{m} \int_\gamma \sum_{n=-\infty}^{-1} a_{n,k}(z - z_k)^n\,dz \\
&= \sum_{k=1}^{m} \int_\gamma \left(\sum_{n=-\infty}^{-2} a_{n,k}(z - z_k)^n \right) + \left(a_{-1,k}(z - z_k)^{-1} \right) dz \\
&= \sum_{k=1}^{m} \int_\gamma a_{-1,k} \frac{1}{z - z_k}\,dz \\
&= \sum_{k=1}^{m} a_{-1,k} \int_\gamma \frac{1}{z - z_k}\,dz.
\end{aligned}$$

Note that $W(\gamma, z_k) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-z_k} \, dz$ and $\text{Res}[f, z_k] = a_{-1,\,k}$ by definition, and so substituting these expressions to our current equation gives us

$$\int_\gamma f(z) \, dz = 2\pi i \sum_{k=1}^{m} W(\gamma, z_k) \text{Res}[f, z_k]$$

where z_1 to z_m are the isolated singularities of $f(z)$, as desired. ■