MATH103: Complex Analysis **Fall 2023**

Homework 6

David Yang

Chapter V (Power Series) Problems.

Section V.4 (Power Series Expansion of an Analytic Function), Problem 4

Suppose $f(z)$ is analytic at $z = 0$ and satisfies $f(z) = z + f(z)^2$. What is the radius of convergence of the power series expansion of $f(z)$ about $z = 0$?

Solution. For $f(z)$ to satisfy $f(z) = z + f(z)^2$, it must also satisfy

$$f(z)^2 - f(z) + z = 0.$$

Solving this equation for $f(z)$ using the Quadratic Formula, we find that

$$f(z) = \frac{1 \pm \sqrt{1-4z}}{2}.$$

Let $g(z)$ be the solution (one of $\frac{1+\sqrt{1-4z}}{2}$ and $\frac{1-\sqrt{1-4z}}{2}$) that satisfies $f(0) = g(0)$. By definition, both $f(z)$ and $g(z)$ satisfy $f(z) = z + f(z)^2$ and $g(z) = z + g(z)^2$, so taking the derivatives, we find that

$$0 = 2f(z)f'(z) - f'(z) + 1 \text{ and } 0 = 2g(z)g'(z) - g'(z) + 1.$$

Solving for $f'(z)$ and $g'(z)$, we get that

$$f'(z) = \frac{1}{1-2f(z)} \text{ and } g'(z) = \frac{1}{1-2g(z)}.$$

Since by construction, $f(0) = g(0)$, we must also have that $f'(0) = g'(0)$, and we can follow this same process to conclude that $f^{(n)}(0) = g^{(n)}(0)$ for any positive integer n.

Thus, since the derivatives of f and g are the same at $z = 0$, their power series are the same and they must have the same radius of convergence. To determine the radius of convergence of the power series expansion of $f(z)$ about $z = 0$, then, we can simply determine the radius of convergence of $g(z)$ about $z = 0$.

Note that

$$g'(z) = \frac{\mp 1}{\sqrt{1-4z}}$$

where the $\mp$ corresponds to the fact that $g(z) = \frac{1+\sqrt{1-4z}}{2}$ or $\frac{1-\sqrt{1-4z}}{2}$, depending on the value of $f(0)$. In either case, note that the derivative is not defined at $z = \frac{1}{4}$. Thus, since the radius of convergence is the distance to the nearest singularity (from $z = 0$), we conclude that the radius of convergence of the power series expansion of $f(z)$ about $z = 0$ is $\boxed{\frac{1}{4}}$. ∎

Section V.5 (Power Series Expansion at Infinity), Problem 4

Let E be a bounded subset of the complex plane $\mathbb{C}$ over which area integrals can be defined, and set

$$f(w) = \iint_E \frac{dx\,dy}{w-z}, \quad w \in \mathbb{C} \setminus E$$

where $z = x+iy$. Show that $f(w)$ is analytic at ∞, and find a formula for the coefficients of the power series of $f(w)$ at ∞ in descending powers of w.

Solution. To show that $f(w)$ is analytic at ∞, we can show that $g(w) = f\left(\frac{1}{w}\right)$ is analytic at $w = 0$. By definition,

$$\begin{aligned} g(w) = f\left(\frac{1}{w}\right) &= \iint_E \frac{1}{\frac{1}{w} - z}\,dx\,dy \\ &= \iint_E \frac{w}{1-zw}\,dx\,dy. \end{aligned}$$

Note that $\frac{w}{1-zw}$ is analytic at $w = 0$, since $\frac{d}{dw}\left(\frac{w}{1-zw}\right) = \frac{(1-wz)-w(1-z)}{(1-wz)^2}$ is continuous at $w = 0$. Thus, since the integrand is analytic, we know that $g(w)$ is analytic at $w = 0$. Equivalently, $f(w)$ is analytic at ∞ as desired.

To find a formula for the coefficients of the power series of $f(w)$ at ∞ in descending powers of w, we will begin by rewriting the integrand in the form of a geometric series sum: note that

$$\begin{aligned} f(w) &= \iint_E \frac{1}{w-z}\,dx\,dy \\ &= \iint_E \frac{\frac{1}{w}}{1-\frac{z}{w}}\,dx\,dy. \end{aligned}$$

Experessing the integrand as the sum of a geometric series, we get that

$$f(w) = \iint_E \sum_{n=0}^{\infty} \left(\frac{1}{w}\right)\left(\frac{z}{w}\right)^n dx\,dy.$$

Since the integral and sum can be interchanged, we find that this is equivalent to

$$\begin{aligned} f(w) &= \sum_{n=0}^{\infty} \iint_E \left(\frac{1}{w}\right)\left(\frac{z}{w}\right)^n dx\,dy \\ &= \sum_{n=0}^{\infty} \left(\iint_E z^n\,dx\,dy\right) \frac{1}{w}\left(\frac{1}{w}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\iint_E z^n\,dx\,dy\right) \frac{1}{w^{n+1}}. \end{aligned}$$

Thus, the coefficient of $\frac{1}{w^{n+1}}$ in the power series expansion of $f(w)$ at ∞ is $\boxed{\iint_E z_n\,dx\,dy}$. ■