

Homework 5

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Chapter IV (Complex Integration and Analyticity) Problems.

Section IV.4 (The Cauchy Integral Formula), Problem 4

Let D be a bounded domain with smooth boundary ∂D , and let $z_0 \in D$. Using the Cauchy integral formula, show that there is a constant C such that

$$|f(z_0)| \leq C \sup \{|f(z)| : z \in \partial D\}$$

for any function $f(z)$ analytic on $D \cup \partial D$. By applying this estimate to $f(z)^n$, taking n th roots, and letting $n \rightarrow \infty$, show that the estimate holds with $C = 1$. **Remark.** This provides an alternative proof of the maximum principle for analytic functions.

Solution. Since $f(z)$ is analytic on $D \cup \partial D$ and D is a bounded domain with smooth boundary, we know by Cauchy's Integral Formula that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0} dw$$

and so consequently,

$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0} dw \right|$$

Furthermore, since $\left| \frac{1}{2\pi i} \right| = \frac{1}{2\pi}$ and $|f(w)| \leq \sup \{|f(z)| : z \in \partial D\}$ for $w \in \partial D$, we know that

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0} dw \right| \\ &= \frac{1}{2\pi} \left| \int_{\partial D} \frac{f(w)}{w - z_0} dw \right| \\ &\leq \frac{1}{2\pi} \sup \{|f(z)| : z \in \partial D\} \left| \int_{\partial D} \frac{1}{w - z_0} dw \right|. \end{aligned}$$

However, note that $\frac{1}{w - z_0}$ is bounded, since $w \in \partial D$ and $z_0 \in D$. Consequently, the term $\left| \int_{\partial D} \frac{1}{w - z_0} dw \right|$ is bounded, say by a constant M . Thus, we have that

$$|f(z_0)| \leq C \sup \{|f(z)| : z \in \partial D\}$$

for a constant $C = \frac{M}{2\pi}$, where M is defined above.

We can now apply this estimate to the function $f(z)^n$, which is analytic since $f(z)$ is analytic. We get that

$$|f(z_0)^n| \leq C \sup \{|f(z)^n| : z \in \partial D\}$$

Note that $|f(z_0)^n| = |f(z_0)|^n$ and $\sup \{|f(z)^n| : z \in \partial D\} \leq (\sup \{|f(z)| : z \in \partial D\})^n$, and so we know that our estimate is equivalent to

$$|f(z_0)|^n \leq C (\sup \{|f(z)| : z \in \partial D\})^n.$$

Taking the n th root of both sides of our inequality, we find that

$$|f(z_0)| \leq C^{\frac{1}{n}} \sup \{|f(z)| : z \in \partial D\}.$$

Note that $\lim_{n \rightarrow \infty} C^{\frac{1}{n}} = 1$ for any nonzero constant C . Thus, we find that our estimate, as $n \rightarrow \infty$, becomes

$$|f(z_0)| \leq \sup \{|f(z)| : z \in \partial D\}.$$

Thus, our estimate holds for $C = 1$ and we also have an alternative proof of the maximum principle for analytic functions. ■

Suppose that $f(z)$ is an entire function such that $f(z)/z^n$ is bounded for $|z| \geq R$. Show that $f(z)$ is a polynomial of degree at most n . What can be said if $f(z)/z^n$ is bounded on the entire complex plane?

Solution. Let D be a disk of radius R centered at the origin. Since $\frac{f(z)}{z^n}$ is bounded for $|z| \geq R$, we know that $\frac{f(z)}{z^n}$ is bounded on the boundary of D , where $|z| = R$. Consequently, we know that

$$\left| \frac{f(z_0)}{z_0^n} \right| \leq M$$

for all $z_0 \in \partial D$.

We can multiply both sides by $|z_0|^n$ to get that

$$|f(z_0)| \leq M|z_0|^n \leq MR^n$$

for all $z_0 \in \partial D$ (where the last step follows from the fact that $|z_0| = R$).

Since $f(z)$ is entire, it is analytic on D . Furthermore, it is bounded by MR^n on ∂D ; thus, by the Maximum Principle, we know that

$$|f(z)| \leq MR^n$$

for all $z \in D$.

Similarly, since $f(z)$ is entire, it is analytic everywhere. Thus, using the fact that $|f(z)| \leq MR^n$ for all $|z| = R$ and applying the Cauchy Estimates Theorem to a disk $|z - z_0| < R$, we know that

$$\begin{aligned} \left| f^{(n+1)}(z_0) \right| &\leq \frac{(n+1)!}{R^{n+1}} MR^n \\ &= \frac{M(n+1)!}{R}. \end{aligned}$$

Thus, we find that

$$\lim_{R \rightarrow \infty} \left| f^{(n+1)}(z_0) \right| = \lim_{R \rightarrow \infty} \frac{M(n+1)!}{R} = 0.$$

Since the $(n+1)^{\text{st}}$ derivative of f at z_0 must be zero, we know that $f(z)$ is a polynomial of degree at most n , as desired.

We know that $f(z)$ is a polynomial of degree at most n . By definition,

$$\frac{f(z)}{z^n} = a_{n-1} + \frac{a_{n-2}}{z} + \cdots + \frac{a_0}{z^n}$$

for constants $a_0, \dots, a_{n-1} \in \mathbb{C}$. Note that we must have that $a_{n-2} = \cdots = a_0 = 0$, since if any of these constants were nonzero, the corresponding term in $\frac{f(z)}{z^n}$ would not be bounded for z near 0.

Thus, if $f(z)/z^n$ is bounded on the entire complex plane, we must have the form

$$f(z) = cz^n$$

for $c \in \mathbb{C}$.

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