MATH103: Complex Analysis **Fall 2023**

Homework 4

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Chapter III (Line Integrals and Harmonic Functions) Problems.

Section III.3 (Harmonic Conjugates), Problem 3

Let $D = \{a < |z| < b\} \setminus (-b, -a)$, an annulus slit along the neagtive real axis. Show that any harmonic function on D has a harmonic conjugate on D. *Suggestion.* Fix c between a and b, and define $v(z)$ explicitly as a line integral along the path consisting of the straight line from c to $|z|$ followed by the circular arc from $|z|$ to z. Or map the slit annulus to a rectangle by $w = \text{Log}z$.

Solution. ■

Section III.4 (The Mean Value Property), Problem 4

Formulate the mean value property for a function on a domain in $\mathbb{R}^3$, and show that any harmonic function has the mean value property. *Hint.* For $A \in \mathbb{R}^3$ amd $r > 0$, let B_r be the ball of radius r centered at A, with volume element $d\tau$, and let ∂B_r be its boundary sphere, with area element $d\sigma$ and unit outward normal vector n. Apply the Gauss divergence theorem

$$\int\int_{\partial B_r} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int\int\int_{B_r} \nabla \cdot \mathbf{F}\, d\tau$$

to $\mathbf{F} = \triangle u$.

Solution. ■

Section III.5 (The Maximum Principle), Problem 3

Use the maximum principle to prove the fundamental theorem of algebra, that any polynomial $p(z)$ of degree $n \geq 1$ has a zero, by applying the maximum principle to $1/p(z)$ on a disk of a large radius.

Solution. ■