MATH103: Complex Analysis **Fall 2023**

Homework 9

David Yang

Chapter VIII (The Logarithmic Integral) Problems.

Section VIII.4 (Open Mapping and Inverse Function Theorems), Problem 1

Suppose D is a bounded domain with piecewise smooth boudnary. Let $f(z)$ be meromorphic and $g(z)$ analytic on D. Suppose that both $f(z)$ and $g(z)$ extend analytically across the boundary of D, and that $f(z) \neq 0$ on ∂D. Show that

$$\frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} m_j g(z_j)$$

where $z_1, \ldots, z_n$ are the zeros and poles of $f(z)$ and m_j is the order of $f(z)$ at z_j.

Note that $g(z)\frac{f'(z)}{f(z)}$ is analytic $D \cup \partial D$ except for a finite number of isolated singularities at $z_1, \ldots, z_n$. Consequently, by the Residue Theorem that

$$\oint_{\partial D} g(z)\frac{f'(z)}{f(z)}\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\left[g(z)\frac{f'(z)}{f(z)}, z_j\right]$$

Consider a given singularity z_j which is either a zero or pole of ordfer m_j at $f(z)$. By definition, we have that

$$f(z) = (z - z_j)^{m_j} h(z)$$

for a function $h(z)$ satisfying $h(z_j) \neq 0$ and $h(z)$ analytic at z_j. By the Chain Rule, we also find that

$$f'(z) = m_j(z - z_j)^{m_j - 1} h(z) + (z - z_j)^{m_j} h'(z)$$

and so

Section VIII.6 (Winding Numbers), Problem 6

Let γ be a closed path in a domain D such that $W(,\gamma,\xi) = 0$ for all $\xi \notin D$. Suppose that $f(z)$ is analytic on D except possibly at finite number of isolated singularities $z_1, \ldots, z_m \in D \setminus \Gamma$. Show that

$$\int_\gamma f(z)\,dz = 2\pi i \sum W(\gamma, z_k)\mathrm{Res}[f, z_k].$$