MATH103: Complex Analysis **Fall 2023**

Homework 2

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Chapter II (Analytic Functions) Problems.

Section II.1 (Review of Basic Analysis), II.1.14

Let $h(t)$ be a continuous complex-valued function on the unit interval $[0, 1]$, and consider

$$H(z) = \int_0^1 \frac{h(t)}{t - z}\, dt.$$

Where is $H(z)$ defined? Where is $H(z)$ continuous? Justify your asnwer. *Hint.* Use the fact that if $|f(t) - g(t)| < \epsilon$ for $0 \le t \le 1$, then $\int_0^1 |f(t) - g(t)|\, dt < \epsilon$.

Solution. $H(z) = \int_0^1 \frac{h(t)}{t-z}$ is defined only when the integrand is defined; this happens only when the denominator of the fraction $\frac{h(t)}{t-z}$ is nonzero. Put simply, we need $t - z \neq 0$ or $z \neq t$. Since by definition $t \in [0, 1]$, $H(z)$ is defined for $z \in \mathbb{C} \setminus [0, 1]$.

We claim that $H(z)$ is continuous for all $z \in \mathbb{C} \setminus [0, 1]$ (by definition, it can only be continuous where it is defined, and so we aim to show that $H(z)$ is continuous at all points where it is defined). To do so, we will appeal to the limit definition of continuity, that $H(z)$ is continuous at z_0 if

$$\lim_{z \to z_o} H(z) = H(z_0),$$

To make use of the hint, let us define $f(t) = \frac{h(t)}{t-z}$ and $g(t) = \frac{h(t)}{t-z_0}$ for any $z, z_0 \in \mathbb{C} \setminus [0, 1]$. Then

$$|f(t) - g(t)| = \left| \frac{h(t)}{t - z} - \frac{h(t)}{t - z_0} \right| = \left| \frac{h(t)(z - z_0)}{(t - z)(t - z_0)} \right|$$

Since $h(t)$ is defined on the compact interval $[0, 1]$, it has a maximum value, which we will denote M. Equivalently, $h(t) \le M$ for all $t \in [0, 1]$. Thus, substituting this back into our above equation and using the fact that $|ab| = |a||b|$, we get

$$\begin{aligned} |f(t) - g(t)| = \left| \frac{h(t)(z - z_0)}{(t - z)(t - z_0)} \right| &< \left| \frac{M(z - z_0)}{(t - z)(t - z_0)} \right|. \\ &= \left| \frac{(z - z_0)}{(t - z)(t - z_0)} \right| |M| \end{aligned}$$

We claim that

$$\lim_{z \to z_0} \left(\left| \frac{(z - z_0)}{(t - z)(t - z_0)} \right| |M| \right) = 0.$$

To see this, note that as $z \to z_0$, the denominator $(t-z)(t-z_0)$ approaches $(t-z_0)(t-z_0) = (t-z_0)^2$. Thus, rewriting the above limit, we have

$$\lim_{z \to z_0} \left(\left| \frac{(z - z_0)}{(t-z)(t-z_0)} \right| |M| \right) = \lim_{z \to z_0} \left(\left| \frac{(z - z_0)}{(t-z_0)^2} \right| |M| \right).$$

Note that since by definition, $z_0 \notin [0,1]$, z_0 cannot get arbitrarily close to t. On the other hand, the numerator $z - z_0$ tends towards 0 as z approaches z_0. Thus,

$$\lim_{z \to z_0} \left(\left| \frac{(z - z_0)}{(t-z_0)^2} \right| |M| \right) = 0.$$

By the hint, we know that since $|f(t) - g(t)| < \epsilon$ for $0 \leq t \leq 1$, then $\int_0^1 |f(t) - g(t)|\, dt < \epsilon$. Equivalently,

$$\lim_{z \to z_0} \int_0^1 \left| \frac{h(t)}{t - z} - \frac{h(t)}{t - z_0} \right| = 0.$$

Furthermore, note that by an absolute value property of integrals, we know that

$$\begin{aligned} \int_0^1 \left| \frac{h(t)}{t - z} - \frac{h(t)}{t - z_0} \right| &\geq \left| \int_0^1 \frac{h(t)}{t - z} - \frac{h(t)}{t - z_0} \right| \\ &= \left| \int_0^1 \frac{h(t)}{t - z} - \int_0^1 \frac{h(t)}{t - z_0} \right| \\ &= |H(z) - H(z_0)| \end{aligned}$$

Put succinctly, we know that

$$H(z) - H(z_0) \leq \int_0^1 \left| \frac{h(t)}{t - z} - \frac{h(t)}{t - z_0} \right|.$$

Thus, since $\lim\limits_{z \to z_0} \int_0^1 \left| \frac{h(t)}{t-z} - \frac{h(t)}{t-z_0} \right| = 0$, we know that

$$\lim_{z \to z_0} |H(z) - H(z_0)| = 0.$$

for any $z_0 \in \mathbb{C} \setminus [0,1]$ (where H is defined). Thus, by the limit definition of continuity, $H(z)$ is continuous everywhere it is defined. ■