Homework 1

David Yang

Chapter I (The Complex Plane and Elementary Functions) Problems.

Section I.3 (Stereographic Projection), I.3.4

Show that a rotation of the sphere of $180°$ about the X-axis corresponds under stereographic projection to the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$.

Solution. Let $P = (X, Y, Z)$ be a point on the unit sphere. After a a $180°$ rotation of the point P on the unit sphere about the X-axis, P is sent to the point $P' = (X, -Y, -Z)$.

Consider the result of P and P' under stereographic projection. By definition, stereographic projection sends P to the point

$$\frac{X}{1-Z} + \frac{Y}{1-Z}i$$

and the point P' to the point

$$\frac{X}{1-(-Z)} + \frac{-Y}{1-(-Z)}i = \frac{X}{1+Z} - \frac{Y}{1+Z}i$$

on the extended complex plane $\mathbb{C}^*$.

We claim that P' is the result of P under the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$; note that

$$\left(\frac{X}{1-Z} + \frac{Y}{1-Z}i\right)\left(\frac{X}{1+Z} - \frac{Y}{1+Z}i\right)$$
$$= \frac{X^2}{(1-Z)(1+Z)} - \frac{XY}{(1-Z)(1+Z)} + \frac{XY}{(1-Z)(1+Z)} - \frac{Y^2}{(1-Z)(1+Z)}i^2.$$

By using the identity $i^2 = -1$, canceling out terms, and simplifying, we find that this is

$$\frac{X^2}{(1-Z)(1+Z)} + \frac{Y^2}{(1-Z)(1+Z)} = \frac{X^2+Y^2}{1-Z^2}.$$

However, since $P = (X, Y, Z)$ is a point on the unit sphere, we know that $X^2 + Y^2 + Z^2 = 1$, so $1 - Z^2 = X^2 + Y^2$. Thus, we know that

$$\left(\frac{X}{1-Z} + \frac{Y}{1-Z}i\right)\left(\frac{X}{1+Z} - \frac{Y}{1+Z}i\right) = \frac{X^2+Y^2}{1-Z^2} = 1.$$

which tells us that the resulting points of P and P' under stereographic projection are complex inverses.

Thus, a rotation of the sphere of $180°$ about the X-axis corresponds under stereographic projection to the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$. ■

Section I.6 (The Logarithm Function), I.6.2

Sketch the image under the map $w = \operatorname{Log} z$ of each of the following figures:

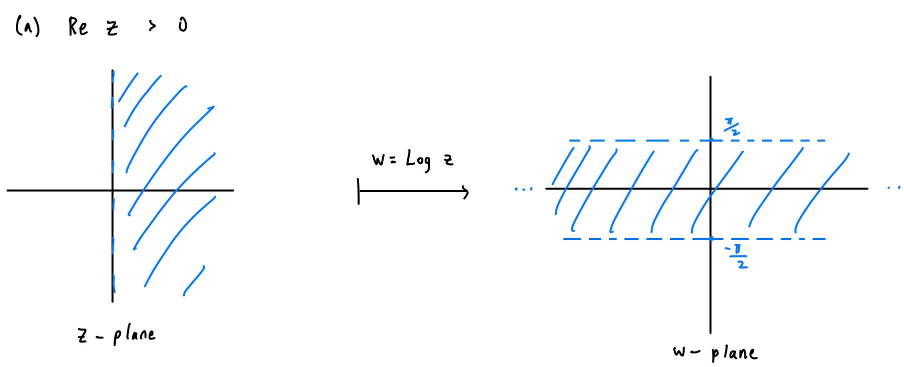
a) **the right half-plane** $\operatorname{Re} z > 0$

We know that

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z.$$

For points in the right half-plane, $-\frac{\pi}{2} < \operatorname{Arg} z < \frac{\pi}{2}$. Thus, the image in the w-plane under the logarithm function satisfies $\frac{-\pi}{2} < \operatorname{Im} w < \frac{\pi}{2}$.

On the other hand, there is no restriction on $|z|$; all values of $|z| > 0$ are in the right half-plane, and so $\operatorname{Re} w \in (-\infty, \infty)$.



c) **the unit circle** $|z| = 1$

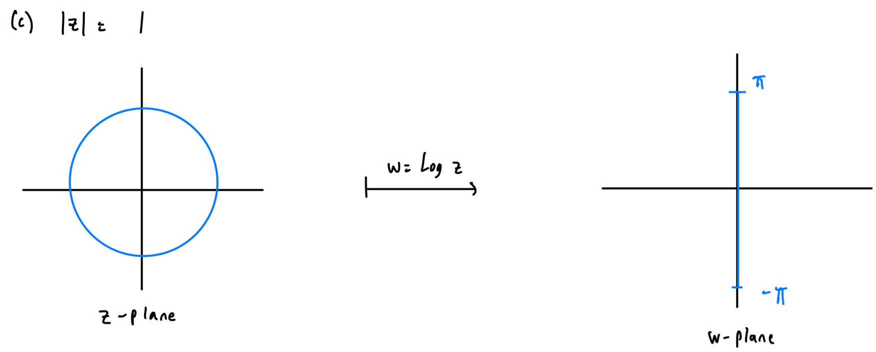
We know that

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z.$$

On the unit circle, $|z| = 1$, so

$$w = \operatorname{Log} z = \log(1) + i \operatorname{Arg} z = i \operatorname{Arg} z.$$

On the unit circle, $\operatorname{Arg} z$ takes on all values in $[-\pi, \pi]$, so the image is simply the line segment $\{iy : y \in [-\pi, \pi]\}$.



e) **the horizontal line** $y = e$

We know that

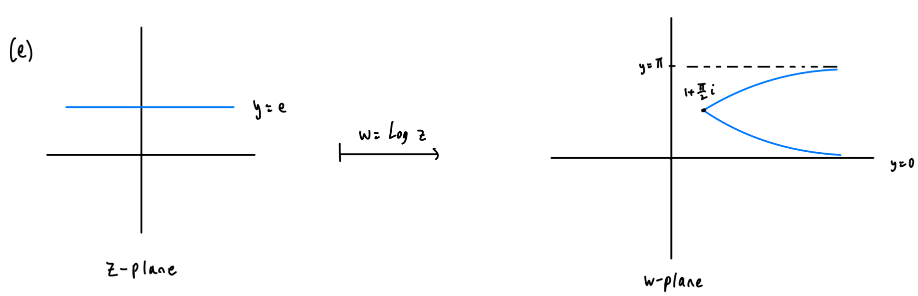
$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z.$$

Since $y = e$, note that the minimum value for $|z| = e$, which occurs when $\operatorname{Re} z = 0$. For $z = e$,

$$w = \operatorname{Log} z = \log e + \operatorname{Arg} e = 1 + \frac{\pi}{2} i.$$

Consider the behavior of w as z moves along the right half of the line $y = e$ (i.e. $\operatorname{Re} z > 0$). As z moves along this section, $|z|$ increases without bound and $\operatorname{Arg} z$ approaches $\frac{\pi}{2}$. Furthermore, we know that $|z|$ (and $\log z$, which is in turn the real part of the image w) grows quicker the sooner we move along this section (when $\operatorname{Re} z$ is small, changes in $|z|$ are large relative to the changes in $|z|$ when $\operatorname{Re} z$ is large).

We can apply a similar line of reasoning to determine the behavior of the left half of the line, where $\operatorname{Re} z < 0$ and $\operatorname{Im} z = e$. In this case, $\operatorname{Arg} z$ approaches 0, and so we arrive at the following image (the image has a reflective symmetry at $y = \frac{\pi}{2}$ in the w-plane):



Let S denote the two slits along the imaginary axis in the complex plane, one running from i to $+i\infty$, the other running from $-i$ to $-i\infty$.

a) **Show that $\frac{1+iz}{1-iz}$ lies on the negative real axis $(-\infty, 0)$ if and only if $z \in S$.**

Solution. We will begin by proving the forward direction and showing that if $\frac{1+iz}{1-iz}$ lies on the negative real axis $(-\infty, 0)$ then $z \in S$.

Let $\frac{1+iz}{1-iz}$ lie on the negative real axis, meaning that

$$\frac{1+iz}{1-iz} = r$$

for real $r \in (-\infty, 0)$. Multiplying both sides by $1 - iz$, we get that

$$1 + iz = r(1 - iz).$$

Moving the imaginary terms to one side and the real terms to the other, we get that

$$zi(1 + r) = r - 1$$

and solving for z gives us

$$z = \frac{1-r}{1+r} i.$$

Note that when r approaches 0 from the left, z approaches i, and as r approaches -1 from the right, z runs from i to

$$\lim_{r \to -1^+} \frac{1-r}{1+r} i = i\infty$$

which is the right slit along the imaginary axis in the complex plane.

On the other hand, as $r \to -\infty$, we know that z approaches

$$\lim_{r \to -\infty} \frac{1-r}{1+r} i = -i$$

and as z approaches -1 from the left, z runs from $-i$ to

$$\lim_{r \to -1^-} \frac{1-r}{1+r} i = -i\infty$$

which is the left slit along the imaginary axis in the complex plane.

Thus, if $\frac{1+iz}{1-iz}$ lies on the negative real axis $(-\infty, 0)$ then $z \in S$.

To prove the reverse direction, we want to show that if $z \in S$, then $\frac{1+iz}{1-iz}$ lies on the negative real axis $(-\infty, 0)$.

By definition, if $z \in S$, then $z = ci$ for some $c \in \mathbb{R}$ satisfying $c \in (-\infty, -1) \cup (1, \infty)$. Plugging this value for z into $\frac{1+iz}{1-iz}$, we get that

$$\frac{1+iz}{1-iz} = \frac{1+i(ci)}{1-i(ci)} = \frac{1-c}{1+c}$$

Note that when c approaches 1 from the left, this expression approaches 0. On the other hand, the expression $\frac{1-c}{1+c}$ is positive if and only if both $1-c$ and $1+c$ are negative, or if both $1-c$ and $1+c$ are positive. Note that the former case cannot occur as if $1-c < 0$, then $c > 1$ which would make $1+c$ positive. Similarly, the latter case can only occur when $-1 < c < 1$, which violates the condition that $c \in (-\infty, -1) \cup (1, \infty)$.

Thus, if $z \in S$, then $\frac{1+iz}{1-iz}$ must lie on the negative real axis $(-\infty, 0)$.

Since we have proved both directions of the if and only if, we can conclude that $\frac{1+iz}{1-iz}$ lies on the negative real axis $(-\infty, 0)$ if and only if $z \in S$. ■

b) **Show that the principal branch**

$$\mathrm{Tan}^{-1} z = \frac{1}{2i} \mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$$

maps the slit plane $\mathbb{C} \setminus S$ one-to-one onto the vertical strip $\{|\mathrm{Re}\, w| < \frac{\pi}{2}\}$.

Solution. We will first show that this map is one-to-one. Let $a, b \in \mathbb{C} \setminus S$ and assume that $\mathrm{Tan}^{-1}(a) = \mathrm{Tan}^{-1}(b)$. We will show that $a = b$. If $\mathrm{Tan}^{-1}(a) = \mathrm{Tan}^{-1}(b)$, we know that

$$\frac{1}{2i} \mathrm{Log}\left(\frac{1+ia}{1-ia}\right) = \frac{1}{2i} \mathrm{Log}\left(\frac{1+ib}{1-ib}\right).$$

Multiplying both sides by $2i$ and applying the exponential function (which we can do since neither a and b are in S, so by part (a), $\frac{1+ia}{1-ia}$ and $\frac{1+ib}{1-ib}$ will not lie on the negative real axis) to both sides, we get that

$$\frac{1+ia}{1-ia} = \frac{1+ib}{1-ib}.$$

Cross multiplying, we get that

$$(1+ia)(1-ib) = (1+ib)(1-ia).$$

Expanding and simplifying, we get that

$$(ab+1) + i(a-b) = (ab+1) + i(b-a).$$

Subtracting both sides by $ab+1$ and dividing by i, we get that $a - b = b - a$, meaning $a = b$. Thus, this map is one-to-one.

It remains to show that the map is onto the vertical strip. Let w be a point on the vertical strip, i.e. $w \in \{|\mathrm{Re}\, w| < \frac{\pi}{2}\}$. Consider $z = \mathrm{Tan}(w)$. By definition, we have

$$z = \mathrm{Tan}(w) = \frac{\sin w}{\cos w} = \frac{\frac{e^{iw}-e^{-iw}}{2i}}{\frac{e^{iw}+e^{-iw}}{2}}.$$

Simplifying, we get that

$$z = \mathrm{Tan}(w) = \frac{1}{i}\left(\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}}\right).$$

Multiplying the numerator and denominator of this fraction by e^{iw}, we get

$$z = \frac{1}{i}\left(\frac{e^{2iw}-1}{e^{2iw}+1}\right).$$

We will show that this value of z is both on the slit plane $\mathbb{C} \setminus S$ and that this maps to w under the principal branch.

First, we will show that $z = i\,\frac{1-e^{2iw}}{1+e^{2iw}}$ is on the slit plane. Note that from part (a), we know that $z \in S$ if and only if $\frac{1+iz}{1-iz}$ is on the negative real axis. To show that $z \notin S$, we will show that $\frac{1+iz}{1-iz}$ is not on the real axis.

We have that

$$\frac{1+iz}{1-iz} = \frac{1+i\left(\frac{1}{i}\left(\frac{e^{2iw}-1}{e^{2iw}+1}\right)\right)}{1-i\left(\frac{1}{i}\left(\frac{e^{2iw}-1}{e^{2iw}+1}\right)\right)} = \frac{1+\left(\frac{e^{2iw}-1}{e^{2iw}+1}\right)}{1-\left(\frac{e^{2iw}-1}{e^{2iw}+1}\right)}$$

Simplifying further by combining the fractions on the numerator and denominator, we get that

$$\frac{1+iz}{1-iz} = \frac{\frac{2e^{2iw}}{e^{2iw}+1}}{\frac{2}{e^{2iw}+1}} = e^{2iw}.$$

Let $w = a + bi$. Then we have that

$$\frac{1+iz}{1-iz} = e^{2i(a+bi)} = e^{-2b}\left(\cos(2a) + i\sin(2a)\right).$$

Observe that $\frac{1+iz}{1-iz}$ can only lie on the negative real axis when both $\sin(2a) = 0$ and $\cos(2a) < 0$. However, by definition, $a = |\mathrm{Re}\, w| < \frac{\pi}{2}$, and so these cannot both be true. Thus, $\frac{1+iz}{1-iz}$ does not lie on the negative real axis, and consequently, $z \notin S$ so z is on the slit plane $\mathbb{C} \setminus S$.

The final step is to show that the principal branch does map $z = i\,\frac{1-e^{2iw}}{1+e^{2iw}}$ onto a point w on the vertical strip. To do so, we will plug in our value of z into the formula for $\mathrm{Tan}^{-1} z$ and show that we get w. We have that

$$\mathrm{Tan}^{-1} z = \frac{1}{2i}\mathrm{Log}\left(\frac{1+i\cdot i\,\frac{1-e^{2iw}}{1+e^{2iw}}}{1-i\cdot i\,\frac{1-e^{2iw}}{1+e^{2iw}}}\right).$$

Simplifying, we get

$$\mathrm{Tan}^{-1}z = \frac{1}{2i}\mathrm{Log}\left(\frac{1-\frac{1-e^{2iw}}{1+e^{2iw}}}{1+\frac{1-e^{2iw}}{1+e^{2iw}}}\right) = \frac{1}{2i}\mathrm{Log}\left(\frac{\frac{2e^{2iw}}{1+e^{2iw}}}{\frac{2}{1+e^{2iw}}}\right) = \frac{1}{2i}\mathrm{Log}e^{2iw} = w$$

as desired. We conclude that for any point w on the vertical strip $\{|\mathrm{Re}\, w| < \frac{\pi}{2}\}$, there is a point $z \in \mathbb{C} \setminus S$ such that $\mathrm{Tan}^{-1}(z) = w$, meaning that the principal branch is onto.

Thus, we know that the principal branch maps the slit plane one-to-one onto the vertical strip. ■