MATH103: Complex Analysis **Fall 2023**

Homework 3

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Chapter II (Analytic Functions) Problems.

Section II.6 (Conformal Mappings), II.6.6

a) **Determine where the function $f(z) = z + 1/z$ is conformal and where it is not conformal.**

Solution. We know that $f(z)$ is conformal at z_0 if it is analytic at z_0 where $f'(z_0) \neq 0$. Since $f(0)$ is not defined, we know f cannot be conformal at the origin. Similarly, we see that

$$f'(z) = 1 - \frac{1}{z^2}$$

and so $f'(z) \neq 0$ when $z \neq \pm 1$. Since $f(z) \neq 0$ at all other points, f is also analytic at all other points. Thus, $f(z) = z + 1/z$ is not conformal for $z = 0, \pm 1$ and is conformal at all other points. ∎

b) **Show that for each w, there are at most two values z for which $f(z) = w$.**

Solution. Fix some complex w. The values z for which $f(z) = w$ must by definition satisfy

$$z + \frac{1}{z} = w.$$

Multiplying both sides by z (which we can do as $z \neq 0$) and rearranging, we get that

$$z^2 - zw + 1 = 0.$$

Since this is a quadratic with respect to z, there are at most two values z that solve this equation, as desired. (In fact, by Vieta's Formulas, we know that the roots z_1 and z_2 of $f(z) = w$ must satisfy $z_1 z_2 = 1$.) ∎

c) **Show that if $r > 1$, $f(z)$ maps the circle $\{|z| = r\}$ onto an ellipse, and that $f(z)$ maps the circle $\{|z| = 1/r\}$ onto the same ellipse.**

Solution. We will first show that if $r > 1$, $f(z)$ maps the circle $\{|z| = r\}$ onto an ellipse. Writing z in polar form, we know that $z = re^{i\theta}$. Thus, we know that

$$f(z) = z + 1/z = re^{i\theta} + \frac{1}{r}e^{-i\theta}.$$

Substituting $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$, into this expression, we find that

$$\begin{aligned} f(z) &= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= r\left(\cos\theta + i\sin\theta\right) + \frac{1}{r}\left(\cos\theta - i\sin\theta\right) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

This is simply the polar form for an ellipse! Thus, we see that the circle $\{|z| = r\}$ is mapped onto an ellipse with equation

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1$$

where u, v represent the real and imaginary components of the images of $z = x + yi$ under $f(z)$.

Observe that since

$$f(z) = re^{i\theta} + \frac{1}{r}e^{-i\theta} = \frac{1}{r}e^{i\theta} + \frac{1}{\left(\frac{1}{r}\right)}e^{i\theta}$$

we can conclude that $\{|z| = \frac{1}{r}\}$ maps onto the same ellipse, as expected. ■

d) **Show that $f(z)$ is one-to-one on the exterior domain $D = \{|z| > 1\}$.**

Solution. Let z_1 be a solution to $f(z) = w$, where $z_1 \in D$ so $|z_1| > 1$. From part (b), we found that the roots z_1, z_2 of the equation $f(z) = w$ must satisfy $z_1 z_2 = 1$, or equivalently, $z_1 = \frac{1}{z_2}$.

Since z_1, z_2 are reciprocals of each other, the other root to $f(z) - w = 0$ must have magnitude

$$|z_2| = \frac{1}{|z_1|} < 1$$

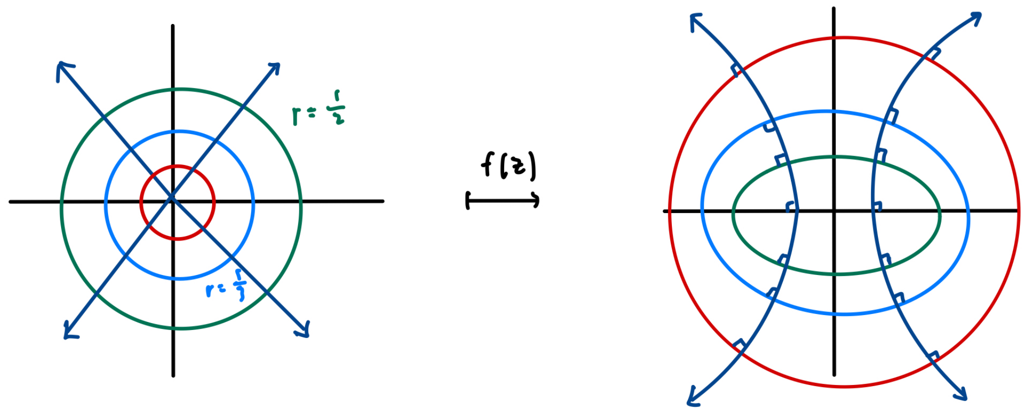
as $|z_1| > 1$. Thus, we know that the other root z_2 must lie in the interior domain, and so $f(z)$ is one-to-one on the exterior domain D, as desired. ■

e) **Determine the image of D under $f(z)$. Sketch the images under $f(z)$ of the circles $\{|z| = r\}$ for $r < 1$, and sketch also the images of the parts of the rays $\{\arg z = \beta\}$ lying in D.**

Solution. From part (c), we found that for $r > 1$, $f(z)$ maps the circles $\{|z| = r\}$ and $\{|z| = \frac{1}{r}\}$ onto the same ellipse. Consequently, the image of D under $f(z)$ is the image of $\{|z| = r\}$ such that $r \neq 1$.

The image of $\{|z| = 1\}$ under $f(z)$ is $f(z) = 2\cos\theta$ where θ is the angle of z with respect to the positive x-axis; so $f(z)$ maps points satisfying $|z| = 1$ to the $[-2, 2]$ portion on the real axis. The image of D under $f(z)$ is all other points, namely $\boxed{\mathbb{C} \setminus [-2, 2]}$.

Finally, as we determined in part (c), the images of $f(z)$ of the circles $\{|z| = r\}$ for $r < 1$ correspond to ellipses, which are more stretched out in the x-direction at first but approach a circle as r decreases (since $|r + 1/r| > |r - 1/r|$ but by less and less as $r \to 0$). The rays $\arg z = \beta$ lying in D intersect circles $\{|z| = r\}$ orthogonally and thus their images under $f(z)$ correspond to parabolas (due to the "limiting" nature of the ellipses approaching a circle) intersecting the resulting ellipses orthogonally.



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Section II.7 (Fractional Linear Transformations), II.7.6

Show that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin.

Solution. By definition, any straight line is defined by $Ax + By = C$ for some complex numbers A, B, C. We will show that if $C \neq 0$ (meaning the line does not through the origin), the image of this line under inversion is a circle, and if $C = 0$ (meaning the line passes through the origin), the image of this line under inversion is a line.

Let $z = x + iy$ be any point on our original straight line

$$Ax + By = C.$$

Dividing both sides of this equation by $x^2 + y^2$, we know that the line is similarly defined by the equation

$$\frac{Ax}{x^2+y^2} + \frac{By}{x^2+y^2} = \frac{C}{x^2+y^2}.$$

By definition, the image of this point under inversion is some point z' where

$$z' = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i.$$

We see that the result of z after inversion is some point $z' = u+iv$, where $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$, and

$$u^2 + v^2 = \left(\frac{x}{x^2+y^2}\right)^2 + \left(-\frac{y}{x^2+y^2}\right)^2 = \frac{1}{x^2+y^2}.$$

Substituting each of these expressions in to the equation of our straight line $\frac{Ax}{x^2+y^2} + \frac{By}{x^2+y^2} = \frac{C}{x^2+y^2}$, we get the equation

$$Au - Bv = C(u^2 + v^2).$$

By inspection, if $C = 0$, then we get $Au - Bv = 0$, which is simply the equation of a straight line. Consequently, we conclude that if a line passes through the origin, its image under inversion is still a straight line.

On the other hand, if $C \neq 0$, let us move all the terms to one side and group related terms to get

$$(Cu^2 - Au) + (Cv^2 + Bv) = 0.$$

Completing the square, we find that

$$C\left(u - \frac{A}{2C}\right)^2 + C\left(v + \frac{B}{2C}\right)^2 = \frac{A^2+B^2}{4C^2}.$$

When $C \neq 0$, this is simply the equation of a circle, centered at $\frac{A}{2C} - \frac{B}{2C}i$ with radius $\frac{1}{2C}\sqrt{A^2+B^2}$. Consequently, we conclude that if a line does not pass through the origin, its image under inversion is a circle.

Combining our two cases together, we find that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin, as desired. ■