MATH103: Complex Analysis **Fall 2023**

Homework 7

David Yang

Chapter VI (Laurent Series and Isolated Singularities) and Chapter VII (The Residue Calculus) Problems

Section VI.2 (Isolated Singularities of an Analytic Function), Problem 12

Show that if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity of $e^{f(z)}$.

Solution. ■

Section VII.1 (The Residue Theorme), Problem 2

Calculate the residue at each isolated singularity in the complex plane of the following functions.

a) $e^{1/z}$

Solution. Note that the isolated singularity of $e^{1/z}$ is at $z = 0$. The Laurent Series of $e^{1/z}$ at $z = 0$ is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \dots$$

By definition, the residue of $e^{1/z}$ at the isolated singularity $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion at $z = 0$, so

$$\text{Res}\left[e^{1/z}, 0\right] = \boxed{1}.$$

■

b) $\tan z$

Solution. Note that the isolated singularities of $\tan z = \frac{\sin z}{\cos z}$ occur when $\cos z = 0$, so $z = \frac{\pi}{2} + n\pi$ for any $n \in \mathbb{Z}$.

Since $\cos z$ has a simple zero at each of these singularities (they are simple since its derivative, $-\sin z$, is nonzero at the singularities), and both $\cos z$ and $\sin z$ are analytic at the singularities, then we know by Rule 3 that

$$\text{Res}\left[\frac{\sin z}{\cos z}, s_n\right] = \frac{\sin s_n}{-\sin s_n} = \boxed{-1}$$

for each singularity $s_n = \frac{\pi}{2} + n\pi$. ■

c) $\frac{z}{(z^2+1)^2}$

Solution. Note that the isolated singularities of $\frac{z}{(z^2+1)^2}$ occur when $z^2+1=0$, so $z=\pm i$. Furthermore, since

$$\frac{1}{\left(\frac{z}{(z^2+1)^2}\right)} = \frac{(z^2+1)^2}{z} = \frac{(z+i)^2(z-i)^2}{z}$$

has zeros of order 2 at the singularities $z=\pm i$, we know that $\frac{z}{(z^2+1)^2}$ has double poles at $\pm i$.

By Rule 2, we have that

$$\begin{aligned}
\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, i\right] &= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{z}{z^2+1}\right] \\
&= \lim_{z\to i} \frac{d}{dz}\left[\frac{z}{(z+i)^2}\right] \\
&= \lim_{z\to i} \frac{-z+i}{(z+i)^3} \\
&= 0.
\end{aligned}$$

Similarly, by Rule 2, we have that

$$\begin{aligned}
\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, -i\right] &= \lim_{z\to -i} \frac{d}{dz}\left[(z+i)^2 \frac{z}{z^2+1}\right] \\
&= \lim_{z\to -i} \frac{d}{dz}\left[\frac{z}{(z-i)^2}\right] \\
&= \lim_{z\to -i} -\frac{z+i}{(z-i)^3} \\
&= 0.
\end{aligned}$$

Thus, we conclude that $\boxed{\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, i\right] = 0 \text{ and } \operatorname{Res}\left[\frac{z}{(z^2+1)^2}, -i\right] = 0}$. ■

d) $\frac{1}{z^2+z}$

Solution. Note that the isolated singularities of $\frac{1}{z^2+z} = \frac{1}{z(z+1)}$ occur at $z=0,-1$. Furthermore, note that

$$\frac{1}{z^2+z} = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}.$$

Note that the Laurent expansion of this expression at $z=0$ is

$$\frac{1}{z} - \frac{1}{z+1} = \frac{1}{z} - \frac{1}{1-(-z)} = \frac{1}{z} + \sum_{n=0}^{\infty}(-z)^k.$$

By definition, the residue at the isolated singularity $z=0$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion at $z=0$, so

$$\operatorname{Res}\left[\frac{1}{z^2+z}, 0\right] = 1.$$

By a similar line of reasoning, note that the Laurent expansion of this expression at $z = -1$ is

$$\frac{1}{z} - \frac{1}{z+1} = \frac{-1}{1-(z+1)} - \frac{1}{z+1} = -\frac{1}{z+1} - \sum_{n=0}^{\infty}(z+1)^k.$$

By definition, the residue at the isolated singularity $z = -1$ is the coefficient of $\frac{1}{z+1}$ in the Laurent expansion at $z = -1$, so

$$\operatorname{Res}\left[\frac{1}{z^2+z}, -1\right] = -1.$$

Thus, we find that $\boxed{\operatorname{Res}\left[\frac{1}{z^2+z}, 0\right] = 1 \text{ and } \operatorname{Res}\left[\frac{1}{z^2+z}, -1\right] = -1}$. ■