MATH103: Complex Analysis **Fall 2023**

Homework 1

David Yang

Chapter I (The Complex Plane and Elementary Functions) Problems.

Section I.3 (Stereographic Projection), I.3.4

Show that a rotation of the sphere of $180°$ about the X-axis corresponds under stereographic projection to the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$.

Solution. Let $P = (X, Y, Z)$ be a point on the unit sphere. After a a $180°$ rotation of the point P on the unit sphere about the X-axis, P is sent to the point $P' = (X, -Y, -Z)$.

Consider the result of P and P' under stereographic projection. By definition, stereographic projection sends P to the point

$$\frac{X}{1-Z} + \frac{Y}{1-Z}i$$

and the point P' to the point

$$\frac{X}{1-(-Z)} + \frac{-Y}{1-(-Z)}i = \frac{X}{1+Z} - \frac{Y}{1+Z}i$$

on the extended complex plane $\mathbb{C}^*$.

We claim that P' is the result of P under the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$; note that

$$\left(\frac{X}{1-Z} + \frac{Y}{1-Z}i\right)\left(\frac{X}{1+Z} - \frac{Y}{1+Z}i\right)$$

$$= \frac{X^2}{(1-Z)(1+Z)} - \frac{XY}{(1-Z)(1+Z)} + \frac{XY}{(1-Z)(1+Z)} - \frac{Y^2}{(1-Z)(1+Z)}i^2.$$

By using the identity $i^2 = -1$, canceling out terms, and simplifying, we find that this is

$$\frac{X^2}{(1-Z)(1+Z)} + \frac{Y^2}{(1-Z)(1+Z)} = \frac{X^2+Y^2}{1-Z^2}.$$

However, since $P = (X, Y, Z)$ is a point on the unit sphere, we know that $X^2 + Y^2 + Z^2 = 1$, so $1 - Z^2 = X^2 + Y^2$. Thus, we know that

$$\left(\frac{X}{1-Z} + \frac{Y}{1-Z}i\right)\left(\frac{X}{1+Z} - \frac{Y}{1+Z}i\right) = \frac{X^2+Y^2}{1-Z^2} = 1.$$

which tells us that the resulting points of P and P' under stereographic projection are complex inverses.

Thus, a rotation of the sphere of $180°$ about the X-axis corresponds under stereographic projection to the inversion $z \mapsto \frac{1}{z}$ of $\mathbb{C}$. ■

Section I.8 (Trigonometric and Hyperbolic Functions), I.8.5

Let S denote the two slits along the imaginary axis in the complex plane, one running from i to $+i\infty$, the other running from $-i$ to $-i\infty$.

a) **Show that $\frac{1+iz}{1-iz}$ lies on the negative real axis $(-\infty, 0]$ if and only if $z \in S$.**

b) **Show that the principal branch**

$$\operatorname{Tan}^{-1} z = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right)$$

maps the slit plane $\mathbb{C} \setminus S$ one-to-one onto the vertical strip $\{|\operatorname{Re} w| < \frac{\pi}{2}\}$.