MATH103: Complex Analysis **Fall 2023**

Homework 11

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Chapter VIII (The Logarithmic Integral) Problems.

Section VIII.8 (Simply Connected Domains), Problem 4

Show that a domain D in the complex plane is simply connected if and only if any analytic function $f(z)$ on D that does not vanish at any point of D has an analytic logarithm on D. *Hint.* If $f(z) \neq 0$ on D, consider the function

$$G(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw.$$

Solution. We will begin by proving the forward implication. Suppose that D is a simply connected domain in the complex plane. By property (ii) of the Theorem on page 254, we know that every closed differential on D is exact. Consider an analytic function $f(z)$ on D that does not vanish at any point of D, i.e. $f(z) \neq 0$ on D, and the function

$$G(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw.$$

Since $f(w)$ is analytic on D, so is $f'(w)$. Furthermore, f does not vanish at any point of D, so $f(w) \neq 0$. Thus, $\frac{f'(w)}{f(w)}$ is analytic for all $w \in D$ and consequently, $G(z)$ is analytic on D.

Furthermore, we know that

$$\begin{aligned} G(z) &= \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw \\ &= \int_{z_0}^{z} d\log(f(w)) \end{aligned}$$

since every closed differential on D is exact. Thus, we have an analytic logarithm of $f(z)$ on D, as desired.

For the reverse implication, we assume that any analytic function f on D that does not vanish at any point on D has an analytic logarithm on D. Consider the function $f(w) = w - z_0$, for some $z_0 \in \mathbb{C} \setminus D$, which does not vanish at any point on D.

Consider, for any closed path γ in D,

$$\begin{aligned} G(z) &= \frac{1}{2\pi i} \int_{\gamma} d\log(f(w)) \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(w)}{f(w)}\, dw. \end{aligned}$$

Substituting $f'(w) = 1$ and $f(w) = w - z_0$ into the integrand, we find that

$$G(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z_0}\, dw = W(\gamma, z_0) = 0.$$

where the equivalence to the winding number follows by definition. Thus, by property (iv) of the Theorem on page 254, we find that D is simply connected, as desired. ■

Extra Problem

Assume that f is analytic and $|f(z)| < 1$ on the set $\{|z| \leq 1\}$. Use Rouche's Theorem to show that f has a fixed point.

Solution. Consider the function $g(z) = -z$ and the disk $D = \{|z| < 1\}$. Note that

$$|f(z)| < |g(z)| = |z|$$

for $z \in \partial D$, since $|z| = 1$ on ∂D.

By Rouche's Theorem, we know that $g(z) = -z$ and $f(z) + g(z) = f(z) - z$ have the same number of zeros in D. Since $g(z) = -z$ has one zero at $z = 0$, we know that $f(z) - z$ has a zero in D, meaning $f(z_0) = z_0$ for some $z_0 \in D$. Equivalently, z_0 is a fixed point of $f(z)$, as desired.. ■