

Homework 3

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*Chapter II (Analytic Functions) Problems.*Section II.6 (Conformal Mappings), II.6.6

- a) **Determine where the function $f(z) = z + 1/z$ is conformal and where it is not conformal.**

We know that $f(z)$ is conformal at z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$. Since $f(0)$ is not defined, we know f cannot be conformal at the origin. Similarly, we see that

$$f'(z) = 1 - \frac{1}{z^2}$$

and so $f'(z) \neq 0$ when $z \neq \pm 1$. Since $f(z) \neq 0$ at all other points, f is also analytic at all other points. Thus, $f(z) = z + 1/z$ is not conformal for $z = 0, \pm 1$ and is conformal at all other points.

- b) **Show that for each w , there are at most two values z for which $f(z) = w$.**

Fix some complex w . The values z for which $f(z) = w$ must by definition satisfy

$$z + \frac{1}{z} = w.$$

Multiplying both sides by z (which we can do as $z \neq 0$) and rearranging, we get that

$$z^2 - zw + 1 = 0.$$

Since this a quadratic with respect to z , there are at most two values z that solve this equation, as desired.

In fact, by Vieta's Formulas, we know that the roots z_1 and z_2 of $f(z) = w$ must satisfy $z_1 z_2 = 1$.

- c) **Show that if $r > 1$, $f(z)$ maps the circle $\{|z| = r\}$ onto an ellipse, and that $f(z)$ maps the circle $\{|z| = 1/r\}$ onto the same ellipse.**

We will first show that if $r > 1$, $f(z)$ maps the circle $\{|z| = r\}$ onto an ellipse. Writing z in polar form, we know that $z = re^{i\theta}$. Thus, we know that

$$f(z) = z + 1/z = re^{i\theta} + \frac{1}{r}e^{-i\theta}.$$

Substituting $e^{i\theta} = \cos \theta + i \sin \theta$ and $e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$, into this expression, we find that

$$\begin{aligned} f(z) &= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= r(\cos \theta + i \sin \theta) + \frac{1}{r}(\cos \theta - i \sin \theta) \\ &= \left(r + \frac{1}{r}\right) \cos \theta + i \left(r - \frac{1}{r}\right) \sin \theta. \end{aligned}$$

This is simply the polar form for an ellipse! Thus, we see that the circle $\{|z| = r\}$ is mapped onto an ellipse with equation

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1$$

where u, v represent the real and imaginary components of the images of $z = x + yi$ under $f(z)$.

Observe that since

$$f(z) = re^{i\theta} + \frac{1}{r}e^{-i\theta} = \frac{1}{r}e^{i\theta} + \frac{1}{\left(\frac{1}{r}\right)}e^{i\theta}$$

we can conclude that $\{|z| = \frac{1}{r}\}$ maps onto the same ellipse, as expected.

- d) **Show that $f(z)$ is one-to-one on the exterior domain $D = \{|z| = 1\}$.**
- e) **Determine the image of D under $f(z)$. Sketch the images under $f(z)$ of the circles $\{|z| = r\}$ for $r < 1$, and sketch also the images of the parts of the rays $\{\arg z = \beta\}$ lying in D .**

Show that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin.

Solution. By definition, any straight line is defined by $Ax + By = C$ for some complex numbers A, B, C . We will show that if $C \neq 0$ (meaning the line does not through the origin), the image of this line under inversion is a circle, and if $C = 0$ (meaning the line passes through the origin), the image of this line under inversion is a line.

Let $z = x + iy$ be any point on our original straight line

$$Ax + By = C.$$

Dividing both sides of this equation by $x^2 + y^2$, we know that the line is similarly defined by the equation

$$\frac{Ax}{x^2 + y^2} + \frac{By}{x^2 + y^2} = \frac{C}{x^2 + y^2}.$$

By definition, the image of this point under inversion is some point z' where

$$z' = \frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i.$$

We see that the result of z after inversion is some point $z' = u + iv$, where $u = \frac{x}{x^2 + y^2}$ and $v = -\frac{y}{x^2 + y^2}$, and

$$u^2 + v^2 = \left(\frac{x}{x^2 + y^2}\right)^2 + \left(-\frac{y}{x^2 + y^2}\right)^2 = \frac{1}{x^2 + y^2}.$$

Substituting each of these expressions in to the equation of our straight line $\frac{Ax}{x^2 + y^2} + \frac{By}{x^2 + y^2} = \frac{C}{x^2 + y^2}$, we get the equation

$$Au - Bv = C(u^2 + v^2).$$

By inspection, if $C = 0$, then we get $Au - Bv = 0$, which is simply the equation of a straight line. Consequently, we conclude that if a line passes through the origin, its image under inversion is still a straight line.

On the other hand, if $C \neq 0$, let us move all the terms to one side and group related terms to get

$$(Cu^2 - Au) + (Cv^2 + Bv) = 0.$$

Completing the square, we find that

$$C\left(u - \frac{A}{2C}\right)^2 + C\left(v + \frac{B}{2C}\right)^2 = \frac{A^2 + B^2}{4C^2}.$$

When $C \neq 0$, this is the simply the equation of a circle, centered at $\frac{A}{2C} - \frac{B}{2C}i$ with radius $\frac{1}{2C}\sqrt{A^2 + B^2}$. Consequently, we conclude that if a line does not pass through the origin, its image under inversion is a circle.

Combining our two cases together, we find that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin, as desired. ■