MATH103: Complex Analysis **Fall 2023**

Homework 7

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Chapter V (Power Series) Problems.

Section V.7 (The Zeros of an Analytic Function), Problem 11

Show that if $f(z)$ is a nonconstant analytic function on a domain D, then the image under $f(z)$ of any open set is open. *Remark.* This is the open mapping theorem for analytic functions. The proof is easy when $f'(z) \neq 0$, since the Jacobian of $f(z)$ coincides with $|f'(z)|^2$. Use Exercise 9 to deal with the points where $f'(z)$ is zero.

Solution. Let S be an open set in D, and let V be the image of S under f. Consider any $w \in V$. By definition, there exists some $z_0 \in S$ such that $f(z_0) = w$.

We will separate our work into two cases, when $f'(z_0) \neq 0$ and when $f'(z_0) = 0$. Let us begin with the first case, when $f'(z_0) \neq 0$. Since f is analytic in D, $z_0 \in S \subset D$, and $f'(z_0) \neq 0$, we know by the Inverse Function Theorem that there is a disk of radius r centered at z_0, which we can denote $D_r(z_0)$, such that $f(D_r(z_0))$ is open. By definition, since $D_r(z_0) \subset S$, $f(D_r(z_0)) \subset V$. Consequently, we know there is a disk around w that is fully contained in V for any arbitrary $w \in V$, and thus, by definition, V is open. We conclude that the image of $f(z)$ of any open set is open for points z_0 where $f'(z_0) \neq 0$.

It remains to show that the open mapping theorem holds for points z_0 where $f'(z_0) = 0$. Consider $g(z) = f(z) - f(z_0)$. Clearly, z_0 is a zero of $g(z)$, as

$$g(z_0) = f(z_0) - f(z_0) = 0.$$

Furthermore, we know that z_0 is a zero of finite order, since $f(z)$ is not constant and thus is not equal to $f(z_0)$. Note also that $g'(z_0) = f'(z_0) - f'(z_0) = 0$. Thus, we can conclude that z_0 is a zero of $g(z)$ of order $N \geq 2$.

We will now use our result from Exercise 9 (V.7.9). Note that since $g(z) = f(z) - f(z_0)$ is analytic (as $f(z)$ is analytic and $-f(z_0)$ is simply a translation by a constant) with zero of finite order $N \geq 2$ at z_0, we know from Exercise 9 that

$$g(z) = h(z)^N$$

for some $h(z)$ analytic near z_0 satisfying $h'(z_0) \neq 0$. Note that since $h'(z_0) \neq 0$, we know that locally around z_0, $h^{-1}(z)$ exists (from our previous case). By construction, then, $g^{-1}(z) = h^{-1}(\sqrt[m]{z})$ exists. Thus, since $g^{-1}(z)$ exists locally, we know that $g(z)$ is an open mapping. To conclude, note that

$$f(z) = g(z) + f(z_0)$$

by construction. Since $g(z)$ is an open mapping and $f(z)$ is a translation of $g(z)$ by the constant $f(z_0)$, $f(z)$ is also an open mapping for points z_0 where $f'(z_0) = 0$.

Since the image under $f(z)$ of an open set S is open for points $z_0 \in S$ where either $f'(z_0) \neq 0$ or $f'(z_0) = 0$, we conclude that the image under $f(z)$ of any open set is open, as desired. ■

Section V.8 (Analytic Continuation), Problem 3

Show that each branch of $\sqrt{z}$ can be continued analytically along any path γ in $\mathbb{C}\setminus\{0\}$, and show that the radius of convergence of the power series $f_t(z)$ representing the continuation is $|\gamma(t)|$. Show that $\sqrt{z}$ cannot be continued analytically along any path containing 0.

Solution. We will first determine the derivatives of $f(z) = \sqrt{z}$. Note that $f'(z) = \frac{1}{2\sqrt{z}}, f''(z) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) z^{-3/2}$, and in general, for $n \geq 1$,

$$f^{(n)}(z) = \frac{(-1)^{n+1}(2n-3)!!}{2^n} z^{\frac{-(2n-1)}{2}}.$$

Thus, the analytic continuation of $f(z) = \sqrt{z}$ along some path γ in $\mathbb{C}\setminus\{0\}$ can be expressed by the following power series:

$$\begin{aligned} f_t(\gamma(t)) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\gamma(t))}{n!}(z-\gamma(t))^n \\ &= f(\gamma(t)) + \sum_{n=1}^{\infty} \frac{f^{(n)}(\gamma(t))}{n!}(z-\gamma(t))^n \\ &= f(\gamma(t)) + \sum_{n=1}^{\infty} \left[(-1)^{n+1}\frac{(2n-3)!!}{2^n \cdot n!}(\gamma(t))^{-\frac{(2n-1)}{2}}\right](z-\gamma(t))^n. \end{aligned}$$

Since the analytic continuation of $f(z)$ is of this form and must be unique if it exists, we know that $\sqrt{z}$ cannot be analytically continued along any path through 0, since the derivatives of $f(z) = \sqrt{z}$ are not defined at $z = 0$. For any path γ in $\mathbb{C}\setminus 0$, though, the analytic continuation is defined as above, and so each branch of $\sqrt{z}$ can be continued analytically along any path γ in $\mathbb{C}\setminus\{0\}$.

We will use the ratio test to determine the radius of convergence of the power series $f_t(z)$. Note that

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{(2k-3)!!}{2^k\cdot k!}(\gamma(t))^{-\frac{(2k-1)}{2}}}{(-1)^{k+2}\frac{(2k-1)!!}{2^{k+1}\cdot(k+1)!}(\gamma(t))^{-\frac{(2k+1)}{2}}}\right| \\ &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{(2k-3)!!}{2^k\cdot k!}(\gamma(t))^{-\frac{(2k-1)}{2}}}{(-1)^{k+2}\frac{(2k-1)(2k-3)!!}{2\cdot 2^k\cdot(k+1)(k)!}(\gamma(t))^{-\frac{(2k+1)}{2}}}\right|. \end{aligned}$$

Simplifying, we find that

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| &= \lim_{k\to\infty}\left|-1\cdot\frac{2(k+1)}{(2k-1)}\gamma(t)\right| \\ &= \lim_{k\to\infty}\left|\frac{2k+2}{2k-1}\right||\gamma(t)| \\ &= |\gamma(t)|. \end{aligned}$$

Thus, by the ratio test, we find that the radius of convergence of the power series $f_t(z)$ representing the continuation is $|\gamma(t)|$, as desired.

Finally, note that if the path $\gamma(t)$ contains 0, then the radius of convergence at 0 in that path is 0; thus, $\sqrt{z}$ cannot be continued analytically along any path containing 0, as desired. $\blacksquare$