MATH103: Complex Analysis **Fall 2023**

Homework 3

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Chapter II (Analytic Functions) Problems.

Section II.6 (Conformal Mappings), II.6.6

Determine where the function $f(z) = z + 1/z$ is conformal and where it is not conformal. Show that for each w, there are at most two values z for which $f(z) = w$. Show that if $r > 1$, $f(z)$ maps the circle $\{|z| = r\}$ onto an ellipse, and that $f(z)$ maps the circle $\{|z| = 1/r\}$ onto the same ellipse. Show that $f(z)$ is one-to-one on the exterior domain $D = \{|z| = 1\}$. Determine the image of D under $f(z)$. Sketch the images under $f(z)$ of the circles $\{|z| = r\}$ for $r < 1$, and sketch also the images of the parts of the rays $\{\arg z = \beta\}$ lying in D.

Section II.7 (Fractional Linear Transformations), II.7.6

Show that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin.

Solution. By definition, any straight line is defined by $Ax + By = C$ for some complex numbers A, B, C. We will show that if $C \neq 0$ (meaning the line does not through the origin), the image of this line under inversion is a circle, and if $C = 0$ (meaning the line passes through the origin), the image of this line under inversion is a line.

Let $z = x + iy$ be any point on our original straight line

$$Ax + By = C.$$

Dividing both sides of this equation by $x^2 + y^2$, we know that the line is similarly defined by the equation

$$\frac{Ax}{x^2+y^2} + \frac{By}{x^2+y^2} = \frac{C}{x^2+y^2}.$$

By definition, the image of this point under inversion is some point z' where

$$z' = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i.$$

We see that the result of z after inversion is some point $z' = u+iv$, where $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$, and

$$u^2 + v^2 = \left(\frac{x}{x^2+y^2}\right)^2 + \left(-\frac{y}{x^2+y^2}\right)^2 = \frac{1}{x^2+y^2}.$$

Substituting each of these expressions in to the equation of our straight line $\frac{Ax}{x^2+y^2} + \frac{By}{x^2+y^2} = \frac{C}{x^2+y^2}$, we get the equation

$$Au - Bv = C(u^2 + v^2).$$

By inspection, if $C = 0$, then we get $Au - Bv = 0$, which is simply the equation of a straight line. Consequently, we conclude that if a line passes through the origin, its image under inversion is still a straight line.

On the other hand, if $C \neq 0$, let us move all the terms to one side and group related terms to get

$$(Cu^2 - Au) + (Cv^2 + Bv) = 0.$$

Completing the square, we find that

$$C\left(u - \frac{A}{2C}\right)^2 + C\left(v + \frac{B}{2C}\right)^2 = \frac{A^2+B^2}{4C^2}.$$

When $C \neq 0$, this is the simply the equation of a circle, centered at $\frac{A}{2C} - \frac{B}{2C}i$ with radius $\frac{1}{2C}\sqrt{A^2+B^2}$. Consequently, we conclude that if a line does not pass through the origin, its image under inversion is a circle.

Combining our two cases together, we find that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin, as desired. ■