MATH103: Complex Analysis **Fall 2023**

Homework 12

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Chapter XI (Conformal Mapping) Problems.

Section XI.5 (Compactness of Families of Functions), Problem 7

Let D be a bounded domain, and let $f(z)$ be an analytic function from D into D. Show that if $z_0 \in D$ is a fixed point for $f(z)$, then $|f'(z_0)| \leq 1$.

Solution. Note that since $f(z)$ is an analytic function from D into D, the n^{th} iterate of $f(z)$, represented as

$$f_n(z) = f(f(\cdots f(z)\cdots)) = f(f_{n-1}(z))$$

is similarly analytic from D to D, for any $n \geq 1$.

Consider the value of the n^{th} iterate of f at the fixed point z_0 of f, i.e. $f_n(z_0)$, for arbitrary n. By the Chain Rule, we have

$$\begin{aligned}\frac{d}{dz}\left(f_n(z_0)\right) &= f'(f_{n-1}(z_0))f'_{n-1}(z_0) \\ &= f'(z_0)f'_{n-1}(z_0) \\ &= f'(z_0)f'(z_0)f'_{n-2}(z_0) \\ &\vdots \\ &= \left(f'(z_0)\right)^n.\end{aligned}$$

Furthermore, since f is an analytic function on the bounded domain D, its derivative must also be analytic and bounded on D. Taking the limit as $n \to \infty$, we must have that

$$\lim_{n\to\infty} \frac{d}{dz}\left(f_n(z_0)\right) = \left(f'(z_0)\right)^n$$

is bounded, which must mean that $|f'(z_0)| \leq 1$, as desired. ■

Extra Problem

Find a conformal map that takes $\{0 < \arg z < \frac{\pi}{2},\ 0 < |z| < 1\}$ **onto** $\{|z| < 1\}$**.**

Solution. We define a sequence of conformal maps ξ, ρ, w, x, and y, such that the composition of them takes $\{0 < \arg z < \frac{\pi}{2},\ 0 < |z| < 1\}$, the first quadrant of the unit disk, onto $\{|z| < 1\}$.

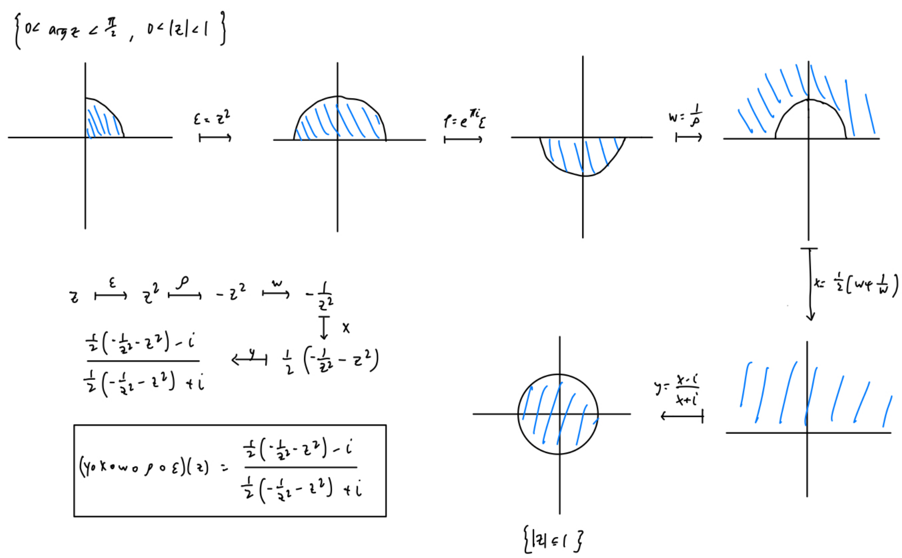
The map $\xi = z^2$ takes the first quadrant of the unit disk $\{0 < \arg z < \frac{\pi}{2},\ 0 < |z| < 1\}$ to the upper half unit disk; this map is one-to-one and onto and is thus conformal.

The rotation $\rho = e^{\pi i}\xi = -\xi$ maps the upper half unit disk to the bottom half unit disk, and is conformal. Similarly, the inversion $w = \frac{1}{\rho}$ is conformal, and maps the bottom half unit disk to the part of the upper half-plane outside the unit circle.

Next, by the exercise on page 292 of Gamelin, the map $x = \frac{1}{2}\left(w + \frac{1}{w}\right)$ is a conformal map from the part of the upper half-plane outside the unit circle to the entire upper half-plane.

Finally, the conformal Cayley Transform map $y = \frac{x-i}{x+i}$ will take the upper half plane to the unit disk.

Each of these conformal maps can be visualized as follows:



Since the composition of conformal maps is conformal, the composition of these maps,

$$(y \circ x \circ w \circ \rho \circ \xi)(z) = \boxed{\frac{\frac{1}{2}\left(-\frac{1}{z^2} - z^2\right) - i}{\frac{1}{2}\left(-\frac{1}{z^2} - z^2\right) + i}},$$

is a conformal map from $\{0 < \arg z < \frac{\pi}{2},\ 0 < |z| < 1\}$ onto $\{|z| < 1\}$, as desired. ■