MATH66: Stochastic and Numerical Methods **Fall 2023**

Homework 2

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Problems from Numerical Analysis (Sauer), Chapter 3.

Section 3.1 (Data and Interpolating Functions), Problem 5

a) **Find a polynomial $P(x)$ of degree 3 or less whose graph passes through the four data points $(-2,8),(0,4),(1,2),(3,-2)$.**

Solution. We can construct a polynomial using Lagrange interpolation:

$$\begin{aligned} P(x) &= 8\frac{(x-0)(x-1)(x-3)}{(-2-0)(-2-1)(-2-3)} + 4\frac{(x-(-2))(x-1)(x-3)}{(0-(-2))(0-1)(0-3)} \\ &\quad + 2\frac{(x-(-2))(x-0)(x-3)}{(1-(-2))(1-0)(1-3)} + (-2)\frac{(x-(-2))(x-0)(x-1)}{(3-(-2))(3-0)(3-1)}. \end{aligned}$$

This simplifies to $\boxed{P(x) = 4-2x}$. ■

b) **Describe any other polynomials of degree 4 or less which pass through the four points in part (a).**

Solution. By the Lagrange Interpolation theorem, $P(x) = 4-2x$ is the unique polynomial of degree less than 4, but any other polynomial that passes through the given point will be of the form

$$\boxed{\tilde{P}(x) = 4-2x+c(x+2)x(x-1)(x-3)}$$

for any constant c; this polynomial is constructed from the fact that the polynomial found in (a) interpolates for the given data points. ■

Section 3.1 (Data and Interpolating Functions), Problem 8

Let $P(x)$ be the degree 9 polynomial that takes the value 112 at $x = 1$, takes the value 2 at $x = 10$, and equals zero for $x = 2, \ldots, 9$. Calculate $P(0)$.

Solution. We can construct such a polynomial using Lagrange interpolation:

$$P(x) = 112\frac{(x-2)\ldots(x-10)}{(1-2)\ldots(1-10)} + 2\frac{(x-1)\ldots(x-9)}{(10-1)\ldots(10-9)} + [0 \text{ terms from } P(2) = \cdots = P(9) = 0.]$$

Simplifying, we find that

$$P(x) = -112\frac{(x-2)\ldots(x-10)}{9!} + 2\frac{(x-1)\ldots(x-9)}{9!}.$$

Thus, plugging in $x = 0$, we find that

$$P(0) = -112\frac{-(10)!}{9!} + 2\frac{-(9)!}{9!} = 1120 - 2 = \boxed{1118}.$$

■

Section 3.1 (Data and Interpolating Functions), Problem 12

Can a degree 3 polynomial intersect a degree 4 polynomial in exactly five points? Explain.

Solution. No. By Lagrange's Interpolation theorem, there is exactly one degree 4 or less polynomial passing through five given points; thus, there cannot be a degree 3 polynomial and a degree 4 polynomial passing through the same five points. ■

Section 3.1 (Data and Interpolating Functions), Problem 17

The estimated mean atmospheric concentration of carbon dioxide in earth's atmosphere is given in the table that follows, in parts per million by volume. Find the degree 3 interpolating polynomial of the data and use it to estimate the CO_2 concentration in (a) 1950 and (b) 2050. (The actual concentration in 1950 was 310 ppm).

Solution. We find the degree 3 interpolating polynomial using Lagrange interpolation:

$$\begin{aligned} P(x) = 280&\frac{(x-1850)(x-1900)(x-2000)}{(1800-1850)(1800-1900)(1800-2000)} + 283\frac{(x-1800)(x-1900)(x-2000)}{(1850-1800)(1850-1900)(1850-2000)} \\ &+ 291\frac{(x-1800)(x-1850)(x-2000)}{(1900-1800)(1900-1850)(1900-2000)} + 370\frac{(x-1800)(x-1850)(x-1900)}{(2000-1800)(2000-1850)(2000-1900)} \end{aligned}$$

Plugging in $x = 1950$, we find that

$$\begin{aligned}
P(1950) &= 280\frac{(100)(50)(-50)}{(-50)(-100)(-200)} + 283\frac{150(50)(-50)}{(50)(-50)(-150)} \\
&\quad + 291\frac{(150)(100)(-50)}{(100)(50)(-100)} + 370\frac{(150)(100)(50)}{(200)(150)(100)} \\
&= \boxed{316 \text{ ppm}}
\end{aligned}$$

and

$$\begin{aligned}
P(2050) &= 280\frac{(200)(150)(50)}{(-50)(-100)(-200)} + 283\frac{250(150)(50)}{(50)(-50)(-150)} \\
&\quad + 291\frac{(250)(200)(50)}{(100)(50)(-100)} + 370\frac{(250)(200)(150)}{(200)(150)(100)} \\
&= \boxed{465 \text{ ppm}}.
\end{aligned}$$

■

Section 3.2 (Interpolation Error), Problem 2

a) **Given the data points $(1,0),(2,\ln 2),(4,\ln 4)$, find the degree 2 interpolating polynomial.**

Solution. The degree 2 interpolating polynomial, from Lagrange's interpolation formula, is

$$P(x) = 0 + \ln 2\frac{(x-1)(x-4)}{(2-1)(2-4)} + \ln 4\frac{(x-1)(x-2)}{(4-1)(4-2)}$$

Simplifying, we get

$$\boxed{P(x) = -\ln 2\frac{(x-1)(x-4)}{2} + \ln 4\frac{(x-1)(x-2)}{6}}.$$

■

b) **Use the result of (a) to approximate $\ln 3$.**

Solution. Plugging in $x = 3$, we get that

$$P(3) = -\ln 2\frac{(3-1)(3-4)}{2} + \ln 4\frac{(3-1)(3-2)}{6} \approx \boxed{1.155}.$$

■

c) **Use Theorem 3.3 to give an error bound for the approximation in part (b).**

Solution. By Theorem 3.3 (the Error Bounding Theorem), we know that the interpolation error for second degree approximation $P(x)$ is

$$f(x) - P(x) = \frac{(x-1)(x-2)(x-4)}{3!}f'''(c)$$

where $f(x) = \ln(x)$, and so $f'''(x) = \frac{2}{x^3}$. Since $1 \le c \le 4$, we know $f'''(c) = \frac{2}{c^3} \le 2$, and so an upper bound of our error at $x = 3$ is

$$|f(3) - P(3)| \le \left|\frac{(3-1)(3-2)(3-4)}{3!}\cdot 2\right| = \boxed{\frac{1}{3}}.$$

■

d) **Compare the actual error to your error bound.**

Solution. The actual error of our approximation for $\ln(3)$ is

$$|\ln(3) - P(3)| \approx |\ln(3) - 1.155| \approx \boxed{0.0564},$$

which is smaller than our error bound found in part (c). ■

Section 3.2 (Interpolation Error), Problem 6

Assume that the polynomial $P_5(x)$ interpolates a function $f(x)$ at the six data points $(x_i, f(x_i))$ with x-coordinates $x_1 = 0, x_2 = 0.2, x_3 = 0.4, x_4 = 0.6, x_5 = 0.8, x_6 = 1$. Assume that the interpolation error at $x = 0.3$ is $|f(0.3) - P_5(0.3)| = 0.01$. Estimate the new interpolation error $|f(0.3) - P_7(0.3)|$ that would result if two additional interpolation points $(x_6, y_6) = (0.1, f(0.1))$ and $(x_7, y_7) = (0.5, f(0.5))$ are added. What assumptions have you made to produce this estimate?

Solution. We know by the Error Bounding Theorem that the interpolation error for the approximation $P_5(x)$ is approximately

$$|f(x) - P_5(x)| = \left| \frac{x(x-0.2)(x-0.4)(x-0.6)(x-0.8)(x-1)}{6!} f^{(6)}(c_1) \right|$$

where $0 \le c_1 \le 1$, since we are given 6 data points. Adding two interpolation points $(x_6, y_6) = (0.1, f(0.1))$ and $(x_7, y_7) = (0.5, f(0.5))$ gives us a slightly different error bound; we have

$$|f(x) - P_7(x)| = \left| \frac{x(x-0.1)(x-0.2)(x-0.4)(x-0.5)(x-0.6)(x-0.8)(x-1)}{8!} f^{(8)}(c_2) \right|$$

where again $0 \le c_2 \le 1$. Observe that we can estimate the error of $P_7(x)$ using the relative ratios between these two error bounds:

$$\frac{|f(x) - P_7(x)|}{|f(x) - P_5(x)|} = \left| \frac{(x-0.1)(x-0.5)}{7 \cdot 8} \right| \left| \frac{f^{(8)}(c_2)}{f^{(6)}(c_1)} \right|$$

Assuming that we have no extra information about $f^{(6)}(c_1)$ and $f^{(8)}(c_2)$, we estimate the relative ratio of their errors to be

$$\frac{|f(x) - P_7(x)|}{|f(x) - P_5(x)|} \approx \left| \frac{(x-0.1)(x-0.5)}{7 \cdot 8} \right|,$$

or equivalently,

$$|f(x) - P_7(x)| \approx |f(x) - P_5(x)| \left| \frac{(x-0.1)(x-0.5)}{7 \cdot 8} \right|,$$

Thus, plugging in $x = 0.3$ and using the fact that $|f(0.3) - P_5(0.3)| = 0.01$, we get that

$$|f(x) - P_7(x)| \approx 0.01 \left| \frac{(0.3-0.1)(0.3-0.5)}{56} \right| \approx \boxed{7 \times 10^{-6}}.$$

To reiterate the assumptions we used to produce this estimate, we assumed that we have no information about the bounds of the derivatives of f. ■

Problems from Numerical Analysis (Sauer), Chapter 5.

Section 5.2 (Newton-Cotes Formulas for Numerical Integration), Problem 4(a)

Apply the composite Simpson's Rule with $m = 1, 2,$ and 4 panels to the integral $\int_0^1 xe^x \, dx$ and report the errors. Repeat with composite trapezoid.

Solution. For this problem, I adapted the DeepNote notebook code we did for N2: Quadrature.

For the composite Simpson's rule with $1, 2,$ and 4 panels, the reported errors were approximately

$$\boxed{0.0026, 1.69 \times 10^{-4}, 1.065 \times 10^{-5}.}$$

For the composite trapezoid rule with 1, 2, and 4 panels, the reported errors were approximately

$$\boxed{0.359, 0.09175, 0.023}.$$

As expected, we see that as we add more panels, the errors get smaller, and the composite Simpson's Rule error is also smaller than the composite trapezoid rule for the same panel number; this matches our expectations since composite Simpson's is a higher order approximation. ■

Section 5.2 (Newton-Cotes Formulas for Numerical Integration), Problem 12

Find c_1, c_2, c_3 such that the rule

$$\int_0^1 f(x)\,dx \approx c_1 f(0) + c_2 f(0.5) + c_3 f(1)$$

has degree of precision greater than one. (Hint: Substitute $f(x) = 1, x$, and x^2. Do you recognize the method that results?)

Solution. We will first plug in $f(x) = 1$:

$$\int_0^1 1\,dx = 1 = c_1 + c_2 + c_3.$$

Next, we plug in $f(x) = x$:

$$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

and

$$c_1 f(0) + c_2 f(0.5) + c_3 f(1) = 0.5c_2 + c_3 = \frac{1}{2}.$$

Finally, for $f(x) = x^2$, we have

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

and

$$c_2 f(0.5) + c_3 f(1) = 0.25c_2 + c_3 = \frac{1}{3}.$$

The rule has degree of precision greater than one when these equations are satisfied; using the final two equations, we find that

$$c_2 = \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4}} = \frac{2}{3}.$$

This tells us that $c_3 = \frac{1}{6}$, and from the first equation, we get that

$$c_1 = 1 - c_2 - c_3 = \frac{1}{6}.$$

Thus, we get the rule

$$\int_0^1 f(x)\,dx \approx \frac{1}{6}f(0) + \frac{2}{3}f(0.5) + \frac{1}{6}f(1)$$

or equivalently,

$$\boxed{\int_0^1 f(x)\,dx \approx \frac{f(0) + 4f(0.5) + f(1)}{6}}$$

which is simply Simpson's Rule. ∎