MATH66: Stochastic and Numerical Methods **Fall 2023**

Homework 2

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Problems from Numerical Analysis (Sauer), Chapter 3.

Section 3.1 (Data and Interpolating Functions), Problem 5

a) **Find a polynomial $P(x)$ of degree 3 or less whose graph passes through the four data points $(-2,8),(0,4),(1,2),(3,-2)$.**

Solution. We can construct a polynomial using Lagrange interpolation:

$$\begin{aligned} P(x) = 8\frac{(x-0)(x-1)(x-3)}{(-2-0)(-2-1)(-2-3)} + 4\frac{(x-(-2))(x-1)(x-3)}{(0-(-2))(0-1)(0-3)} \\ + 2\frac{(x-(-2))(x-0)(x-3)}{(1-(-2))(1-0)(1-3)} + (-2)\frac{(x-(-2))(x-0)(x-1)}{(3-(-2))(3-0)(3-1)}. \end{aligned}$$

This simplifies to $\boxed{P(x) = 4-2x}$. ■

b) **Describe any other polynomials of degree 4 or less which pass through the four points in part (a).**

Solution. By the Lagrange Interpolation theorem, $P(x) = 4-2x$ is the unique polynomial of degree less than 4, but any other polynomial that passes through the given point will be of the form

$$\boxed{\tilde{P}(x) = 4-2x + c(x+2)x(x-1)(x-3)}$$

for any constant c; this polynomial is constructed from the fact that the polynomial found in (a) interpolates for the given data points. ■

Section 3.1 (Data and Interpolating Functions), Problem 8

Let $P(x)$ be the degree 9 polynomial that takes the value 112 at $x = 1$, takes the value 2 at $x = 10$, and equals zero for $x = 2, \dots, 9$. Calculate $P(0)$.

Solution. We can construct such a polynomial using Lagrange interpolation:

$$P(x) = 112\frac{(x-2)\dots(x-10)}{(1-2)\dots(1-10)} + 2\frac{(x-1)\dots(x-9)}{(10-1)\dots(10-9)} + [0 \text{ terms from } P(2) = \dots = P(9) = 0.]$$

Simplifying, we find that

$$P(x) = -112\frac{(x-2)\dots(x-10)}{9!} + 2\frac{(x-1)\dots(x-9)}{9!}.$$

Thus, plugging in $x = 0$, we find that

$$P(0) = -112\frac{-(10)!}{9!} + 2\frac{-(9)!}{9!} = 1120 - 2 = \boxed{1118}.$$

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Section 3.1 (Data and Interpolating Functions), Problem 12

Can a degree 3 polynomial intersect a degree 4 polynomial in exactly five points? Explain.

Solution. No. By Lagrange's Interpolation theorem, there is exactly one degree 4 or less polynomial passing through five given points; thus, there cannot be a degree 3 polynomial and a degree 4 polynomial passing through the same five points. ■

Section 3.1 (Data and Interpolating Functions), Problem 17

The estimated mean atmospheric concentration of carbon dioxide in earth's atmosphere is given in the table that follows, in parts per million by volume. Find the degree 3 interpolating polynomial of the data and use it to estimate the CO_2 concentration in (a) 1950 and (b) 2050. (The actual concentration in 1950 was 310 ppm).

Solution. We find the degree 3 interpolating polynomial using Lagrange interpolation:

$$\begin{aligned} P(x) = 280&\frac{(x-1850)(x-1900)(x-2000)}{(1800-1850)(1800-1900)(1800-2000)} + 283\frac{(x-1800)(x-1900)(x-2000)}{(1850-1800)(1850-1900)(1850-2000)} \\ &+ 291\frac{(x-1800)(x-1850)(x-2000)}{(1900-1800)(1900-1850)(1900-2000)} + 370\frac{(x-1800)(x-1850)(x-1900)}{(2000-1800)(2000-1850)(2000-1900)} \end{aligned}$$

Plugging in $x = 1950$, we find that

$$\begin{aligned} P(1950) &= 280\frac{(100)(50)(-50)}{(-50)(-100)(-200)} + 283\frac{150(50)(-50)}{(50)(-50)(-150)} \\ &+ 291\frac{(150)(100)(-50)}{(100)(50)(-100)} + 370\frac{(150)(100)(50)}{(200)(150)(100)} \\ &= \boxed{316 \text{ ppm}} \end{aligned}$$

and

$$\begin{aligned} P(2050) &= 280\frac{(200)(150)(50)}{(-50)(-100)(-200)} + 283\frac{250(150)(50)}{(50)(-50)(-150)} \\ &+ 291\frac{(250)(200)(50)}{(100)(50)(-100)} + 370\frac{(250)(200)(150)}{(200)(150)(100)} \\ &= \boxed{465 \text{ ppm}}. \end{aligned}$$

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Section 3.2 (Interpolation Error), Problem 2

Section 3.2 (Interpolation Error), Problem 6