MATH66: Stochastic and Numerical Methods **Fall 2023**

Homework 4

David Yang

Homework 4 Problems

1. **(The Lipschitz Condition): Consider the ODE**

$$y' = \frac{3}{2}y^{1/3} \textbf{ with } y(0) = 0.$$

a) **Apply the Forward Euler method to this problem and report what you find. Does the value of h (the step size) matter? Are the values you computed a reasonable approximate solution for this IVP?**

Solution. Applying the Forward Euler method to the ODE with $y(0) = 0$ gives

$$y_1 = y_0 + h \cdot f(t_0, y_0) = 0 + h \cdot 0 = 0$$

where we see that the value of y in the next iteration is independent of the step size h. Consequently, since $y' = \frac{3}{2}y^{1/3}$ depends only on the y-variable, $f(t, y) = f(t, 0) = 0$ for any value of t.

Thus, the approximate solution we get to the IVP is $\boxed{y(t) = 0}$, which is a valid solution for the given ODE. ■

b) **Find a solution to this IVP using the separation of variables technique.**

Solution. We can similarly solve the IVP using separation of variables. Dividing both sides by $y^{1/3}$, we get that

$$\frac{1}{y^{1/3}}y' = \frac{3}{2}.$$

Integrating both sides, we get that

$$\int \frac{1}{y^{1/3}}\, dy = \int \frac{3}{2}\, dt$$
$$\frac{3}{2}y^{2/3} = \frac{3}{2}t$$

Dividing both sides by $\frac{3}{2}$ and raising both sides to the power of $\frac{3}{2}$, we get the solution

$$y = t^{\frac{3}{2}}.$$

Since we do see that this solution satisfies the intial value $y(0) = 0$, we find that the solution to the IVP is $\boxed{y' = t^{\frac{3}{2}}}$. ■

c) **Evaluate the y-derivative of $f(y) = \frac{3}{2}y^{1/3}$ and argue that this function is not Lipschitz continuous in any interval that contains $y = 0$.**

Solution. The y-derivative of $f(y)$ is

$$f'(y) = \frac{3}{2} \cdot \frac{1}{3}y^{-2/3} = \frac{1}{2y^{2/3}}.$$

This is not Lipschitz continuous in any interval containing $y = 0$ since as y_1, y_2 get arbitrarily close to 0, $|f(t, y_1) - f(t, y_2)|$ grows without bound compared to $|y_1 - y_2|$. ■

d) **Apply a relevant theorem discussed in class to make sense of your answers in (a) and (b). Is the Forward Euler method invalid for this problem? Why or why not?**

Solution. We want to make use of our local existence and uniqueness theorem for first order IVPs. However, since f is not Lipschitz continuous in any interval containing $y = 0$, we cannot say that there is a unique solution to some interval about $y = 0$, explaining the two distinct solutions in (a) and (b).

The Forward Euler method is not invalid for this problem; it provides the trivial solution which shows that the ODE need not have a unique solution when Lipschitz continuity is not satisfied. ■

Consider the initial value ODE problem:

$$y' = 2(t+1)y, \text{ with initial value } y(0) = 1.$$

Use the separation of variables technique to find a solution formula for this ODE. You will use this exact solution to measure errors for the following numerical calculations. For each of the following methods listed, do the following:

a) **Approximate the value of $y(0.1)$ by calculating one-step of each numerical method using $h = 0.1$.**
b) **Approximate the value of $y(0.01)$ by calculating one-step of each numerical method using $h = 0.01$.**
c) **Calculate the one-step error (local truncation error) in each approximatuion and report the order of the method.**

We will first solve for the exact solution using separation of variables. Dividing both sides by y, we have that

$$\frac{1}{y}y' = 2(t+1).$$

Integrating both sides, we find that

$$\ln(y) = t^2 + 2t + C.$$

Since $y(0) = 1$, we know that $\ln(1) = 0^2 + 2(0) + C$, so $C = 0$. Thus, our solution is

$$\boxed{y = e^{t^2+2t}}$$

and we will use this exact solution to measure the errors for each of the following methods:

2. **Forward Euler method**

a) The Forward Euler method approximates $y(0.1)$ as

$$y(0.1) = y(0) + 0.1(f(0,1)) = 1 + 0.1(2(0+1)1) = \boxed{1.2}$$

b) The Forward Euler method approximates $y(0.01)$ as

$$y(0.01) = y(0) + 0.01(f(0,1)) = 1 + 0.01(2(0+1)1) = \boxed{1.02}$$

c) The local truncation error in each approximation are $\left|1.2 - e^{0.1^2+2(0.1)}\right|$ and $\left|1.02 - e^{0.01^2+2(0.01)}\right|$. Thus, the order of the method is

$$\log_{10}\left(\frac{\left|1.2 - e^{0.1^2+2(0.1)}\right|}{\left|1.02 - e^{0.01^2+2(0.01)}\right|}\right) \approx \boxed{2}.$$

3. **Backward Euler method**

   a) The Backward Euler method approximates $y(0.1)$ as

   $$y(0.1) = y(0) + 0.1(f(0.1, y(0.1))) = 1 + 0.1(2(0.1 + 1)y(0.1)).$$

   Solving for $y(0.1)$, we get that

   $$y(0.1) = \frac{1}{1 - 0.1(2(1 + 0.1))} \approx \boxed{1.282}$$

   b) The Backward Euler method approximates $y(0.01)$ as

   $$y(0.01) = y(0) + 0.01(f(0.01, y(0.01))) = 1 + 0.01(2(0.01 + 1)y(0.01)).$$

   Solving for $y(0.01)$, we get that

   $$y(0.01) = \frac{1}{1 - 0.01(2(1 + 0.01))} \approx \boxed{1.0206}$$

   Thus, the order of the method is

   $$\log_{10}\left(\frac{\left|1.282 - e^{0.1^2+2(0.1)}\right|}{\left|1.0206 - e^{0.01^2+2(0.01)}\right|}\right) \approx \boxed{2}.$$

4. **Trapezoid method**

   a) The Trapezoid method approximates $y(0.1)$ as

   $$y(0.1) = y(0) + 0.1\left(\frac{1}{2}(f(0,1) + f(0.1, y(0.1)))\right) = 1 + 0.1(1 + (0.1 + 1)y(0.1)).$$

   Solving for $y(0.1)$, we get that

   $$y(0.1) = \frac{1 + 0.1}{1 - 0.1(1.1)} \approx \boxed{1.236}$$

   b) The Trapezoid method approximates $y(0.01)$ as

   $$y(0.01) = y(0) + 0.01\left(\frac{1}{2}(f(0,1) + f(0.01, y(0.01)))\right) = 1 + 0.01(1 + (0.01 + 1)y(0.1)).$$

   Solving for $y(0.01)$, we get that

   $$y(0.01) = \frac{1 + 0.01}{1 - 0.01(1.01)} \approx \boxed{1.020305}$$

   c) Thus, the order of the method is

   $$\log_{10}\left(\frac{\left|1.236 - e^{0.1^2+2(0.1)}\right|}{\left|1.020305 - e^{0.01^2+2(0.01)}\right|}\right) \approx \boxed{3}.$$

5. **Explicit trapezoid method (Heun's Method)**

a) Heun's method approximates $y(0.1)$ as

$$y(0.1) = y(0) + 0.1 \cdot \frac{1}{2}(f(0,1) + f(0.1, 1 + 0.1f(0,1)))$$

Plugging in $f(0,1) = 2(0+1)1 = 2$, we get

$$y(0.1) = 1 + 0.05(2 + 2(0.1+1)(1+0.1(2))) = \boxed{1.236}$$

b) Heun's method approximates $y(0.01)$ as

$$y(0.01) = y(0) + 0.01 \cdot \frac{1}{2}(f(0,1) + f(0.01, 1 + 0.01f(0,1)))$$

Plugging in $f(0,1) = 2$ and $f(0.01, 1 + 0.01f(0,1)) = f(0.01, 1.02) = 2.0604$, we get

$$y(0.01) = 1 + 0.005(2 + 2.0604) = \boxed{1.020302}.$$

c) The local truncation error in each approximation are $\left|1.232 - e^{0.1^2+2(0.1)}\right|$ and $\left|1.020302 - e^{0.01^2+2(0.01)}\right|$. Thus, the order of the method is

$$\log_{10}\left(\frac{\left|1.232 - e^{0.1^2+2(0.1)}\right|}{\left|1.020302 - e^{0.01^2+2(0.01)}\right|}\right) \approx \boxed{3}.$$

6. **Taylor-2 method**

Note that $f_t(t,y) = 2y$ and $f_y(t,y) = 2t+2$.

a) The Taylor-2 method approximates $y(0.1)$ as

$$y(0.1) = y(0) + 0.1\left(f(0,1) + \frac{0.1}{2}\left(f_t(0,1) + f(0,1)f_y(0,1)\right)\right)$$

Plugging in $f(0,1) = 2(0+1)1 = 2$, $f_t(0,1) = 2$, and $f_y(0,1) = 2$, we get

$$y(0.1) = 1 + 0.1\left(2 + \frac{0.1}{2}(2 + 2 \cdot 2)\right) = \boxed{1.23}$$

b) The Taylor-2 method approximates $y(0.01)$ as

$$y(0.01) = y(0) + 0.01\left(f(0,1) + \frac{0.01}{2}\left(f_t(0,1) + f(0,1)f_y(0,1)\right)\right)$$

Plugging in $f(0,1) = 2(0+1)1 = 2$, $f_t(0,1) = 2$, and $f_y(0,1) = 2$, we get

$$y(0.01) = 1 + 0.01\left(2 + \frac{0.01}{2}(2 + 2 \cdot 2)\right) = \boxed{1.0203}$$

c) The local truncation error in each approximation are $\left|1.23 - e^{0.1^2+2(0.1)}\right|$ and $\left|1.0203 - e^{0.01^2+2(0.01)}\right|$. Thus, the order of the method is

$$\log_{10}\left(\frac{\left|1.23 - e^{0.1^2+2(0.1)}\right|}{\left|1.0203 - e^{0.01^2+2(0.01)}\right|}\right) \approx \boxed{3}.$$

7. **Show the local truncation error of the Backward Euler method is h^2.**

Solution. Assuming that y'' is continuous, the exact solution at $t_{i+1} = t_i + h$, by Taylor's Theorem, is

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3).$$

The Backward Euler method estimates

$$y_{i+1} = y_i + h(f(t_{i+1}, y_{i+1})).$$

Using a multivariate Taylor expansion about t_i, y_i, we can write $f(t_{i+1}, y_{i+1})$ as

$$\begin{aligned} f(t_{i+1}, y_{i+1}) &= f(t_i + h, y_i) \\ &= f(t_i, y_i) + h(f_t(t_i, y_i)) + (y_{i+1} - y_i)f_y(t_i, y_i). \end{aligned}$$

From our previous Taylor expansion, we know that $y_{i+1} - y_i = hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)$ and so substituting this in gives us

$$\begin{aligned} f(t_{i+1}, y_{i+1}) &= f(t_i, y_i) + h(f_t(t_i, y_i)) + (y_{i+1} - y_i)f_y(t_i, y_i) \\ &= f(t_i, y_i) + hf_t(t_i, y_i) + \left(hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)\right) f_y(t_i, y_i) \\ &= y'(t_i) + hf_t(t_i, y_i) + \left(hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)\right) f_y(t_i, y_i). \end{aligned}$$

Thus, the local truncation error of the Backward Euler method is

$$\ell_{i+1} = y(t_{i+1}) - [y(t_i) + hf(t_{i+1}, y_{i+1})]$$

$$= \left(y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)\right) - \left[y(t_i) + h\left(y'(t_i) + hf_t(t_i, y_i) + O(h)f_y(t_i, y_i)\right)\right]$$

$$= O(h^2)$$

and so our local truncation error has order h^2. ■

8. **Show that the 4th order RK method (Eq** 6.50, **page** 330) **has** h^5 **error when applied to the ODE** $y' = ay$.

Solution. The 4th order RK method is of the form

$$w_{i+1} = w_i + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

where

$$\begin{aligned}
s_1 &= f(t_i, w_i) = aw_i \\
s_2 &= f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}s_1\right) = (aw_i + O(h))O(h) \\
s_3 &= f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}s_2\right) = (aw_i + O(h^2))O(h) \\
s_4 &= f\left(t_i + h, w_i + hs_3\right) = (aw_i + O(h^3))O(h).
\end{aligned}$$

Plugging these back into the 4th order RK form, we find that

$$w_{i+1} = w_i + \frac{h}{6}(aw_i + O(h^4)).$$

Thus, we find that $w_{i+1} - y(i) = O(h^5)$, and so the 4th order RK method has h^5 error when applied to $y' = ay$. ■