David Yang

due 12/6

Math 66 - Fall 2023 - Goldwyn

HW 9 - due Wednesday 12/6

Problems labelled with * will checked for correctness, all others checked for effort / completeness.

Computational exercises:

Instructions: You do NOT need to work with a partner. You do NOT need to submit any code through the google form. You MUST report your computer answers in your written HW. If possible, provide some brief explanations/descriptions of what you did in your computations.

✓**Computational exercise #1:** Refer to the Stochastic Simulation deepnote notebook we worked on in-class before the break. Perform simulations of the Bernoulli random walk to estimate expected value and variance of a random walk process after 100 time steps.

> Typo correction: In the deepnote notebook I wrote that the expected value of the random walk process is $E[W_n] = np$. Please correct this to be $E[W_n] = n(p - q)$.

✓**Computational exercise #2*:** Refer to the Stochastic Simulation deepnote notebook we worked on in-class before the break. Write simulation code for the Gambler's Ruin problem and use this code to estimate the probability of "ruin" for a game with win probability $p = 0.4$ and maximum possible winnings of $N = \$7$.

Book exercises:

Problem and chapter numbers refer to *Introduction to Stochastic Processes with R* by Dobrow.
A pdf of Chapter 2 of Dobrow, including exercise pages, is available on moodle.
You may use a calculator, python, or similar, as needed to carry out arithmetic and matrix-vector or matrix-matrix calculations

Practice basic probability calculations: **2.1**, **2.2**, **2.4***, **2.5**

Practice formulating Markov chains: **2.12**, **2.14***, **2.15**

Computational Exercises

1. Simulations of Bernoulli random walk to estimate e.v. and variance after 100 time steps

- I ran 100 random walk simulations, storing the end value in an array (for $p = 0.7$)
- I then found the e.v and variance of the array

$E[W_{100}] \approx 40.78$ Expected $E[W_{100}] = 100(0.7 - 0.3) = 40$

$Var[W_{100}] \approx 80.23$ $Var[W_{100}] = 4(100)(0.7)(0.3) = 84$

My results are close to the actual e.v. and variance.

2. Simulation of Gambler's Ruin problem with $p = 0.4$, $N = \$7$.

- I ran 100,000 simulations of the Gambler's Ruin problem, using different starting values from \$1 to \$6.

I counted the # wins for each, dividing by total # of runs (100,000).

For $p = 0.4$, $N = \$7$ (max winnings), I found

$$P(\text{win} \mid \text{start} = i) = \begin{cases} 0.03191 & i = 1 \\ 0.07785 & i = 2 \\ 0.14742 & i = 3 \\ 0.25217 & i = 4 \\ 0.41096 & i = 5 \\ 0.64304 & i = 6 \end{cases}$$

Doobrow Chapter 2 Exercises

2.1:

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.3 & 0.6 \\ 2 & 0 & 0.4 & 0.6 \\ 3 & 0.3 & 0.2 & 0.5 \end{array} \qquad \alpha = (0.2, 0.3, 0.5)$$

(a) $P(X_7 = 3 \mid X_6 = 2)$ is a 1-step probability, so

$= P_{23} = \boxed{0.6}$

(b) $P(X_9 = 2 \mid X_1 = 2, X_5 = 1, X_7 = 3)$

$= P(X_9 = 2 \mid X_7 = 3)$ as only the most recent state "matters)

$= (P^{9-7})_{(3,2)} = P^2_{(3,2)}$

Since $P^2 = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}\begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.19 & 0.27 & 0.54 \\ 2 & 0.18 & 0.28 & 0.54 \\ 3 & 0.18 & 0.27 & 0.55 \end{array}$

$P^2_{(3,2)} = \boxed{0.27}$

(c) $P(X_0 = 3 \mid X_1 = 1) = \dfrac{P(X_1 = 1, X_0 = 3)}{P(X_1 = 1)}$

$= \dfrac{P(X_0 = 3)\, P_{31}}{(0.2)(0.1) + (0.3)(0) + (0.5)(0.3)}$

$= \dfrac{(0.5)(0.3)}{(0.2)(0.1) + (0.5)(0.3)} = \boxed{\dfrac{15}{17}}$

(d) $E(X_2)$

Note that $P^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.19 & 0.27 & 0.54 \\ 2 & 0.18 & 0.28 & 0.54 \\ 3 & 0.18 & 0.27 & 0.55 \end{array}$.

Thus, $E(X_2) = \sum_{j=1}^{3} j \sum_{i=1}^{3} P^2_{(i,j)}\, P(X_0 = i)$

$= 1\left(P^2_{1,1} P(X_0=1) + P^2_{2,1} P(X_0=2) + P^2_{3,1} P(X_0=3)\right) + 2\left(P^2_{1,2} P(X_0=1) + P^2_{2,2} P(X_0=2) + P^2_{3,2} P(X_0=3)\right)$

$3\left(P^2_{1,3} P(X_0=1) + P^2_{2,3} P(X_0=2) + P^2_{3,3} P(X_0=3)\right)$

$= 1\left(0.19(0.2) + 0.18(0.3) + 0.18(0.5)\right) + 2\left(0.27(0.2) + 0.28(0.3) + 0.27(0.5)\right) + 3\left(0.54(0.2) + 0.54(0.3) + 0.55(0.5)\right)$

$= \boxed{2.363}$

2.2

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}, \quad \alpha = (1/2,\ 0,\ 1/2)$$

(a) $P(X_2 = 1 \mid X_1 = 3)$

$= P_{31}$ by one-step probability rule

$= \boxed{1/3}$

(b) $P(X_1 = 3, X_2 = 1)$

$= P(X_2 = 1, X_1 = 3)$. Since $P(X_2 = 1 \mid X_1 = 3) = \dfrac{P(X_2=1, X_1=3)}{P(X_1=3)}$, we have $P(X_2=1, X_1=3) = P(X_2=1 \mid X_1=3)\, P(X_1=3)$.

$= P(X_2 = 1 \mid X_1 = 3)\, P(X_1 = 3)$.

Note that $P(X_1 = 3) = \sum_{i=1}^{3} P_{i,3}\, P(X_0 = i)$

$= P_{1,3} P(X_0 = 1) + P_{2,3} P(X_0 = 2) + P_{3,3} P(X_0 = 3)$

$= 1/2\,(1/2) + 0\,(0) + \frac{1}{3}\left(\frac{1}{2}\right)$

$= \boxed{\dfrac{5}{12}}$

(c) $P(X_1 = 3 \mid X_2 = 1) = \dfrac{P(X_1 = 3, X_2 = 1)}{P(X_2 = 1)}$

We can determine $P(X_2 = 1)$ using P^2.

$$P^2 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 0 & 1/2 & 1/2 \\ 4/9 & 5/18 & 5/18 \end{pmatrix} \end{array}$$

$P(X_2 = 1) = \sum_{i=1}^{3} P^2_{i,2}\, P(X_0 = i)$

$= P^2_{1,2} P(X_0 = 1) + P^2_{2,2} P(X_0 = 2) + P^2_{3,2} P(X_0 = 3)$

$= 1/6\,(1/2) + 1/2\,(0) + \frac{5}{18}\left(\frac{1}{2}\right)$

$= \boxed{2/9}$

(d) $P(X_9 = 1 \mid X_1 = 3, X_4 = 1, X_7 = 2)$

$= P(X_9 = 1 \mid X_7 = 2)$ as only final step "matters" by Markov Property

$= P^{(9-7)}_{(2,1)} = P^2_{(2,1)} = \boxed{0}$

2.4

$$P = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array} \qquad \alpha = (\alpha_1, \alpha_2)$$

(a) 2-step transition matrix is simply P^2

$$P^2 = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} (1-p)^2 + pq & (1-p)p + p(1-q) \\ q(1-p) + (1-q)q & pq + (1-q)^2 \end{pmatrix} \end{array}$$

(b) distribution of X_1

$$P(X_1 = a) = P(X_1 = a)\, P_{aa} + P(X_1 = b)\, P_{ba}$$
$$= \alpha_1 (1-p) + \alpha_2 q.$$

$$P(X_1 = b) = P(X_1 = b)\, P_{ab} + P(X_1 = b)\, P_{bb}$$
$$= \alpha_1 p + \alpha_2 (1-q).$$

and we notice $P(X_1 = a) + P(X_1 = b)$
$= \alpha_1 + \alpha_2 = 1$, as desired.

2.5 random walk on $\{0, \dots, 3\}$ with $q = \frac{1}{4}$, $p = \frac{3}{4}$ (q = left, p = right)

(a) The transition matrix is

$$\boxed{P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}}$$

for this random walk with reflecting boundaries.

(b) $P(X_7 = 1 \mid X_0 = 3, X_2 = 2, X_4 = 2)$

$= P(X_7 = 1 \mid X_4 = 2)$ by the Markov Property

$= (P^{7-4})_{(2,1)} = P^3_{(2,1)}$.

Since $P^3 = \begin{bmatrix} 0 & \frac{7}{16} & 0 & \frac{9}{16} \\ \frac{7}{64} & 0 & \frac{57}{64} & 0 \\ 0 & \frac{19}{64} & 0 & \frac{45}{64} \\ \frac{1}{16} & 0 & \frac{15}{16} & 0 \end{bmatrix}$, $P^3_{(2,1)} = \boxed{0}$.

(c) $P(X_3=1,\ X_5=3)$

$= P(X_5=3,\ X_3=1)$.

Since $P(X_5=3 \mid X_3=1) = \dfrac{P(X_5=3,\ X_3=1)}{P(X_3=1)}$, we know $P(X_5=3,\ X_3=1) = \underbrace{P(X_5=3 \mid X_3=1)}_{P^{(5-3)}_{1,3}} P(X_3=1)$.

Since $P^2 = \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{7}{16} & 0 & \frac{9}{16} \\ \frac{1}{16} & 0 & \frac{15}{16} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix}$, $\quad P(X_5=3 \mid X_3=1) = P^2_{(1,3)} = \dfrac{3}{4}$

Furthermore, using P^3, we know

$$P(X_3=1) = \sum_{i=0}^{3} P^3_{i,1}\ P(X_0=i)$$

$$= P^3_{0,3}\ P(X_0=0) + P^3_{1,3}\ P(X_0=1) + P^3_{2,3}\ P(X_0=2) + P^3_{3,3}\ P(X_0=3).$$

Since initial distribution is uniform, $P(X_0=0) = P(X_0=1) = P(X_0=2) = P(X_0=3) = \frac{1}{4}$, so

$$P(X_3=1) = \frac{9}{16}\left(\frac{1}{4}\right) + 0\left(\frac{1}{4}\right) + \frac{45}{64}\left(\frac{1}{4}\right) + 0\left(\frac{1}{4}\right)$$

$$= \frac{81}{256}$$

Thus, $P(X_5=3,\ X_3=1) = P(X_5=3 \mid X_3=1)\ P(X_3=1)$.

$$= P^2_{(1,3)}\ P(X_3=1)$$

$$= \frac{3}{4} \cdot \frac{81}{256} = \boxed{\frac{243}{1024}}$$

2.12

2.12 Two urns contain k balls each. Initially, the balls in the left urn are all red and the balls in the right urn are all blue. At each step, pick a ball at random from each urn and exchange them. Let X_n be the number of blue balls in the left urn. (Note that necessarily $X_0 = 0$ and $X_1 = 1$.) Argue that the process is a Markov chain. Find the transition matrix. This model is called the Bernoulli–Laplace model of diffusion and was introduced by Daniel Bernoulli in 1769 as a model for the flow of two incompressible liquids between two containers.

This process is a Markov Chain because the number of blue balls in the left urn in the next state (X_{n+1}) depends only on the number of blue balls currently in the left urn (X_n) and does not depend on the value at any previous time step, $(X_{n-1}, \ldots, X_1, X_0)$ i.e. the deep history of the number of blue balls in the left urn is not relevant.
Thus, the sequence of X_i satisfy the Markov property so the process is a Markov chain.

If the left urn has i blue balls, it will have $k-i$ red balls (and the right urn has $k-i$ blue and i red balls).
After a step, there can either be

- 1 less blue ball in the left urn (blue from left exchanged with red from right
  with probability $\left(\frac{i}{k}\right)\left(\frac{i}{k}\right) = \frac{i^2}{k^2}$
- same # blue balls in left urn (blue from left exchanged with blue from right OR red $\longleftrightarrow$ red)
  with probability $2\left(\frac{i}{k}\right)\left(\frac{k-i}{k}\right) = \frac{2i(k-i)}{k^2}$
- 1 more blue ball in left urn (red from left exchanged with blue from right)
  with probability $\left(\frac{k-i}{k}\right)\left(\frac{k-i}{k}\right) = \frac{(k-i)^2}{k^2}$

(there are special cases for when $X_i = 0$ or $X_i = k$).

Thus, the transition matrix is as follows:

$$\begin{array}{cc}
 & \begin{matrix} 0 & \cdots & \cdots & i & & k \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ k \end{matrix} &
\begin{bmatrix}
0 & 1 & \cdots & \cdots & 0 & 0 \\
\frac{1}{k^2} & \frac{2(k-1)}{k^2} & \frac{(k-1)^2}{k^2} & \cdots & 0 & 0 \\
 & \ddots & & & & 0 \\
 & & \ddots & & & \vdots \\
0 & \frac{i^2}{k^2} & \frac{2i(k-i)}{k^2} & \frac{(k-i)^2}{k^2} & & \vdots \\
 & & & & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 & 0
\end{bmatrix}
\end{array}$$

2.14

2.14 There are k songs on Mary's music player. The player is set to *shuffle* mode, which plays songs uniformly at random, sampling with replacement. Thus, repeats are possible Let X_n denote the number of *unique* songs that have been heard after the n th play.

(a)Show that $X_0, X_1, \ldots$ is a Markov chain and give the transition matrix.

(b)If Mary has four songs on her music player, find the probability that all songs are heard after six plays.

(a) This process is a Markov Chain because the number of unique songs heard after the $(n+1)$-th play (X_{n+1}) depends <u>only</u> on the number of unit songs heard after n plays (X_n) <u>and</u> does not depend on the number at any previous time step $(X_{n-1}, \ldots, X_1, X_0)$ i.e. the deep history of the number of unique songs hear is not relevant.
Thus, the sequence of X_i satisfy the Markov property so it is a Markov chain.

The sequence $X_0, X_1, \ldots$ (number of unique songs after some time steps) is a non-decreasing sequence.

- in the transition matrix P, $P_{i,i+1} = \dfrac{(k-i)}{k}$ for $0 \le i \le k-1$ (there are k-1 unheard songs, and each has prob. $\frac{1}{k}$ to be picked as we sample uniformly with replacement)
  $P_{i,i} = \dfrac{i}{k}$ (a song we've already heard is picked)
- $P_{k,k} = 1$, $P_{k,j} = 0$ for all $j \ne k$

The transition matrix P is thus

$$\boxed{\begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & k-1 & k \\ \hline 0 & 0 & 1 & 0 & \cdots & & 0 \\ 1 & 0 & \frac{1}{k} & \frac{k-1}{k} & 0 \cdots & 0 & \vdots \\ 2 & & & \frac{2}{k} & \frac{k-2}{k} & \cdots & \vdots \\ 3 & & & & \frac{3}{k} \quad \frac{k-3}{k} & \cdots & \vdots \\ \vdots & & 0 & & \ddots & \cdots & 0 \\ k-1 & & & & & \frac{k-1}{k} & \frac{1}{k} \\ k & 0 & 0 & 0 & \cdots & & 1 \end{array}}$$

(b) The transition matrix for $k=4$ is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{4} & \frac{2}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^6 = \begin{bmatrix} 0 & 1/1024 & \frac{93}{1024} & \frac{135}{256} & \frac{195}{512} \\ 0 & \frac{1}{4096} & \frac{189}{4096} & \frac{903}{2048} & \frac{525}{1024} \\ 0 & 0 & \frac{1}{64} & \frac{665}{2048} & \frac{1351}{2048} \\ 0 & 0 & 0 & \frac{729}{4096} & \frac{3367}{4096} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The probability all four songs are heard after six plays is $P(X_6 = 4) = P(X_6 = 4 \mid X_0 = 0) = P^{(6-0)}_{(0,4)} = P^6_{(0,4)}$.

$$= \boxed{\frac{195}{512}}$$

2.15

2.15 Assume that $X_0, X_1, \dots$ is a two-state Markov chain on $S = \{0, 1\}$ with transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}.$$

The present state of the chain only depends on the previous state. One can model a bivariate process that looks back two time periods by the following construction. Let $Z_n = (X_{n-1}, X_n)$, for $n \geq 1$. The sequence $Z_1, Z_2, \dots$ is a Markov chain with state space $S \times S = \{(0,0), (0,1), (1,0), (1,1)\}$. Give the transition matrix of the new chain.

The transition matrix of the chain $Z_1, \dots$ contains the probabilities from states (i,j) to (k,l) for $i,j,k,l \in [0,1]$.

- Note that $P(Z_{i+1} = (k,l) \mid Z_i = (i,j))$
  - $= 0$ if $k \neq j$
  - $= P_{(k,l)}$ if $k = j$.

Thus, the transition matrix Q for the chain $Z_1, \dots$ is

$$Q = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array}\begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \end{bmatrix} \end{array}$$