MATH66: Stochastic and Numerical Methods **Fall 2023**

Homework 1

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Problems from Numerical Analysis (Sauer), Chapter 1.

Example 1.15, page 61

Apply Newton's Method to $f(x) = 4x^4 - 6x^2 - 11/4$ **with starting guess** $x_0 = \frac{1}{2}$.

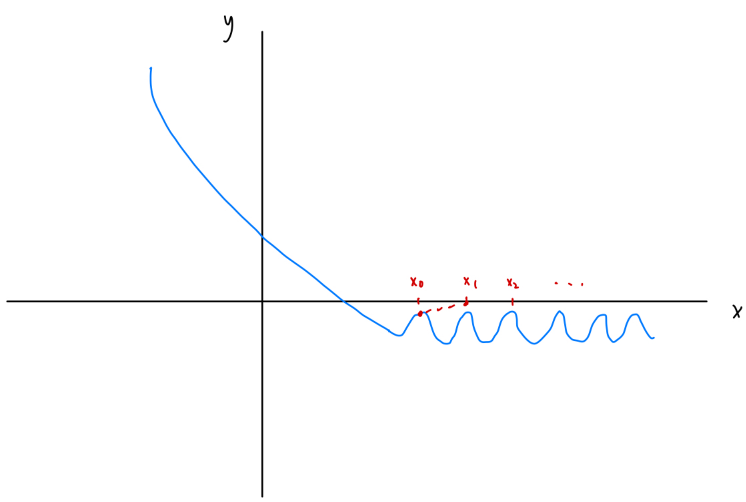
Solution. The sequence of numbers produced by Newton's method for starting guess $x_0 = \frac{1}{2}$ alternates between $\frac{1}{2}$ and $-\frac{1}{2}$:

$$x_0 = \frac{1}{2},\ x_1 = -\frac{1}{2},\ x_2 = \frac{1}{2}, \ldots$$

Since neither $\pm\frac{1}{2}$ is a root of $f(x)$, the sequence produced by Newton's method fails to converge to a root. ■

Section 1.4 (Newton's Method), Problem 6

Sketch a function f and initial guess for which Newton's Method diverges.



Solution. Newton's Method would diverge with the marked initial guess; the further iterations continue along the positive x axis and move away from the root. ■

Section 1.4 (Newton's Method), Problem 8

Prove that Newton's Method applied to $f(x) = ax + b$ converges in one step.

Solution. Let x_0 be the initial guess for Newton's method, where $x_0 \neq -\frac{b}{a}$ (assumed to not be the root). We know that $f'(x_0) = a$.

Newton's method after one iteration says that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{ax_0 + b}{a} = -\frac{b}{a}.$$

Note that $f(x_1) = 0$; thus, Newton's method applied to $f(x) = ax + b$ converges in one step. ■

Section 1.5 (Root-Finding without Derivatives), Problem 1(c)

Apply two steps of the Secant Method to $e^x + \sin x = 4$ with initial guesses $x_0 = 1, x_1 = 2$.

Solution. First, we rearrange the equation to reform it as a root-finding problem: $f(x) = e^x + \sin x - 4$.

After one step of the Secant Method, we get

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 2 - \frac{f(2)(2-1)}{f(2) - f(1)}.$$

Plugging in $f(2) = e^2 + \sin(2) - 4 \approx 4.298$ and $f(1) = e^1 + \sin(1) - 4 \approx -0.44$, we get that

$$x_2 \approx 2 - \frac{4.298(2-1)}{4.298 - (-0.44)} \approx 1.093.$$

The second step of the Secant Method gives the following:

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} \approx 1.093 - \frac{f(1.093)(1.093 - 2)}{f(1.093) - f(2)}.$$

Plugging in the approximations $f(1.093) \approx e^{1.093} + \sin(1.093) - 4 \approx -0.129$ and $f(2) \approx 4.298$, we get that

$$x_3 \approx 1.093 - \frac{-0.129(0.129 - 2)}{-0.129 - 4.298} \approx \boxed{1.1193}.$$

As we can see, after two steps, the result from Newton's method approaches the true root, $r \approx 1.13$. ■