David Yang

due 12/13

Math 66 - Fall 2023 - Goldwyn

HW 10 - due Wednesday 12/13

Please do the following and expect this material to be covered on the final exam:
Skills to practice: Calculate the fundamental matrix and interpret/apply this matrix in applications (expected time until absorption, absorption probabilities, etc.)

1. * Use the Fundamental Matrix to analyze the Gambler's ruin problem that you simulated last week. In particular, for a Gambler's Ruin problem with $N = 7$ and $p = 0.4$, calculate the "probability of ruin" from each initial state. This is, report $P(\text{ruin}|X_0 = i)$ for $i = 1, \ldots, N-1$. Compare to your numerical approximations from last week's assignment and confirm your work is/was correct. Report your results (including some explanation of the calculations you did). You do **not** need to submit code.
2. Exercise: 3.13
3. Exercise: 3.28*
4. Exercise: 3.52
5. Exercise: 3.54
6. Exercise: 3.59*
7. Exercise: 3.63. Use and adapt the code provided in the Markov Chain deepnote notebook to complete this problem (link available on "Week 14, Thursday" on moodle page. Report your results (including some explanation of the calculations you did). You do **not** need to submit code.

> Note about linear algebra calculations: You may use technology if you choose, to help with matrix calculations. But **be advised**: You will need to do things like matrix-matrix multiply, solve systems of equations, find matrix inverse, etc. on the final exam.
>
> Note about the weighted graph in 3.59: Interpret these numbers as transition probabilities according to the description on page 50. From the graph for this exercise, the transition probabilities are $P(X_1 = a|X_0 = a) = 1/6, P(X_1 = b|X_0 = a) = 1/6, P(X_1 = d|X_0 = a) = 4/6, P(X_1 = b|X_0 = a) = 1/6, P(X_1 = b|X_0 = b) = 3/6$, and so on. In general, the transition probabilities are equal to the weight on each edge divided by the sum of all weights coming out of a given vertex.

Optional exercises to for your interest and education. The material in the following problems will **not** be covered on the final exam:

Exercises 3.4, 3.10, 3.25

Gambler's Ruin Problem

The transition matrix for the Gambler's Ruin Problem with $N=7,\ p=0.4$ is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

States 0 and 7 are absorbing. We can set up our transition matrix as

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 0 \\ 7 \end{array}\begin{array}{c} \begin{array}{cccccc|cc} Q\ 1 & 2 & 3 & 4 & 5 & 6 & 0 & 7\ R \end{array} \\ \left[\begin{array}{cccccc|cc} 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

. We can calculate $F = (I-Q)^{-1}$ and then

$$FR = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 7 \end{array} \\ \begin{bmatrix} 0.969 & 0.031 \\ 0.923 & 0.077 \\ 0.852 & 0.148 \\ 0.748 & 0.252 \\ 0.590 & 0.410 \\ 0.354 & 0.646 \end{bmatrix} \end{array}$$

HW 9 calculations

$$P(\text{win} \mid \text{start} = i) = \begin{cases} 0.03191 & i=1 \\ 0.07785 & i=2 \\ 0.14742 & i=3 \\ 0.25217 & i=4 \\ 0.41096 & i=5 \\ 0.64304 & i=6 \end{cases}$$

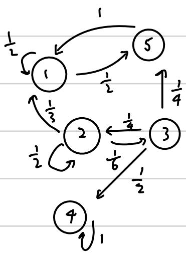
The probability of ruin is the first column of FR.

$$P(\text{ruin} \mid \text{start} = \text{row } i) = \text{entry } i \text{ of } \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \text{ruin} \\ \begin{bmatrix} 0.969 \\ 0.923 \\ 0.852 \\ 0.748 \\ 0.590 \\ 0.354 \end{bmatrix} \end{array}$$

These match the probability of ruin calculated in HW9 $(1 - P(\text{win} \mid \text{start} = i))$.

3.13: Communication classes of a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1/6 & 0 & 0 \\ 0 & 1/4 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



The communication classes are

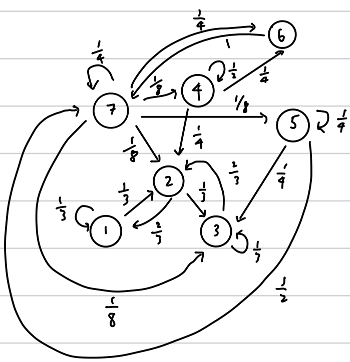
$$\boxed{\underbrace{\{1, 5\}}_{\text{recurring}}, \quad \underbrace{\{2, 3\}}_{\text{transient}}, \quad \underbrace{\{4\}}_{\text{recurring}}}$$

canonical form:

$$\boxed{\begin{array}{c} \\ 2 \\ 3 \\ 1 \\ 5 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 3 & 1 & 5 & 4 \end{array} \\ \left[\begin{array}{cc|ccc} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ \hline 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}}$$

3.28 Markov chain with

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 2/3 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$



The communication classes are $\underbrace{\{1, 2, 3\}}_{\text{recurrent}}$, $\underbrace{\{4, 5, 6, 7\}}_{\text{transient}}$

$$\boxed{\lim_{n\to\infty} P_{ij}^{\,n} = \begin{cases} \frac{1}{3} & \text{if } j = 1, 2, \text{ or } 3 \\ 0 & \text{otherwise} \quad (\text{transient}) \end{cases}}$$

(recurrent, submatrix of P for states 1,2,3 has rows & columns summing to 1 → limiting distribution is uniform)

3.52

(a) We can set up the transition matrix for snakes & ladders:

$$P = \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 & 0 & \frac14 & 0 & 0 \\ 1 & 0 & 0 & 0 & \frac12 & \frac14 & 0 & 0 & \frac14 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 \\ 7 & 0 & 0 & 0 & 0 & \frac14 & 0 & 0 & \frac12 & 0 & \frac14 \\ 8 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(assuming if overflow is rolled, we stay on current square)

& we need exact roll to "win"

The only absorbing state is state 9.

So our matrix

$$Q = \begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 & 0 & \frac14 & 0 \\ 1 & 0 & 0 & 0 & \frac12 & \frac14 & 0 & 0 & \frac14 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 \\ 4 & 0 & 0 & 0 & \frac14 & \frac14 & 0 & \frac14 & \frac14 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 \\ 7 & 0 & 0 & 0 & 0 & \frac14 & 0 & 0 & \frac12 & 0 \\ 8 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$

, and the fundamental matrix is $F = (I - Q)^{-1}$.

The expected duration of the game is $(F\vec{1})_1 = \boxed{8.625}$

(b) Since state 3 can now be treated as an "absorbing state", we have the new transition matrix

$$P' = \begin{array}{c|cccccccc|cc} & 0 & 1 & 2 & 4 & 5 & 6 & 7 & 8 & 3 & 9 \\ \hline 0 & 0 & \frac14 & 0 & \frac14 & 0 & 0 & \frac14 & 0 & \frac14 & 0 \\ 1 & 0 & 0 & 0 & \frac14 & 0 & 0 & \frac14 & 0 & \frac12 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 & 0 & \frac14 \\ 7 & 0 & 0 & 0 & \frac14 & 0 & 0 & \frac12 & 0 & 0 & \frac14 \\ 8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 3 & 0 & 0 & 0 & \frac14 & 0 & \frac14 & \frac14 & 0 & \frac14 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Q' R'

with $F' = (I - Q')^{-1}$.

Note that the probability of hitting state 3 before state 9, if we start at state 6, is

$$(F'R')_{(\text{state } 6,\ \text{end state } 3)} = \begin{array}{c|cc} & 3 & 9 \\ \hline 0 & 0.609375 & 0.390625 \\ 1 & 0.6875 & 0.3125 \\ 2 & 0.25 & 0.75 \\ 4 & 0.5 & 0.5 \\ 5 & 1 & 0 \\ 6 & 0.25 & 0.75 \\ 7 & 0.25 & 0.75 \\ 8 & 0.5 & 0.5 \end{array} = \boxed{0.25}$$

3.54

The transition matrix for the mouse in the maze is

$$P = \begin{array}{cc} & \begin{array}{ccccccccc} A & B & C & D & E & F & G & H & I \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

where

$$Q = \begin{array}{cc} & \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \end{array}$$

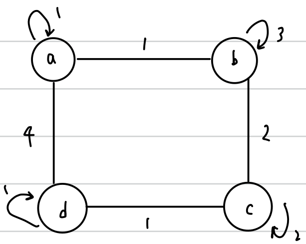
Thus, Expected # visits to other rooms before cheese

$$= \left((I - Q)^{-1} \vec{1} \right)_1 = \boxed{44.5}$$

(b) The expected number of revisits to room A is

$$\Big(\underbrace{(I - Q)^{-1}}_{F} \Big)_{(1,1)} = \boxed{7.5}\ .$$

3.59



has transition matrix

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1/6 & 1/6 & 0 & \frac{2}{3} \\ 1/6 & 1/2 & 1/3 & 0 \\ 0 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{3} & 0 & 1/6 & 1/6 \end{bmatrix} \end{array}$$

(a) Using the Markov Chain deepnote code, we can find the stationary distribution π which solves $(I - P^T)\pi = 0$

$\pi \approx (0.2608957,\ 0.26086957,\ 0.2173917,\ 0.2608957)$

The expected # of steps to return to state a is

$$\frac{1}{0.2608957} \approx \boxed{3.833}$$

(b) Treating b as an absorbing state, our transition matrix is

$$P = \begin{array}{c} \\ a \\ c \\ d \\ b \end{array} \begin{array}{c} \begin{array}{ccc|c} a & c & d & b \end{array} \\ \left[\begin{array}{ccc|c} 1/6 & 0 & \frac{2}{3} & 1/6 \\ 0 & \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{3} & 1/6 & 1/6 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Q

so our fundamental matrix $F = (I - Q)^{-1}$

$$\approx \begin{bmatrix} 3.818 & 0.909 & 3.272 \\ 1.0909 & 2.04545 & 1.3636 \\ 3.2727 & 1.1363 & 4.0909 \end{bmatrix}$$

The expected # of steps to first hit b given we start at a is simply $(F\vec{1})_1$, the sum of the entries in the first row, which is $3.818 + 0.909 + 3.272 = \boxed{8}$.

(c) We can now treat both b and c as absorbing states, so our transition matrix is

$$P = \begin{array}{c} \\ a \\ d \\ c \\ b \end{array} \begin{array}{c} \begin{array}{cc|cc} a & d & c & b \end{array} \\ \left[\begin{array}{cc|cc} 1/6 & \frac{2}{3} & 0 & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Q R

where $F = (I - Q)^{-1} = \begin{bmatrix} \frac{5}{6} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{5}{6} \end{bmatrix}^{-1} = \dfrac{1}{(\frac{5}{6})^2 - (\frac{-2}{3})^2} \begin{bmatrix} \frac{5}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{10}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{10}{3} \end{bmatrix}$

Thus, $FR = \begin{bmatrix} \frac{10}{3} & \frac{8}{3} \\ \frac{10}{3} & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix} = \begin{array}{c} \\ a \\ d \end{array} \begin{array}{c} \begin{array}{cc} c & b \end{array} \\ \begin{bmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{5}{9} & \frac{4}{9} \end{bmatrix} \end{array}$

Thus, the probability the walk hits b before c if we start at a is $\boxed{\frac{5}{9}}$.

3.63

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.756 & 0.113 & 0.129 & 0.002 & 0 & 0 & 0 \\ 0.174 & 0.821 & 0.004 & 0.001 & 0 & 0 & 0 \\ 0.141 & 0.001 & 0.776 & 0.082 & 0 & 0 & 0 \\ 0.003 & 0 & 0.192 & 0.753 & 0.052 & 0 & 0 \\ 0 & 0 & 0.002 & 0.227 & 0.735 & 0.036 & 0 \\ 0 & 0 & 0 & 0.007 & 0.367 & 0.604 & 0.022 \\ 0 & 0 & 0 & 0 & 0.053 & 0.158 & 0.789 \end{bmatrix} \end{array}$$

(a)

(i) To find the limiting distribution, we take P to a large power, and taking the first (or any) row. This gives

$$\lambda \approx [0.338 \quad 0.215 \quad 0.319 \quad 0.141 \quad 0.035 \quad 0.003 \quad 0.0003]$$

(ii) To find the stationary distribution, we solve $(I - P^T)\pi = 0$ and normalize, giving

$$\pi \approx [0.0326 \quad 0.207 \quad 0.305 \quad 0.132 \quad 0.03 \quad 0.002 \quad 0.003]$$

note: I got negative value for stationary distribution, state 7 (likely due to rounding errors in notebook) but approximated using limiting distribution

(b) long-run proportion of time in each state:

$$\boxed{[0.32 \quad 0.21 \quad 0.318 \quad 0.124 \quad 0.021 \quad 0.0006 \quad 0.00006]}$$

(using code to "normalize" (a)(ii))