David Yang

due 11/15

Math 66 - Fall 2023 - Goldwyn

HW 7 - due Wednesday 11/15

Problem numbers refer to the **Exercises** sections in the Sauer textbook, and NOT the Computer Problems sections.

Please do the following:

- **4.1** #8. Set up and solve using the **normal equations**. Plot your results (you may use desmos, or python, or similar)

  RMSE (root mean square error) is $\|\mathbf{r}\|/\sqrt{m}$ where $\mathbf{r}$ is the residual (backward error) of the least squares calculation and m is the size of $\mathbf{b}$

- **4.2** #2(b). Set up and solve using the **normal equations**. Plot your results (you may use desmos, or python, or similar).

  You may use python (or similar) to set up the normal equations, if you would like. You should find it straightforward to solve the normal equations (by hand).

- **4.3** #2(b)
- **4.3** #8(b). [* problem, graded for correctness]

  Show your work calculating Q and R, and use backward substitution to solve the relevant system of linear equations.

- **Optional**: **4.3** #11

Chapter 4: Least Squares Problems

Section 4.1 (Least Squares and the Normal Equations)

4.1.8: Find the best line through each set of points, find RMSE:

(a) $(0,0), (1,3), (2,3), (5,6)$

Setting up normal equations, we have

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}} \underset{x}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix}}$$

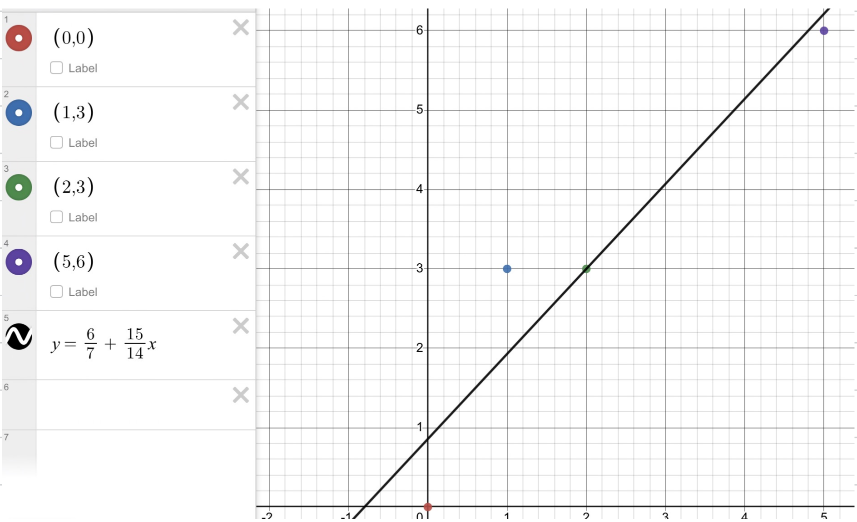
The normal equations $(A^T A \vec{x} = A^T \vec{b})$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 4 & 8 \\ 8 & 30 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 39 \end{bmatrix}$$

Solving, we get that $c_1 = \frac{6}{7}$, $c_2 = \frac{15}{14}$ so the best fit-line has equation $\boxed{y = \frac{6}{7} + \frac{15}{14}x}$

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}} \underset{\hat{x}}{\begin{bmatrix} \frac{6}{7} \\ \frac{15}{14} \end{bmatrix}} = \underset{\hat{b}}{\begin{bmatrix} 6/7 \\ 27/14 \\ 3 \\ 87/14 \end{bmatrix}}, \quad \text{with} \quad \underset{b}{\begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix}} - \underset{\hat{b}}{\begin{bmatrix} 6/7 \\ 27/14 \\ 3 \\ 87/14 \end{bmatrix}} = \underset{\text{resid. error} = r}{\begin{bmatrix} -6/7 \\ 15/14 \\ 0 \\ -3/14 \end{bmatrix}}$$

so the RMSE is $\frac{\|r\|}{\sqrt{4}} \approx \boxed{0.694}$



(b) $(1,2),\ (3,2),\ (4,1),\ (6,3)$

Setting up normal equations, we have

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}}_{b}$$

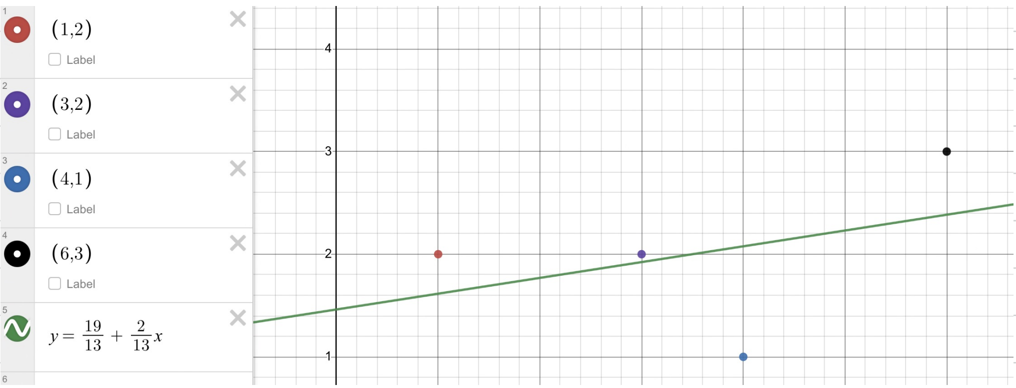
The normal equations $(A^T A \vec{x} = A^T \vec{b})$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 4 & 8 \\ 8 & 30 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 30 \end{bmatrix}$$

Solving, we get that $c_1 = \frac{19}{13}$ $c_2 = \frac{2}{13}$ so the best fit-line has equation $\boxed{y = \frac{19}{13} + \frac{2}{13}x}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{19}{13} \\ \frac{2}{13} \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 21/13 \\ 25/13 \\ 27/13 \\ 31/13 \end{bmatrix}}_{\hat{b}}, \quad \text{with} \quad \underbrace{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}}_{b} - \underbrace{\begin{bmatrix} 21/13 \\ 25/13 \\ 27/13 \\ 31/13 \end{bmatrix}}_{\hat{b}} = \underbrace{\begin{bmatrix} 5/13 \\ 1/13 \\ -14/13 \\ 8/13 \end{bmatrix}}_{\text{resid. error} = r}$$

so the RMSE is $\frac{\|r\|}{\sqrt{4}} \approx \boxed{0.65}$



(c) (0,5), (1,3), (2,3), (3,1)

Setting up normal equations, we have

$$\underbrace{\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}c_1\\c_2\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}5\\3\\3\\1\end{bmatrix}}_{b}$$

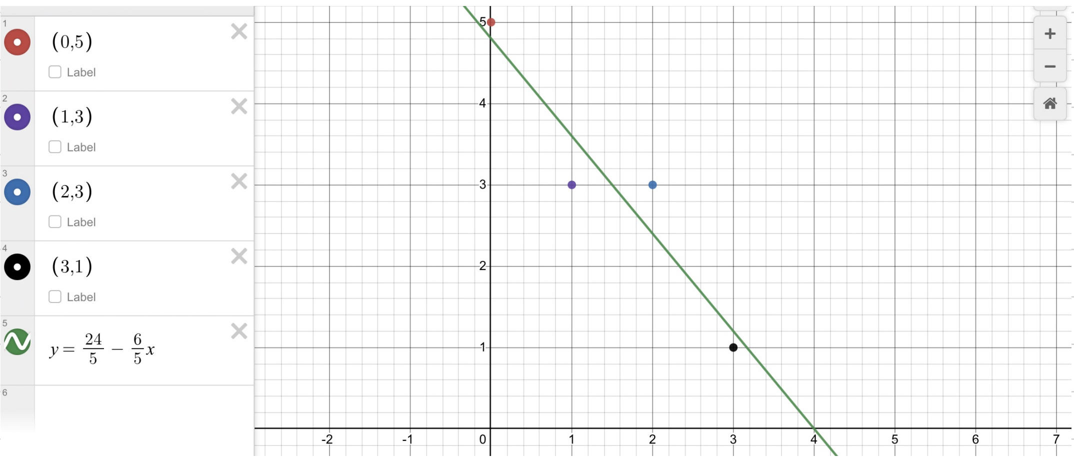
The normal equations ($A^T A \vec{x} = A^T \vec{b}$)

$$\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}5\\3\\3\\1\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}4&6\\6&14\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}12\\12\end{bmatrix}$$

Solving, we get that $c_1 = 24/5$ $c_2 = -6/5$ so the best fit-line has equation $\boxed{y = \frac{24}{5} - \frac{6}{5}x}$

$$\underbrace{\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}24/5\\-6/5\end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix}24/5\\18/5\\12/5\\6/5\end{bmatrix}}_{\hat{b}}, \quad \text{with} \quad \underbrace{\begin{bmatrix}5\\3\\3\\1\end{bmatrix}}_{b} - \underbrace{\begin{bmatrix}24/5\\18/5\\12/5\\6/5\end{bmatrix}}_{\hat{b}} = \underbrace{\begin{bmatrix}1/5\\-3/5\\3/5\\-1/5\end{bmatrix}}_{\text{resid. error} = r}$$

so the RMSE is $\dfrac{\|r\|}{\sqrt{4}} \approx \boxed{0.447}$



Section 4.2: A Survey of Models

t	y
0	4
1/6	2
1/3	0
1/2	-5
2/3	-1
5/6	3

4.2.2(b) : Fit the data to the models $F_3(t) = c_1 + c_2\cos(2\pi t) + c_3\sin(2\pi t)$

$F_4(t) = c_1 + c_2\cos(2\pi t) + c_3\sin(2\pi t) + c_4\cos(4\pi t)$

Find the 2-norm errors and compare the fits of F_3 and F_4.

For F_3, we set up the inconsistent equation $Ax = b$, with

$$A = \begin{bmatrix} 1 & \cos(0) & \sin(0) \\ 1 & \cos(2\pi/6) & \sin(\frac{2\pi}{6}) \\ 1 & \cos(2\pi/3) & \sin(2\pi/3) \\ 1 & \cos(2\pi/2) & \sin(2\pi/2) \\ 1 & \cos(4\pi/3) & \sin(4\pi/3) \\ 1 & \cos(10\pi/6) & \sin(10\pi/6) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1/2 & \sqrt{3}/2 \\ 1 & -1/2 & \sqrt{3}/2 \\ 1 & -1 & 0 \\ 1 & -1/2 & -\sqrt{3}/2 \\ 1 & 1/2 & -\sqrt{3}/2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -5 \\ -1 \\ 3 \end{bmatrix}$$

The normal equation $A^TA\vec{x} = A^T\vec{b}$ gives

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}.$$

Solving gives $c_1 = \frac{3}{6} = \frac{1}{2}$, $c_2 = \frac{12}{3} = 4$, and $c_3 = 0$, so the curve

$$\boxed{F_3(t) = \tfrac{1}{2} + 4\cos(2\pi t)}$$

To calculate the residual error and RMSE, we have

$$\underset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1/2 & \sqrt{3}/2 \\ 1 & -1/2 & \sqrt{3}/2 \\ 1 & -1 & 0 \\ 1 & -1/2 & -\sqrt{3}/2 \\ 1 & 1/2 & -\sqrt{3}/2 \end{bmatrix}} \underset{\hat{x}}{\begin{bmatrix} 1/2 \\ 4 \\ 0 \end{bmatrix}} = \underset{\hat{b}}{\begin{bmatrix} 9/2 \\ 5/2 \\ -3/2 \\ -7/2 \\ -3/2 \\ 5/2 \end{bmatrix}} \quad \text{with} \quad \underset{b}{\begin{bmatrix} 4 \\ 2 \\ 0 \\ -5 \\ -1 \\ 3 \end{bmatrix}} - \underset{\hat{b}}{\begin{bmatrix} 9/2 \\ 5/2 \\ -3/2 \\ -7/2 \\ -3/2 \\ 5/2 \end{bmatrix}} = \underset{\text{resid error} = r}{\begin{bmatrix} -1/2 \\ -1/2 \\ 3/2 \\ -3/2 \\ -1/2 \\ 1/2 \end{bmatrix}}$$

So the RMSE error is $\frac{\|r\|}{\sqrt{6}} \approx \boxed{0.957}$ for F_3

For F_4, we set up the inconsistent equation $Ax = b$, with

$$A = \begin{bmatrix} 1 & \cos(0) & \sin(0) & \cos(0) \\ 1 & \cos(2\pi/6) & \sin(\frac{2\pi}{6}) & \cos(4\pi/6) \\ 1 & \cos(2\pi/3) & \sin(2\pi/3) & \cos(4\pi/3) \\ 1 & \cos(2\pi/2) & \sin(2\pi/2) & \cos(4\pi/2) \\ 1 & \cos(4\pi/3) & \sin(4\pi/3) & \cos(8\pi/3) \\ 1 & \cos(10\pi/6) & \sin(10\pi/6) & \cos(20\pi/6) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1/2 & \sqrt{3}/2 & -1/2 \\ 1 & -1/2 & \sqrt{3}/2 & -1/2 \\ 1 & -1 & 0 & 1 \\ 1 & -1/2 & -\sqrt{3}/2 & -1/2 \\ 1 & 1/2 & -\sqrt{3}/2 & -1/2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -5 \\ -1 \\ 3 \end{bmatrix}$$

The normal equation $A^TA\vec{x} = A^T\vec{b}$ gives

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \\ -3 \end{bmatrix}.$$

Solving gives $c_1 = \frac{3}{6} = \frac{1}{2}$, $c_2 = \frac{12}{3} = 4$, $c_3 = 0$, and $c_4 = \frac{3}{3} = 1$, so

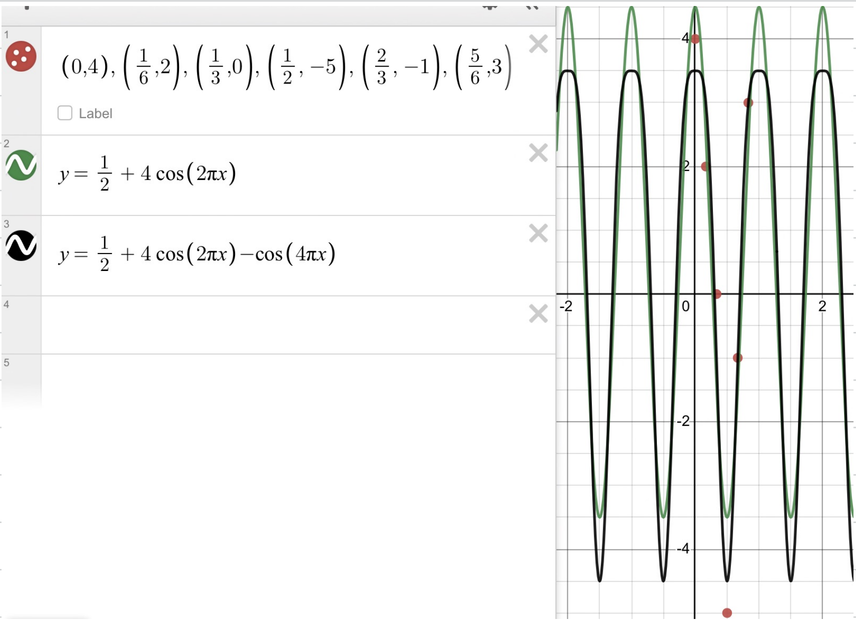
$$\boxed{F_4(t) = \tfrac{1}{2} + 4\cos(2\pi t) - \cos(4\pi t)}$$

To calculate the residual error and RMSE, we have

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1/2 & \sqrt{3}/2 & -1/2 \\ 1 & -1/2 & \sqrt{3}/2 & -1/2 \\ 1 & -1 & 0 & 1 \\ 1 & -1/2 & -\sqrt{3}/2 & -1/2 \\ 1 & 1/2 & -\sqrt{3}/2 & -1/2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1/2 \\ 4 \\ 0 \\ -1 \end{bmatrix}}_{\hat{x}} = \underbrace{\begin{bmatrix} 7/2 \\ 3 \\ -1 \\ -9/2 \\ -1 \\ 3 \end{bmatrix}}_{\hat{b}} \quad \text{with} \quad \underbrace{\begin{bmatrix} 4 \\ 2 \\ 0 \\ -5 \\ -1 \\ 3 \end{bmatrix}}_{b} - \underbrace{\begin{bmatrix} 7/2 \\ 3 \\ -1 \\ -9/2 \\ -1 \\ 3 \end{bmatrix}}_{\hat{b}} = \underbrace{\begin{bmatrix} 1/2 \\ -1 \\ 1 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}}_{\substack{\text{resid error} \\ = r}}$$

So the RMSE error is $\dfrac{\|r\|}{\sqrt{6}} \approx \boxed{0.645}$ for F_4.

It makes sense that the RMSE error for F_4 is less than that for F_3 as we have an extra term to "reduce error", giving a "tighter fit" curve which will yield a smaller RMSE.



It was fun to tinker around in Mathematica for this assignment.

Section 4.3: QR Factorization

2. Apply Gram-Schmidt Factorization to find full QR factorization of

(b) $\begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}$

Set $y_1 = A_1 = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$. Then the first unit vector $q_1 = \dfrac{y_1}{\|y_1\|} = \dfrac{1}{6}\begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$.

To find the second unit vector, set

$$y_2 = A_2 - q_1 q_1^T A_2$$

$= \begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ -3 \end{bmatrix}$ So the second unit vector is $q_2 = \dfrac{y_2}{\|y_2\|} = \dfrac{1}{9}\begin{bmatrix} -6 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$.

We have $r_{11} = \|y_1\| = 6$, $r_{12} = q_1^T A_2 = -3$, and $r_{22} = \|y_2\| = 9$, so $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$, $R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$ becomes

$$\boxed{\underbrace{\begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -2/3 & -2/3 \\ -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 6 & -3 \\ 0 & 9 \end{bmatrix}}_{R}}.$$

8. (b) Find the QR factorization and use it to solve the least squares problem

$$\underbrace{\begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}}_{b}$$

Set $y_1 = A_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$. Then the first unit vector $q_1 = \dfrac{y_1}{\|y_1\|} = \dfrac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 1/3 \end{bmatrix}$.

To find the second unit vector, set

$$y_2 = A_2 - q_1 q_1^T A_2$$

$$= \begin{bmatrix} 4 \\ -1 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 2 \end{bmatrix}$$

So the second unit vector is $q_2 = \dfrac{y_2}{\|y_2\|} = \dfrac{1}{3\sqrt{2}}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/3 \\ -\sqrt{2}/6 \\ -\sqrt{2}/2 \\ \sqrt{2}/3 \end{bmatrix}$

We have $r_{11} = \|y_1\| = 3$, $r_{12} = q_1^T A_2 = 3$, and $r_{22} = \|y_2\| = 3\sqrt{2}$, so $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$, $R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$ becomes

$$\underbrace{\begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2/3 & \sqrt{2}/3 \\ 0 & -\sqrt{2}/6 \\ 2/3 & -\sqrt{2}/2 \\ 1/3 & \sqrt{2}/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 3 \\ 0 & 3\sqrt{2} \end{bmatrix}}_{R}$$

To solve $A\vec{x} = \vec{b}$, we will first solve $Rx = Q^T b$. This becomes

$$\begin{bmatrix} 3 & 3 \\ 0 & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & 2/3 & 1/3 \\ \sqrt{2}/3 & -\sqrt{2}/6 & -\sqrt{2}/2 & \sqrt{2}/3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -\dfrac{3\sqrt{2}}{2} \end{bmatrix}.$$

Using backward substitution, we get $x_2 = \dfrac{\left(-\frac{3\sqrt{2}}{2}\right)}{3\sqrt{2}} = -\dfrac{1}{2}$ and $x_1 = \dfrac{1 - 3x_2}{3} = \dfrac{5}{6}$

Thus, the solution to the Least-Squares problem is $\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/6 \\ -1/2 \end{bmatrix}}$