

## Homework 4

David Yang

## Homework 4 Problems

## 1. (The Lipschitz Condition): Consider the ODE

$$y' = \frac{3}{2}y^{1/3} \text{ with } y(0) = 0.$$

- a) **Apply the Forward Euler method to this problem and report what you find. Does the value of  $h$  (the step size) matter? Are the values you computed a reasonable approximate solution for this IVP?**

*Solution.* Applying the Forward Euler method to the ODE with  $y(0) = 0$  gives

$$y_1 = y_0 + h \cdot f(t_0, y_0) = 0 + h \cdot 0 = 0$$

where we see that the value of  $y$  in the next iteration is independent of the step size  $h$ . Consequently, since  $y' = \frac{3}{2}y^{1/3}$  depends only on the  $y$ -variable,  $f(t, y) = f(t, 0) = 0$  for any value of  $t$ .

Thus, the approximate solution we get to the IVP is  $y(t) = 0$ , which is a valid solution for the given ODE. ■

- b) **Find a solution to this IVP using the separation of variables technique.**

*Solution.* We can similarly solve the IVP using separation of variables. Dividing both sides by  $y^{1/3}$ , we get that

$$\frac{1}{y^{1/3}} y' = \frac{3}{2}.$$

Integrating both sides, we get that

$$\begin{aligned} \int \frac{1}{y^{1/3}} dy &= \int \frac{3}{2} dt \\ \frac{3}{2} y^{2/3} &= \frac{3}{2} t \end{aligned}$$

Dividing both sides by  $\frac{3}{2}$  and raising both sides to the power of  $\frac{3}{2}$ , we get the solution

$$y = t^{\frac{3}{2}}.$$

Since we do see that this solution satisfies the initial value  $y(0) = 0$ , we find that the solution to the IVP is  $y' = t^{\frac{3}{2}}$ . ■

- c) **Evaluate the  $y$ -derivative of  $f(y) = \frac{3}{2}y^{1/3}$  and argue that this function is not Lipschitz continuous in any interval that contains  $y = 0$ .**

*Solution.* The  $y$ -derivative of  $f(y)$  is

$$f'(y) = \frac{3}{2} \cdot \frac{1}{3}y^{-2/3} = \frac{1}{2y^{2/3}}.$$

This is not Lipschitz continuous in any interval containing  $y = 0$  since as  $y_1, y_2$  get arbitrarily close to 0,  $|f(t, y_1) - f(t, y_2)|$  grows without bound compared to  $|y_1 - y_2|$ . ■

- d) **Apply a relevant theorem discussed in class to make sense of your answers in (a) and (b). Is the Forward Euler method invalid for this problem? Why or why not?**

*Solution.* We want to make use of our local existence and uniqueness theorem for first order IVPs. However, since  $f$  is not Lipschitz continuous in any interval containing  $y = 0$ , we cannot say that there is a unique solution to some interval about  $y = 0$ , explaining the two distinct solutions in (a) and (b).

The Forward Euler method is not invalid for this problem; it provides the trivial solution which shows that the ODE need not have a unique solution when Lipschitz continuity is not satisfied. ■

Consider the initial value ODE problem:

$$y' = 2(t + 1)y, \text{ with initial value } y(0) = 1.$$

Use the separation of variables technique to find a solution formula for this ODE. You will use this exact solution to measure errors for the following numerical calculations. For each of the following methods listed, do the following:

- a) Approximate the value of  $y(0.1)$  by calculating one-step of each numerical method using  $h = 0.1$ .
- b) Approximate the value of  $y(0.01)$  by calculating one-step of each numerical method using  $h = 0.01$ .
- c) Calculate the one-step error (local truncation error) in each approximation and report the order of the method.

We will first solve for the exact solution using separation of variables. Dividing both sides by  $y$ , we have that

$$\frac{1}{y}y' = 2(t + 1).$$

Integrating both sides, we find that

$$\ln(y) = t^2 + 2t + C.$$

Since  $y(0) = 1$ , we know that  $\ln(1) = 0^2 + 2(0) + C$ , so  $C = 0$ . Thus, our solution is

$$y = e^{t^2+2t}$$

and we will use this exact solution to measure the errors for each of the following methods:

## 2. Forward Euler method

- a) The Forward Euler method approximates  $y(0.1)$  as

$$y(0.1) = y(0) + 0.1(f(0, 1)) = 1 + 0.1(2(0 + 1)1) = \boxed{1.2}$$

- b) The Forward Euler method approximates  $y(0.01)$  as

$$y(0.01) = y(0) + 0.01(f(0, 1)) = 1 + 0.01(2(0 + 1)1) = \boxed{1.02}$$

- c) The local truncation error in each approximation are  $\left|1.2 - e^{0.1^2+2(0.1)}\right|$  and  $\left|1.02 - e^{0.01^2+2(0.01)}\right|$ . Thus, the order of the method is

$$\log_{10} \left( \frac{\left|1.2 - e^{0.1^2+2(0.1)}\right|}{\left|1.02 - e^{0.01^2+2(0.01)}\right|} \right) \approx \boxed{2}.$$

## 3. Backward Euler method

## 4. Trapezoid method

## 5. Explicit trapezoid method (Heun's Method)

a) Heun's method approximates  $y(0.1)$  as

$$y(0.1) = y(0) + 0.1 \cdot \frac{1}{2}(f(0, 1) + f(0.1, 1 + 0.1f(0, 1)))$$

Plugging in  $f(0, 1) = 2(0 + 1)1 = 2$ , we get

$$y(0.1) = 1 + 0.05(2 + 2(0.1 + 1)(1 + 0.1(2))) = \boxed{1.232}$$

b) Heun's method approximates  $y(0.01)$  as

$$y(0.01) = y(0) + 0.01 \cdot \frac{1}{2}(f(0, 1) + f(0.01, 1 + 0.01f(0, 1)))$$

Plugging in  $f(0, 1) = 2$  and  $f(0.01, 1 + 0.01f(0, 1)) = f(0.01, 1.02) = 2.0604$ , we get

$$y(0.01) = 1 + 0.005(2 + 2.0604) = \boxed{1.020302}.$$

c) The local truncation error in each approximation are  $\left|1.232 - e^{0.1^2+2(0.1)}\right|$  and  $\left|1.020302 - e^{0.01^2+2(0.01)}\right|$ .  
Thus, the order of the method is

$$\log_{10} \left( \frac{\left|1.232 - e^{0.1^2+2(0.1)}\right|}{\left|1.020302 - e^{0.01^2+2(0.01)}\right|} \right) \approx \boxed{3}.$$

## 6. Taylor-2 method

7. **Show the local truncation error of the Backward Euler method is  $h^2$ .**

*Solution.* Assuming that  $y''$  is continuous, the exact solution at  $t_{i+1} = t_i + h$ , by Taylor's Theorem, is

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3).$$

The Backward Euler method estimates

$$y_{i+1} = y_i + h(f(t_{i+1}, y_{i+1})).$$

Using a multivariate Taylor expansion about  $t_i, y_i$ , we can write  $f(t_{i+1}, y_{i+1})$  as

$$\begin{aligned} f(t_{i+1}, y_{i+1}) &= f(t_i + h, y_i) \\ &= f(t_i, y_i) + h(f_t(t_i, y_i)) + (y_{i+1} - y_i)f_y(t_i, y_i). \end{aligned}$$

From our previous Taylor expansion, we know that  $y_{i+1} - y_i = hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3)$  and so substituting this in gives us

$$\begin{aligned} f(t_{i+1}, y_{i+1}) &= f(t_i, y_i) + h(f_t(t_i, y_i)) + (y_{i+1} - y_i)f_y(t_i, y_i) \\ &= f(t_i, y_i) + hf_t(t_i, y_i) + \left( hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3) \right) f_y(t_i, y_i) \\ &= y'(t_i) + hf_t(t_i, y_i) + \left( hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3) \right) f_y(t_i, y_i). \end{aligned}$$

Thus, the local truncation error of the Backward Euler method is

$$\begin{aligned} \ell_{i+1} &= y(t_{i+1}) - [y(t_i) + hf(t_{i+1}, y_{i+1})] \\ &= \left( y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3) \right) - [y(t_i) + h(y'(t_i) + hf_t(t_i, y_i) + O(h)f_y(t_i, y_i))] \\ &= O(h^2) \end{aligned}$$

and so our local truncation error has order  $h^2$ . ■

8. Show that the 4th order RK method (Eq 6.50, page 330) has  $h^5$  error when applied to the ODE  $y' = ay$ .