David Yang

due 11/22

Math 66 - Fall 2023 - Goldwyn

HW 8 - due Wednesday 11/22

Please do the following:

1. Consider a Bernoulli process of length 5:
$$\{X_1, X_2, X_3, X_4, X_5\}$$
with $P(X_i = 1) = p$ for each i. What is the probability of observing a realization of this process in which there are exactly three "successes" (occurrences of $X_i = 1$) and these occur on consecutive time steps?

2. * Let T_1 be the time of first success for a Bernoulli process. Recall we derived the mass function for this random variable in class, it is a geometric random variable. Derive the fact that $E[T_1] = 1/p$

   Hints:

   - Begin from the definition $E[T_1] = \sum_{n=1}^{\infty} nP\{T_1 = n\}$
   - Plug in the formula for $P\{T_1 = n\}$ (an expression in terms of p and q, derived in class)
   - To complete the problem: relating the series $\sum_{n=1}^{\infty} nP\{T_1 = n\}$ to the derivative of a geometric series.
   - *Reminder:* For $|x| < 1$, the geometric series is absolutely convergent with $\sum_{n=0}^{\infty} x^n = 1/(1-x)$.

   Challenge (optional): If you can't get enough of working with geometric series, try a similar analysis to derive $Var(T_1) = q/p^2$

   For the following two problems:

   Use the properties of *independent* and *stationary* increments.

   Please write down an exact expression (do not simplify more than you want to, do not use a calculator to find numerical values).

3. Recall that the probability that there are k successes in n time steps of a Bernoulli process is the Binomial distribution
$$P\{N_n = k\} = \binom{n}{k} p^k q^{n-k}$$
Find the probability of the event $P\{N_3 = 1, N_6 = 4, N_{10} = 7\}$

4. * Recall that the probability that the k^{th} event of a Bernoulli process occurs at time n is
$$P\{T_k = n\} = \binom{n-1}{k-1} p^k q^{n-k}$$
Find the probability of the event $P\{T_1 = 3, T_5 = 9, T_7 = 17\}$

This problem is about the Poisson process, a continuous time stochastic process

5. *Poisson process.* The counting process N_n for the Bernoulli process counts the number of success in n steps of the process. The Poisson process can be thought of as a version of this in continuous time, call it N_t, which counts the number of "successes" in time t, where t takes is real-valued and non-negative. The defining properties of the Poisson process are:

   - A Poisson process is defined by the parameter λ that is the rate of successes (also called "arrrivals")
   - The increments of N_t are independent and stationary. As a consequence,
   $$P(N_{t+s} - N_s = k) = P(N_t = k) \quad \text{for all } t > 0$$
   - The number of arrivals in an interval of length t has a Poisson distribution:
   $$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

   (a) Find $P(T_1 \leq t)$, the cumulative distribution function for T_1.
   Tip: How can $P(T_1 > t)$ be expressed using the Poisson distribution for N_t, the number of events by time t?

   (b) Take a derivative with respect to t of the function in you found in (a) to find the probability density function for T_1.
   Comment: The distribution and density functions in (a) and (b) define the **exponential distribution**.

   (c) Find $E[T_1]$.
   Tip: Notice that T_1 takes only non-negative values. Applying the definition of expected value for continuous random variables, you must evaluate:
   $$E[T_1] = \int_0^\infty t f(t)\ dt$$
   where $f(t)$ is the probability density function you found in (b).

6. A random variable X is called **memoryless** with respect to t if it satisfies the following:
   $$P(X > s + t \mid X > s) = P(X > t) \quad \text{for all } s \text{ with } t \neq 0$$

   (a) Show that T_1 is memoryless for the case that T_1 represents the first success of a Bernoulli process

   (b) Show that T_1 is memoryless for the case that T_1 represents the first success of a Poisson process

   Remark: Recall that T_1 for the Bernoulli process is a geometric random variables (discrete), and T_1 for the Poisson process is an exponential random variable (continuous). It is a fact that geometric and exponential random variables are the only memoryless random variables.

1. Consider a Bernoulli process of length 5:

$$\{X_1, X_2, X_3, X_4, X_5\}$$

with $P(X_i = 1) = p$ for each i. What is the probability of observing a realization of this process in which there are exactly three "successes" (occurrences of $X_i = 1$) and these occur on consecutive time steps?

We can get exactly three "successes" which occur at consecutive time steps in 3 ways:

X_1, X_2, X_3 ; X_2, X_3, X_4 ; X_3, X_4, X_5

Since each X_i is independent, each of these realizations occur with probability

$(P(X_i=1))^3 (P(X_i=0))^2 = p^3(1-p)^2$ as we need 3 successes and 2 failures.

Thus, the total probability is $\boxed{3p^3(1-p)^2}$

2. * Let T_1 be the time of first success for a Bernoulli process. Recall we derived the mass function for this random variable in class, it is a geometric random variable. Derive the fact that $E[T_1] = 1/p$

Hints:

- Begin from the definition $E[T_1] = \sum_{n=1}^{\infty} nP\{T_1 = n\}$
- Plug in the formula for $P\{T_1 = n\}$ (an expression in terms of p and q, derived in class)
- To complete the problem: relating the series $\sum_{n=1}^{\infty} nP\{T_1 = n\}$ to the derivative of a geometric series.
- *Reminder:* For $|x| < 1$, the geometric series is absolutely convergent with $\sum_{n=0}^{\infty} x^n = 1/(1-x)$.

Challenge (optional): If you can't get enough of working with geometric series, try a similar analysis to derive $Var(T_1) = q/p^2$

T_1 = time of first success in Bernoulli process

$P\{T_1 = n\} = q^{n-1}p$ ("failures" in first n-1 occurrences, success in n^{th})

$$E(T_1) = \sum_{n=1}^{\infty} n P\{T_1 = n\}$$

$$= \sum_{n=1}^{\infty} n q^{n-1} p = p \sum_{n=1}^{\infty} n q^{n-1} = p \sum_{n=1}^{\infty} n(1-p)^{n-1} \quad \text{as } q = 1-p.$$

Note that $\sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{-d}{dp} \sum_{n=1}^{\infty} (1-p)^n$ (negative due to chain rule)

$$= \frac{-d}{dp}\left(\frac{1-p}{1-(1-p)}\right)$$

$$= \frac{-d}{dp}\left(\frac{1-p}{p}\right) = \frac{-d}{dp}\left(\frac{1}{p} - 1\right) = \frac{-d}{dp}\left(\frac{1}{p}\right) + \frac{d}{dp}(-1) = \frac{1}{p^2}$$

Thus, $p \sum_{n=1}^{\infty} n(1-p)^{n-1} = p\left(\frac{1}{p^2}\right) = \frac{1}{p}$, so $\boxed{E[T_1] = \frac{1}{p}}$ as desired.

For the following two problems:

Use the properties of *independent* and *stationary* increments.
Please write down an exact expression (do not simplify more than you want to, do not use a calculator to find numerical values).

3. Recall that the probability that there are k successes in n time steps of a Bernoulli process is the Binomial distribution
$$P\{N_n = k\} = \binom{n}{k} p^k q^{n-k}$$
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3. Using increments, we know

$$P\{N_3 = 1,\ N_6 = 4,\ N_{10} = 7\}$$
$$= P\{N_3 = 1,\ N_6 - N_3 = 3,\ N_{10} - N_5 = 3\}$$
independence property of increments
$$= P\{N_3 = 1\}\, P\{N_6 - N_3 = 3\}\, P\{N_{10} - N_5 = 3\}$$
Stationary property of increments
$$= P\{N_3 = 1\}\, P\{N_3 = 3\}\, P\{N_5 = 3\}$$

We can calculate these individually using the given formula.

$$P\{N_3 = 1\} = \binom{3}{1} p^2 q$$
$$P\{N_3 = 3\} = \binom{3}{3} p^3$$
$$P\{N_5 = 3\} = \binom{5}{3} p^3 q^2$$

$$\Rightarrow P\{N_3 = 1,\ N_6 = 4,\ N_{10} = 7\} = \binom{3}{1}\binom{3}{3}\binom{5}{3} p^{2+3+3} q^{1+0+2}$$
$$= \boxed{\binom{3}{1}\binom{3}{3}\binom{5}{3} p^8 q^3}$$

4. Using increments, we know

$$P\{T_1 = 3,\ T_5 = 9,\ T_7 = 17\}$$
$$= P\{T_1 = 3,\ T_5 - T_1 = 6,\ T_7 - T_5 = 12\}$$
independence property of increments
$$= P\{T_1 = 3\}\, P\{T_5 - T_1 = 6\}\, P\{T_7 - T_5 = 12\}$$
Stationary property of increments
$$= P\{T_1 = 3\}\, P\{T_4 = 6\}\, P\{T_2 = 12\}$$

We can calculate these individually using the given formula.

$$P\{T_1 = 3\} = \binom{2-1}{1-1} p^1 q^2 = \binom{1}{0} p^1 q^2$$
$$P\{T_4 = 6\} = \binom{6-1}{4-1} p^4 q^2 = \binom{5}{3} p^4 q^2$$
$$P\{T_2 = 12\} = \binom{12-1}{2-1} p^2 q^{10} = \binom{11}{1} p^2 q^{10}$$

$$\Rightarrow P\{T_1 = 3,\ T_5 = 9,\ T_7 = 17\} = \binom{1}{0}\binom{5}{3}\binom{11}{1} p^{1+4+2} q^{2+2+10}$$
$$= \boxed{\binom{1}{0}\binom{5}{3}\binom{11}{1} p^7 q^{14}}$$

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   (a) Find $P(T_1 \leq t)$, the cumulative distribution function for T_1.
   Tip: How can $P(T_1 > t)$ be expressed using the Poisson distribution for N_t, the number of events by time t?

   (b) Take a derivative with respect to t of the function in you found in (a) to find the probability density function for T_1.
   Comment: The distribution and density functions in (a) and (b) define the **exponential distribution**.

   (c) Find $E[T_1]$.
   Tip: Notice that T_1 takes only non-negative values. Applying the definition of expected value for continuous random variables, you must evaluate:
   $$E[T_1] = \int_0^\infty t f(t)\, dt$$
   where $f(t)$ is the probability density function you found in (b).

(a) T_1 is the time of the first sucess in Poisson process

$$P(T_1 \leq t) = 1 - P(T_1 > t)$$

$T_1 > t$ means there are no successes yet at time t, so $N_t = 0$.

$$= 1 - P(N_t = 0)$$

$$= 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t}$$

$$= 1 - e^{-\lambda t}$$

Thus, $\boxed{P(T_1 \leq t) = 1 - e^{-\lambda t}}$ for $t > 0$ is the CDF for T_1.

(b) $$P(T_1 = t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right)$$

$$= \lambda e^{-\lambda t}$$

so the PDF of T_1 (the exponential distribution) is

$\boxed{P(T_1 = t) = \lambda e^{-\lambda t} \quad \text{for } t > 0}$

(c)

$$E[T_1] = \int_0^\infty t\, f(t)\, dt$$
$$= \int_0^\infty t \lambda e^{-\lambda t}\, dt$$

Using integration by parts, let $u = t$, $dv = \lambda e^{-\lambda t}\, dt$ so
$du = dt$, $v = -e^{-\lambda t}$

$$\int_0^\infty \underset{u}{t}\, \underset{dv}{\lambda e^{-\lambda t}\, dt} = \Big[\underset{uv}{-t e^{-\lambda t}}\Big]_0^\infty - \int_0^\infty \underset{v}{-e^{-\lambda t}}\, \underset{du}{dt}$$

(Note: λ = rate of success > 0)

$$= 0 - \left[\frac{1}{\lambda} e^{-\lambda t}\right]_0^\infty$$
$$= 0 - \left(0 - \frac{1}{\lambda}\right) = \frac{1}{\lambda}.$$

Thus, $\boxed{E[T_1] = \frac{1}{\lambda}}$

6. A random variable X is called **memoryless** with respect to t if it satisfies the following:

$$P(X > s+t \mid X > s) = P(X > t) \quad \text{for all } s \text{ with } t \neq 0$$

(a) Show that T_1 is memoryless for the case that T_1 represents the first success of a Bernoulli process

(b) Show that T_1 is memoryless for the case that T_1 represents the first success of a Poisson process

Remark: Recall that T_1 for the Bernoulli process is a geometric random variables (discrete), and T_1 for the Poisson process is an exponential random variable (continuous). It is a fact that geometric and exponential random variables are the only memoryless random variables.

(a) Let T_1 be the first success of a Bernoulli process.

$P(T_1 > s+t \mid T_1 > s)$

$= \dfrac{P(T_1 > s,\ T_1 > s+t)}{P(T_1 > s)}$ by definition of conditional probability

$= \dfrac{P(T_1 > s+t)}{P(T_1 > s)}$ since if $T_1 > s+t$, $T_1 > s$.

$= \dfrac{P(N_{s+t} = 0)}{P(N_s = 0)} = \dfrac{\binom{s+t}{0} q^{s+t}}{\binom{s}{0} q^s} = q^t.$

But note that $q^t = P(N_t = 0) = P(T_1 > t)$ as well (using similar logic as above).

Thus,

$P(T_1 > s+t \mid T_1 > s) = P(T_1 > t)$, so T_1 (first success in Bernoulli process) is memoryless. ■

(b) Let T_1 be the first success of a Poisson process.

$P(T_1 > s+t \mid T_1 > s)$

$= \dfrac{P(T_1 > s,\ T_1 > s+t)}{P(T_1 > s)}$ by definition of conditional probability

$= \dfrac{P(T_1 > s+t)}{P(T_1 > s)}$ since if $T_1 > s+t$, $T_1 > s$.

$= \dfrac{1 - P(T_1 \leq s+t)}{1 - P(T_1 \leq s)}$. Using CDF derived in 5(a), we have $P(T_1 \leq s+t) = 1 - e^{-\lambda(s+t)}$, $P(T_1 \leq s) = 1 - e^{-\lambda s}$.

so $P(T_1 > s+t \mid T_1 > s) = \dfrac{1 - P(T_1 \leq s+t)}{1 - P(T_1 \leq s)}$

$= \dfrac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-\lambda s})}$ by plugging in above expressions for $P(T_1 \leq s+t)$, $P(T_1 \leq s)$

$= \dfrac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$

Note that $e^{-\lambda t} = 1 - (1 - e^{-\lambda t})$

$= 1 - P(T_1 \leq t)$

$= P(T_1 > t)$ by the same logic as above.

Thus,

$P(T_1 > s+t \mid T_1 > s) = P(T_1 > t)$, so T_1 (first success in Poisson process) is memoryless. ■