David Yang due 11/1/23

Math 66 - Fall 2023 - Goldwyn

HW 5 - due Wednesday 11/1

(1) Count the number of floating point operations (flops) in the following calculations. $\mathbf{v}$ and $\mathbf{w}$ are $n \times 1$ vectors, A is an $m \times n$ matrix, and B is a $n \times p$ matrix

(a) vector sum:

$$\mathbf{v} + \mathbf{w}$$

(b) inner product (dot product):

$$\langle \mathbf{v}, \mathbf{w} \rangle$$

alternate notations for the inner product of vectors are

$$\mathbf{v} \cdot \mathbf{w} \qquad \text{and} \qquad \mathbf{v}^T \mathbf{w}$$

(c) matrix-vector multiplication:

$$A\mathbf{x}$$

Tip: Recall that the i^{th} element of the vector $A\mathbf{x}$ is the inner product of the i^{th} row of A with the vector $\mathbf{x}$. (Apply your answer from b).

(d) matrix-matrix multiplication:

$$AB$$

Tip: Recall that the i^{th} column of AB is the product of the matrix A with the i^{th} column of B. (Apply your answer from c).

(2) For the matrix

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

(a) Find a lower triangular matrix L and an upper triangular matrix U so that $A = LU$

(b) Use L and U and the methods of forward and backward substitution to solve $A\mathbf{x} = \mathbf{b}$ for the vector

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

(3) (*) For the matrix

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$$

(a) Find a lower triangular matrix L and an upper triangular matrix U so that $A = LU$

(b) Use L and U and the methods of forward and backward substitution to solve $A\mathbf{x} = \mathbf{b}$ for the vector

$$\mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

(4) If A is invertible and there exist a lower triangular matrix L and an upper triangular matrix U so that $A = LU$, then L and U must also be invertible. Explain why (briefly).

Tip: Make use of properties of matrix determinants.

(5) In this problem and the next problem you will explore constructing interpolating polynomials as a solution to a linear algebra problem.

Consider the following as interpolation points:

$$(x_1, y_1) = (1, 1) \quad \text{and} \quad (x_2, y_2) = (2, 3) \quad \text{and} \quad (x_3, y_3) = (3, 6)$$

(a) Construct a 3×3 matrix X with rows of the form $[1 \ x_i \ x_i^2]$ for $i = 1, 2, 3$

(b) Solve the system $X\mathbf{c} = \mathbf{y}$ (by hand)

You may use this as extra practice for LU factorization, or if you would like you can use "standard" Gaussian elimination steps

(c) Use your answer to (b) to write a formula for the (unique) quadratic polynomial that passes through the three points given above.

(d) **[optional]** Sketch the polynomial or use plotting software (python, desmos, etc.) to visualize your answer.

(6) (*) The matrix X in #5 is an example of a **Vandermonde matrix**. Vandermonde matrices have the form

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

Other methods can be used to calculate interpolating polynomials. We discussed one such method earlier in the semester (the Lagrange formula for polynomial interpolation).

Another method polynomial form that can be used is **Newton's form** of a polynomial.

Newton's form for the interpolating polynomial through $(x_1, y_1), \ldots, (x_n, y_n)$ is:

$$y = a_0 + a_1(x - x_1) + a_2(x - x_1)(x - x_2) + \cdots + a_{n-1} \prod_{i=1}^{n-1}(x - x_i) + a_n \prod_{i=1}^{n}(x - x_i)$$

where the $\prod$ symbol means the product of many terms (interpret similarly to $\sum$ for summation).

Using the interpolation points in #5 and Newton's form with $n = 2$ (quadratic polynomial) do the following:

(a) For each interpolation point, plug in the x and y values into Newton's form of the polynomial. By doing this, create a system of three equations that are linear in the unknown coefficients a_0, a_1, and a_2.

(b) Express these equations in matrix-vector form. What special structure do you notice in the matrix?(!) and why is this structure desirable?

(c) Solve for the coefficients and write down the interpolating polynomial in Newton's form

(d) **[optional]** Algebraically or graphically: confirm this polynomial is identical to your answer to #5, as is required by the theorem on uniqueness of polynomial interpolation.

Remark: This form of polynomial interpolation can also be solved using **divided difference** formulas. Interested students can read about this in Chapter 3.1.2 (not required for the course).

1. Number of flops, when $\vec{v}, \vec{w}$ are $n \times 1$ vectors, A: $m \times n$ matrix, B: $n \times p$ matrix

(a) Vector sum $\vec{v} + \vec{w}$ has $\boxed{n \text{ flops}}$ since there is 1 flop for each sum in the n components of $\vec{v} + \vec{w}$.

(b) $\vec{v} \cdot \vec{w}$ has $2n-1$ flops.

- n flops, where there is 1 per product in the n components
- $n-1$ flops to sum the n products

Thus, there are $n + (n-1) = \boxed{2n-1 \text{ total flops}}$.

(c) $A\vec{x}$ (assuming $\vec{x}$ is $n \times 1$) requires $m(2n-1)$ flops

Since the i^{th} element of $A\vec{x}$ is the inner product of the i^{th} row of A with $\vec{x}$, and this inner product requires $2n-1$ flops by part (b), the total # of flops is $\boxed{(m)(2n-1)}$.

inner products (1 for each row)

flops per inner product

(d) AB requires $mp(2n-1)$ flops.

Since the i^{th} column of AB is the product of A $(m \times n)$ with the i^{th} column of B $(n \times 1)$, which requires $m(2n-1)$ flops by part (c), the total # of flops is

$$\underbrace{p}_{\text{\# columns in } AB} \cdot \underbrace{m(2n-1)}_{\text{flops per column of } AB} = \boxed{mp(2n-1)}.$$

2. (a)

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

Using Gaussian Elimination:

$(-2R_1 + R_2 \to R_2)$ implemented by $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ gives $L_1 A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$

$(-\frac{1}{2}R_1 + R_3 \to R_3)$ implemented by $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$ gives $L_2 L_1 A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{5}{2} \end{bmatrix}$

$(-R_2 + R_3 \to R_3)$ implemented by $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ gives $L_3 L_2 L_1 A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \equiv U.$

Since $L_3 L_2 L_1 A = U$, $L = L_1^{-1} L_2^{-1} L_3^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}$$

Thus, $A = LU$ for $\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}}$

(b) Using part (a), $A = LU$, so $A\vec{x} = \vec{b}$ is the same as $LU\vec{x} = \vec{b}$, for $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

We first solve $L\vec{y} = \vec{b}$ using forward substitution:

$$L\vec{y} = \vec{b} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = 2, \quad y_2 = 1 - 2y_1 = -3, \quad y_3 = 3 - \tfrac{1}{2}y_1 - y_2 = 5$$

Thus, $\vec{y} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$.

It remains to solve $U\vec{x} = \vec{y}$, for $\vec{x}$.

$$U\vec{x} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}.$$

Using backward substitution,

$$x_3 = \frac{5}{(\frac{5}{2})} = 2, \quad x_2 = -3, \quad x_1 = \frac{2 - 2x_2 - 3x_3}{2} = 1$$

$\Rightarrow \boxed{\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}}$ and we can check $\begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, as desired.

3. (a)

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$$

Using Gaussian Elimination:

$(R_1 + R_2 \to R_2)$ implemented by $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ gives $L_1 A = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{bmatrix}$

$(-2R_1 + R_3 \to R_3)$ implemented by $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ gives $L_2 L_1 A = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix}$

$(5R_2 + R_3 \to R_3)$ implemented by $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ gives $L_3 L_2 L_1 A = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \equiv U.$

Since $L_3 L_2 L_1 A = U$, $L = L_1^{-1} L_2^{-1} L_3^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

Thus, $A = LU$ for $\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}}$

(b) Using part (a), $A = LU$, so $A\vec{x} = \vec{b}$ is the same as $LU\vec{x} = \vec{b}$, for $\vec{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$.

We first solve $L\vec{y} = \vec{b}$ using forward substitution:

$$L\vec{y} = \vec{b} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

$\Rightarrow$ $y_1 = -7$, $y_2 = 5 + y_1 = -2$, $y_3 = 2 - 2y_1 + 5y_2 = 6$

Thus, $\vec{y} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$.

It remains to solve $U\vec{x} = \vec{y}$, for $\vec{x}$.

$$U\vec{x} = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}.$$

Using backward substitution,

$x_3 = -6$, $x_2 = \dfrac{-2 + x_3}{-2} = 4$, $x_1 = \dfrac{-7 + 7x_2 + 2x_3}{3} = 3$

$\Rightarrow$ $\boxed{\vec{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}}$ and we can check $\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$, as desired.

4. If A is invertible, then $\det(A) \neq 0$.

Since $\exists\, L$ lower triangular and $\exists\, U$ upper triangular such that $A = LU$, and since the determinant is multiplicative, we know

$A = LU$

$\Rightarrow \det(A) = \det(LU) = \det(L)\det(U)$. Since $\det(A) \neq 0$, $\det(L) \neq 0$ and $\det(U) \neq 0$ so L, U are by definition invertible. ■

5. (a)

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

(I did not use LU here to save some time)

(b)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 6 \end{array}\right]$$

$$\underset{(-R_1 + R_2 \to R_2)}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 1 & 3 & 9 & 6 \end{array}\right]} \longrightarrow \underset{(-R_1 + R_3 \to R_3)}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 8 & 5 \end{array}\right]} \longrightarrow \underset{(-2R_2 + R_3 \to R_3)}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 1 \end{array}\right]}$$

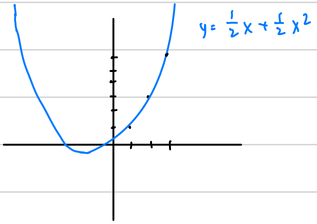
Backward substitution: $c_3 = \frac{1}{2}$, $c_2 = 2 - 3c_3 = \frac{1}{2}$, $c_1 = 1 - c_2 - c_3 = 0$

So $\vec{c} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

(c) Thus, the unique quadratic polynomial passing through the 3 points is

$$\boxed{\frac{1}{2}x^2 + \frac{1}{2}x = y}$$

(d)



6. Newton's form for interpolating polynomial:

$$y = a_0 + a_1(x-x_1) + \dots + a_{n-1}\prod_{i=1}^{n-1}(x-x_i) + a_n\prod_{i=1}^{n}(x-x_i).$$

(a) Newton's form for the interpolating polynomial through points $(1,1)$ and $(2,3)$ is

$$y = a_0 + a_1(x-1) + a_2(x-1)(x-2).$$

Plugging in the points $(1,1)$, $(2,3)$, $(3,6)$ into this equation, we get

$$\begin{cases} 1 = a_0 + a_1(1-1) + a_2(1-1)(1-2) \\ 3 = a_0 + a_1(2-1) + a_2(2-1)(2-2) \\ 6 = a_0 + a_1(3-1) + a_2(3-1)(3-2) \end{cases} \Rightarrow \begin{cases} 1 = a_0 \\ 3 = a_0 + a_1 \\ 6 = a_0 + 2a_1 + 2a_2 \end{cases}$$

(b) The above system can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

This matrix is "nice" since it is a lower triangular matrix, making it very easy to solve the system using forward substitution.

(c) Solve using Forward Substitution:

$$1a_0 = 1 \Rightarrow a_0 = 1$$

$$a_1 = 3 - 1a_0 = 2$$

$$a_2 = \tfrac{1}{2}(6 - a_0 - 2a_1) = \tfrac{1}{2}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{1}{2} \end{bmatrix}.$$

Plugging these constants in, we get that the interpolating polynomial in Newton's form is

$$\boxed{y = 1 + 2(x-1) + \tfrac{1}{2}(x-1)(x-2)}.$$

(d) Simplifying, we see

$$y = 1 + (2x-2) + \tfrac{1}{2}(x^2 - 3x + 2)$$

$$= 1 + (2x-2) + \tfrac{1}{2}x^2 - \tfrac{3}{2}x + 1$$

$$= \tfrac{1}{2}x^2 + \tfrac{1}{2}x,$$

which is the same polynomial as what we found previously (matching what we expect by the Thm of uniqueness on polynomial interpolation.